Characterizing Trees for Lambda-mu Terms

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Abstract. We give the conditions to characterize Böhm tree structures which represent terms of the Lambda-mu calculus. This result answers a question stated in Saurin's FLOPS paper.

1 Introduction

In [2], Saurin extends the Böhm trees and the expanded Böhm trees, called Nakajima trees in [2], to the $\Lambda\mu$-terms. It is a beautiful theoretical result that we can obtain the (expanded) Böhm trees for $\Lambda\mu$ just by extending the bound of lengths of prefixes and arguments to ω^2, whereas it is ω for the λ-calculus. An open problem stated in [2] is on characterization of the tree structures which represent some $\Lambda\mu$-terms. For the λ-calculus, this problem was solved by Nakajima in [1], where he gave some conditions for the expanded Böhm trees and showed that they give the exact characterization of trees which represents some λ-terms.

In this paper, we extended Nakajima's result to $\Lambda\mu$, and solve Saurin's open problem. The idea of the characterizing conditions are almost the same as [1], and they require that the information of each node is computable, and that all of nodes except for a finite part are obtained by η-expansion. The conditions for $\Lambda\mu$-calculus become much more complicated than the case of λ-calculus, because we have to manage the correspondence of μ-variables in prefixes and bodies.

2 Expanded Böhm Trees of $\Lambda\mu$-Terms

Definition 1. The $\Lambda\mu$-terms are defined as follows:

$$t, s ::= x \mid \lambda x.t \mid (t)s \mid \mu\alpha.t \mid (t)\alpha$$

$\Sigma_{\Lambda\mu}$ is the set of all $\Lambda\mu$-terms. $\Sigma^{c}_{\Lambda\mu}$ is the set of all closed $\Lambda\mu$-terms. In this paper, we always suppose every $\Lambda\mu$-term is closed with respect to μ-variables.

The reduction rules for the $\Lambda\mu$-calculus are the following.

$$\begin{aligned}
(\lambda x.t)s &\to_{\beta_T} t[x := s] \\
(\mu\alpha.t)\beta &\to_{\beta_S} t[\alpha := \beta] \\
\mu\alpha.(t)\alpha &\to_{\eta_S} t && (\alpha \notin FV(t)) \\
\mu\alpha.t &\to_{\mathrm{fst}} \lambda x.\mu\alpha.t[(v)\alpha := (v)x\alpha]
\end{aligned}$$

As in [2], the stream head normal form (shnf) of a $\Lambda\mu$-term is

$$\lambda \boldsymbol{x}^0 \mu\alpha^0 \cdots \lambda \boldsymbol{x}^{n-1} \mu\alpha^{n-1}.(y)\boldsymbol{t}^0\beta^0 \cdots \boldsymbol{t}^{m-1}\beta^{m-1},$$

where each $\boldsymbol{x}^i$ and $\boldsymbol{t}^j$ are finite sequences of λ-variables and $\Lambda\mu$-terms. For simplicity, we write $t \to_h^* h$ for the head reduction followed by zero- or one-step η_S-reduction to a shnf h.

Saurin showed in [2] that the Böhm trees are adapted to the $\Lambda\mu$-calculus by extending the width from ω to ω^2. We consider fully η-expanded form, and hence each node uniformly has ω^2 children. We call such trees *expanded Böhm trees* for the $\Lambda\mu$-calculus, which are originally defined by coinduction as follows:

$$\mathfrak{T} ::= \bot \mid \Lambda(x_i)_{i\in\omega^2}.(y)(\mathfrak{T}_j)_{j\in\omega^2},$$

where $\Lambda(x_i)_{i\in\omega^2}.(y)$ is called a prefix. Positions of nodes in trees are expressed by finite lists of elements of ω^2.

Definition 2. Δ is the set of all finite lists consisting of elements of ω^2, that is, $[\,] \in \Delta$ (empty list), and if $\delta \in \Delta$ and $\mu \in \omega^2$, then $\delta :: \mu \in \Delta$. $\delta \le \delta'$ means that δ is an initial segment of δ'. $\delta < \delta'$ means $\delta \le \delta'$ and $\delta \ne \delta'$.

By renaming bound variables, we suppose that bound λ-variables in prefixes are uniformly indexed by an element of $\Delta \times \omega^2$ depending on the position where they are abstracted, that is, we define the set of all bound λ variables as $BV_\lambda = \{x_\delta^\mu \mid \delta \in \Delta, \mu \in \omega^2\}$, and the prefix at the position δ is fixed as $\lambda \boldsymbol{x}_\delta^0 \boldsymbol{x}_\delta^1 \cdots\cdots .(y)$ with some head variable y, where $\boldsymbol{x}_\delta^i = x_\delta^{\omega\cdot i} x_\delta^{\omega\cdot i+1} x_\delta^{\omega\cdot i+2} \cdots$. We use this notation $\boldsymbol{x}_\delta^i$ in the following, and another notation $\boldsymbol{x}_\delta^{i,<j} = x_\delta^{\omega\cdot i} x_\delta^{\omega\cdot i+1} \cdots x_\delta^{\omega\cdot i+j-1}$.

