Partitions, Representations and Modular Forms

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September 2025

This is a placeholder for now as of 10/17. Better drafts coming soon!

1 Ordinary Representations

1.1

1.2 Induction and Restriction, or Raising and Lowering

This allows us to compute the dimension of the irreducible representation afforded by ρ_λ fairly directly. Since the restriction operators commute in an obvious way, we have that $\operatorname{Res}_1^n \rho_\lambda = \operatorname{Res}_1^2 \operatorname{Res}_2^3 \cdots \operatorname{Res}_{n-1}^n \rho_\lambda$. From our previous theorem, we can describe these in a familiar way,

$$\operatorname{Res}_{n-1}^n \rho_\lambda = \bigoplus_{\substack{\mu \vdash n-1 \\ \mu \prec \lambda}} \rho_\mu,$$

from which one can derive

$$\operatorname{Res}_{n-k}^n \rho_\lambda = \bigoplus_{\substack{\mu_k \vdash n-k \\ \mu_k \prec \cdots \prec \mu_1 \prec \lambda}} \rho_{\mu_k}.$$

So, since S_1 only has one irreducible representation of dimension 1 and Res preserves the dimension of representations, we have that $\operatorname{Res}_1^n \rho_\lambda = \rho_{(1)}^{\dim \rho_\lambda}$, and comparing to our earlier expression, this is the number of ascending chains of partitions with $(1) = \mu_1 \prec \cdots \prec \mu_n = \lambda$. Now, if we imagine starting with the Young diagram of μ_1 and for each μ_i, labeling the new square that we added with i, since we can only add a square at a location in the Young diagram if there are already squares to the left and above it, one can see that the numbers must be increasing left to right along a row and increasing down a column. So, each chain corresponds to such a filling of λ with the numbers 1 to n (called a standard Young Tableau), and vice versa. So, $\dim \rho_\lambda$ is given by the number of standard Young Tableaux of shape λ which is given by the famed Hook-length formula

$$H(\lambda) = \frac{n!}{\prod_{a \in \lambda} h(a)}.$$

If a is a square with coordinates (i, j) in λ, then $h(a) = 1 + (\lambda_i - i) + (\lambda'_j - j)$, or more simply, the number of squares to the right or below it, including a.

As a bit of fun, we can look at $\operatorname{Res}_1^n \operatorname{Ind}_1^n \rho_{(1)}$. Using a bit of Mackey theory, we have that $\operatorname{Res}_1^n \operatorname{Ind}_1^n \rho_{(1)} = \bigoplus_{\pi \in S_1 \backslash S_n / S_1} \operatorname{Ind}^n_{\pi^{-1} S_1 \pi \cap S_1}(\pi \rho_{(1)} \pi^{-1}) = \bigoplus_{\pi \in S_n} \rho_{(1)} = \rho_{(1)}^{n!}$. However, if we compute it directly, we get

$$\operatorname{Res}_1^n \operatorname{Ind}_1^n \rho_{(1)} = \operatorname{Res}_1^n \left(\bigoplus_{\lambda \vdash n} \rho_\lambda^{\dim \rho_\lambda} \right) = \bigoplus_{\lambda \vdash n} \rho_1^{(\dim \rho_\lambda)^2} = \rho_1^{n!}.$$

Thus, $\sum_{\lambda \vdash n} (\dim \rho_\lambda)^2 = n! = |S_n|$, a standard result in basic representation theory. Here, there is actually nothing special about S_n, for $\{e\} \subseteq G$, the induced representation is the regular representation (of which each irreducible occurs $\langle \chi_{reg}, \chi_i \rangle = \frac{1}{|G|} \sum_g \overline{\chi_{reg}(g)} \chi_i(g) = \frac{\chi_{reg}(1)}{|G|} \chi_i(1) = \dim \rho_i$ many times), and the restriction of ρ_i is going to be the trivial representation of $\{e\}$ with multiplicity $\dim \rho_i$ while the regular representation has dimension $|G|$.

