1 Biinfinite sequences

Consider all biinfinite sequences of the form $w = \overline{0}s\overline{1}$ where s is some finite word in the alphabet $\{0,1\}$ and $\overline{0}, \overline{1}$ represents an infinite sequence of 0s and 1s respectively. We can absorb leading 0s of s into $\overline{0}$ and similarly for trailing 1s into $\overline{1}$, so that there are identities such as $w = \overline{0}0101101\overline{1} = \overline{0}101101\overline{1} = \overline{0}10110111\overline{1}$. We can visualize these sequences by dragging a pen across paper with for every 0, we move it one unit up and for every 1, we move it one unit to the right. This ends up drawing the boundary of a partition!

One can verify that inserting a 0 will include the part being drawn, possibly again (if thus far we've encountered k 1s, we're drawing the part k, and if we encounter a 0 here, we include the part k), while inserting a 1 will increase the length of the part. Furthermore, replacing a 01 with a 10 will end up adding a single square, and similarly if there is a 0 and k places after is a 1 and we swap the two, we add k squares.

We now give our sequences some indexing, as one is often wont to do. As opposed to the first instinct of simply absorbing as much as possible from the middle string and indexing from there, we instead define w_0 to be the unique index such that the number of w_i with $i < 0$ with $w_i = 1$ is the same as the number of $i \geq 0$ with $w_i = 0$. This indexing scheme ends up being more convenient for a number of things. We often delineate the separation between negative and non-negative indices with a vertical bar. The following example shows both the indices (usually omitted) and the bar.

$$\overline{0}101101\overline{1} = \overline{0}1_{-2}0_{-1}|1_0 1_1 0_2 \overline{1}.$$

For every number $t > 0$, there's a natural way to split w into t many biinfinite words w^j by having w^j consisting of the subsequence $\{w_i\}$ with $i \equiv j \mod t$. For example (taking indices modulo 3):

$$w = \overline{0}1_1 1_2 0_0 0_1 1_2 | 0_0 1_1 0_2 1_0 0_1 \overline{1}$$
$$w^0 = \overline{01},\ w^1 = \overline{0}1010\overline{1},\ w^2 = \overline{0}110\overline{1}$$

If we define u to have $u_{3i} = w_{3i+3}$, $u_{3i+1} = w_{3i-2}$ and $u_{3i+2} = w_{3i+2}$ (i.e. copying w, then shifting w^0 over to the left and w^1 over to the right), we can check that the indexing convention is still upheld and that $u^j = w^j$ for all j. So, in order to upgrade our map to be a bijection, we essentially need to preserve information about where each of the w^js start relative to each other. This is encapsulated by the t-core.

Definition 1. *For the biinfinite word $w \sim \lambda$, swap every instance of $w_i = 1, w_{i+t} = 0$ with $w_i = 0, w_{i+t} = 1$ until no such swaps are possible and call this word w^c. Let $w^c \sim \lambda^c$. We define λ^c to be the $t-$core of λ.*

It is nice to note that throughout this process of swapping, indexing stays consistent (i.e. if w' is the result after a swap with the proper indexing, $w_j = w'_j$, for all $j \neq i, i+t$), as can be seen by considering a few cases. For $i, i+t < 0$ (and $i, i+t \geq 0$), it doesn't alter the number of 0s or 1s to the left (or right) of the bar, so balance is maintained. If $i < 0$ and $i + t \geq 0$, then we decrease the number of 1s to the left of the bar and the number of 0s to the right of the bar by one each, so balance is maintained and the indexing doesn't change.

The following is a more standard way of defining the $t-$core.

Lemma 2. *The $t-$core λ^c is the result of taking λ and repeatedly removing all $t-$rim hooks until no such removal is possible. Furthermore, the order that $t-$rim hooks are removed doesn't matter.*

Proof. The first statement amounts to proving that a swap corresponds to removing a $t-$rim hook. We prove this by induction on t. This statement is obviously true for $t = 1$ as 10 corresponds to a square and 01 is just removing that square. Now, if $w_i \cdots w_{i+t} = \underbrace{1 \cdots 1}_{t}0$, then it is easy to see that $0\underbrace{1 \cdots 1}_{t}$ just removes t squares. On the other hand, if there's some $0 < a < t$ with $w_{i+a} = 0$, then we have that $(w_i, w_{i+a}, w_t) = (1, 0, 0)$ being replaced by $(0, 1, 0)$ and then $(0, 0, 1)$ is going to be a removal of an $a-$rim hook and then a $(t-a)-$rim hook. The key observation here is that both of these rim hooks are going to share an edge which intersects the border so that it forms a $t-$rim hook.