In order to consider the fst-reduction, we have to remember some information on bound μ-variables during the definition of the expanded Böhm trees of $\Lambda\mu$-terms. In the following definition, $\phi(\delta, k) = l$ means that the prefix of the shnf at the position δ contains a subterm of the form $\cdots \lambda \boldsymbol{x}_\delta^{k,<l} \mu\alpha_\delta^k \cdots$. Similarly to BV_λ, we fix the name of bound μ-variables depending on the position where they are abstracted. The set of bound μ-variables is $BV_\mu = \{\alpha_\delta^k \mid \delta \in \Delta, k \in \omega\}$.

Then, names of head variables at each position and existence of shnf are sufficient to characterize expanded Böhm trees.

Definition 3. An *expanded Böhm tree* for $\Lambda\mu$-calculus is a mapping $\mathfrak{T}$ from Δ to $BV_\lambda \cup \{\bot\}$ such that $\mathfrak{T}(\delta) = \bot$ and $\delta' > \delta$ imply $\mathfrak{T}(\delta') = \bot$. The set of the expanded Böhm trees is denoted by $\Lambda\mu$-$\mathfrak{BT}^+$. We write $\mathfrak{T}(\delta) \uparrow$ to mean $\mathfrak{T}(\delta) = \bot$, and $\mathfrak{T}(\delta) \downarrow$ otherwise.

We can intuitively understand this definition as follows: $\mathfrak{T}(\delta) = x_{\delta'}^\mu$ means that the head variable in the prefix at δ is the μ-th variable in the prefix at δ', and $\mathfrak{T}(\delta) \uparrow$ means that the node at δ is $\bot$, which represents an unsolvable term, and we suppose that all nodes below $\bot$ are indexed by $\bot$ for simplicity.

Definition 4. For $t \in \Sigma_{\Lambda\mu}^c$, we define the *expanded Böhm tree* $\mathfrak{BT}_t^+$ of t with auxiliary partial mappings $t_{(\cdot)} : \Delta \to \Sigma_{\Lambda\mu} \cup \{\bot\}$ and $\phi_t : \Delta \times \omega \longrightarrow \omega$ as follows:

(0) $\mathfrak{T}$ is recursive, and there exist the following five partial recursive mappings:

- $\mathsf{p}_\mu^{\mathfrak{T}}, \mathsf{b}_\mu^{\mathfrak{T}} : \Delta \longrightarrow \omega$, the domains of which are $\{\delta \in \Delta \mid \mathfrak{T}(\delta)\downarrow\}$,
- $\mathsf{p}_\lambda^{\mathfrak{T}}, \mathsf{b}_\lambda^{\mathfrak{T}} : \Delta \times \omega \longrightarrow \omega$, and $\mathsf{Bd}_\mu^{\mathfrak{T}} : \Delta \times \omega \longrightarrow \Delta \times \omega$, the domains of which are $\{\langle \delta, k\rangle \in \Delta \times \omega \mid \mathfrak{T}(\delta)\downarrow\}$.

(1) $\mathfrak{T}(\delta) = x_{\delta'}^{\mu} \Longrightarrow \delta' \leq \delta$ and $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle \Longrightarrow \delta' \leq \delta$
(2) $\mathfrak{T}(\delta :: (\omega \cdot k + l)) = x_{\delta'}^{\omega \cdot k' + l'}$ & $l < \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k) \Longrightarrow l' < \mathsf{p}_\lambda^{\mathfrak{T}}(\delta', k')$
(3) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ & $k < \mathsf{b}_\mu^{\mathfrak{T}}(\delta) \Longrightarrow k' < \mathsf{p}_\mu^{\mathfrak{T}}(\delta')$
(4) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ & $l \geq \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k) \Longrightarrow$

- $\mathfrak{T}(\delta :: (\omega \cdot k + l)) = x_{\delta'}^{\omega \cdot k' + (\mathsf{p}_\lambda^{\mathfrak{T}}(\delta', k') + l - \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k))}$
- $\delta'' \geq \delta :: (\omega \cdot k + l) \Longrightarrow \mathfrak{T}(\delta'' :: \mu) = x_{\delta''}^{\mu}$ for any μ & $\mathsf{p}_\mu^{\mathfrak{T}}(\delta'') = 0$ & $\mathsf{b}_\mu^{\mathfrak{T}}(\delta'') = 0$

(5) $\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, \mathsf{p}_\mu^{\mathfrak{T}}(\delta) + n) = 0$ and $\mathsf{b}_\lambda^{\mathfrak{T}}(\delta, \mathsf{b}_\mu^{\mathfrak{T}}(\delta) + n) = 0$ for any $n \in \omega$
(6) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, \mathsf{b}_\mu^{\mathfrak{T}}(\delta) + n) = \langle \delta, \mathsf{p}_\mu^{\mathfrak{T}}(\delta) + n\rangle$ for any $n \in \omega$

