An isomorphism between cut-elimination procedure and proof reduction

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Abstract. This paper introduces a cut-elimination procedure of the intuitionistic sequent calculus and shows that it is isomorphic to the proof reduction of the intuitionistic natural deduction with general elimination and explicit substitution. It also proves strong normalization and Church-Rosser property of the cut-elimination procedure by projecting the sequent calculus to the natural deduction with general elimination without explicit substitution.

1 Introduction

The Curry-Howard isomorphism between proof reduction and program computation is a useful tool to study logical systems and calculus systems. The correspondence has been investigated for the intuitionistic natural deduction and the λ-calculus, but that for the sequent calculus has not been studied enough and this research area is still developing.

To clarify the computational meaning of the sequent calculus, one of the most favorable approach is to study the relationship between the sequent calculus and the natural deduction, because the computational aspect of the natural deduction is relatively clear by the Curry-Howard isomorphism. Gentzen, who introduced the sequent calculus and the natural deduction, gave translations from proofs of each system to those of the other [5]. Prawitz gave a many-one mapping from proofs of the sequent calculus to those of the natural deduction [11]. Zucker studied on a correspondence between the cut-elimination in the sequent calculus and the proof reduction in the natural deduction [16]. Herbelin introduced a term reduction system for a variant of sequent calculus, called LJT [6], and gave a one-to-one correspondence between cut-free proofs in his sequent calculus and normal proofs in the natural deduction. He also showed his cut-elimination steps includes propagation steps of explicit substitution [1]. Herbelin style formulation of the sequent calculus has been widely studied [4, 9]. For the original style sequent calculus, Urban and Bierman proposed a cut-elimination procedure for the classical sequent calculus [13, 14], and proved its strong normalization. In particular, Urban [13] investigated a local-step cut-elimination procedure of Gentzen style sequent calculus, where “local-step” means that each cut-elimination step is a local transformation of proofs. He gave translations between sequent calculus and natural deduction in intuitionistic and classical cases, but neither of them is an isomorphism. Kikuchi introduced a term assignment for the intuitionistic sequent calculus and its local step cut-elimination procedure [10]. He defined a subclass of proofs of the sequent calculus, called *pure terms*, which corresponds to proofs of the natural deduction. He also showed the cut-elimination can simulates the proof reduction of pure terms. However, strong normalization of the cut-elimination was not shown in [10]. Von Plato gave a correspondence between the sequent calculus and the natural deduction with general elimination rules [15]. However, the relationship between cut-elimination steps and proof reduction steps was not studied.

This paper gives a cut-elimination procedure for a Gentzen style intuitionistic sequent calculus LJ and an isomorphism between it and a proof normalization for the natural deduction with general elimination and explicit substitution. The computational meaning of the natural deduction with general elimination rules is relatively easy to understand. Indeed, we can find a reduction preserving continuation passing style (CPS) translation which gives an interpretation of our system in the well-known simply typed λ-calculus. This paper also proves strong normalization (SN) and Church-Rosser property (CR) of the cut-elimination of LJ. Though our system can be seen as a confluent subsystem of Urban's classical sequent calculus [13], SN of our system is proved by another method with a modified CPS-translation. In the chapter 7 of [12], we can find Dragalin's simple proof of SN of a sequent calculus. We cannot apply his proof to our sequent calculus due to the rule (π) corresponding to the permutative conversion.

First, we introduce a cut-elimination procedure of LJ and systems $\mathsf{LJ_p}$ and Λ_g. $\mathsf{LJ_p}$ is a subsystem of LJ which includes only a particular type of cuts, called *principal cuts* (*p-cuts*). $\mathsf{LJ_p}$ is not closed under the cut-elimination procedure of LJ, so we define *cut-elimination strategies* for cut-elimination of p-cuts. Λ_g is a simply typed λ-calculus with general elimination and permutative conversion. By the Curry-Howard isomorphism, Λ_g corresponds to the natural deduction with the general elimination rules. We show that $\mathsf{LJ_p}$ and Λ_g are isomorphic as reduction systems, where p-cuts and left-rules correspond to general eliminations. In particular, this isomorphism also gives a one-to-one correspondence between cut-free LJ-proofs and normal Λ_g-proofs. Secondly, we show SN and CR of LJ. These are proved by reducing to those of Λ_g. Joachimski and Matthes [8] proved SN of Λ_g by an inductive characterization of the set of SN terms. In this paper, we give another proof by Ikeda and Nakazawa's CGPS-translation method [7]. Then, SN of LJ is proved by Bloo's method [2]. Finally, we define Λ_{gx}, which is Λ_g with explicit substitution, and show that LJ is isomorphic to Λ_{gx} modulo a term quotient and that Λ_{gx} enjoys SN and CR. The figure 1 summarizes the relationship of systems in this paper, where Λ is the simply typed λ-calculus, Λ^p_{gx} is the modified Λ_{gx}, and horizontal arrows represent projection maps.

2 Definitions of systems

In this section, we define the intuitionistic sequent calculus with a cut-elimination procedure and the simply typed λ-calculus with the general elimination rules.

$$
\begin{tikzcd}
 & \mathsf{LJ} \arrow[r] \arrow[d, phantom, "\wr"] & \mathsf{LJ_p} \arrow[d, phantom, "\wr"] & \\
\Lambda_{gx} \arrow[r] & \Lambda^{p}_{gx} & \Lambda_{g} \arrow[r] & \Lambda
\end{tikzcd}
$$

Fig. 1. Relationship of systems

2.1 LJ: Intuitionistic sequent calculus

Definition 1 (LJ)**.** LJ *consists of the following.*

1. *We suppose that there are countable* atomic formulas. Formulas *(denoted by A, $B\ldots$) are defined as*
$$A ::= p \mid A \to A,$$
where p denotes an atomic formula.
2. Term variables *are denoted by x, y, z ....* Pseudo-terms *(denoted by* M, N, $\mathsf{P}\ldots$*) are defined as*
$$\mathsf{M} ::= x \mid \underline{\lambda} x.\mathsf{M} \mid x[\![\mathsf{M}, x.\mathsf{M}]\!] \mid \mathsf{M} \curlyvee^{x} \mathsf{M}.$$
In terms $\underline{\lambda} x.\mathsf{M}$, $y[\![\mathsf{N}, x.\mathsf{M}]\!]$ and $\mathsf{N} \curlyvee^{x} \mathsf{M}$, variable occurrences of x in M are bound. Renaming bound variables is admitted as usual. We write $\underline{x}[\![\mathsf{M}, y.\mathsf{P}]\!]$ for $x[\![\mathsf{M}, y.\mathsf{P}]\!]$ if the variable x does not freely occur in $[\![\mathsf{M}, y.\mathsf{P}]\!]$ following [9].
3. Contexts *(denoted by Γ, $\Delta\ldots$) are sets of formulas labeled by term variables, such as $\Gamma = \{A_1^{x_1}, A_2^{x_2} \cdots\}$. We also write A^x for the singleton $\{A^x\}$.* Judgments *have the form of $\Gamma \vdash \mathsf{M} : A$.* Proofs *are defined by the inference rules in the figure 2, where $\cup$ and $\setminus$ are usual set-theoretical operators, union and difference. Each context in derivation trees has to meet the following condition: for any two distinct elements A^x, B^y in the context, x and y are distinct variables. A pseudo-term M is an* LJ-term *iff $\Gamma \vdash \mathsf{M} : A$ is derivable for some Γ and A.*

$$\frac{}{A^x \vdash x : A}\ (\mathsf{Ax}) \qquad \frac{\Gamma \vdash \mathsf{M} : A \quad \Delta \vdash \mathsf{N} : B}{\Gamma \cup (\Delta \setminus A^x) \vdash \mathsf{M} \curlyvee^{x} \mathsf{N} : B}\ (\mathsf{Cut})$$

$$\frac{\Gamma \vdash M : B}{\Gamma \setminus A^x \vdash \underline{\lambda} x.\mathsf{M} : A \to B}\ (\to \mathsf{R}) \quad \frac{\Gamma \vdash \mathsf{M} : A \quad \Delta \vdash \mathsf{P} : C}{\Gamma \cup (\Delta \setminus B^y) \cup (A \to B)^x \vdash x[\![\mathsf{M}, y.\mathsf{P}]\!] : C}\ (\to \mathsf{L})$$

Fig. 2. Inference rules of LJ

Note 1. In LJ, structural rules are admitted implicitly. In fact, neither Γ in $(\to \mathsf{R})$ nor Δ in (Cut) have to contain A^x, so the weakening is admissible. In $(\to \mathsf{L})$, Γ and Δ may contain common elements, and Γ or Δ may contain $(A \to B)^x$, so the contraction is also admissible.

