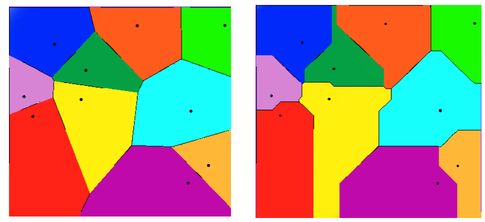
Computational Geometry

Voronoi Diagrams, Delaunay Triangulations, and Convex Hull

Convexity of the Voronoi Regions

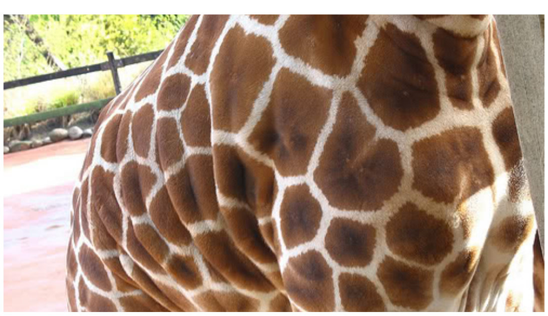
- Heavily dependent on distance metric



Spider Web

Leaf

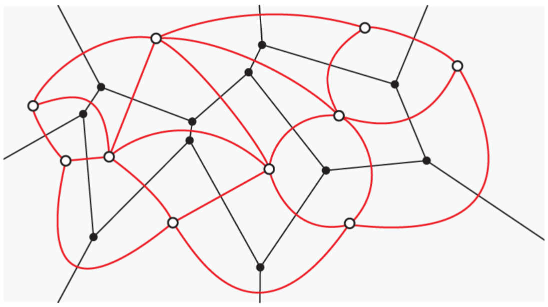
Giraffe



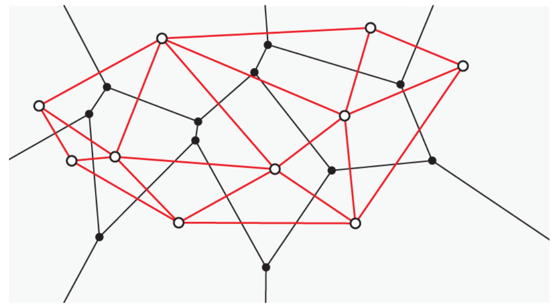
3D

- Nature by Numbers
- Animated 3D Voronoi

The Dual Graph

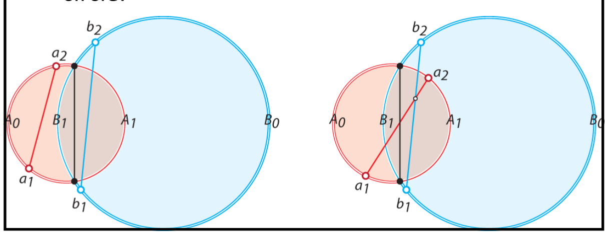


Straight-line Dual Graph



Lemma

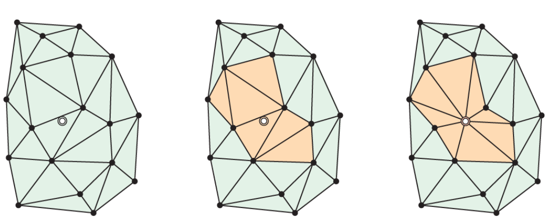
- Let A and B be two circles with chords that properly cross. Then at least one endpoints of one circle's chord is strictly inside the other circle.



Theorem

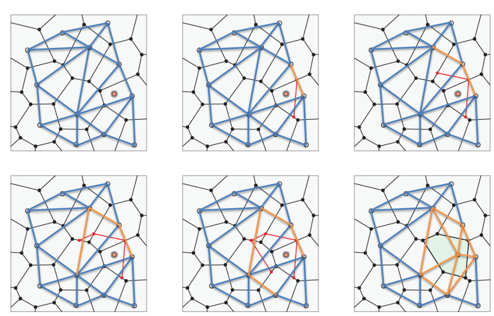
- The straight-line dual graph of *Vor*(*S*) is planar
- The straight-line dual graph of *Vor*(*S*) is a triangulation of *S* when *S* is in general position.
- The dual triangulation of *Vor*(*S*) is the Delaunay triangulation of *S.*

Incremental Delaunay via the Dual Graph



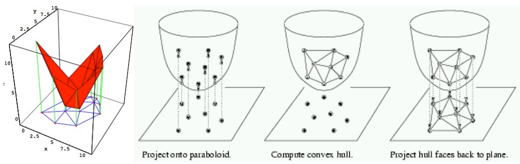
(a) (b) (c)