Fig. 1. Characterizing Conditions

- $t_{[]} = t$
- If t_δ has no shnf, $\mathfrak{BT}_t^+(\delta) = \bot$, and $t_{\delta'}$ for any $\delta < \delta'$ and $\phi_t(\delta'', k)$ for any $\delta \leq \delta''$ and $k \in \omega$ are undefined.
- If $t_\delta \to_h^* \lambda \boldsymbol{x}_\delta^{0,<i_0} \mu \alpha_\delta^0 \cdots \lambda \boldsymbol{x}_\delta^{n-1,<i_{n-1}} \mu \alpha_\delta^{n-1}.(y)\boldsymbol{t}^0 \alpha_{\delta_0}^{j_0} \cdots \boldsymbol{t}^{m-1} \alpha_{\delta_{m-1}}^{j_{m-1}}$, then

$$\mathfrak{BT}_t^+(\delta) = y$$

$$\phi_t(\delta, k) = \begin{cases} i_k & (k < n) \\ 0 & (k \geq n) \end{cases}$$

$$t_{\delta :: (\omega \cdot k + l)} = \begin{cases} t_k^l & (k < m \ \&\ \boldsymbol{t}_k = t_k^0 \cdots t_k^{i_k - 1} \ \&\ l < i_k) \\ x_{\delta_k}^{\omega \cdot j_k + (\phi_t(\delta_k, j_k) + l - i_k)} & (k < m \ \&\ \boldsymbol{t}_k = t_k^0 \cdots t_k^{i_k - 1} \ \&\ l \geq i_k) \\ x_\delta^{\omega \cdot (k - m + n) + l} & (k \geq m) \end{cases}$$

Note that $\delta_k \leq \delta$ holds for $0 \leq k < m$ since $\alpha_{\delta_k}^{j_k}$ is a bound μ-variable in t.

3 Characterization

3.1 Characterization of Expanded Böhm Trees for $\Lambda\mu$-Terms

For each $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{BT}^+$, we consider the conditions in Figure 1. The intuitive meaning of the partial mappings is the following: $\mathsf{p}_\mu^{\mathfrak{T}}(\delta)$ is the number of μ-abstractions in the prefix of the shnf of t_δ. $\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, k)$ is the number of λ-abstractions surrounding the k-th μ-variable in the prefix of the shnf of t_δ. $\mathsf{b}_\mu^{\mathfrak{T}}(\delta)$ is the number of μ-variables in the body part of the shnf of t_δ. $\mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k)$ is the number of term arguments delimited by the k-th μ-variable in the body part of the shnf of t_δ. $\mathsf{b}_\lambda^{\mathfrak{T}}$ corresponds to the function ϕ_t. $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ means that the k-th μ-variable in the body part of the shnf of t_δ is bound at k'-th μ-abstraction in the prefix of the shnf of t'_δ.

The conditions are intuitively explained as follows. (1) means that each variable is bound at a position outside of it. (2) and (3) require that, if a variable occurs in a body part of a shnf, it is bound at a prefix which is not in an η-expanded part, which means a part obtained by the η_S-expansion and the fst-reduction. (4) means that, if a λ-variable occurs in an η-expanded part, it is obtained a fst-reduction for a corresponding μ-variable, and all of the nodes below it are in η-expanded parts. (5) means that, if a μ-variable is in an η-expanded part, there is no λ-variable accompanying the μ-variable. (6) means that, if a μ-variable occurs in an η-expanded part, the μ-variables following it are obtained by the η_S-expansion.

Theorem 1. *For $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{BT}^+$, there exists a closed $\Lambda\mu$-term t such that $\mathfrak{T} = \mathfrak{BT}_t^+$ iff $\mathfrak{T}$ satisfies all of the conditions in Figure 1.*

Proof. (The detailed proof is in Section A.)

Let $\mathfrak{T} = \mathfrak{BT}_t^+$. By definition, when $\mathfrak{T}(\delta) = y$, we have

$$t_\delta \to_h^* \lambda \boldsymbol{x}_\delta^{0,<i_0} \mu\alpha_\delta^0 \cdots \mu\alpha_\delta^{k-2} \lambda x_\delta^{k-1,<i_{k-1}} \mu\alpha_\delta^{k-1}. \\ (y)t^0 \cdots t^{j_0} \alpha_{\delta_0}^{j_0} \cdots \alpha_{\delta_{l-2}}^{j_{l-2}} t^{\omega\cdot(l-1)} \cdots t^{\omega\cdot(l-1)+j_{l-1}-1} \alpha_{\delta_{l-1}}^{j_{l-1}}.$$

Then we define the partial mappings in the condition (0) as follows:

$$\mathsf{p}_\mu^{\mathfrak{T}}(\delta) = k \qquad \mathsf{b}_\mu^{\mathfrak{T}}(\delta) = l$$

$$\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, n) = \begin{cases} i_n & (n < k) \\ 0 & (n \geq k) \end{cases} \qquad \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, n) = \begin{cases} j_n & (n < l) \\ 0 & (n \geq l) \end{cases}$$

$$\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, n) = \begin{cases} \langle \delta_n, j_n \rangle & (n < l) \\ \langle \delta, n - l + k \rangle & (n \geq l). \end{cases}$$

They are undefined when $\mathfrak{T}(\delta) \uparrow$. They are recursive by definition. It is easy to see that $\mathfrak{T}$ satisfies the conditions (0) through (6).