Definition 2 (Principal cuts). *A cut* $\mathsf{M} \curlyvee^x \mathsf{N}$ *is a* principal cut *(or* p-cut*) iff* N *has the form of* $\underline{x}[\![\mathsf{N}_1, y.\mathsf{N}_2]\!]$ *and* M *is not a variable. We write* $\mathsf{M}[\![\mathsf{N}_1, y.\mathsf{N}_2]\!]$ *for the p-cut* $\mathsf{M} \curlyvee^x \underline{x}[\![\mathsf{N}_1, y.\mathsf{N}_2]\!]$*. If a cut is not principal, it is a* non-principal cut *(or* n-cut*). We write* $\mathsf{M} \curlyvee^x_n \mathsf{N}$ *if the cut is an n-cut.* $\mathsf{LJ_p}$ *denotes the set of terms of* LJ *whose subterms of the form* $\mathsf{M} \curlyvee^x \mathsf{N}$ *are all p-cuts.*

Note 2. An expression $\mathsf{M}[\![\mathsf{N}, x.\mathsf{P}]\!]$ denotes either a left-rule application or a p-cut depending on M is a variable or not. This notation is not ambiguous because $\mathsf{M} \curlyvee^x \underline{x}[\![\mathsf{N}, y.\mathsf{P}]\!]$ is a p-cut iff M is not a variable.

Definition 3 (Cut-elimination procedure). *Rules of the* cut-elimination procedure *of* LJ *are in the figure 3. We suppose that y is not free in* P *of (β), and z is not free in* $\underline{x}[\![\mathsf{N}', z'.\mathsf{P}']\!]$ *of (π) by renaming bound variables. Note that the cut* $\mathsf{P} \curlyvee^x \underline{x}[\![\mathsf{N}', z'.\mathsf{P}']\!]$ *in the right-hand side of (π) is not a p-cut if* P *is a variable.* $\Rightarrow_\beta$ *denotes the one-step cut-elimination defined as the congruence relation including the rule (β).* $\Rightarrow^+_\beta$ *and* $\Rightarrow^*_\beta$ *denotes the transitive closure and the reflexive*

$$
\begin{array}{llr}
(\beta) & (\underline{\lambda} y.\mathsf{M})[\![\mathsf{N}, z.\mathsf{P}]\!] \Rightarrow_\beta \mathsf{N} \curlyvee^y (\mathsf{M} \curlyvee^z \mathsf{P}) & \\
(\pi) & \mathsf{M}[\![\mathsf{N}, z.\mathsf{P}]\!][\![\mathsf{N}', z'.\mathsf{P}']\!] \Rightarrow_\pi \mathsf{M}[\![\mathsf{N}, z.\mathsf{P} \curlyvee^x \underline{x}[\![\mathsf{N}', z'.\mathsf{P}']\!]]\!] & \\
(\mathsf{x1}) & \mathsf{M} \curlyvee^x_n y \Rightarrow_{\mathsf{x}} y & (x \not\equiv y) \\
(\mathsf{x2}) & \mathsf{M} \curlyvee^x_n x \Rightarrow_{\mathsf{x}} \mathsf{M} & \\
(\mathsf{x3}) & \mathsf{M} \curlyvee^x_n \underline{\lambda} y.\mathsf{N} \Rightarrow_{\mathsf{x}} \underline{\lambda} y.(\mathsf{M} \curlyvee^x \mathsf{N}) & \\
(\mathsf{x4}) & \mathsf{M} \curlyvee^x_n y[\![\mathsf{N}, z.\mathsf{P}]\!] \Rightarrow_{\mathsf{x}} y[\![\mathsf{M} \curlyvee^x \mathsf{N}, z.\mathsf{M} \curlyvee^x \mathsf{P}]\!] & (x \not\equiv y) \\
(\mathsf{x5}) & \mathsf{M} \curlyvee^x_n x[\![\mathsf{N}, z.\mathsf{P}]\!] \Rightarrow_{\mathsf{x}} \mathsf{M}[\![\mathsf{M} \curlyvee^x \mathsf{N}, z.\mathsf{M} \curlyvee^x \mathsf{P}]\!] & (x \in FV([\![\mathsf{N}, z.\mathsf{P}]\!])) \\
(\mathsf{x6}) & \mathsf{M} \curlyvee^x_n \mathsf{Q}[\![\mathsf{N}, z.\mathsf{P}]\!] \Rightarrow_{\mathsf{x}} (\mathsf{M} \curlyvee^x \mathsf{Q})[\![\mathsf{M} \curlyvee^x \mathsf{N}, z.\mathsf{M} \curlyvee^x \mathsf{P}]\!] & (\mathsf{Q} \text{ is not a variable}) \\
(\mathsf{x7}) & y \curlyvee^x_n \underline{x}[\![\mathsf{N}, z.\mathsf{P}]\!] \Rightarrow_{\mathsf{x}} y[\![\mathsf{N}, z.\mathsf{P}]\!] &
\end{array}
$$

Fig. 3. Cut-elimination rules of LJ

transitive closure of $\Rightarrow_\beta$ *respectively. For π- and* x*-rules, relations are similarly defined.* $\Rightarrow$ *denotes the union of* $\Rightarrow_\beta$, $\Rightarrow_\pi$ *and* $\Rightarrow_{\mathsf{x}}$. $\Rightarrow^+$ *and* $\Rightarrow^*$ *are similarly defined. An* LJ*-term* M *is* $\Rightarrow$-normal *iff there is no term* N *such that* $\mathsf{M} \Rightarrow \mathsf{N}$.

Each p-cut is a redex of either (β) or (π), which corresponds to a redex of β-reduction or permutative conversion in the natural deduction. On the other hand, each n-cut is a redex of some x-rule, which corresponds to the propagation of explicit substitutions, if we understand an n-cut $\mathsf{M} \curlyvee^x \mathsf{P}$ as an explicit substitution $\langle \mathsf{M}/x \rangle \mathsf{P}$. Note that $\mathsf{LJ_p}$ is not closed under (β) and (π) because a result of them may contain n-cuts.