The last statement isn't too hard to see from the word construction. Let s be a finite segment of w (inheriting w's indexing) such that $w = \bar{0}s\bar{1}$. We have that $w_i = 1, w_{i+t} = 0$ if and only if $s_i = 1, s_{i+t} = 0$ and furthermore due to the structure $w = \bar{0}s\bar{1}$, any swaps that are done on w can be done solely on s. Now, fix i, then the s_j with $j \equiv i \mod t$ take on the values 0 and 1 finitely many times, say a, and b respectively. We can see that swaps leave the number of occurrences invariant and they will continue to be possible until the subword formed by these s_j looks like $\underbrace{0 \cdots 0}_{a}\underbrace{1 \cdots 1}_{b}$. This end result doesn't depend on the order that we conducted the swaps of s, or in other words, the order of $t-$rim hook removals doesn't matter. □

And the following is perhaps the most common definition:

Proposition 3. *The $t-$core λ^c is the result of taking λ and repeatedly removing all $kt-$ hooks until no such removal is possible. Furthermore, if there is a $kt-$(rim) hook, then there must exist some $t-$(rim) hook.*

Proof. Suppose that we have a $kt-$hook, then we can write $w = \bar{0}a1b0c\bar{1}$ for some (possibly empty) strings a, c. The main thing is that the sequence for a $kt-$hook is going to be $\bar{0}1\underbrace{0 \cdots 0}_{x}\underbrace{1 \cdots 1}_{y}0\bar{1}$ where $x+y+1 = kt$ and for b to "fit within" the kt hook, it must be the same length as $\underbrace{0 \cdots 0}_{x}\underbrace{1 \cdots 1}_{y}$, which is $kt - 1$. So, we have that the $kt-$hook gives us some $w_i = 1, w_{i+kt} = 0$ and the resulting sequence when removing the kt hook is the same as swapping it, $\bar{0}a0b1c\bar{1}$. Thus, the statements follow. □

Now, we wish to go in the other direction and go from our words w^c (the core), $w^1, \ldots, w^t$ to a single word. To do this, we go in a bit of a roundabout fashion.

Lemma 4. *Let $t \geq 1$ be fixed. There is a bijection between $t-$cores λ^c and integer vectors $v = (n_0, \ldots, n_{t-1})$ with $\sum n_i = 0$.*

Proof. First, for λ^c, take the word w^c and define $v = (n_1, \ldots, n_t)$ be such that n_i is the smallest k such that $w^c_{i+kt} = 1$. From the way that indexing is defined, there is an equal number of 1s with negative indices as there are 0s with non-negative indices, so if $n_i > 0$, the indices $j \equiv i \mod t$ contribute n_i 0s appearing with non-negative indices (and no 1s with negative indices since it is a t-core). Similarly, if $n_i < 0$, it contributes n_1 1s appearing with negative indices. Thus, $\sum_{n_i \geq 0} n_i = -\sum_{n_i < 0} n_i$, and the conclusion follows.

Similarly, for any length t vector $(n_0, \ldots, n_{t-1})$ summing to 0, we define w^c as $w^c_{i+kt} = 0$ whenever $k < n_i$ and $w^c_{i+kt} = 1$ otherwise. Since $w^c_i = 1, w^c_{i+t} = 0$ never happens, this means that w^c is indeed a core. It is clear that these two algorithms are inverses to each other. □

Theorem 5. *(Littlewood Decomposition) For every $t \geq 1$, there is a bijection between partitions λ and $t+1$ partitions $\lambda^c, (\lambda^0, \ldots, \lambda^{t-1})$ where λ^c is a $t-$core.*

Proof. For a given λ, we take the core and the $t-$quotient for our $t+1$ partitions.

Now, for given $\lambda^c, (\lambda^0, \ldots, \lambda^{t-1})$, we take the associated vector of λ^c, $(n_0, \ldots, n_{t-1})$, and define the word w by $w_{i+kt} = w^i_{k-n_i}$ where w^i is λ^i's word. To see that w has the proper indexing, we note that shifting a sequence rightwards by n_i will cause there to be n_i more 0s to have non-negative indices than 1s with negative indices, but since $\sum n_i = 0$, these effects cancel in w.

Checking that these algorithms are inverses for each other isn't too bad. Suppose $\lambda^c, (\lambda^0, \ldots, \lambda^{t-1}) \mapsto \lambda$, then since the indexing of w is proper, it is easy to see that the $t-$quotient of λ is $\lambda^0, \ldots, \lambda^{t-1}$. As for the $t-$core of λ, we have that the process of swapping $w_j = 1, w_{j+t} = 0$ pairs for $w_j = 0, w_{j+t} = 1$ pairs amongst the w_{i+kt} effectively trivializes each w^i. So, we might as well specialize to $w^i = \bar{0}\bar{1}$, but this amounts to the prior bijection between $t-$cores and zero-sum vectors of length t. So, $\lambda^c, (\lambda^0, \ldots) \mapsto \lambda \mapsto \lambda^c, (\lambda^0, \ldots)$. All we need to do is show that $\lambda \mapsto \lambda^c, (\lambda^0, \ldots)$ is injective. Suppose that $\lambda, \lambda' \mapsto \lambda^c, (\lambda^0, \ldots)$, and λ, λ' have corresponding words w, w'. The $t-$quotients being the same implies that there exists a_i such that $w_{i+kt} = w'_{i+(k+a_i)t}$ for all k. However, the cores, or rather, their vectors, cannot be the same unless each a_i take on the same value, but from the indexing scheme, it forces $a_i = 0$. $\square$