For the other direction, suppose that $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{BT}^+$ satisfies all of the conditions in Figure 1, and we will construct a term $t^{\mathfrak{T}}$ such that $\mathfrak{BT}_{t^{\mathfrak{T}}}^+ = \mathfrak{T}$. In the following, we omit the superscript $\mathfrak{T}$ for each mapping and term.

We have the encodings of elements of ω, ω^2, Δ, BV, and pairs of them in the λ-calculus. These encodings are overlined. By (0), we have λ-representations of $\mathfrak{T}$ and the five partial recursive functions: $\overline{\mathfrak{T}}$, $\overline{\mathsf{p}_\mu}$, $\overline{\mathsf{p}_\lambda}$, $\overline{\mathsf{b}_\mu}$, $\overline{\mathsf{b}_\lambda}$, and $\overline{\mathsf{Bd}_\mu}$. Furthermore we can assume the existence of the following λ-term π:

$$\pi\, \overline{\delta} \to_h^* \begin{cases} \lambda z.z & (\mathfrak{T}(\delta) \downarrow) \\ \text{has no hnf} & (\mathfrak{T}(\delta) \uparrow) \end{cases}$$

We define association lists L_λ and L_μ to map correspondences between actual bound variables and their encodings.

$$\overline{\mathsf{init}}_\lambda = \lambda z.z \qquad [\langle \delta, \mu \rangle \mapsto y]@L_\lambda = \lambda z.(\mathsf{if}\ z = \overline{x_\delta^\mu}\ \mathsf{then}\ y\ \mathsf{else}\ Lz\ \mathsf{fi})$$

$$\overline{\mathsf{init}}_\mu = \lambda z.z \qquad [\langle \delta, k \rangle \mapsto \alpha]@L_\mu = \lambda pz.(\mathsf{if}\ p = \langle \overline{\delta}, \overline{k} \rangle\ \mathsf{then}\ (z)\alpha\ \mathsf{else}\ L_\mu pz\ \mathsf{fi})$$

The term t is recursively defined as follows:

$$
\begin{aligned}
t &= \Theta\overline{[\,]}\,\overline{\mathsf{init}_\lambda}\,\overline{\mathsf{init}_\mu} \\
\Theta\,\overline{\delta}L_\lambda L_\mu &= \pi\,\overline{\delta}\,(F\overline{\delta}\,\overline{0}\,\overline{0}\,(\overline{\mathfrak{T}}\,\overline{\delta})L_\lambda L_\mu) \\
F\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} G\overline{\delta}\,(\overline{\mathsf{b}_\mu}\,\overline{\delta})\,\overline{0}VL_\lambda L_\mu & (\mathsf{p}_\mu(\delta) \geq k) \\ \mu\alpha.F\overline{\delta}\,\overline{k+1}\,\overline{0}VL_\lambda([\langle\delta,k\rangle \mapsto \alpha]@L_\mu) & (\mathsf{p}_\lambda(\delta,k) \leq l) \\ \lambda z.F\overline{\delta}\,\overline{k}\,\overline{l+1}V([\langle\delta,\omega\cdot k+l\rangle \mapsto z]@L_\lambda)L_\mu & (\text{otherwise}) \end{cases} \\
G\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} L_\lambda V & (k = 0\ \&\ l = 0) \\ L_\mu(\overline{\mathsf{Bd}_\mu}\,\overline{\delta}\,\overline{k-1})(G\overline{\delta}\,\overline{k-1}\,(\overline{\mathsf{b}_\lambda}\,\overline{\delta}\,\overline{k-1})VL_\lambda L_\mu) & (k > 0\ \&\ l = 0) \\ (G\overline{\delta}\,\overline{k}\,\overline{l-1}VL_\lambda L_\mu)(\Theta\overline{\delta :: (\omega\cdot k+(l-1))}L_\lambda L_\mu) & (l > 0) \end{cases}
\end{aligned}
$$

Then, we can see that $\mathfrak{T}(\delta) = \mathfrak{BT}^+_t(\delta)$ for any $\delta \in \Delta$. $\square$

3.2 Free $\boldsymbol{\lambda}$-Variables

The discussion in the previous subsection can be extended to $\Lambda\mu$-terms with free λ-variables. We suppose the set of free λ-variables FV_λ, which is disjoint from BV_λ. The codomain of $\Lambda\mu$-$\mathfrak{BT}^+$ is extended to $BV_\lambda \cup FV_\lambda \cup \{\bot\}$. We define $FV_\lambda(\mathfrak{T}) = \{z \in FV_\lambda \mid \mathfrak{T}(\delta) = z \text{ for some } \delta\}$, and we require the following additional condition.