Note 3. This cut-elimination can be seen as a refinement of LJ^t [3, 4]. The cut-elimination corresponding to x-reductions of LJ is treated as a meta-operation of substitution in LJ^t. On the other hand, the cut-elimination in this paper consists of local transformations of proofs. For example, (β) and (π) are the following:

$$\dfrac{\dfrac{\overset{\vdots\ \mathsf{M}}{A^y \vdash B}}{\vdash A \to B}\,(\to \mathsf{R}) \qquad \dfrac{\overset{\vdots\ \mathsf{N}}{\vdash A} \quad \overset{\vdots\ \mathsf{P}}{B^z \vdash C}}{(A \to B)^x \vdash C}\,(\to \mathsf{L})}{\vdash C}\,(\mathsf{Cut}) \quad \Rightarrow_\beta \quad \dfrac{\overset{\vdots\ \mathsf{N}}{\vdash A} \qquad \dfrac{\overset{\vdots\ \mathsf{M}}{A^y \vdash B} \quad \overset{\vdots\ \mathsf{P}}{B^z \vdash C}}{A^y \vdash C}\,(\mathsf{Cut})}{\vdash C}\,(\mathsf{Cut}),$$

$$\dfrac{\dfrac{\overset{\vdots\ \mathsf{M}}{\vdash A_1 \to A_2} \qquad \dfrac{\overset{\vdots\ \mathsf{N}}{\vdash A_1} \quad \overset{\vdots\ \mathsf{P}}{A_2^z \vdash B_1 \to B_2}}{(A_1 \to A_2)^y \vdash B_1 \to B_2}\,(\to \mathsf{L})}{\vdash B_1 \to B_2}\,(\mathsf{Cut}) \qquad \overset{\vdots\ \underline{x}\llbracket \mathsf{N}', z'.\mathsf{P}' \rrbracket}{(B_1 \to B_2)^x \vdash C}}{\vdash C}\,(\mathsf{Cut})$$

$$\Rightarrow_\pi \quad \dfrac{\overset{\vdots\ \mathsf{M}}{\vdash A_1 \to A_2} \qquad \dfrac{\overset{\vdots\ \mathsf{N}}{\vdash A_1} \quad \dfrac{\overset{\vdots\ \mathsf{P}}{A_2^z \vdash B_1 \to B_2} \quad \overset{\vdots\ \underline{x}\llbracket \mathsf{N}', z'.\mathsf{P}' \rrbracket}{(B_1 \to B_2)^x \vdash C}}{A_2^z \vdash C}\,(\mathsf{Cut})}{(A_1 \to A_2)^y \vdash C}\,(\to \mathsf{L})}{\vdash C}\,(\mathsf{Cut}).$$

We can divide the π-step into two steps such as

$$\begin{aligned}\mathsf{M}\llbracket \mathsf{N}, z.\mathsf{P} \rrbracket \llbracket \mathsf{N}', z'.\mathsf{P}' \rrbracket &\overset{(\star)}{\Rightarrow} \mathsf{M} \curlyvee^y (\underline{y}\llbracket \mathsf{N}, z.\mathsf{P} \rrbracket \llbracket \mathsf{N}', z.\mathsf{P}' \rrbracket) \\ &\Rightarrow \mathsf{M}\llbracket \mathsf{N}, z.\mathsf{P} \curlyvee^x \underline{x}\llbracket \mathsf{N}', z'.\mathsf{P}' \rrbracket \rrbracket,\end{aligned}$$

by a provisional cut-elimination step $(\star)$. But the middle term is reduced to the left-hand side by the x-rules, so if we admit $(\star)$, the cut-elimination is not SN.

Note 4. Our set of rules is similar to rules introduced by Kikuchi [10]. The rules (x1) through (x5), (x7) and (β) are the same as (1) through (6) and (*Beta*) of [10] respectively. ($Perm_1$) of [10] corresponds to $(\star)$. To avoid the loop noted above, in the rule ($Perm_2$) of [10], which is corresponding to (x6), Q is restricted to $\underline{\lambda}$-abstractions. SN is, however, not proved in [10]. If we choose the rules (π) and (x6), SN can be proved as shown in the following. Moreover our system does not contain the rule (7) of [10], since there is not its counterpart in the natural deduction.

Proposition 1 (Subject reduction). *If $\Gamma \vdash \mathsf{M} : A$ and $\mathsf{M} \Rightarrow \mathsf{N}$ hold, there exists Γ' such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \mathsf{M} : A$ hold.*

Proof. By induction on $\mathsf{M} \Rightarrow \mathsf{N}$.

2.2 Λ_g: λ-calculus with general elimination rules

In this subsection, we define the simply typed λ-calculus Λ and its variant Λ_g with *general elimination rules*, and show that Λ_g is a generalization of Λ.

Definition 4 **(Λ).** *Formulas,* term variables *and* contexts *of Λ are the same as* LJ. Pseudo-terms *(denoted by M, N, $P \ldots$) are defined as*

$$M ::= x \mid \lambda x.M \mid MM.$$

Bound variables and capture-avoiding substitution $[M/x]N$ are defined as usual. Inference rules and reduction rules are in the figure 4. Λ-terms *and relations $\to$ and $\to^+$ are defined similarly to* LJ.

$$\frac{}{A^x \vdash x : A}\ (\mathsf{Ax})$$

$$\frac{\Gamma \vdash M : B}{\Gamma \setminus A^x \vdash \lambda x.M : A \to B}\ (\to \mathsf{I}) \qquad \frac{\Gamma \vdash M : A \to B \quad \Delta \vdash N : A}{\Gamma \cup \Delta \vdash MN : B}\ (\to \mathsf{E})$$

$$(\beta)\ (\lambda x.M)N \to_\beta [N/x]M$$

Fig. 4. Inference rules and reduction rules of Λ

Definition 5 **(Λ_g).** *Formulas,* term variables *and* contexts *of Λ_g are the same as those of* LJ. Pseudo-terms *(denoted by M, N, $P \ldots$) are defined as*

$$M ::= x \mid \lambda x.M \mid M[M, x.M].$$

In $\lambda x.P$ and $M[N, x.P]$, variable occurrences of x in P are bound. Capture-avoiding substitution $[M/x]N$ is defined as usual. Inference rules and reduction rules are in the figure 5. We suppose that x is not free in P of (β) and that x is not free in the subexpression $[N', x'.P']$ of (π) by renaming bound variables. The π-reduction is called permutative conversion. Λ_g-terms *and relations $\to$, $\to_\beta$, $\to^+$ and so on are defined similarly to* LJ.

$$\frac{}{A^x \vdash x : A}\ (\mathsf{Ax})$$

$$\frac{\Gamma \vdash M : B}{\Gamma \setminus A^x \vdash \lambda x.M : A \to B}\ (\to \mathsf{I}) \qquad \frac{\Gamma \vdash M : A \to B \quad \Delta_1 \vdash N : A \quad \Delta_2 \vdash P : C}{\Gamma \cup \Delta_1 \cup (\Delta_2 \setminus B^x) \vdash M[N, x.P] : C}\ (\to \mathsf{E})$$

$$\begin{array}{ll} (\beta) & (\lambda x.M)[N, y.P] \to_\beta [N/x][M/y]P \\ (\pi) & M[N, x.P][N', x'.P'] \to_\pi M[N, x.P[N', x'.P']] \end{array}$$

Fig. 5. Inference rules and reduction rules of Λ_g

Definition 6. *We define two maps φ from Λ to Λ_g and ψ from Λ_g to Λ as*

$$\begin{aligned} \varphi(x) &\equiv x, & \psi(x) &\equiv x, \\ \varphi(\lambda x.M) &\equiv \lambda x.\varphi(M), & \psi(\lambda x.M) &\equiv \lambda x.\psi(M), \\ \varphi(MN) &\equiv \varphi(M)[\varphi(N), x.x], & \psi(M[N, x.P]) &\equiv [\psi(M)\psi(N)/x]\psi(P). \end{aligned}$$

Proposition 2. *1. For any Λ-term M, $\psi(\varphi(M)) \equiv M$.*
2. For any Λ-terms M and N, if $M \to N$, then we have $\varphi(M) \to_\beta \varphi(N)$.
3. For any Λ_g-terms M and N, if $M \to N$, then we have $\psi(M) \to^* \psi(N)$. In particular, if $M \to_\pi N$, then $\psi(M) \equiv \psi(N)$.

Proof. Straightforward.

Note 5. The image of φ is characterized as

$$M ::= \ x \mid \lambda x.M \mid M[M, x.x].$$

Their π-normal forms are

$$M ::= \ V \mid V\varepsilon, \quad \text{where } V ::= x \mid \lambda x.M \quad \text{and } \varepsilon ::= \ [M, x.x] \mid [M, x.\underline{x}\varepsilon],$$

which correspond to pure terms in [10]. It is shown in [10] that normal pure terms correspond to normal Λ-terms. The class of pure terms is not closed under substitution, so the definition of β-reduction on pure terms needs some technical meta operation in [10]. In our paper, instead, the image of φ is closed under β-reduction of Λ_g, so the correspondence between it and Λ is much simpler.

