Instructor's Solution Manual
Introduction to Electrodynamics
Fourth Edition

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Contents

Preface

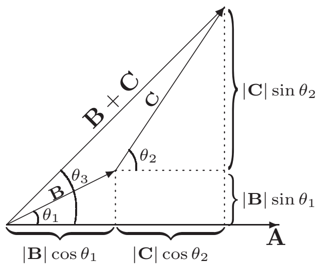
Although I wrote these solutions, much of the typesetting was done by Jonah Gollub, Christopher Lee, and James Terwilliger (any mistakes are, of course, entirely their fault). Chris also did many of the figures, and I would like to thank him particularly for all his help. If you find errors, please let me know (griffith@reed.edu).

David Griffiths

Chapter 1

Vector Analysis

Problem 1.1

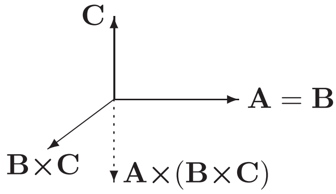


(a) From the diagram, $|\mathbf{B}+\mathbf{C}|\cos\theta_3 = |\mathbf{B}|\cos\theta_1 + |\mathbf{C}|\cos\theta_2$. Multiply by $|\mathbf{A}|$.
$|\mathbf{A}||\mathbf{B}+\mathbf{C}|\cos\theta_3 = |\mathbf{A}||\mathbf{B}|\cos\theta_1 + |\mathbf{A}||\mathbf{C}|\cos\theta_2$.
So: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$. (Dot product is distributive)

Similarly: $|\mathbf{B}+\mathbf{C}|\sin\theta_3 = |\mathbf{B}|\sin\theta_1 + |\mathbf{C}|\sin\theta_2$. Mulitply by $|\mathbf{A}|\,\hat{\mathbf{n}}$.
$|\mathbf{A}||\mathbf{B}+\mathbf{C}|\sin\theta_3\,\hat{\mathbf{n}} = |\mathbf{A}||\mathbf{B}|\sin\theta_1\,\hat{\mathbf{n}} + |\mathbf{A}||\mathbf{C}|\sin\theta_2\,\hat{\mathbf{n}}$.
If $\hat{\mathbf{n}}$ is the unit vector pointing out of the page, it follows that
$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$. (Cross product is distributive)

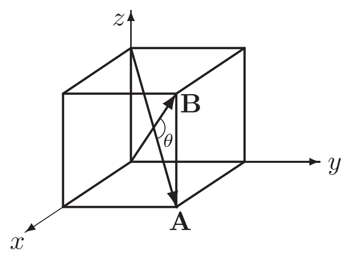
(b) For the general case, see G. E. Hay's *Vector and Tensor Analysis*, Chapter 1, Section 7 (dot product) and Section 8 (cross product)

Problem 1.2



The triple cross-product is *not* in general associative. For example, suppose $\mathbf{A} = \mathbf{B}$ and $\mathbf{C}$ is perpendicular to $\mathbf{A}$, as in the diagram. Then $(\mathbf{B}\times\mathbf{C})$ points out-of-the-page, and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ points *down*, and has magnitude ABC. But $(\mathbf{A}\times\mathbf{B}) = \mathbf{0}$, so $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{0} \neq \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Problem 1.3



$\mathbf{A} = +1\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} - 1\,\hat{\mathbf{z}}$; $A = \sqrt{3}$; $\mathbf{B} = 1\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}$; $B = \sqrt{3}$.