(7) $\#FV_\lambda(\mathfrak{T}) < \omega$

Then, the encoding of the variables are extended to

$$
\overline{y} = \begin{cases} \overline{y} & (y \in BV_\lambda) \\ y & (y \in FV_\lambda(\mathfrak{T})), \end{cases}
$$

which can be defined due to the condition (7). Notice that for any association list L_λ of λ-variables and $y \in FV_\lambda$, we have $L_\lambda\, y \to_h^* \overline{\mathsf{init}_\lambda}\, y \to_h y$.

References

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2. A. Saurin. Standardization and Böhm trees for $\Lambda\mu$-calculus. In Blume, M., Kobayashi, N., and Vidal, G., editors, *Tenth International Symposium on Functional and Logic Programming (FLOPS 2010)*, volume 6009 of *LNCS*, pages 134–149. Springer, 2010.

A Proof of Main Theorem

Under the condition (0), we use the following terminology: δ is said to be in the *expanded part* if $\delta = \delta' :: (\omega \cdot k + l)$ such that $\mathfrak{T}(\delta') \downarrow$ and $l \geq \mathsf{b}_\lambda^\mathfrak{T}(\delta', k)$, and δ is said to be in the *finite part* if $\mathfrak{T}(\delta') \downarrow$ for any $\delta' < \delta$ and δ is not in the expanded part. In particular, $[\,]$ is always in the finite part.

Lemma 1. *Suppose that $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{BT}^+$ satisfies all of the conditions.*

1. If δ is in the expanded part and $\delta' > \delta$, then δ' is in the expanded part.

2. If δ is in the finite part and $\delta' < \delta$, then δ' is in the finite part.

3. If δ is in the expanded part, then for any k we have $\mathsf{p}_\mu^\mathfrak{T}(\delta) = \mathsf{b}_\mu^\mathfrak{T}(\delta) = \mathsf{p}_\lambda^\mathfrak{T}(\delta, k) = \mathsf{b}_\lambda^\mathfrak{T}(\delta, k) = 0$, and $\mathsf{Bd}_\mu^\mathfrak{T}(\delta, k) = \langle \delta, k \rangle$.

Proof. 1. Since δ is in the expanded part, we have $\delta = \delta_1 :: (\omega \cdot k_1 + l_1)$ such that $\mathfrak{T}(\delta_1) \downarrow$ and $l_1 \geq \mathsf{b}_\lambda^\mathfrak{T}(\delta_1, k_1)$. Let $\delta' = \delta_2 :: (\omega \cdot k_2 + l_2) < \delta$. Since $\delta_2 \geq \delta_1 :: (\omega \cdot k_1 + l_1)$, by (4), we have $\mathfrak{T}(\delta_2) \downarrow$ and $\mathsf{b}_\mu^\mathfrak{T}(\delta_2) = 0$. By (5), we have $\mathsf{b}_\lambda^\mathfrak{T}(\delta_2, n) = 0$ for any n, and hence $l_2 \geq \mathsf{b}_\lambda^\mathfrak{T}(\delta_2, k_2)$. Therefore, δ' is in the expanded part.

2. If δ' is in the expanded part, then δ must be in the expanded part by 1. If $\mathfrak{T}(\delta') \uparrow$, then $\mathfrak{T}(\delta)$ must be $\perp$.

3. By (4), we have $\mathsf{p}_\mu^\mathfrak{T}(\delta) = \mathsf{b}_\mu^\mathfrak{T}(\delta) = 0$. Then, we have $\mathsf{p}_\lambda^\mathfrak{T}(\delta, k) = \mathsf{b}_\lambda^\mathfrak{T}(\delta, k) = 0$ by (5), and $\mathsf{Bd}_\mu^\mathfrak{T}(\delta, k) = \langle \delta, k \rangle$.

Theorem 2. *For a partial map $\mathfrak{T} : \Delta \longrightarrow BV$, $\mathfrak{T}$ is an expanded Böhm tree of a closed $\Lambda\mu$-term iff $\mathfrak{T}$ satisfies all of the conditions in Figure 1.*

Proof. Let $\mathfrak{T} = \mathfrak{BT}_+(t)$. By the definition of $\mathfrak{BT}_+$, when $\mathfrak{T}(\delta) = y$, we have

$$\begin{aligned} t_\delta \to_h^* & \lambda \boldsymbol{x}_\delta^{0,<i_0} \mu\alpha_\delta^0 \cdots \mu\alpha_\delta^{k-2} \lambda x_\delta^{k-1,<i_{k-1}} \mu\alpha_\delta^{k-1}. \\ & \quad (y) t^0 \cdots t^{j_0} \beta^0 \cdots \beta^{l-2} t^{\omega\cdot(l-1)} \cdots t^{\omega\cdot(l-1)+j_{l-1}-1} \beta^{l-1}. \end{aligned}$$

