Отчёт по ЭВМ

Задача: $\boxed{u_t(t,x) = u_{xx}(t,x) + f(t,x)}$

Кр. условия: $u(t,0)=0$, $u(t,1)=0$, $u(0,x) = u^0(x)$

Сетка: 

$$\frac{1+\frac{h}{2}}{N} = h \iff h = \frac{1}{N-0.5}$$

Найдём решение в виде $u(t,x) = X(x)T(t)$

$u(t,0) = X(0)T(t) = 0 \Rightarrow X(0)=0$

$u(t,1) = X(1)T(t) = 0 \Rightarrow X(1) = 0$

$X''T = XT' \iff \frac{X''}{X} = \frac{T'}{T} = -\lambda$; $\frac{T'}{T} = \lambda \iff T = ce^{-\lambda t}$

$X'' = -\lambda X \iff X = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x)$

$X(0) = c_2 = 0$

$X(1) = c_1 \sin(\sqrt{\lambda}) = 0 \iff \sqrt{\lambda} = \pi k, k \in \mathbb{Z} \iff \lambda = \pi^2 k^2, k \in \mathbb{Z}$

$\Rightarrow \underline{u(t,x) = \bar{C} \sin(\pi k x) e^{-\pi^2 k^2 t}}$

$\Rightarrow u(t,x) = \sum\limits_{n=1}^{\infty} c_n \sin(\pi k x) e^{-\pi^2 k^2 t}$

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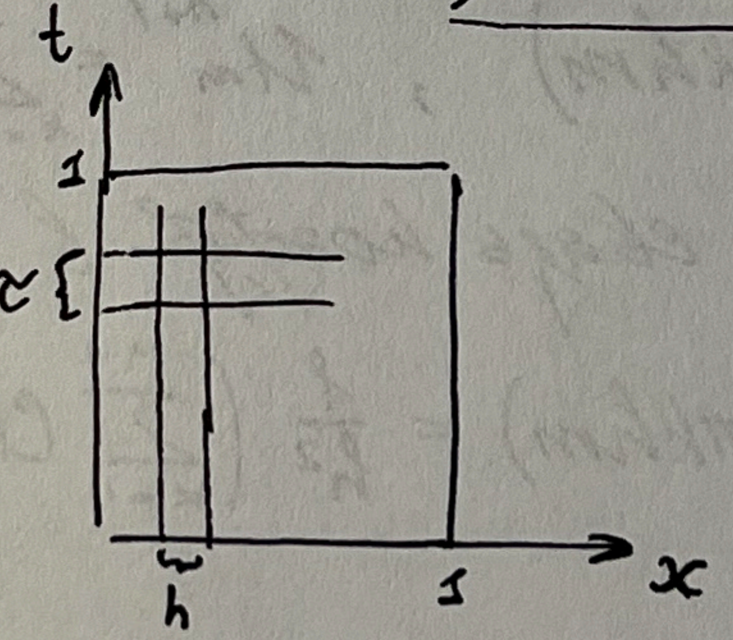
Дискретизация задачи



$$\frac{u_m^{n+1} - u_m^n}{\tau} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + f_m^n$$

$m = 1, .., M-1$

$n = 1, .., N-1$

Кр. условия: $\begin{cases} u_M^n = 0 \\ \frac{u_0^n + u_1^n}{2} = 0 \end{cases} \rightsquigarrow \delta^n$

$u_m^{n+1} = \frac{\tau}{h^2} u_{m+1}^n + \left(1 - \frac{2\tau}{h^2}\right) u_m^n + \frac{\tau}{h^2} u_{m-1}^n + \tau f_m^n$