$\mathbf{A}\cdot\mathbf{B} = +1 + 1 - 1 = 1 = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta \Rightarrow \cos\theta = \frac{1}{3}$.

$$\boxed{\theta = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 70.5288^\circ}$$

Problem 1.4

The cross-product of any two vectors in the plane will give a vector perpendicular to the plane. For example, we might pick the base ($\mathbf{A}$) and the left side ($\mathbf{B}$):

$\mathbf{A} = -1\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$; $\mathbf{B} = -1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}.$$

This has the right *direction*, but the wrong *magnitude*. To make a *unit* vector out of it, simply divide by its length:

$|\mathbf{A}\times\mathbf{B}| = \sqrt{36+9+4} = 7.$ $\quad\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \boxed{\frac{6}{7}\hat{\mathbf{x}} + \frac{3}{7}\hat{\mathbf{y}} + \frac{2}{7}\hat{\mathbf{z}}}.$

Problem 1.5

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$= \hat{\mathbf{x}}[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] + \hat{\mathbf{y}}() + \hat{\mathbf{z}}()$

(I'll just check the x-component; the others go the same way)

$= \hat{\mathbf{x}}(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z) + \hat{\mathbf{y}}() + \hat{\mathbf{z}}().$

$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = [B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)]\,\hat{\mathbf{x}} + ()\,\hat{\mathbf{y}} + ()\,\hat{\mathbf{z}}$
$= \hat{\mathbf{x}}(A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x) + \hat{\mathbf{y}}() + \hat{\mathbf{z}}()$. They agree.

Problem 1.6

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) + \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}(\mathbf{C}\cdot\mathbf{B}) + \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) - \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) = \mathbf{0}.$
So: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) - (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$

If this is zero, then either $\mathbf{A}$ is parallel to $\mathbf{C}$ (including the case in which they point in *opposite* directions, or one is zero), or else $\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{A} = 0$, in which case $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$ (including the case $\mathbf{B} = \mathbf{0}$.)

Conclusion: $\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \iff \text{either } \mathbf{A} \text{ is parallel to } \mathbf{C}\text{, or } \mathbf{B} \text{ is perpendicular to } \mathbf{A} \text{ and } \mathbf{C}.}$

Problem 1.7

$$\boldsymbol{\mathscr{r}} = (4\,\hat{\mathbf{x}} + 6\,\hat{\mathbf{y}} + 8\,\hat{\mathbf{z}}) - (2\,\hat{\mathbf{x}} + 8\,\hat{\mathbf{y}} + 7\,\hat{\mathbf{z}}) = \boxed{2\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + \hat{\mathbf{z}}}$$

$$\mathscr{r} = \sqrt{4+4+1} = \boxed{3}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \boxed{\frac{2}{3}\hat{\mathbf{x}} - \frac{2}{3}\hat{\mathbf{y}} + \frac{1}{3}\hat{\mathbf{z}}}$$

Problem 1.8

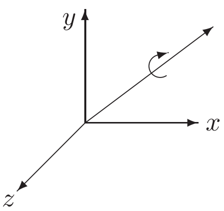
(a) $\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = (\cos\phi A_y + \sin\phi A_z)(\cos\phi B_y + \sin\phi B_z) + (-\sin\phi A_y + \cos\phi A_z)(-\sin\phi B_y + \cos\phi B_z)$
$= \cos^2\phi A_yB_y + \sin\phi\cos\phi(A_yB_z + A_zB_y) + \sin^2\phi A_zB_z + \sin^2\phi A_yB_y - \sin\phi\cos\phi(A_yB_z + A_zB_y) + \cos^2\phi A_zB_z$
$= (\cos^2\phi + \sin^2\phi)A_yB_y + (\sin^2\phi + \cos^2\phi)A_zB_z = A_yB_y + A_zB_z.$ ✓

(b) $(\overline{A}_x)^2 + (\overline{A}_y)^2 + (\overline{A}_z)^2 = \Sigma_{i=1}^3\overline{A}_i\overline{A}_i = \Sigma_{i=1}^3\left(\Sigma_{j=1}^3 R_{ij}A_j\right)\left(\Sigma_{k=1}^3 R_{ik}A_k\right) = \Sigma_{j,k}\left(\Sigma_i R_{ij}R_{ik}\right)A_jA_k.$