Then we define the partial maps in the condition (0) as follows:

$$\mathsf{p}_\mu^\mathfrak{T}(\delta) = k \qquad \mathsf{b}_\mu^\mathfrak{T}(\delta) = l$$

$$\mathsf{p}_\lambda^\mathfrak{T}(\delta, n) = \begin{cases} i_n & (n < k) \\ 0 & (n \geq k) \end{cases} \qquad \mathsf{b}_\lambda^\mathfrak{T}(\delta, n) = \begin{cases} j_n & (n < l) \\ 0 & (n \geq l) \end{cases}$$

$$\mathsf{Bd}_\mu^\mathfrak{T}(\delta, n) = \begin{cases} \langle \delta', m \rangle & (n < l \ \&\ \beta^n \equiv \alpha_{\delta'}^m) \\ \langle \delta, n - l + k \rangle & (n \geq l). \end{cases}$$

They are undefined when $\mathfrak{T}(\delta) \uparrow$. They are recursive by definition. It is easy to see that $\mathfrak{T}$ satisfies the condition (0) through (6).

For the other direction, suppose that $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{BT}^+$ satisfies all of the conditions in Figure 1, and we will construct a term $t^\mathfrak{T}$ such that $\mathfrak{BT}_{t^\mathfrak{T}}^+ = \mathfrak{T}$. In the following, we omit the superscript $\mathfrak{T}$ for each map and term.

(I) *Construction of t.* We have the encodings of elements of ω, ω^2, Δ, BV, and pairs of them in the λ-calculus. These encodings are denoted by overline.

By (0), we have λ-representations of $\mathfrak{T}$ and the five partial recursive functions: $\overline{\mathfrak{T}}$, $\overline{\mathsf{p}_\mu}$, $\overline{\mathsf{p}_\lambda}$, $\overline{\mathsf{b}_\mu}$, $\overline{\mathsf{b}_\lambda}$, and $\overline{\mathsf{Bd}_\mu}$. Furthermore we can assume the existence of the following λ-term π:

$$\pi\,\overline{\delta} \to_h^* \begin{cases} \lambda z.z & (\mathfrak{T}(\delta)\downarrow) \\ \text{has no hnf} & (\mathfrak{T}(\delta)\uparrow) \end{cases}$$

We define association lists L_λ and L_μ to memory correspondence between actual variables and their encodings.

$$\begin{array}{ll} \overline{\mathsf{init}}_\lambda = \lambda z.z & [\langle\delta,\mu\rangle \mapsto y]@L_\lambda = \lambda z.(\mathsf{if}\ z = \overline{x_\delta^\mu}\ \mathsf{then}\ y\ \mathsf{else}\ Lz\ \mathsf{fi}) \\ \overline{\mathsf{init}}_\mu = \lambda z.z & [\langle\delta,k\rangle \mapsto \alpha]@L_\mu = \lambda pz.(\mathsf{if}\ p = \langle\overline{\delta},\overline{k}\rangle\ \mathsf{then}\ (z)\alpha\ \mathsf{else}\ L_\mu pz\ \mathsf{fi}) \end{array}$$

Then, the term t is recursively defined as follows:

$$\begin{aligned} t &= \Theta\overline{[\,]}\,\overline{\mathsf{init}}_\lambda\,\overline{\mathsf{init}}_\mu \\ \Theta\,\overline{\delta}L_\lambda L_\mu &= \pi\,\overline{\delta}\,(F\overline{\delta}\,\overline{0}\,\overline{0}\,(\overline{\mathfrak{T}}\,\overline{\delta})L_\lambda L_\mu) \\ F\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} G\overline{\delta}\,(\overline{\mathsf{b}_\mu}\,\overline{\delta})\,\overline{0}VL_\lambda L_\mu & (\mathsf{p}_\mu(\delta) \geq k) \\ \mu\alpha.F\overline{\delta}\,\overline{k+1}\,\overline{0}VL_\lambda([\langle\delta,k\rangle \mapsto \alpha]@L_\mu) & (\mathsf{p}_\lambda(\delta,k) \leq l) \\ \lambda z.F\overline{\delta}\,\overline{k}\,\overline{l+1}V([\langle\delta,\omega\cdot k+l\rangle \mapsto z]@L_\lambda)L_\mu & (\text{otherwise}) \end{cases} \\ G\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} L_\lambda V & (k = 0\ \&\ l = 0) \\ L_\mu(\overline{\mathsf{Bd}_\mu}\,\overline{\delta}\,\overline{k-1})(G\overline{\delta}\,\overline{k-1}\,(\overline{\mathsf{b}_\lambda}\,\overline{\delta}\,\overline{k-1})VL_\lambda L_\mu) & (k > 0\ \&\ l = 0) \\ (G\overline{\delta}\,\overline{k}\,\overline{l-1}VL_\lambda L_\mu)(\Theta\overline{\delta :: (\omega\cdot k+(l-1))}L_\lambda L_\mu) & (l > 0) \end{cases} \end{aligned}$$