2.3 Isomorphism between $\mathsf{LJ_p}$ and Λ_g as sets of terms

Definition 7. *We define two maps M^* from $\mathsf{LJ_p}$ to Λ_g and M_* from Λ_g to $\mathsf{LJ_p}$ as follows.*

$$\begin{aligned} x^* &\equiv x & x_* &\equiv x \\ (\underline{\lambda} x.\mathsf{M})^* &\equiv \lambda x.\mathsf{M}^* & (\lambda x.M)_* &\equiv \underline{\lambda} x.M_* \\ (\mathsf{M}[\![\mathsf{N}, x.\mathsf{P}]\!])^* &\equiv \mathsf{M}^*[\mathsf{N}^*, x.\mathsf{P}^*] & (M[N, x.P])_* &\equiv M_*[\![N_*, x.P_*]\!] \end{aligned}$$

These maps are almost the same as identity maps except the $\mathsf{LJ_p}$-term $\mathsf{M}[\![\mathsf{N}, x.\mathsf{P}]\!]$ is an abbreviation of the p-cut $\mathsf{M} \curlyvee^x \underline{x}[\![\mathsf{N}, x.\mathsf{P}]\!]$ if M is not a variable. In particular, between cut-free LJ-terms and $\beta\pi$-normal Λ_g-terms.

Proposition 3. *1. For any $\mathsf{LJ_p}$-term M, we have $(\mathsf{M}^*)_* \equiv \mathsf{M}$. For any Λ_g-term M, we have $(M_*)^* \equiv M$.*
2. If $\Gamma \vdash \mathsf{M} : A$ is derivable in $\mathsf{LJ_p}$, then $\Gamma \vdash \mathsf{M}^* : A$ is derivable in Λ_g. If $\Gamma \vdash M : A$ is derivable in Λ_g, then $\Gamma \vdash M_* : A$ is derivable in $\mathsf{LJ_p}$.
3. For any cut-free $\mathsf{LJ_p}$-term M, M^* is normal. For any normal Λ_g-term M, M_* is cut-free.

Proof. By induction on M and M. For 3, note that the $\beta\pi$-normal terms in Λ_g have the form of either x or $x[M, y.P]$.

3 Correspondence between cut-elimination and proof reduction

As shown in the previous section, $\mathsf{LJ_p}$ and Λ_g are almost identical as sets of terms. In fact, this correspondence can be extended to the cut-elimination and the reduction of λ-calculus. $\mathsf{LJ_p}$ is not closed under the cut-elimination procedure, so we introduce *cut-elimination strategies* for $\mathsf{LJ_p}$.

3.1 Projection from LJ to $\mathsf{LJ_p}$

First we define a projection from LJ to $\mathsf{LJ_p}$. It is nothing but the normalization function with respect to the x-reduction.

Definition 8 (Projection).

1. Pseudo-substitution $(\!|\mathsf{M}/x|\!)\mathsf{N}$ *for* $\mathsf{LJ_p}$*-terms* M *and* N *is defined as follows.*

$$(\!|\mathsf{M}/x|\!)y \equiv y \quad (x \not\equiv y) \qquad (\!|\mathsf{M}/x|\!)(\underline{\lambda} y.\mathsf{N}) \equiv \underline{\lambda} y.(\!|\mathsf{M}/x|\!)\mathsf{N}$$
$$(\!|\mathsf{M}/x|\!)x \equiv \mathsf{M} \qquad (\!|\mathsf{M}/x|\!)(\mathsf{Q}[\![\mathsf{N}, y.\mathsf{P}]\!]) \equiv ((\!|\mathsf{M}/x|\!)\mathsf{Q})[\![(\!|\mathsf{M}/x|\!)\mathsf{N}, y.(\!|\mathsf{M}/x|\!)\mathsf{P}]\!]$$

Note that the right hand side of the last equation is either a left-rule application or a p-cut depending on whether $(\!|\mathsf{M}/x|\!)\mathsf{Q}$ *is a variable or not.*

2. Projection M^{x} *from* LJ *to* $\mathsf{LJ_p}$ *is defined as follows.*

$$x^{\mathsf{x}} \equiv x \qquad (\mathsf{M}[\![\mathsf{N}, x.\mathsf{P}]\!])^{\mathsf{x}} \equiv \mathsf{M}^{\mathsf{x}}[\![\mathsf{N}^{\mathsf{x}}, x.\mathsf{P}^{\mathsf{x}}]\!]$$
$$(\underline{\lambda} x.\mathsf{M})^{\mathsf{x}} \equiv \underline{\lambda} x.\mathsf{M}^{\mathsf{x}} \qquad (\mathsf{M} \curlyvee^x_n \mathsf{N})^{\mathsf{x}} \equiv (\!|\mathsf{M}^{\mathsf{x}}/x|\!)\mathsf{N}^{\mathsf{x}}$$

Lemma 1. *1. For any* $\mathsf{LJ_p}$*-terms* M *and* N*, we have* $((\!|\mathsf{M}/x|\!)\mathsf{N})^* \equiv [\mathsf{M}^*/x]\mathsf{N}^*$*.*
2. For any Λ_g*-terms* M *and* N*, we have* $([M/x]N)_* \equiv (\!|M_*/x|\!)N_*$*.*

Proof. By induction on N and N respectively.

Lemma 2. *For any* LJ*-term* M*,* M^{x} *is a* $\mathsf{LJ_p}$*-term and we have* $\mathsf{M} \Rightarrow^*_{\mathsf{x}} \mathsf{M}^{\mathsf{x}}$*.*

Proof. For any $\mathsf{LJ_p}$-terms M and N, $\mathsf{M} \curlyvee^x \mathsf{N} \Rightarrow^*_{\mathsf{x}} (\!|\mathsf{M}/x|\!)\mathsf{N}$ is proved by induction on N. And then, $\mathsf{M} \Rightarrow^*_{\mathsf{x}} \mathsf{M}^{\mathsf{x}}$ is proved by induction on M.

In the following, we show that each step of the cut-elimination is projected to the reduction steps of Λ_g. To describe the claim more precisely, we prepare an auxiliary notion, which was introduced in [2].

Definition 9. *A subterm occurrence* N *in* M *is* void *iff there is a subterm* $\mathsf{P} \curlyvee^x_n \mathsf{Q}$ *of* M *such that* N *occurs in* P *and* $x \notin FV(\mathsf{Q}^{\mathsf{x}})$*. A cut-elimination step* $\mathsf{M} \Rightarrow \mathsf{N}$ *is* void *iff the redex of* M *is void. We write* $\mathsf{M} \overset{\mathsf{v}}{\Rightarrow} \mathsf{N}$ *when the step is void.*

Proposition 4. *Let the symbol* $\bullet$ *be either* β *or* π*. For any* LJ*-terms* M *and* N*, if* $\mathsf{M} \Rightarrow_\bullet \mathsf{N}$ *holds and it is not void, then we have* $(\mathsf{M}^{\mathsf{x}})^* \to^+_\bullet (\mathsf{N}^{\mathsf{x}})^*$ *in* Λ_g*. In particular, when* M *is an* $\mathsf{LJ_p}$*-term, we have* $(\mathsf{M}^{\mathsf{x}})^* \to_\bullet (\mathsf{N}^{\mathsf{x}})^*$*. If either* $\mathsf{M} \Rightarrow_{\mathsf{x}} \mathsf{N}$ *or* $\mathsf{M} \overset{\mathsf{v}}{\Rightarrow} \mathsf{N}$ *holds, then we have* $\mathsf{M}^{\mathsf{x}} \equiv \mathsf{N}^{\mathsf{x}}$*.*

Proof. By induction on $\mathsf{M} \Rightarrow \mathsf{N}$. We prove only the case where $\mathsf{M} \equiv \mathsf{M}_0 \curlyvee^x \mathsf{M}_1$, $\mathsf{N} \equiv \mathsf{N}_0 \curlyvee^x \mathsf{M}_1$, $\mathsf{M}_0 \Rightarrow \mathsf{N}_0$ and $x \notin FV(\mathsf{M_1}^{\mathsf{x}})$, that is, $\mathsf{M} \stackrel{\mathsf{v}}{\Rightarrow} \mathsf{N}$. We have $\mathsf{M}^{\mathsf{x}} \equiv (\!|\mathsf{M}_0^{\mathsf{x}}/x|\!)\mathsf{M}_1^{\mathsf{x}}$ and $\mathsf{N}^{\mathsf{x}} \equiv (\!|\mathsf{N}_0^{\mathsf{x}}/x|\!)\mathsf{M}_1^{\mathsf{x}}$, which are identical since $x \notin FV(\mathsf{M}_1^{\mathsf{x}})$. Other cases are easily proved by the Lemma 1.