Добьёмся порядка аппроксимации на решении для кр. усл-я: $\underline{O}(\tau^2) + \underline{O}(h^2)$

$$\left|\frac{u(t_n,x_1) + u(t_n,x_0)}{2} - \delta^n\right| = \left|\frac{1}{2}\left(u(t_n, 0+\tfrac{h}{2}) + u(t_n, 0-\tfrac{h}{2})\right) - \delta^n\right| = \left|\cancelto{0}{u(t_n,0)} + \frac{h^2}{4} u_{xx}(t_n,0) + \bar{\bar{o}}(h^3) - \delta^n\right|$$

$$\Rightarrow \delta^n = \frac{h^2}{4} u_{xx}(t_n,0) = \frac{h^2}{4}\left(u_t(t_n,0) - f(t_n,0)\right) = \frac{h^2}{4}\left(\frac{u(t_n+\tau) - u(t_n,0)}{\tau} + \bar{o}(\tau) - f(t_n,0)\right)$$

$$\frac{u_1^n + u_0^n}{2} = \frac{h^2}{2}\left(\frac{u_0^{n+1} - u_0^n}{\tau} - f(t_n,0)\right) = \delta^n \iff u_0^{n+1} = \tau\left(\frac{2}{h^2}(u_1^n + u_0^n) + f(t_n,0)\right) + u_0^n$$

$$\Rightarrow \left|\frac{u(t_n,x_1) + u(t_n,x_0)}{2} - \delta^n\right| \leq \underline{O}(h^2\tau) \leq \underline{O}(h^4) + \underline{O}(\tau^2) \leq \underline{O}(h^2) + \underline{O}(\tau^2)$$

• Аппроксимация на решении схемы:

$$u_m^{n+1} = u(t_{n+1}, x_m) = u(t_n, x_m) + \tau u_t(t_n, x_m) + \frac{\tau^2}{2} u_{tt}(t_n, x_m) + \frac{\tau^3}{6} u_{ttt}(t_n, x_m) + \frac{\tau^4}{24} u_{tttt}(t, x_m) + \bar{o}(\tau^5)$$

$$u_m^n = u(t_n, x_m)$$

$$u_{m+1}^n = u(t_n, x_{m+1}) = u(t_n, x_m) + h u_x(t_n, x_m) + \frac{h^2}{2} u_{xx}(t_n, x_m) + \frac{h^3}{6} u_{xxx}(t_n, x_m) + \frac{h^4}{24} u_{xxxx}(t_n, x_m) + \bar{o}(h^5)$$

$$u_{m-1}^n = u(t_n, x_{m-1}) = u(t_n, x_m) - h u_x(t_n, x_m) + \frac{h^2}{2} u_{xx}(t_n, x_m) - \frac{h^3}{6} u_{xxx}(t_n, x_m) + \frac{h^4}{24} u(t_n, x_m) + \bar{o}(h^5)$$

1) $$\frac{u_m^{n+1} - u_m^n}{\tau} = u_t(t_n, x_m) + \frac{\tau}{2} u_{tt}(t_n, x_m) + \frac{\tau^2}{6} u_{ttt}(t_n, x_m) + \frac{\tau^3}{24} u_{tttt}(t_n, x_m) + \bar{o}(t^4)$$

2) $$\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} = u_{xx}(t_n, x_m) + \frac{h^2}{12} u_{xxxx}(t_n, x_m) + \bar{o}(h^3)$$

Итого: $$\left\| \frac{u_m^{n+1} - u_m^n}{\tau} - \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} - f(t_n, x_m) \right\| =$$

$$= \| u_t(t_n, x_m) + \underline{O}(\tau) - u_{xx}(t_n, x_m) + \underline{O}(h^2) - f(t_n, x_m) \| \leq \underline{O}(\tau^2 + h^2)$$

Нормировка правых частей: $|f(t_n, x_m) - f_m^n| \to 0 \ (=0)$

$\Rightarrow$ порядок аппрокс-и на решении $p = 2$.