(II) We prove the following by induction on δ: $t_\delta = \Theta\,\overline{\delta}\,L_\lambda^{\delta-}\,L_\mu^{\delta-}$ for any δ in the finite part, where $L_\lambda^{\delta-}$ is constructed from $\overline{\mathsf{init}}_\lambda$ by adding $[\langle\delta',\omega\cdot k+l\rangle \mapsto x_{\delta'}^{\omega\cdot k+l}]$ for any $\delta' < \delta$ and $k,l \in \omega$ such that $k < \mathsf{p}_\mu(\delta)$ and $l < \mathsf{p}_\lambda(\delta,k)$, and $L_\mu^{\delta-}$ is constructed from $\overline{\mathsf{init}}_\mu$ by adding $[\langle\delta',k\rangle \mapsto \alpha_{\delta'}^k]$ for any $\delta' < \delta$ and $k < \mathsf{p}_\mu(\delta')$.

(Case $[\,]$) By definition.

(Case $\delta :: \mu$) Let μ be $\omega\cdot k+l$. By the conditions, δ is also in the finite part. By IH, we have $t_\delta = \Theta\,\overline{\delta}\,L_\lambda^{\delta-}\,L_\mu^{\delta-}$. Since $\delta :: \mu$ is in the finite part, we have $\mathfrak{T}(\delta)\downarrow$ and $l < \mathsf{b}_\lambda(\delta,k)$. By $\pi\,\overline{\delta} \to_h^* \lambda z.z$, we have

$$\begin{aligned} t_\delta &\to_h^* F\overline{\delta}\,\overline{0}\,\overline{0}\,(\overline{\mathfrak{T}}\,\overline{\delta})L_\lambda^{\delta-}L_\mu^{\delta-} \\ &\to_h^* \lambda\boldsymbol{x}_\delta^{0,<\mathsf{p}_\lambda(\delta,0)}\mu\alpha_\delta^0\lambda\boldsymbol{x}_\delta^{1,<\mathsf{p}_\lambda(\delta,1)}\mu\alpha_\delta^1\cdots\mu\alpha_\delta^{\mathsf{p}_\mu(\delta)-1}.G\overline{\delta}\,(\overline{\mathsf{b}_\mu}\,\overline{\delta})\,\overline{0}(\overline{\mathfrak{T}}\overline{\delta})L_\lambda^\delta L_\mu^\delta, \end{aligned}$$

where

- L_λ^δ is constructed from $L_\lambda^{\delta-}$ by adding $[\langle\delta,\omega\cdot i+j\rangle \mapsto x_\delta^{\omega\cdot i+j}]$ for any i,j such that $i < \mathsf{p}_\mu(\delta)$ and $j < \mathsf{p}_\lambda(\delta,i)$, and
- L_μ^δ is constructed from $L_\mu^{\delta-}$ by adding $[\langle\delta,i\rangle \mapsto \alpha_\delta^i]$ for any $i < \mathsf{p}_\mu(\delta)$.

Then the body part is reduced to $(L_\lambda^\delta(\overline{\mathfrak{T}}\,\overline{\delta}))\boldsymbol{t}^0\beta^0\boldsymbol{t}^1\beta^1\cdots\beta^{\mathsf{b}_\mu(\delta)-1}$ by the head reduction, where

– $\beta^i \equiv \alpha_{\delta'}^{i'}$ when $\mathsf{Bd}_\mu(\delta, i) = \langle \delta', i' \rangle$, where the lookup for L_μ^δ always succeeds by the conditions (1) and (3).
– $\boldsymbol{t}^i \equiv (\Theta\, \overline{\delta :: (\omega \cdot i)}\, L_\lambda^\delta\, L_\mu^\delta) \cdots (\Theta\, \overline{\delta :: (\omega \cdot i + (\mathsf{b}_\lambda(\delta, i) - 1))}\, L_\lambda^\delta\, L_\mu^\delta)$.

For the head variable, $L_\lambda^\delta(\overline{\mathfrak{T}}\,\overline{\delta})$ is reduced to $\mathfrak{T}(\delta)$ since the lookup always succeeds by the conditions (1) and (2). Hence, by the head reduction, t_δ is reduced to the shnf

$$\lambda \boldsymbol{x}_\delta^{0,<\mathsf{p}_\lambda(\delta,0)} \mu \alpha_\delta^0 \cdots \mu \alpha_\delta^{\mathsf{p}_\mu(\delta)-1}.(\mathfrak{T}(\delta))\boldsymbol{t}^0 \beta^0 \boldsymbol{t}^1 \beta^1 \cdots \beta^{\mathsf{b}_\mu(\delta)-1}.$$

Since $l < \mathsf{b}_\lambda(\delta, k)$, $t_{\delta::\mu}$ is the l-th element in $\boldsymbol{t}^k$, which is $(\Theta\, \overline{\delta :: (\omega \cdot k + l)}\, L_\lambda^\delta\, L_\mu^\delta)$.