2 Modular representations

2.1 Motivation

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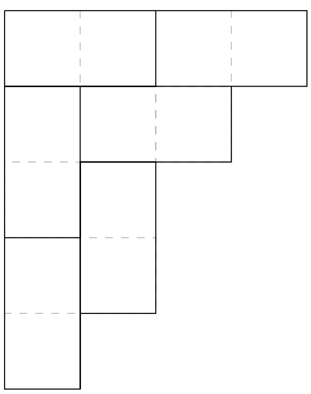
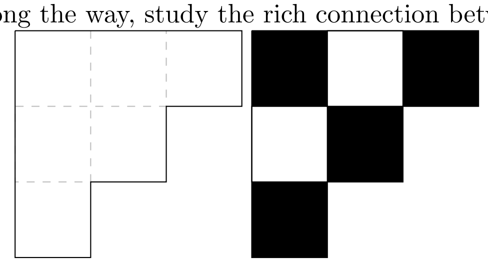
Suppose that you have n boxes on a grid that are pushed up against a corner. Under what conditions can you tile it with 2×1 dominoes? [RP Stanley's Algebraic Combinatorics, Exercise 8.7]



Evidently, we need n to be even. However, we quickly find that this isn't sufficient and some more consideration leads us to the condition that after a chessboard coloring, the number of white and black squares should be the same. It turns out that this is in fact exact. For one direction, one notes that for a domino tiling, an individual domino covers one white and one black square. For the other direction, since the outermost squares must contain a pair of white/black squares (otherwise it's a staircase pattern and has an overall white/black imbalance), one can remove this pair and use induction to show that it can be tiled. Now that we have a condition for it being tileable, a natural question is how many such tileable configurations exist for a given n?

Let $P(x) = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \cdots) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$, then, one can verify that the coefficient of x^n in $P(x)$ is the number of ways to arrange the n boxes so that they're in the corner. And, as it turns out that the number of tileable configurations for n is given by the coefficient of $x^{n/2}$ in $P(x)^2$. This suggests that there is a canonical way to essentially split the tileable configurations of

n boxes into groups of k boxes and $\frac{n}{2} - k$ boxes. We will realize this bijection in its generality, and along the way, study the rich connection between integer partitions and representations of S_n.



3 Introduction

A (proper) partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of decreasing positive integers, and we say that λ is a partition of (or partitions) $\sum \lambda_i = n$ and write this as $\lambda \vdash n$ or $|\lambda| = n$. We can visualize λ as having k rows where the ith row has λ_i boxes, and we often conflate λ with its shape, or Young diagram. We let $p(n)$ denote the number of ways to partition n, and we can rephrase the motivational problem using these terms.

That is, for a partition λ of n it is possible to cover its Young diagram with dominoes when the chessboard coloring has an equal number of black and white squares, and there is a bijection between tileable Young diagrams λ, and pairs of partitions λ_1, λ_2 with $|\lambda_1| + |\lambda_2| = |\lambda|/2$. Consequently, for a given even n, the number of partitions is given by $\sum_{i=0}^{n/2} p(i)p(n/2 - i)$.

Although we don't use it much, one can also populate the squares of a Young diagram with the numbers 1 through n in order to obtain a Young tableau, and for each λ there are $n!$ Young tableaux.