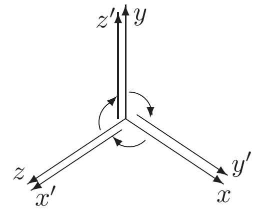
This equals $A_x^2 + A_y^2 + A_z^2$ *provided* $\boxed{\Sigma_{i=1}^3 R_{ij}R_{ik} = \begin{Bmatrix} 1 \; if \; j = k \\ 0 \; if \; j \neq k \end{Bmatrix}}$

Moreover, if R is to preserve lengths for *all* vectors $\mathbf{A}$, then this condition is not only *sufficient* but also *necessary*. For suppose $\mathbf{A} = (1,0,0)$. Then $\Sigma_{j,k}\left(\Sigma_i R_{ij}R_{ik}\right)A_jA_k = \Sigma_i R_{i1}R_{i1}$, and this must equal 1 (since we want $\overline{A}_x^2 + \overline{A}_y^2 + \overline{A}_z^2 = 1$). Likewise, $\Sigma_{i=1}^3 R_{i2}R_{i2} = \Sigma_{i=1}^3 R_{i3}R_{i3} = 1$. To check the case $j \neq k$, choose $\mathbf{A} = (1,1,0)$. Then we want $2 = \Sigma_{j,k}\left(\Sigma_i R_{ij}R_{ik}\right)A_jA_k = \Sigma_i R_{i1}R_{i1} + \Sigma_i R_{i2}R_{i2} + \Sigma_i R_{i1}R_{i2} + \Sigma_i R_{i2}R_{i1}$. But we already know that the first two sums are both 1; the third and fourth are *equal*, so $\Sigma_i R_{i1}R_{i2} = \Sigma_i R_{i2}R_{i1} = 0$, and so on for other unequal combinations of j, k. ✓ In matrix notation: $\tilde{R}R = 1$, where $\tilde{R}$ is the *transpose* of R.

Problem 1.9



Looking down the axis:



A 120° rotation carries the z axis into the y $(=\overline{z})$ axis, y into x $(=\overline{y})$, and x into z $(=\overline{x})$. So $\overline{A}_x = A_z$, $\overline{A}_y = A_x$, $\overline{A}_z = A_y$.

$$\boxed{R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

Problem 1.10

(a) $\boxed{\text{No change.}}$ $(\overline{A}_x = A_x,\ \overline{A}_y = A_y,\ \overline{A}_z = A_z)$

(b) $\boxed{\mathbf{A} \longrightarrow -\mathbf{A},}$ in the sense $(\overline{A}_x = -A_x,\ \overline{A}_y = -A_y,\ \overline{A}_z = -A_z)$

(c) $(\mathbf{A}\times\mathbf{B}) \longrightarrow (-\mathbf{A})\times(-\mathbf{B}) = (\mathbf{A}\times\mathbf{B})$. That is, if $\mathbf{C} = \mathbf{A}\times\mathbf{B}$, $\boxed{\mathbf{C} \longrightarrow \mathbf{C}}$. *No* minus sign, in contrast to behavior of an "ordinary" vector, as given by (b). If $\mathbf{A}$ and $\mathbf{B}$ are *pseudovectors*, then $(\mathbf{A}\times\mathbf{B}) \longrightarrow (\mathbf{A})\times(\mathbf{B}) = (\mathbf{A}\times\mathbf{B})$. So the cross-product of two pseudovectors is again a *pseudovector.* In the cross-product of a vector and a pseudovector, one changes sign, the other doesn't, and therefore the cross-product is itself a *vector.* *Angular momentum* $(\mathbf{L} = \mathbf{r}\times\mathbf{p})$ and *torque* $(\mathbf{N} = \mathbf{r}\times\mathbf{F})$ are pseudovectors.

(d) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \longrightarrow (-\mathbf{A})\cdot((-\mathbf{B})\times(-\mathbf{C})) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. So, if $a = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, then $\boxed{a \longrightarrow -a;}$ a pseudoscalar *changes sign* under inversion of coordinates.