(III) We prove the following by induction on δ: (a) $t_\delta = \mathfrak{T}(\delta)$ for any δ in the expanded part, and (b) $\mathsf{p}_\lambda(\delta, k) = \phi_t(\delta, k)$ for any δ such that $\mathfrak{T}(\delta)\downarrow$ and any k.

(a) By the definition of the expanded part, δ has the form $\delta' :: (\omega \cdot k + l)$ such that $l \geq \mathsf{b}_\lambda(\delta', k)$.

(Case δ' in the finite part) By (II), $t_{\delta'} = \Theta\, \overline{\delta'}\, L_\lambda^{\delta'-}\, L_\mu^{\delta'-}$. Since $\mathfrak{T}(\delta')\downarrow$, $t_{\delta'}$ has a shnf, and its body part is $(\mathfrak{T}(\delta'))\boldsymbol{t}^0\beta^0\boldsymbol{t}^1\beta^1 \cdots \beta^{\mathsf{b}_\mu(\delta')-1}$ as discussed in (II). We consider further case analysis as follows.

(Subcase $k < \mathsf{b}_\mu(\delta')$) For $\mathsf{Bd}_\mu(\delta', k) = \langle \delta'', j'' \rangle$, that means $\beta^k \equiv \alpha_{\delta''}^{j''}$, we have $t_\delta = x_{\delta''}^{\omega \cdot j'' + (\phi_t(\delta'', j'') + l - \mathsf{b}_\lambda(\delta', k))}$ by definition. Since $\delta'' \leq \delta'$, we have $\mathfrak{T}(\delta'')\downarrow$. By IH(b) for δ'', we have $t_\delta = x_{\delta''}^{\omega \cdot j'' + (\mathsf{p}_\lambda(\delta'', j'') + l - \mathsf{b}_\lambda(\delta', k))}$, which is identical to $\mathfrak{T}(\delta)$ by (4).

(Subcase $k \geq \mathsf{b}_\mu(\delta')$) By definition, we have $t_\delta = x_{\delta'}^{\omega \cdot (k - \mathsf{b}_\mu(\delta') + \mathsf{p}_\mu(\delta')) + l}$. We have $\mathsf{Bd}_\mu(\delta', k) = \langle \delta', \mathsf{p}_\mu(\delta') + (k - \mathsf{b}_\mu(\delta')) \rangle$ by (6), and we have $\mathsf{p}_\lambda(\delta', \mathsf{p}_\mu(\delta') + (k - \mathsf{b}_\mu(\delta'))) = 0$ and $\mathsf{b}_\lambda(\delta', k) = 0$ by (5). Hence, we have $t_\delta = \mathfrak{T}(\delta)$ by (4).

(Case δ' in the expanded part) By IH(a), t'_δ is a variable, say y. By definition, we have $t_\delta = x_{\delta'}^{\omega \cdot k + l}$. Since δ' is in the expanded part, we have $\mathfrak{T}(\delta) = x_{\delta'}^{\omega \cdot k + l}$ by Lemma 1.3 and (4).

(b) When δ is in the finite part, we have t_δ has a shnf, and its prefix is $\lambda \boldsymbol{x}_\delta^{0,<\mathsf{p}_\lambda(\delta,0)} \mu \alpha_\delta^0 \lambda \boldsymbol{x}_\delta^{1,<\mathsf{p}_\lambda(\delta,1)} \mu \alpha_\delta^1 \cdots \mu \alpha_\delta^{\mathsf{p}_\mu(\delta)-1}$, and $\mathsf{p}_\lambda(\delta, k) = 0$ for $k \geq \mathsf{p}_\mu(\delta)$. Hence we have $\phi_t(\delta, k) = \mathsf{p}_\lambda(\delta, k)$.

When δ is in the expanded part, by (a), t_δ is a variable, that is, its prefix is empty. By Lemma 1.3, we have $\mathsf{p}_\lambda(\delta, k) = 0$ for any k, so we have $\phi_t(\delta, k) = \mathsf{p}_\lambda(\delta, k) = 0$.

(IV) We prove $\mathfrak{B}\mathfrak{T}_t^+(\delta) = \mathfrak{T}(\delta)$ for any $\delta \in \Delta$.

(Case δ in the finite part) By (II), we have $t_\delta = \Theta\, \overline{\delta}\, L_\lambda^{\delta-}\, L_\mu^{\delta-}$. If $\mathfrak{T}(\delta) = y$, we can see similarly to (II) that t_δ has a shnf the head variable of which is y. If $\mathfrak{T}(\delta)\uparrow$, t_δ has no shnf since $\pi\, \overline{\delta}$ has no shnf. Hence $\mathfrak{B}\mathfrak{T}_t^+(\delta) = \bot$.

(Case $\delta :: \omega \cdot k + l$ in the expanded part) By (III).

Otherwise, $\mathfrak{T}(\delta')\uparrow$ holds for some $\delta' < \delta$, and then $\mathfrak{B}\mathfrak{T}_t^+(\delta) = \mathfrak{T}(\delta) = \bot$.

□