# Computational Geometry

## Area and Triangulation

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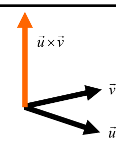
## The Cross Product

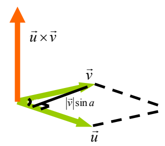



- A vector operation

$$\vec{u}\times\vec{v}=\begin{bmatrix} i & j & k \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}=\begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

- Geometric significance

$$|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin a$$

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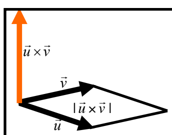
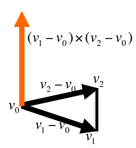
## Area of Triangle




- $A(\triangle) = ½|(v_1-v_0)X(v_2-v_0)|$




$$(v_1-v_0)\times(v_2-v_0)=\begin{bmatrix} i & j & k \\ x_1-x_0 & y_1-y_0 & 0 \\ x_2-x_0 & y_2-y_0 & 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ (x_1-x_0)(y_2-y_0)-(y_1-y_0)(x_2-x_0) \end{bmatrix}$$

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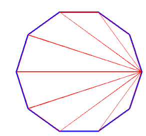
## Area of Polygon

- Convex polygons:



- General?

$$2A(P)=\sum_{i=0}^{n-1}(x_i y_{i+1}-y_i x_{i+1})$$

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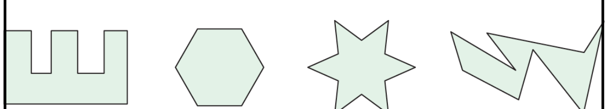
## How Many Distinct Triangulations?



(a) (b) (c) (d)

- A diagonal exists between any two non-adjacent vertices of a polygon *P* iff *P* is a convex polygon

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## Theorem

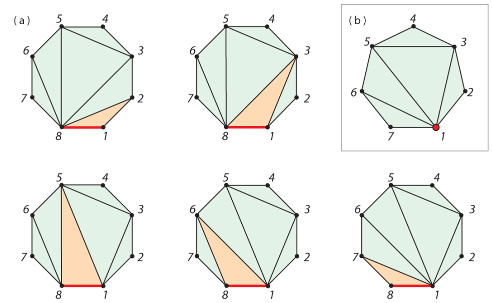
- The number of triangulations of a convex polygon with $n$+2 vertices is the Catalan number: $C_n=\frac{1}{n+1}\binom{2n}{n}=\frac{(2n)!}{(n+1)!n!}$

## The Catalan Numbers

- Well-formed sequences of parentheses
- Stack permutations
- Full binary trees
- Triangulations of convex *n*-gon

| | | | |
|---|---|---|---|
| 0 | 1 | 10 | 16796 |
| 1 | 1 | 11 | 58786 |
| 2 | 2 | 12 | 208012 |
| 3 | 5 | 13 | 742900 |
| 4 | 14 | 14 | 2674440 |
| 5 | 42 | 15 | 9694845 |
| 6 | 132 | 16 | 35357670 |
| 7 | 429 | 17 | 129644790 |
| 8 | 1430 | 18 | 477638700 |
| 9 | 4862 | 19 | 1767263190 |

## Edge Contraction



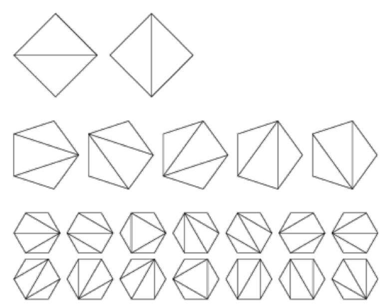

## The Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

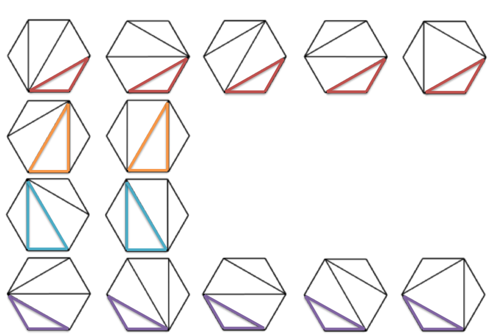
$$C_n = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n}{i}^2$$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

## n=4, 5, 6



## n=6



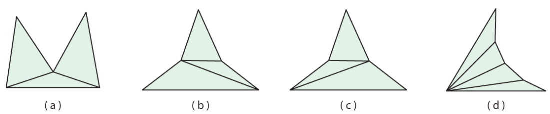
## Recurrence Relations

- $P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_3P_{n-1} + P_n$
- $P_2 = 1$
- $P_{n+1} = P_nP_2 + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_3P_{n-1} + P_2P_n$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \Rightarrow C_{n-1} = \sum_{i=0}^{n-2} C_i C_{n-2-i}$$

$$E_n = \frac{2\times 6\times 10\times \ldots \times (4n-10)}{(n-1)!}$$

## Reflex Vertices?




## Big-O Notation

- Upper bounds on the growth rate of functions and running time of algorithms.
- *n* represents input size, i.e. a list of *n* elements.
- Constants are ignored and only the dominant term matters
- A for-each sequential loop is O(*n*)
- A nested for where inner loop runs *y* times and outer loop runs *x* times is O(xy), or O($n^2$)