Problem 1.11

$$\boxed{\begin{aligned} &(a)\boldsymbol{\nabla} f = 2x\,\hat{\mathbf{x}} + 3y^2\,\hat{\mathbf{y}} + 4z^3\,\hat{\mathbf{z}} \\ &(b)\boldsymbol{\nabla} f = 2xy^3z^4\,\hat{\mathbf{x}} + 3x^2y^2z^4\,\hat{\mathbf{y}} + 4x^2y^3z^3\,\hat{\mathbf{z}} \\ &(c)\boldsymbol{\nabla} f = e^x \sin y \ln z\,\hat{\mathbf{x}} + e^x \cos y \ln z\,\hat{\mathbf{y}} + e^x \sin y (1/z)\,\hat{\mathbf{z}} \end{aligned}}$$

Problem 1.12

(a) $\boldsymbol{\nabla} h = 10[(2y - 6x - 18)\,\hat{\mathbf{x}} + (2x - 8y + 28)\,\hat{\mathbf{y}}]$. $\boldsymbol{\nabla} h = 0$ at summit, so

$$\left.\begin{aligned} &2y - 6x - 18 = 0 \\ &2x - 8y + 28 = 0 \Longrightarrow 6x - 24y + 84 = 0 \end{aligned}\right\} 2y - 18 - 24y + 84 = 0.$$

$22y = 66 \Longrightarrow y = 3 \Longrightarrow 2x - 24 + 28 = 0 \Longrightarrow x = -2.$

Top is $\boxed{\text{3 miles north, 2 miles west, of South Hadley.}}$

(b) Putting in $x = -2$, $y = 3$:
$h = 10(-12 - 12 - 36 + 36 + 84 + 12) = \boxed{720\text{ ft.}}$

(c) Putting in $x = 1$, $y = 1$: $\boldsymbol{\nabla} h = 10[(2 - 6 - 18)\,\hat{\mathbf{x}} + (2 - 8 + 28)\,\hat{\mathbf{y}}] = 10(-22\,\hat{\mathbf{x}} + 22\,\hat{\mathbf{y}}) = 220(-\,\hat{\mathbf{x}} + \hat{\mathbf{y}})$.
$|\boldsymbol{\nabla} h| = 220\sqrt{2} \approx \boxed{311\text{ ft/mile;}}$ direction: $\boxed{\text{northwest.}}$

Problem 1.13

$$\boldsymbol{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}; \quad \mathscr{r} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}.$$

(a) $\boldsymbol{\nabla}(\mathscr{r}^{\,2}) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()\,\hat{\mathbf{z}} = 2(x-x')\,\hat{\mathbf{x}} + 2(y-y')\,\hat{\mathbf{y}} + 2(z-z')\,\hat{\mathbf{z}} = 2\,\boldsymbol{\mathscr{r}}$.

(b) $\boldsymbol{\nabla}(\frac{1}{\mathscr{r}}) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{-\frac{1}{2}}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()^{-\frac{1}{2}}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()^{-\frac{1}{2}}\,\hat{\mathbf{z}}$
$= -\frac{1}{2}()^{-\frac{3}{2}}2(x-x')\,\hat{\mathbf{x}} - \frac{1}{2}()^{-\frac{3}{2}}2(y-y')\,\hat{\mathbf{y}} - \frac{1}{2}()^{-\frac{3}{2}}2(z-z')\,\hat{\mathbf{z}}$
$= -()^{-\frac{3}{2}}[(x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}] = -(1/\mathscr{r}^{\,3})\boldsymbol{\mathscr{r}} = -(1/\mathscr{r}^{\,2})\hat{\boldsymbol{\mathscr{r}}}$.