Proposition 5. *$\Rightarrow_{\mathsf{x}}$ is* SN *and* CR*, so any* LJ*-term* M *has the unique $\Rightarrow_{\mathsf{x}}$-normal form, which is* M^{x}*.*

Proof. For SN, we define $|\mathsf{M}|$ and $\#\mathsf{M}$ as

$$\begin{aligned}
|x| &= 1, & \#x &= 1,\\
|\underline{\lambda} x.\mathsf{M}| &= |\mathsf{M}|, & \#(\underline{\lambda} x.\mathsf{M}) &= \#\mathsf{M} + 1,\\
|x[\![\mathsf{M}, y.\mathsf{N}]\!]| &= |\mathsf{M}| + |\mathsf{N}|, & \#(x[\![\mathsf{M}, y.\mathsf{N}]\!]) &= \#\mathsf{M} + \#\mathsf{N} + 2,\\
|\mathsf{P}[\![\mathsf{M}, y.\mathsf{N}]\!]| &= |\mathsf{P}| + |\mathsf{M}| + |\mathsf{N}|, & \#(\mathsf{P}[\![\mathsf{M}, y.\mathsf{N}]\!]) &= \#\mathsf{P} + \#\mathsf{M} + \#\mathsf{N} + 1,\\
|z \curlyvee^x_n \underline{x}[\![\mathsf{M}, y.\mathsf{N}]\!]| &= |\mathsf{M}| + |\mathsf{N}| + 1, & \#(\mathsf{M} \curlyvee^x_n \mathsf{N}) &= \#\mathsf{M} \cdot \#\mathsf{N},\\
|\mathsf{M} \curlyvee^x_n \mathsf{N}| &= |\mathsf{M}| \cdot \#\mathsf{N} + |\mathsf{N}| \text{ (o.w.)}, & &
\end{aligned}$$

where P is not a variable. We can prove that $\mathsf{M} \Rightarrow_{\mathsf{x}} \mathsf{N}$ implies $|\mathsf{M}| > |\mathsf{N}|$ and $\#\mathsf{M} \geq \#\mathsf{N}$. For CR, suppose that $\mathsf{M} \Rightarrow_{\mathsf{x}}^* \mathsf{M}_1$ and $\mathsf{M} \Rightarrow_{\mathsf{x}}^* \mathsf{M}_2$ holds. By the Proposition 4, $\mathsf{M}^{\mathsf{x}} \equiv \mathsf{M}_1^{\mathsf{x}} \equiv \mathsf{M}_2^{\mathsf{x}}$ holds, so we have $\mathsf{M}_1 \Rightarrow_{\mathsf{x}}^* \mathsf{M}_1^{\mathsf{x}} \equiv \mathsf{M}^{\mathsf{x}}$ and $\mathsf{M}_2 \Rightarrow_{\mathsf{x}}^* \mathsf{M}_2^{\mathsf{x}} \equiv \mathsf{M}^{\mathsf{x}}$ by the Lemma 2.

3.2 Isomorphism between $\mathbf{LJ_p}$ and Λ_g

We define *cut-elimination strategies* on $\mathsf{LJ_p}$corresponding the reductions in Λ_g.

Definition 10 ($\beta\pi$-strategy). *Relation $\mathsf{M} \rightarrow_\beta \mathsf{N}$ on $\mathsf{LJ_p}$ is defined as $\mathsf{M} \Rightarrow_\beta \mathsf{M}'$ and $\mathsf{M}'^{\mathsf{x}} \equiv \mathsf{N}$ for some M'. We call $\rightarrow_\beta$* β-strategy*. Similarly* π-strategy *$\mathsf{M} \rightarrow_\pi \mathsf{N}$ is defined as $\mathsf{M} \Rightarrow_\pi \mathsf{M}'$ and $\mathsf{M}'^{\mathsf{x}} \equiv \mathsf{N}$ for some M'.*

Lemma 3. *Let the symbol $\bullet$ be either β or π. For any $\mathsf{LJ_p}$-terms* M *and* N*, $\mathsf{M} \rightarrow_\bullet \mathsf{N}$ implies $\mathsf{M} \Rightarrow_{\bullet n}^+ \mathsf{N}$ in* LJ*.*

Proof. By the definition of the $\beta\pi$-strategy and the Lemma 2.

Theorem 1. *Let the symbol $\bullet$ be either β or π.*

1. *For any $\mathsf{LJ_p}$-terms* M *and* N*, if $\mathsf{M} \rightarrow_\bullet \mathsf{N}$ holds, then $\mathsf{M}^* \rightarrow_\bullet \mathsf{N}^*$ holds in Λ_g.*
2. *For any Λ_g-terms M and N, if $M \rightarrow_\bullet N$ holds, then $M_* \rightarrow_\bullet N_*$ holds in* $\mathsf{LJ_p}$.

Proof.
1. Suppose $\mathsf{M} \rightarrow_\bullet \mathsf{N}$. By the definition of the strategy, there exists an LJ-term M' such that $\mathsf{M} \Rightarrow_\bullet \mathsf{M}' \Rightarrow_{\mathsf{x}}^* \mathsf{N}$. By the Proposition 4, we have $(\mathsf{M}^{\mathsf{x}})^* \rightarrow_\bullet (\mathsf{N}^{\mathsf{x}})^*$, that is, $\mathsf{M}^* \rightarrow_\bullet \mathsf{N}^*$ since $\mathsf{M}^{\mathsf{x}} \equiv \mathsf{M}$ for any $\mathsf{LJ_p}$-term M.
2. By induction on $M \rightarrow N$.

Corollary 1. *Let the symbol $\bullet$ be either β or π. For any* LJ*-terms* M *and* N*, if $\mathsf{M} \Rightarrow_\bullet \mathsf{N}$ holds and it is not void, then we have $\mathsf{M}^{\mathsf{x}} \rightarrow_\bullet^+ \mathsf{N}^{\mathsf{x}}$ in* $\mathsf{LJ_p}$.

Corollary 2.
1. *For any* LJ*-terms* M *and* N*, if $\mathsf{M} \Rightarrow_\beta \mathsf{N}$ holds, then $\psi((\mathsf{M}^{\mathsf{x}})^*) \rightarrow^* \psi((\mathsf{N}^{\mathsf{x}})^*)$ holds in Λ. If $\mathsf{M} \Rightarrow_{\pi\mathsf{x}} \mathsf{N}$ holds, then $\psi((\mathsf{M}^{\mathsf{x}})^*) \equiv \psi((\mathsf{N}^{\mathsf{x}})^*)$ holds.*
2. *For any Λ-terms M and N, if $M \rightarrow N$ holds, then $(\varphi(M))_* \Rightarrow_{\beta\mathsf{x}}^+ (\varphi(M))_*$ holds in* LJ.

4 Strong normalization and Church-Rosser property

In this section, we prove SN and CR of the cut-elimination procedure for LJ by those of the $\beta\pi$-reduction in Λ_g.

4.1 SN and CR of Λ_g

First, we prove SN and CR of the $\beta\pi$-reduction on Λ_g by a *continuation and garbage passing style (CGPS) translation*, which is a variant of continuation passing style translations and was introduced by Ikeda and Nakazawa [7].