Устойчивость

Схема наз. устойчивой безусловно, если $\exists c_1, c_2$: при $\tau, h \to 0$, имеем

$$\| u_m^N \|_{L_2, h} \leq c_1 \| u_m^0 \|_{L_2, h} + c_2 \| f \|_{\tau, h}$$

Разложим u_m^n по синусам (это ортогональный базис)

$$u_m^n = \sum_{k=1}^{M-1} c_k^{(n)} \sin(\pi k h m), \quad u_m^{n+1} = \sum_{k=1}^{M-1} c_k^{(n+1)} \sin(\pi k h m)$$

$\sin(\pi k h m)\cos(\pi k h) + \cos(\pi k h m)\sin(\pi k h) - 2\sin(\pi k h m) + \sin(\pi k h m)\cos(\pi k h) - \cos(\pi k h m)\sin(\pi k h)$

$= 2\sin(\pi k h m)[\cos(\pi k h) - 1] = -4\sin(\pi k h m)\sin^2\left(\frac{\pi k h m}{2}\right)$

Посмотрим на связь коэфф-в $c_k^{(n)}$ и $c_k^{(n+1)}$:

$$\underbrace{\sum_{k=1}^{M-1} \left(\frac{c_k^{(n+1)} - c_k^{(n)}}{\tau} \right) \sin(\pi k h m)}_{\frac{u_m^{n+1} - u_m^n}{\tau}} = \underbrace{\frac{1}{h^2} \left(\sum_{k=1}^{M-1} c_k^{(n)} \left[\sin(\pi k h(m+1)) - 2\sin(\pi k h m) + \sin(\pi k h(m-1)) \right] \right)}_{\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}}$$

$$\frac{c_k^{(n+1)} - c_k^{(n)}}{\tau} = -\overbrace{\frac{4}{h^2} \sin^2\left(\frac{\pi k h m}{2}\right)}^{\lambda_k} c_k^{(n)} \Leftrightarrow c_k^{(n+1)} = c_k^{(n)}(1 - \tau\lambda) \Rightarrow c_k^{(N)} = (1 - \tau\lambda_k)^N c_k^{(0)}$$

$k = 1, \dots, M-1$

$$\| u_m^0 \| = \left(\sum_{k=1}^{M-1} (u_m^0, u_m^0) \right)^{1/2} = \left(\sum_{k=1}^{M-1} c_k^{(0)2} \cdot \frac{1}{2} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \left(\sum_{k=1}^{M-1} c_k^{(0)2} \right)^{\frac{1}{2}}; \quad \| u_m^N \| = \sqrt{\frac{1}{2}} \left(\sum_{k=1}^{M-1} c_k^{(0)2} \right)^{\frac{1}{2}} \Leftrightarrow \tau \leq \frac{h^2}{2}$$

$$\| u_m^N \| = \sqrt{\frac{1}{2}} \left(\sum_{k=1}^{M-1} c_k^{(N)2} \right)^{\frac{1}{2}} = \sqrt{\frac{1}{2}} \left(\sum_{k=1}^{M-1} (1 - \tau\lambda_k)^{2N} c_k^{(0)2} \right)^{\frac{1}{2}} \leq \| u_m^0 \| \Rightarrow \text{есть}$$