Incremental Voronoi and Delaunay



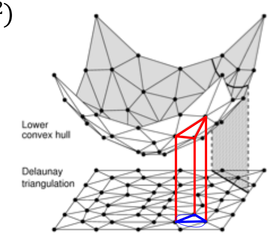
Delaunay and Convex Hull

- Lift sites to a paraboloid ($z = x^2 + y^2$)
- Compute 3D convex hull of points
- Project lower hull faces back to plane



Theorem

- Given a point set S in the plane, the Delaunay triangulation Del(S) is exactly the projection to the xy-plane of the lower convex hull of the points $(x, y, x^2 + y^2)$

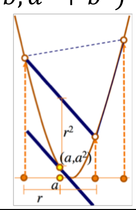


Correctness

- All points on the paraboloid are convex and will be on the lower hull
- The lower hull consists of triangles
- The projection is a triangulation – projected edges do not cross
- To prove
  - The triangulation is Delaunay

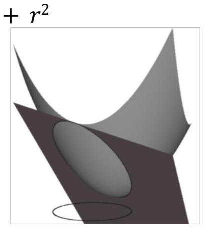
Notes

- Each triangle on the lower hull defines a plane
- These planes intersect the paraboloid
- Drop the plane by some r^2 so that it becomes tangent to the paraboloid at $(a, b, a^2 + b^2)$
- Then the projected vertices of the triangle must lie on a circle of radius r around the point $(a, b, a^2 + b^2)$.



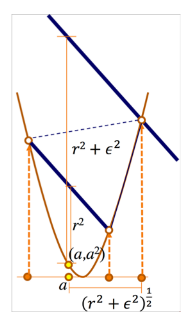
Intersection of Plane and Paraboloid

- Tangent plane to the paraboloid at $(a, b, a^2 + b^2)$
  - $-z = 2ax + 2by - (a^2 + b^2)$
- Shift plane upwards in Z^+ by r^2
  - $-z = 2ax + 2by - (a^2 + b^2) + r^2$
- Plane intersection with the paraboloid is a circle
  - $-z = x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + r^2 \Longrightarrow (x - a)^2 + (y - b)^2 = r^2$



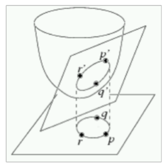
Empty Circle

- Original plane was on the lower hull, so all other points must be above
- Raise the plane until it hits any other point
- Distance from the projection of this other point to (a, b) must be larger than r.
- The circle of radius r around (a, b) is empty



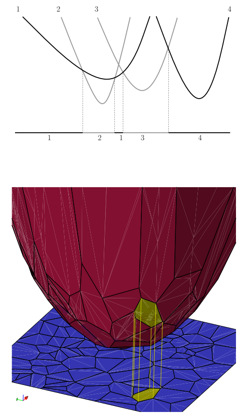
Conclusion

- A lower face f on the convex hull projects to a triangle with a circumcircle of radius r. Since f is on the lower hull, all other sites lie above ($>$ r w.r.t. plane of tangency), and thus project outside of the circle, which then satisfies the empty circle property

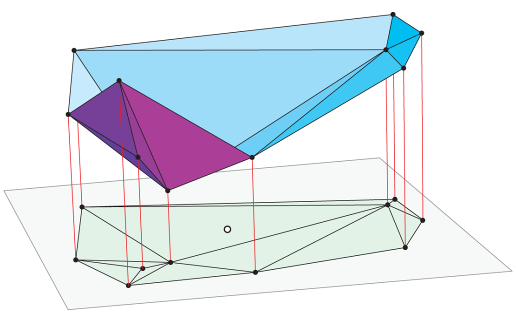


Notes

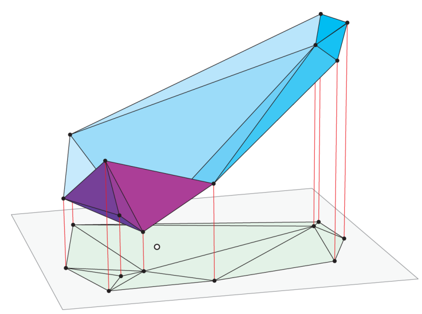
- If the tangent planes are also constructed and their intersections projected, it's the Voronoi diagram



Original



Translation to the Right



Notes

- Compute Delaunay by computing 3D convex hull instead – $O(nlogn)$
- The relationship holds in higher dimensions as well, thus Delaunay tetrahedralizations are typically constructed by constructing 4D convex hulls.