Computational Geometry

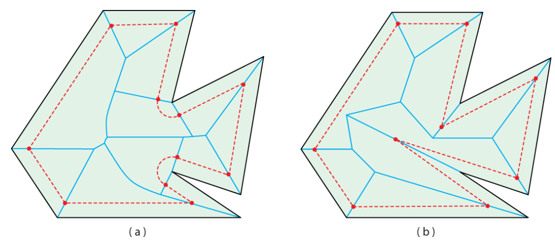
Minkowski Sums

Herman Minkowski 1864-1909

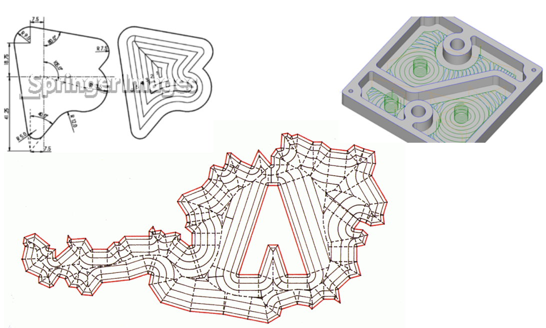
Offset Curve

- Given a smooth curve *C*, the offset curve is the locus of points offset by a constant distance *r* along the curve normal.
- The offset curve can also be defined as the envelope of a family of disks of radius *r* whose centers lie on *C*.

Medial Axis, Straight Skeleton and the Offset Curve



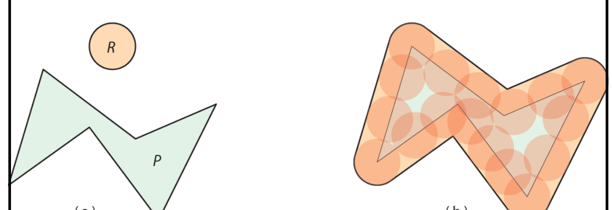
Pocket Machining



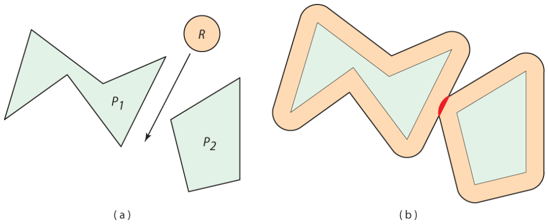
Minkowski Sum

- The Minkowski sum of two sets *A* and *B*

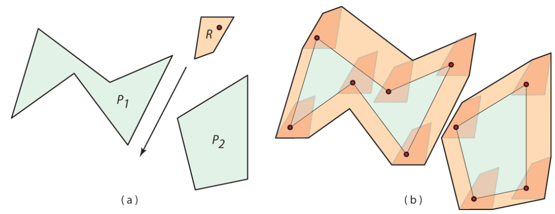
  $A \oplus B = \{x+y \mid x \in A, y \in B\}$



Motion Planning

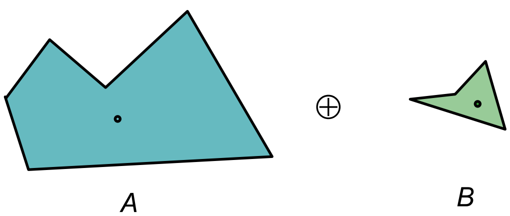


Convex Polygon

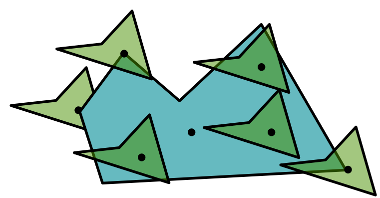


Minkowski Sum

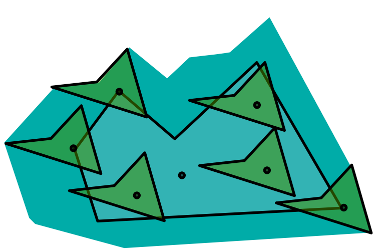
- $A \oplus B$



Minkowski Sum



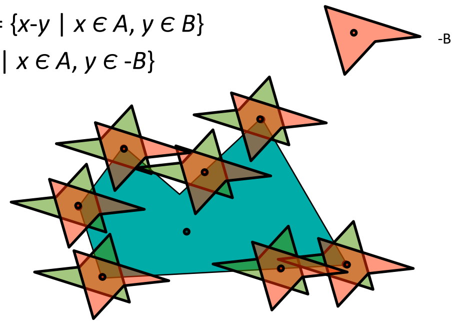
Minkowski Sum



Minkowski Sum

Minkowski Difference

- $A \ominus B = \{x\text{-}y \mid x \in A, y \in B\}$
- $= \{x+y \mid x \in A, y \in \text{-}B\}$