$1 - \tau\lambda_k \leq 1$

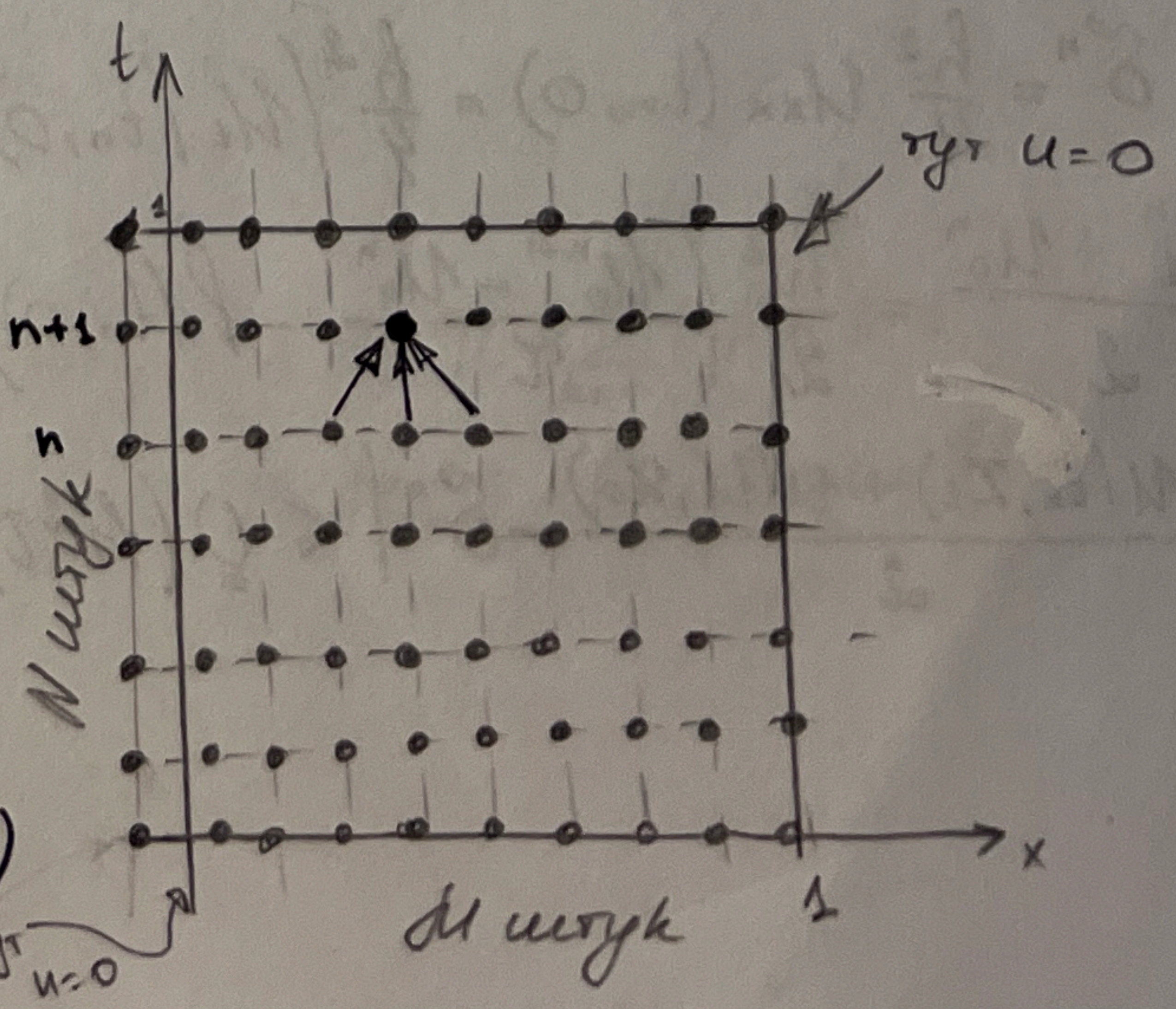
условная устойчивость.

Построение сети

$$\frac{u_m^{n+1} - u_m^n}{\tau} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + f_m^n$$

$$\frac{u_m^{n+1}}{\tau} = \frac{u_{m+1}^n}{h^2} - u_m^n \left(\frac{2}{h^2} - \frac{1}{\tau} \right) + \frac{u_{m-1}^n}{h^2} + f(t_n, x_m)$$

$$u_m^{n+1} = \frac{\tau}{h^2} u_{m+1}^n - u_m^n \left(\frac{2\tau}{h^2} - 1 \right) + u_{m-1}^n \cdot \frac{\tau}{h^2} + \tau f(t_n, x_m)$$