Let S_n be the symmetric group on $\{1, \ldots, n\}$. Suppose that $\pi, \phi \in S_n$. If we write π as a product of disjoint cycles $\prod \pi_i$ with $\pi_i = (x_{i1}\ x_{i2}\ \cdots x_{i\ell_i})$, then we have that conjugation by ϕ gives us $\phi\pi\phi^{-1} = \prod_i \phi\pi_i\phi^{-1} = \prod_i (\phi(x_{i1})\ \phi(x_{i2})\ \cdots\ \phi(x_{i\ell_i}))$. The π_i are disjoint cycles so they commute with each other, and hence, so too must the $\phi\pi_i\phi^{-1}$. Thus, π_i and $\phi\pi_i\phi^{-1}$ have the same cycle length and the conjugacy class $[\pi]$ is going to be determined by the multiset of the lengths of the cycles ℓ_i. We can define the cycle type, $c(\pi)$, as this multiset including contributions from trivial cycles. Furthermore, writing it as $c(\pi) = (\ell_1, \ldots, \ell_k)$, where the ℓ_i are descending, we see that $c(\pi) \vdash n$. For example, if we take $\pi = (1\ 2)(3\ 4)(5) \in S_5$, $c(\pi) = (2, 2, 1) \vdash 5$.

4 Appendix A: Littlewood Decomposition

There are many ways of viewing the Littlewood decomposition and going through the construction, and here we collect a number of them.

4.1 Biinfinite sequences

Consider all biinfinite sequences of the form $w = \overline{0}s\overline{1}$ where s is some finite word in the alphabet $\{0, 1\}$ and $\overline{0}, \overline{1}$ represents an infinite sequence of 0s and 1s respectively. We can absorb leading 0s of s into $\overline{0}$ and similarly for trailing 1s into $\overline{1}$, so that there are identities such as $w = \overline{0}0101101\overline{1} = \overline{0}101101\overline{1} = \overline{0}10110111\overline{1}$. We can visualize these sequences by dragging a pen across paper with for every 0, we move it one unit up and for every 1, we move it one unit to the right. This ends up drawing the boundary of a partition!

One can verify that inserting a 0 will include the part being drawn, possibly again (if thus far we've encountered k 1s, we're drawing the part k, and if we encounter a 0 here, we include the part k), while inserting a 1 will increase the length of the part. Furthermore, replacing a 01 with a 10 will end up adding a single square, and similarly if there is a 0 and k places after is a 1 and we swap the two, we add k squares.

We now give our sequences some indexing, as one is often wont to do. As opposed to the first instinct of simply absorbing as much as possible from the middle string and indexing from there, we instead define w_0 to be the unique index such that the number of w_i with $i < 0$ with $w_i = 1$ is the same as the number of $i \geq 0$ with $w_i = 0$. This indexing scheme ends up being more convenient for a number of things. We often delineate the seperation between negative and non-negative indices with a vertical bar. The following example shows both the indicies (usually omitted) and the bar.

$$\overline{0}10110\overline{1} = \overline{0}1_{-2}0_{-1}|1_0 1_1 0_2\overline{1}.$$

For every number $t > 0$, there's a natural way to split w into t many biinfinite words w^j by having w^j consisting of the subsequence $\{w_i\}$ with $i \equiv j \mod t$. For example (taking indices modulo 3):

$$w = \overline{0}1_1 1_2 0_0 0_1 1_2|0_0 1_1 0_2 1_0 0_1\overline{1}$$
$$w^0 = \overline{01},\ w^1 = \overline{0}1010\overline{1},\ w^2 = \overline{0}110\overline{1}$$

If we define u to have $u_{3i} = w_{3i+3}$, $u_{3i+1} = w_{3i-2}$ and $u_{3i+2} = w_{3i+2}$ (i.e. copying w, then shifting w^0 over to the left and w^1 over to the right), we can check that the indexing convention is still upheld and that $u^j = w^j$ for all j. So, in order to upgrade our map to be a bijection, we essentially need to preserve information about where each of the w^js start relative to each other. We do this by taking w, fixing the indices as we modify it, and whenever $w_i = 1, w_{i+t} = 0$, we swap them such that $w_i = 0, w_{i+t} = 1$ until no such swaps are possible. This gives us a new word w'. It turns out that w' is the $t-$core of w meanwhile the w_i form the $t-$quotient of w.