Minkowski Difference

- Widely used in game engines and path planning for collision detection
- Collision detection amounts to asking if two shapes have any points in common. If they do, then $p \ominus p = 0$ in the Minkowski difference
- If $A \ominus B$ contains the origin, then A and B collide

Minkowski Sum and Difference

- Minkowski sum is used to “fatten” objects
- Minkowski difference is used to produce configuration space obstacles
- $\mathrm{A} \oplus B = B \oplus A$
- $A \ominus B \neq B \ominus A$
- sum and difference produce different shapes unless the moving object is symmetrical

GJK Algorithm

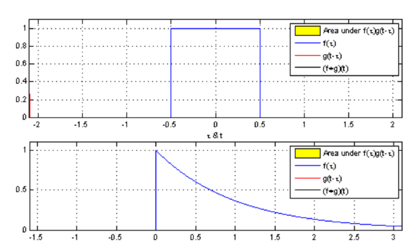
- Gilbert-Johnson-Keerthi (1988)
- Linear time iterative method for computing the distance between two convex sets in m-dimensional space
- Continuous enhancements since, major results in 97 and 04

Computational Geometry

Convolution of Curves

Convolution

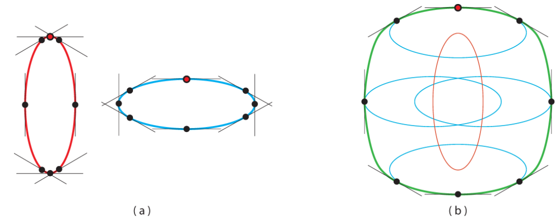
- Convolution of two functions produces a third function that expresses the overlap



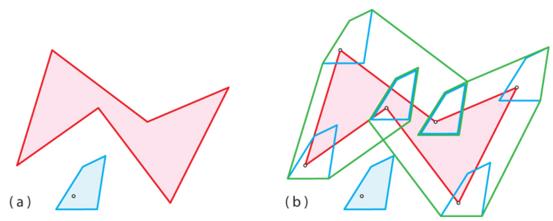
Convolution of Curves

- The Minkowski sum is a set
- We need the boundary curve $\delta(A \bigoplus B)$ only
- The convolution of curves α and β is the curve
  - $\alpha * \beta = \{x+y \mid x \in \alpha, y \in \beta, T_x = T_y\}$
  - T_p is the unit tangent vector at point p

Convolution of Curves

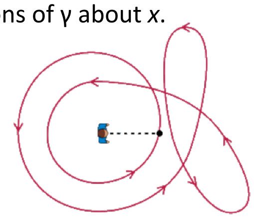


P_1*-R

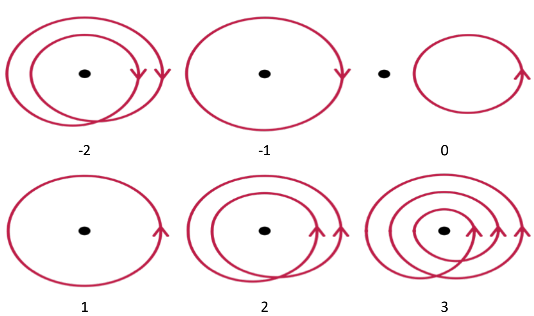


Winding Number

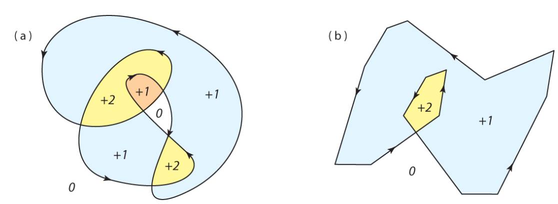
- Let γ be a close planar curve (oriented ccw) and x be a point not on γ. The winding number $\Phi_\gamma(x)$ of γ with respect to x counts the number of full revolutions of γ about x.



Examples



More Examples



Theorem

- The Minkowski sum of two planar polygons A and B is the set of points in the plane with positive winding number with respect to the convolution of *δA* with *δB.*
- $A \oplus B = \{p = \mathrm{R}^2 \mid \Phi_{\delta A*\delta B}(p) > 0\}$

Algorithm

- Compute *δA*δB*
- Identify its convolution cycles
- Retain cycles that have a positive winding number
- Merge these to construct $\delta(A \oplus B)$