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$$\begin{cases} \dfrac{u_m^{n+1}-u_m^n}{\tau} = \dfrac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{h^2} + f_m^{n+1}, & m=1,\dots,M-1 \\ & n = 1,\dots,N-1 \\ u_M^n = 0 \\ \dfrac{u_0^n+u_1^n}{2} = 0 \\ u_m^0 = u_0(x_m), \ m=0,\dots,M \end{cases}$$

Аппроксимация на решении:

$$u(t_n, x_{m\pm1}) = u(t_n,x_m) \pm h u_x(t_n,x_m) + \frac{h^2}{2}u_{xx}(t_n,x_m) \pm \frac{h^3}{6}u_{xxx}(t_n,x_m) + \frac{h^4}{24}u_{xxxx}(t_n,x_m) + \bar{\bar{o}}(h^5)$$

$$u(t_{n+1}, x_m) = u(t_n,x_m) + \tau u_t(t_n,x_m) + \frac{\tau^2}{2}u_{tt}(t_n,x_m) + \frac{\tau^3}{6}u_{ttt}(t_n,x_m) + \frac{\tau^4}{24}u_{tttt}(t_n,x_m) + \bar{\bar{o}}(\tau^5)$$

$$\left|\frac{u_m^{n+1}-u_m^n}{\tau} - \frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{h^2} - f_m^{n+1}\right| = \Big|u_t(t_n,x_m) + \frac{\tau}{2}u_{tt}(t_n,x_m) + \frac{\tau^2}{6}u_{ttt}(t_n,x_m) +$$

$$+ \frac{\tau^3}{24}u_{tttt}(t_n,x_m) + \bar{\bar{o}}(\tau^4) - \left(u_{xx}(t_n,x_m) + \frac{h^2}{12}u_{xxxx}(t_n,x_m) + \bar{\bar{o}}(h^3)\right) - f_m^{n+1}\Big| = 0$$

порядок апп-и на решении $O(h^2+\tau)$

$x=0$: $\dfrac{u_0^n+u_1^n}{2} = \delta^n$ — уже вывели выше

нормировка правых частей: $|f(t_{n+1},x_m) - f_m^{n+1}| \to 0$

Устойчивость

$\{ \|u_m^n\|_{L_2,h} \le C_1\|u_m^0\|_{L_2,h} + C_2\|f\|_{\tau,h} \}$

$$u_m^n = \sum_{k=1}^{M-1} c_k^{(n)}\sin(\pi k h m), \quad u_m^{n+1} = \sum_{k=1}^{M-1} c_k^{(n+1)}\sin(\pi k h m)$$

$$\sum_{k=1}^{M-1}\left(\frac{c_k^{(n+1)}-c_k^{(n)}}{\tau}\right)\sin(\pi k h m) = \left(\sum_{k=1}^{M-1} c_k^{(n+1)}\lambda_k \cdot \sin(\pi k h m)\right)$$

$$\frac{c_k^{(n+1)}-c_k^{(n)}}{\tau} = -\lambda_k c_k^{(n+1)} \iff c_k^{(n+1)}\left(\frac{1}{\tau}+\lambda_k\right) = \frac{c_k^{(n)}}{\tau} \iff c_k^{(n+1)} = c_k^{(n)}\frac{1}{\tau\lambda_k+1}, \quad c_k^{(0)}\frac{1}{(1+\tau\lambda_k)^{n+1}}$$

$$u_m^{n+1} = u_m^n + \frac{\tau}{h^2}\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) + \tau f(t_{n+1},x_m)$$

минимальный из всех

$\exists\ m_0 = \arg\max_m |u_m^{n+1}| : |u_{m_0}^{n+1}| > |u_{m_0-1}^{n+1}|$ и $|u_{m_0}^{n+1}| \ge |u_{m_0+1}^{n+1}|$. Тогда