The CGPS-translation maps Λ_g-terms to Λ-terms, preserving typability and one-or-more step reduction relation. In the following definition, metavariables K and G for Λ-terms denote continuation terms and garbage terms respectively. We introduce garbage parts to map each π-reduction step in Λ_g into dummy β-reduction steps in Λ. Note that, if we ignore the garbage parts denoted by g and G, we can get a CPS-translation of Λ_g.

Definition 11 (CGPS-translation). *Let $\bot$ be a fixed atomic formula, $\neg A \equiv A \to \bot$ for a formula A, and $\langle\!\langle M; N\rangle\!\rangle \equiv (\lambda x.M)N$ for Λ-terms M and N, where x is a fresh variable.* Negative translation $\overline{A}$ *of a formula A is defined as $\neg\bot \to \neg\neg A^\dagger$, where $A^\dagger$ is defined as $p^\dagger \equiv p$ and $(A \to B)^\dagger \equiv \overline{A} \to \overline{B}$.* CGPS-translation $\overline{M}$ *of a Λ_g-term M is defined as a Λ-term $\lambda gk.(M :_g k)$, where g and k are fresh and $M :_G K$ for a Λ_g-term M and Λ-terms G, K is defined as*

$$\begin{aligned} x :_G K &\equiv xGK, \\ \lambda x.M :_G K &\equiv \langle\!\langle K(\lambda x.\overline{M}); G\rangle\!\rangle, \\ M[N, x.P] :_G K &\equiv M :_{\langle\!\langle G; K'\rangle\!\rangle} K' \qquad (K' \equiv \lambda y.(\lambda x.(P :_G K))(y\overline{N})). \end{aligned}$$

Lemma 4. *1. If $\Gamma \vdash M : A$ is derivable in Λ_g, we have $\overline{\Gamma} \vdash \overline{M} : \overline{A}$ in Λ, where $\overline{\Gamma}$ is defined as $\{\overline{A}^x \mid A^x \in \Gamma\}$.*
2. If $M \to N$ holds in Λ_g, we have $\overline{M} \to^+ \overline{N}$ in Λ.

Proof. 1. By induction on M. We need to prove it simultaneously with another claim: $\Gamma \vdash M : A$ in Λ_g implies $\overline{\Gamma}, (\neg A^\dagger)^k, (\neg\bot)^g \vdash (M :_g k) : \bot$ in Λ.
2. By induction on $M \to N$. We use the fact that we have $M :_G N \to^+ M :_G N'$ and $M :_N K \to^+ M :_{N'} K$ for any $N \to N'$, and we have $[\overline{M}/x](P :_G K) \to^* ([M/x]P) :_{[\overline{M}/x]G} [\overline{M}/x]K$.

Proposition 6 (SN of Λ_g). *For any Λ_g-term M, there is no infinite $\beta\pi$-reduction sequence from M.*

Proof. Suppose that $M_0 \to M_1 \to \cdots$ is an infinite reduction sequence from an Λ_g-term M_0. By the Lemma 4, $\overline{M}_0 \to^+ \overline{M}_1 \to^+ \cdots$ is an infinite sequence from the Λ-term M_0, which contradicts the SN of Λ.

Proposition 7 (CR of Λ_g). *For any Λ_g-terms M, M_1 and M_2, if $M \to^* M_1$ and $M \to^* M_2$ hold, then there exists a Λ_g-term M_3 such that $M_1 \to^* M_3$ and $M_2 \to^* M_1$.*

Proof. It is sufficient to prove WCR: if we have $M \to M_1$ and $M \to M_2$, then there exists M_3 such that $M_1 \to^* M_3$ and $M_2 \to^* M_3$ hold. Then, CR is derived from SN and WCR by Newman's lemma. We prove WCR by induction on M. For the case where M has the form of $Q[N, x.P]$, there are three types of forms of $Q[N, x.P] \to M_i$: (1) β-redex $(\lambda y.Q_0)[N, x.P] \to [N/y][Q_0/x]P$, (2) π-redex $Q'[N', x'.P'][N, x.P] \to Q'[N', x'.P'[N, x.P]]$, (3) $Q[N, x.P] \to Q'[N', x.P']$, where either $Q \to Q'$, $N \to N'$ or $P \to P'$. The case where both $M \to M_i$ are the same type of either (1) or (2), we have $M_1 \equiv M_2$. Other cases are easily proved by the induction hypothesis. Note that there is no case where one of $M \to M_i$ is the type (1) and the other is the type (2).

Corollary 3. $\mathsf{LJ_p}$ *enjoys* SN *and* CR *with respect to the* $\beta\pi$*-strategy.*

4.2 SN and CR of LJ

Theorem 2. *The cut-elimination procedure of* LJ *enjoys* CR*.*

Proof. Suppose $\mathsf{M} \Rightarrow^* \mathsf{M}_1$ and $\mathsf{M} \Rightarrow^* \mathsf{M}_2$ hold in LJ. By the Corollary 1, we have $\mathsf{M}^\mathsf{x} \to^* \mathsf{M}_i^\mathsf{x}$ for $i = 1$ and 2. By CR of $\mathsf{LJ_p}$, there is an $\mathsf{LJ_p}$-term M_3 such that $\mathsf{M}_i^\mathsf{x} \to^* \mathsf{M}_3$. And then we have $\mathsf{M}_i \Rightarrow_\mathsf{x}^* \mathsf{M}_i^\mathsf{x} \Rightarrow^* \mathsf{M}_3$ by the Lemma 2 and 3.

SN of LJ is proved by the method which has been applied to SN proofs for calculi with explicit substitution in [2, 9] and so on.

Definition 12. *An* LJ*-term* M *is* decent *iff for any subterm* $\mathsf{N} \curlyvee_n^x \mathsf{P}$ *of* M*,* N *is* SN. Rank $\rho(\mathsf{M})$ *of an* LJ*-term* M *is defined as the maximum length of* $\beta\pi$*-strategy sequence from* M^x*.*

Lemma 5. *1. In any infinite sequence of* $\Rightarrow_\mathsf{x}$ *and* $\stackrel{\mathsf{v}}{\Rightarrow}$*, all reduction steps except for finitely many steps are void.*
2. In any infinite sequence of $\Rightarrow$*, all reduction steps except for finitely many steps are void.*
3. Any decent term M *is* SN *with respect to* $\stackrel{\mathsf{v}}{\Rightarrow}$*.*

Proof. 1. We define the norm $\|\mathsf{M}\|$ and $\flat\mathsf{M}$ for a LJ-term M as follows, where P is a variable.

$$
\begin{array}{rlrl}
\|x\| &= 1 & \flat x &= 1 \\
\|\underline{\lambda} x.\mathsf{M}\| &= \|\mathsf{M}\| & \flat(\underline{\lambda} x.\mathsf{M}) &= \flat\mathsf{M} + 1 \\
\|x[\![\mathsf{M}, y.\mathsf{N}]\!]\| &= \|\mathsf{M}\| + \|\mathsf{N}\| & \flat(x[\![\mathsf{M}, y.\mathsf{N}]\!]) &= \flat\mathsf{M} + \flat\mathsf{N} + 2 \\
\|\mathsf{P}[\![\mathsf{M}, y.\mathsf{N}]\!]\| &= \|\mathsf{P}\| + \|\mathsf{M}\| + \|\mathsf{N}\| & \flat(\mathsf{P}[\![\mathsf{M}, y.\mathsf{N}]\!]) &= \flat\mathsf{P} + \flat\mathsf{M} + \flat\mathsf{N} + 1 \\
\|z \curlyvee_n^x \underline{x}[\![\mathsf{M}, y.\mathsf{N}]\!]\| &= \|\mathsf{M}\| + \|\mathsf{N}\| + 1 & \flat(\mathsf{M} \curlyvee_n^x \mathsf{N}) &= \flat\mathsf{M} \cdot \flat\mathsf{N} \quad (x \in FV(\mathsf{N}^\mathsf{x})) \\
\|\mathsf{M} \curlyvee_n^x \mathsf{N}\| &= \|\mathsf{M}\| \cdot \flat\mathsf{N} + \|\mathsf{N}\| \quad (x \in FV(\mathsf{N}^\mathsf{x})) & \flat(\mathsf{M} \curlyvee_n^x \mathsf{N}) &= \flat\mathsf{N} \quad (x \notin FV(\mathsf{N}^\mathsf{x})) \\
\|\mathsf{M} \curlyvee_n^x \mathsf{N}\| &= \flat\mathsf{N} + \|\mathsf{N}\| \quad (x \notin FV(\mathsf{N}^\mathsf{x})) & &
\end{array}
$$

We can prove that if $\mathsf{M} \Rightarrow_\mathsf{x} \mathsf{N}$ holds and it is not void, then we have $\|\mathsf{M}\| > \|\mathsf{N}\|$ and $\flat\mathsf{M} \geq \flat\mathsf{N}$, and if $\mathsf{M} \stackrel{\mathsf{v}}{\Rightarrow} \mathsf{N}$ holds, then we have $\|\mathsf{M}\| = \|\mathsf{N}\|$.