4.2 Star diagrams

4.3 Frobenius ???

4.4 Abaci

4.5 Beta numbers

5 Appendix B: Asymptotics of Partitions with Trivial t-core

Suppose that you are given a random partition of $n = tm$ and you want to find the odds of it having trivial $t-$core. From the Littlewood decomposition, for fixed n, you can find this relatively easily by dividing $\sum_{a_1+\cdots+a_t=m} \prod p(a_i)$ and dividing it by $p(n)$. However, we can decompose $p(n)$ with similar sums based on the $t-$core λ_c as

$$\sum_{\lambda_c} \sum_{\sum a_i = (|\lambda|-|\lambda_c|)/t} \prod p(a_i).$$

Since $p(n)$ is increasing for all $n \geq 1$, it's clear that the contribution from the empty $t-$core should be larger than the contribution from any $\lambda_c \neq \emptyset$. Although one might expect for large n that small cores $|\lambda_c| \ll n$ may end up having similar contributions to $\lambda_c = \emptyset$, it is difficult to say what the overall effect will be and if the probability of having trivial $t-$core should go to some constant (especially in the case of $t = 2$ where the $2-$cores correspond to triangular numbers) or if it goes to 0 like $O(n^{-c})$ or some $O(e^{-f(n)})$.

Proposition 1. *Let $p_t(tn)$ denote the number of partitions of tn with trivial $t-$core, then*

$$p_t(tn) \sim C_t n^{(-t-3)/4} e^{\pi\sqrt{2tn/3}}$$

where

$$C_t = \frac{1}{2\sqrt{\pi}} \left(\frac{1}{2\pi}\right)^{t/2} \left(\frac{t\pi^2}{6}\right)^{(1+t)/4}.$$

In comparison,

$$p(tn) \sim \frac{1}{4tn\sqrt{3}} e^{\pi\sqrt{2tn/3}}$$

so that $\frac{p_t(tn)}{p(tn)} = O(n^{(-t+1)/4})$.

This is all an immediate result of the following after noting that the relevant generating function is given by $P(x^t)^t = \prod_{i=1}^{\infty} \frac{1}{(1-x^{it})^t}$ (as each block in the quotient corresponds to t many of the original and we're counting tuples of total size m). Furthermore, in the notation of the following result, we have that $D(s) = t\zeta(s)$ so $\alpha = 1, A = t\,\mathrm{res}_{s=1}\,\zeta(s) = t$, and we have the special values $\zeta(0) = -1/2, \zeta'(0) = -\ln(2\pi)/2, \zeta(2) = \pi^2/6$.

Theorem 2. *(Meinardus, 1954, Asymptotische Aussagen über Partionen) Let $a_n \geq 0$ be a sequence of integers and consider the product $f(X) = \prod_{n=1}^{\infty}(1 - X^n)^{-a_n} = 1 + \sum_{n=1}^{\infty} r(n)X^n$ and the associated Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Suppose $D(s)$ converges for all $\Re(s) > \alpha > 0$ where it has a simple pole at α with residue A, and furthermore, it is able to be analytically continued until $\Re(s) = -c_0$ with $0 < c_0 < 1$, then the following holds:*

$$r(n) = Cn^{\chi} e^{(\exp(\frac{\alpha \log n}{\alpha+1})(1+\frac{1}{\alpha})(A\Gamma(\alpha+1)\zeta(\alpha+1))^{1/(\alpha+1)}} (1 + O(n^{-\chi_1}))$$

where

$$\begin{aligned}
C &= e^{D'(0)} (2\pi(1+\alpha))^{-1/2} (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1-2D(0)}{2(1+\alpha)}} \\
\chi &= \frac{2D(0) - 2 - \alpha}{2(1+\alpha)} \\
\chi_1 &= \frac{1}{1+\alpha} \min(c_0/\alpha - \delta/4, 1/2 - \delta)
\end{aligned}$$

for any $\delta > 0$.

TODO: Include assumption about $g(\tau)$