$|2u_{m_0}^{n+1}| > |u_{m_0-1}^{n+1}| + |u_{m_0+1}^{n+1}|$ и $\operatorname{sgn}(2u_{m_0}^{n+1}) = \operatorname{sgn}(2u_{m_0}^{n+1} - u_{m_0-1}^{n+1} - u_{m_0+1}^{n+1})$

$$\Rightarrow \|u_m^{n+1}\| = |u_{m_0}^{n+1}| \le \Big|u_{m_0}^{n+1} + \underset{\frac{\tau}{h^2}}{\hat{c}^0}\underbrace{\left(2u_{m_0}^{n+1} - u_{m_0-1}^{n+1} - u_{m_0+1}^{n+1}\right)}_{\frac{h^2}{\tau}(u_{m_0}^n - u_{m_0}^{n+1}) + \tau f_m^{n+1}\frac{h^2}{\tau}}\Big| = |u_{m_0}^n + \tau f_m^{n+1}|$$

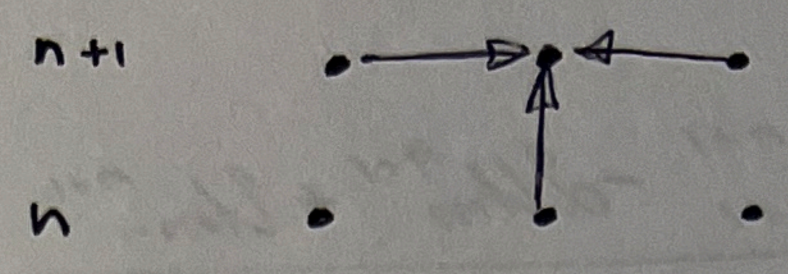
$\Rightarrow$ при $\forall \tau$ и при $\forall h$:

$$\|u_m^{n+1}\| \le \|u_m^n\| + \tau\|f_m^{n+1}\| \le \|u_m^{n-1}\| + \tau(\|f_m^n\| + \|f_m^{n-1}\|) \le \dots \le \|u_m^0\| + \sum_{k=0}^{n}\tau\|f_m^k\| \le$$

$$\le \|u_m^0\| + (n+1)\tau\max_k\|f_m^k\|.$$

$\Rightarrow$ схема устойчива безусловно.

Построение сети



$$u_m^{n+1} = u_m^n + \frac{\tau}{h^2}\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) + \tau f_m^{n+1}$$

$$u_m^n + \frac{\tau}{h^2} u_{m+1}^{n+1} + u_m^{n+1}\left(-\frac{2\tau}{h^2} - 1\right) + \frac{\tau}{h^2} u_{m-1}^{n+1} + \tau f_m^{n+1} = 0$$

$$\frac{1}{h^2} u_{m-1}^{n+1} - \left(\frac{2}{h^2} + \frac{1}{\tau}\right) u_m^{n+1} + \frac{1}{h^2} u_{m+1}^{n+1} = -f_m^{n+1} - \frac{1}{\tau} u_m^n$$

$$-\frac{1}{h^2} u_{m-1}^{n+1} + \left(\frac{2}{h^2} + \frac{1}{\tau}\right) u_m^{n+1} - \frac{1}{h^2} u_{m+1}^{n+1} = f_m^{n+1} + \frac{1}{\tau} u_m^n$$

$$\begin{pmatrix} \frac{2}{h^2} + \frac{1}{\tau} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2} + \frac{1}{\tau} & -\frac{1}{h^2} & & \\ & -\frac{1}{h^2} & \frac{2}{h^2} + \frac{1}{\tau} & -\frac{1}{h^2} & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \frac{2}{h^2} + \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ u_{M-1}^{n+1} \end{pmatrix} = \begin{pmatrix} f_1^{n+1} + \frac{1}{\tau} u_1^n \\ \vdots \\ \vdots \\ f_{M-1}^{n+1} + \frac{1}{\tau} u_{M-1}^n \end{pmatrix}$$

Программная реализация перехода на слой: метод прогонки.