2 Asymptotics of Partitions with Trivial t-core

Suppose that you are given a random partition of $n = tm$ and you want to find the odds of it having trivial $t-$core. From the Littlewood decomposition, for fixed n, you can find this relatively easily by dividing $\sum\limits_{a_1+\cdots+a_t=m} \prod p(a_i)$ and dividing it by $p(n)$. However, we can decompose $p(n)$ with similar sums based on the $t-$core λ_c as

$$\sum_{\lambda_c} \sum_{\sum a_i = (|\lambda| - |\lambda_c|)/t} \prod p(a_i).$$

Since $p(n)$ is increasing for all $n \geq 1$, it's clear that the contribution from the empty $t-$core should be larger than the contribution from any $\lambda_c \neq \emptyset$. Although one might expect for large n that small cores $|\lambda_c| \ll n$ may end up having similar contributions to $\lambda_c = \emptyset$, it is difficult to say what the overall effect will be and if the probability of having trivial $t-$core should go to some constant (especially in the case of $t = 2$ where the $2-$cores correspond to triangular numbers) or if it goes to 0 like $O(n^{-c})$ or some $O(e^{-f(n)})$.

Proposition 1. *Let $p_t(tn)$ denote the number of partitions of tn with trivial $t-$core, then*

$$p_t(tn) \sim C_t n^{(-t-3)/4} e^{\pi\sqrt{2tn/3}}$$

where

$$C_t = \frac{1}{2\sqrt{\pi}} \left(\frac{1}{2\pi}\right)^{t/2} \left(\frac{t\pi^2}{6}\right)^{(1+t)/4}.$$

In comparison,

$$p(tn) \sim \frac{1}{4tn\sqrt{3}} e^{\pi\sqrt{2tn/3}}$$

so that $\frac{p_t(tn)}{p(tn)} = O(n^{(-t+1)/4})$.

This is all an immediate result of the following after noting that the relevant generating function is given by $P(x^t)^t = \prod_{i=1}^{\infty} \frac{1}{(1-x^{it})^t}$ (as each block in the quotient corresponds to t many of the original and we're counting tuples of total size m). Furthermore, in the notation of the following result, we have that $D(s) = t\zeta(s)$ so $\alpha = 1, A = t \operatorname{res}_{s=1} \zeta(s) = t$, and we have the special values $\zeta(0) = -1/2, \zeta'(0) = -\ln(2\pi)/2, \zeta(2) = \pi^2/6$.

Theorem 2. *(Meinardus, 1954, Asymptotische Aussagen über Partionen) Let $a_n \geq 0$ be a sequence of integers and consider the product $f(X) = \prod_{n=1}^{\infty}(1 - X^n)^{-a_n} = 1 + \sum_{n=1}^{\infty} r(n)X^n$ and the associated Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Suppose $D(s)$ converges for all $\Re(s) > \alpha > 0$ where it has a simple pole at α with residue A, and furthermore, it is able to be analytically continued until $\Re(s) = -c_0$ with $0 < c_0 < 1$. Write $\tau = y + 2\pi ix$ and suppose that $g(\tau) = \sum_{n=1}^{\infty} a_n e^{-n\tau}$ for $|\arg \tau| > \pi/4, |x| \leq 1/2$ that for small enough y there exists $\varepsilon, c_2 > 0$ such that $\Re(g(\tau) - g(y)) \leq -c_2 y^{-\varepsilon}$. Assuming all of these conditions, the following holds:*

$$r(n) = Cn^{\chi} e^{(\exp(\frac{\alpha \log n}{\alpha+1})(1+\frac{1}{\alpha})(A\Gamma(\alpha+1)\zeta(\alpha+1))^{1/(\alpha+1)}}(1 + O(n^{-\chi_1}))$$

where

$$\begin{aligned} C &= e^{D'(0)}(2\pi(1+\alpha))^{-1/2}(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1-2D(0)}{2(1+\alpha)}} \\ \chi &= \frac{2D(0) - 2 - \alpha}{2(1+\alpha)} \\ \chi_1 &= \frac{1}{1+\alpha}\min(c_0/\alpha - \delta/4, 1/2 - \delta) \end{aligned}$$

for any $\delta > 0$.