2. Suppose that $\mathsf{M}_0 \Rightarrow \mathsf{M}_1 \Rightarrow \mathsf{M}_2 \Rightarrow \cdots$ is an infinite sequence. By the Corollary 1, we have $\mathsf{M}_0^{\mathsf{x}} \to^* \mathsf{M}_1^{\mathsf{x}} \to^* \mathsf{M}_2^{\mathsf{x}} \to^* \cdots$, where there is an index m such that $\mathsf{M}_i^{\mathsf{x}} \equiv \mathsf{M}_{i+1}^{\mathsf{x}}$ for any $i > m$ by SN of $\mathsf{LJ_p}$. So $\mathsf{M}_i \Rightarrow \mathsf{M}_{i+1}$ is $\Rightarrow_{\mathsf{x}}$ or $\overset{\mathsf{v}}{\Rightarrow}$ for any $i > m$. By 1, the sequence contains only finitely many non-void $\Rightarrow_{\mathsf{x}}$-steps.
3. By induction on M.

Lemma 6. *If* M *is a decent* LJ*-term,* M *is* SN *with respect to* $\Rightarrow$*.*

Proof. By induction on $\rho(\mathsf{M})$. Suppose that, for any N such that $\rho(\mathsf{M}) > \rho(\mathsf{N})$, if N is decent, then N is SN. First, we show that any N such that $\mathsf{M} \Rightarrow \mathsf{N}$ is decent by induction on $\mathsf{M} \Rightarrow \mathsf{N}$. For the case where $\mathsf{M} \equiv (\underline{\lambda} x.\mathsf{M}_1)[\![\mathsf{M}_2, y.\mathsf{M}_3]\!]$ and $\mathsf{N} \equiv \mathsf{M}_2 \curlyvee^x (\mathsf{M}_1 \curlyvee^y \mathsf{M}_3)$, since $\mathsf{M}_1^{\mathsf{x}}$ and $\mathsf{M}_2^{\mathsf{x}}$ are proper subterms of a β-redex $\mathsf{M}^{\mathsf{x}} \equiv (\underline{\lambda} x.\mathsf{M}_1^{\mathsf{x}})[\![\mathsf{M}_2^{\mathsf{x}}, y.\mathsf{M}_3^{\mathsf{x}}]\!]$, we have $\rho(\mathsf{M}) > \rho(\mathsf{M}_1)$ and $\rho(\mathsf{M}) > \rho(\mathsf{M}_2)$. Moreover, M_1 and M_2 are decent since M is decent, so they are SN by the hypothesis of the outer induction. Therefore, N is decent. The cases of π and x-redexes are similarly proved. Other cases are proved by the hypothesis for the inner induction. Secondly, suppose that there is an infinite sequence $\mathsf{M} \Rightarrow \mathsf{M}_1 \Rightarrow \mathsf{M}_2 \Rightarrow \cdots$. By the fact proved above, any M_i is decent. Furthermore, by 2 of the Lemma 5, there is an index m such that $\mathsf{M}_i \overset{\mathsf{v}}{\Rightarrow} \mathsf{M}_{i+1}$ holds for any $i > m$, which contradicts 3 of the Lemma 5.

Theorem 3. *The cut-elimination procedure of* LJ *enjoys* SN*.*

Proof. By induction on LJ-terms. For any M, any proper subterm of M is SN by the induction hypothesis, so M is decent. Therefore, M is SN by the Lemma 6.

5 Sequent calculus and explicit substitutions

In this section, we define another system Λ_{gx}, which is a λ-calculus with general elimination rules and explicit substitutions. We show that LJ is isomorphic to Λ_{gx} modulo a term quotient.

Definition 13 (Λ_{gx}). *Λ_{gx} is defined as an extension of Λ_g.* Pseudo-terms *are extended by* explicit substitutions $\langle M/x\rangle N$*. The only additional inference rule* (X) *and reduction rules of Λ_{gx} are in the figure 6.*

Λ_{gx} is almost the same as LJ under an identification of $\mathsf{M} \curlyvee_n^x \mathsf{N}$ and $\langle M/x\rangle N$. The only difference is that, if M is not a variable, $\langle M/x\rangle(\underline{x}[N, y.P])$ and $M[N, y.P]$ are distinct Λ_{gx}-terms, while the corresponding LJ-terms are identical, since the cut $\mathsf{M} \curlyvee^x \underline{x}[\![\mathsf{N}, y.\mathsf{P}]\!]$ is a p-cut.

Lemma 7. $\Rrightarrow_p$ *is* CR *and* SN*.*

Proof. Each $\Rrightarrow_p$-step decreases size of terms, so SN holds. WCR is easily proved.

$$\frac{\Gamma \vdash M : A \quad \Delta \vdash N : B}{\Gamma \cup (\Delta \setminus A^x) \vdash \langle M/x \rangle N : B} \ (\mathsf{X})$$

(β)	$(\lambda y.M)[N, z.P] \Rrightarrow_\beta \langle N/y \rangle \langle M/z \rangle P$	
(π)	$M[N, z.P][N', z'.P'] \Rrightarrow_\pi M[N, z.\langle P/x \rangle (\underline{x}[N', z'.P'])]$	
$(\mathsf{x}1)$	$\langle M/x \rangle y \Rrightarrow_\mathsf{x} y$	$(x \not\equiv y)$
$(\mathsf{x}2)$	$\langle M/x \rangle x \Rrightarrow_\mathsf{x} M$	
$(\mathsf{x}3)$	$\langle M/x \rangle (\lambda y.N) \Rrightarrow_\mathsf{x} \lambda y.\langle M/x \rangle N$	
$(\mathsf{x}4)$	$\langle M/x \rangle (y[N, z.P]) \Rrightarrow_\mathsf{x} y[\langle M/x \rangle N, z.\langle M/x \rangle P]$	$(x \not\equiv y)$
$(\mathsf{x}5)$	$\langle M/x \rangle (x[N, z.P]) \Rrightarrow_\mathsf{x} M[\langle M/x \rangle N, z.\langle M/x \rangle P]$	$(x \in FV([N, z.P]))$
$(\mathsf{x}6)$	$\langle M/x \rangle (Q[N, z.P]) \Rrightarrow_\mathsf{x} (\langle M/x \rangle Q)[\langle M/x \rangle N, z.\langle M/x \rangle P]$	(Q is not a variable)
$(\mathsf{x}7)$	$\langle y/x \rangle (\underline{x}[N, z.P]) \Rrightarrow_\mathsf{x} y[N, z.P]$	
(p)	$\langle M/x \rangle (\underline{x}[N, z.P]) \Rrightarrow_p M[N, z.P]$	(M is not a variable)

Fig. 6. Additional inference rule and reduction rules of Λ_{gx}

Definition 14. *For Λ_{gx}-term M, M^p denotes its $\Rrightarrow_p$-normal form. Λ^p_{gx} consists of the following. Terms are $\Rrightarrow_p$-normal Λ_{gx}-terms. For Λ^p_{gx}-terms M and N, β-*strategy *$M \Rightarrow_\beta N$ holds iff there exists an Λ_{gx}-term M' such that $M \Rrightarrow_\beta M'$ and $M'^p \equiv N$ hold. π- and* x-strategies *are similarly defined. Furthermore, We extend M^* and M_* to maps between* LJ *and Λ^p_{gx} as follows:*

$$(\mathsf{M} \curlyvee^x_n \mathsf{N})^* \equiv \langle \mathsf{M}^*/x \rangle \mathsf{N}^*, \qquad (\langle M/x \rangle N)_* \equiv M_* \curlyvee^x N_*.$$

The following properties are proved in a straightforward way.

Proposition 8. *1. For any* LJ*-term* M*, we have $(\mathsf{M}^*)_* \equiv \mathsf{M}$. For any Λ^p_{gx}-term M, we have $(M_*)^* \equiv M$.*
2. If $\Gamma \vdash \mathsf{M} : A$ is derivable in LJ*, then $\Gamma \vdash \mathsf{M}^* : A$ is derivable in Λ^p_{gx}. If $\Gamma \vdash M : A$ is derivable in Λ^p_{gx}, then $\Gamma \vdash M_* : A$ is derivable in* LJ*.*

Lemma 8. *1. For any* LJ*-terms M and N, we have $(\mathsf{M} \curlyvee^x \mathsf{N})^* \equiv (\langle \mathsf{M}^*/x \rangle \mathsf{N}^*)^p$.*
2. For any Λ_{gx}-terms M and N, we have $((\langle M/x \rangle N)^p)_* \equiv (M^p)_* \curlyvee^x (N^p)_*$.

Lemma 9. *Let the symbol $\bullet$ be either β, π or* x*.*

1. *For any Λ^p_{gx}-terms M and N, $M \Rightarrow_\bullet N$ implies $M \Rrightarrow^*_{\bullet p} N$.*
2. *For any Λ_{gx}-terms M and N, $M \Rrightarrow_\bullet N$ implies $M^p \Rightarrow_\bullet N^p$, and $M \Rrightarrow_p N$ implies $M^p \equiv N^p$.*

Theorem 4. *Let $\bullet$ be either β, π or* x*.*

1. *For any* LJ*-terms M and N, $\mathsf{M} \Rightarrow_\bullet \mathsf{N}$ in* LJ *implies $\mathsf{M}^* \Rightarrow_\bullet \mathsf{N}^*$ in Λ^p_{gx}.*
2. *For any Λ^p_{gx}-terms M and N, $M \Rightarrow_\bullet N$ in Λ^p_{gx} implies $\mathsf{M}_* \Rightarrow_\bullet \mathsf{N}_*$ in* LJ*.*

Proof. 1. By induction on $\mathsf{M} \Rightarrow \mathsf{N}$. We prove only the case where $\mathsf{M} \equiv (\underline{\lambda} x.\mathsf{Q})[\![\mathsf{N}, z.\mathsf{P}]\!]$ and $\mathsf{N} \equiv \mathsf{N} \curlyvee^x (\mathsf{Q} \curlyvee^z \mathsf{P})$. We have $\mathsf{M}^* \equiv (\lambda x.\mathsf{Q}^*)[\mathsf{N}^*, z.\mathsf{P}^*] \Rightarrow_\beta (\langle \mathsf{N}^*/x \rangle \langle \mathsf{Q}^*/z \rangle \mathsf{P}^*)^p$, which is identical to $(\mathsf{N} \curlyvee^x (\mathsf{Q} \curlyvee^z \mathsf{P}))^*$ by 1 of the Lemma 8.

2. It is similarly proved by induction on $M \Rightarrow N$, using 2 of the Lemma 8.

CR and SN of Λ^p_{gx} immediately follows those of LJ. Furthermore, CR and SN of Λ_{gx} can be easily proved by means of those of Λ^p_{gx}.

Theorem 5. *Λ_{gx} enjoys CR and SN.*

Proof. CR is proved in a similar way to the Theorem 2 by the Lemma 9. Furthermore, by 2 of the Lemma 9, the map $(\cdot)^p$ translates an infinite $\Rrightarrow$-sequence in Λ_{gx} to an infinite $\Rightarrow$-sequence in Λ^p_{gx}, since $\Rrightarrow_p$ is SN.

6 Concluding Remarks

This paper proposes an SN and CR cut-elimination procedure of the intuitionistic sequent calculus which is isomorphic to the proof reduction of the natural deduction with general elimination and explicit substitutions. The discussion in this paper can be extended to other logical connectives. For example, general elimination rules for conjunction and disjunction are

$$\frac{\Gamma \vdash M : A \wedge B \quad \Delta \vdash P : C}{\Gamma \cup (\Delta \setminus \{A^x, B^y\}) \vdash M[(x,y).P] : C}\ (\wedge\mathsf{E}), \qquad \frac{\Gamma \vdash M : A \vee B \quad \Delta_1 \vdash P : C \quad \Delta_2 \vdash Q : C}{\Gamma \cup (\Delta_1 \setminus A^x) \cup (\Delta_2 \setminus B^y) \vdash M[x.P, y.Q] : C}\ (\vee\mathsf{E})$$

and other definitions and proofs can be extended in a straightforward way.

The sequent calculus is well-suited to classical logic because of its beautiful symmetry, so we hope that our result will be extended to classical logic. However, even for the intuitionistic case, the cut-elimination contains much richer computation than our proposal. In fact, our cut-elimination includes only *t-protocol* cut-elimination of LK^{tq} [3]. For example, we can consider the cut-elimination steps following the *q-protocol* such as

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash B_1} \quad \dfrac{\vdots}{B_2 \vdash A}}{B_1 \to B_2 \vdash A}\ (\to\mathsf{L}) \qquad \dfrac{\vdots}{A \vdash C_1 \to C_2}\ (\to\mathsf{R})}{B_1 \to B_2 \vdash C_1 \to C_2}\ (\mathsf{Cut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdots}{\vdash B_1} \quad \dfrac{\dfrac{\vdots}{B_2 \vdash A} \quad \dfrac{\vdots}{A \vdash C_1 \to C_2}\ (\to\mathsf{R})}{B_2 \vdash C_1 \to C_2}\ (\mathsf{Cut})}{B_1 \to B_2 \vdash C_1 \to C_2}\ (\to\mathsf{L}),$$

whose term representation is $x[\![\mathsf{M}, y.\mathsf{P}]\!] \curlyvee^z_n (\underline{\lambda} w.\mathsf{N}) \Rightarrow x[\![\mathsf{M}, y.\mathsf{P} \curlyvee^z_n (\underline{\lambda} w.\mathsf{N})]\!]$, which is not contained in our system. Moreover, we can consider another *orientation* (in [3]) of logical cut-elimination such as $(\underline{\lambda} x.\mathsf{M})[\![\mathsf{N}, y.\mathsf{P}]\!] \Rightarrow (\mathsf{N} \curlyvee^x \mathsf{M}) \curlyvee^y \mathsf{P}$. Adding this rule makes no trouble with the isomorphism between the cut-elimination and the proof reduction. However the SN proof of the cut-elimination in this paper does not work for the new β-rule, since SN of $\mathsf{N} \curlyvee^x \mathsf{M}$ in the contractum is not guaranteed by decency of the redex. Urban gave a strongly normalizable cut-elimination for the classical sequent calculus [13], which admits to permute cuts in both direction by the notion of *labelled cuts*. He also gave a correspondence between the sequent calculus and a classical natural deduction with multiple conclusion, but it is not an isomorphism. It is a future work to extend the isomorphism established in this paper to classical logic and clarify computational meaning of cut-elimination in the classical sequent calculus.

Acknowledgment

The author is grateful to Makoto Tatsuta and Kentaro Kikuchi for fruitful discussion and their valuable comments.

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