(c) $\frac{\partial}{\partial x}(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-1}\frac{\partial \mathscr{r}}{\partial x} = n\,\mathscr{r}^{\,n-1}(\frac{1}{2}\frac{1}{\mathscr{r}}2\,\mathscr{r}_{\,x}) = n\,\mathscr{r}^{\,n-1}\hat{\boldsymbol{\mathscr{r}}}_{\,x}$, so $\boxed{\boldsymbol{\nabla}(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-1}\,\hat{\boldsymbol{\mathscr{r}}}}$

Problem 1.14

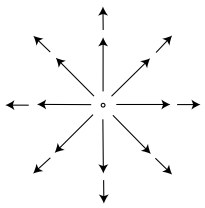
$\overline{y} = +y\,\cos\phi + z\,\sin\phi$; multiply by $\sin\phi$: $\overline{y}\sin\phi = +y\,\sin\phi\cos\phi + z\,\sin^2\phi$.
$\overline{z} = -y\,\sin\phi + z\,\cos\phi$; multiply by $\cos\phi$: $\overline{z}\cos\phi = -y\,\sin\phi\cos\phi + z\,\cos^2\phi$.
Add: $\overline{y}\sin\phi + \overline{z}\cos\phi = z(\sin^2\phi + \cos^2\phi) = z$. Likewise, $\overline{y}\cos\phi - \overline{z}\sin\phi = y$.
So $\frac{\partial y}{\partial \overline{y}} = \cos\phi$; $\frac{\partial y}{\partial \overline{z}} = -\sin\phi$; $\frac{\partial z}{\partial \overline{y}} = \sin\phi$; $\frac{\partial z}{\partial \overline{z}} = \cos\phi$. Therefore

$$\left.\begin{aligned} \overline{(\boldsymbol{\nabla} f)}_y &= \frac{\partial f}{\partial \overline{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{y}} = +\cos\phi(\boldsymbol{\nabla} f)_y + \sin\phi(\boldsymbol{\nabla} f)_z \\ \overline{(\boldsymbol{\nabla} f)}_z &= \frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{z}} = -\sin\phi(\boldsymbol{\nabla} f)_y + \cos\phi(\boldsymbol{\nabla} f)_z \end{aligned}\right\} \text{So } \boldsymbol{\nabla} f \text{ transforms as a vector.} \quad \text{qed}$$

Problem 1.15

$$\boxed{\begin{aligned} &(a)\boldsymbol{\nabla}\cdot\mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 2x + 0 - 2x = 0. \\ &(b)\boldsymbol{\nabla}\cdot\mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz) = y + 2z + 3x. \\ &(c)\boldsymbol{\nabla}\cdot\mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy+z^2) + \frac{\partial}{\partial z}(2yz) = 0 + (2x) + (2y) = 2(x+y) \end{aligned}}$$

Problem 1.16



$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3}) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$
$+ \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$
$= ()^{-\frac{3}{2}} + x(-3/2)()^{-\frac{5}{2}}2x + ()^{-\frac{3}{2}} + y(-3/2)()^{-\frac{5}{2}}2y + ()^{-\frac{3}{2}}$
$+ z(-3/2)()^{-\frac{5}{2}}2z = 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = 3r^{-3} - 3r^{-3} = 0.$

This conclusion is surprising, because, from the diagram, this vector field is obviously diverging away from the origin. How, then, can $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$? The answer is that $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ everywhere *except* at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for $\mathbf{v}$ blows up. In fact, $\boldsymbol{\nabla}\cdot\mathbf{v}$ is *infinite* at that one point, and zero elsewhere, as we shall see in Sect. 1.5.

Problem 1.17

$\overline{v}_y = \cos\phi\, v_y + \sin\phi\, v_z$; $\overline{v}_z = -\sin\phi\, v_y + \cos\phi\, v_z$.

$\frac{\partial \overline{v}_y}{\partial \overline{y}} = \frac{\partial v_y}{\partial \overline{y}}\cos\phi + \frac{\partial v_z}{\partial \overline{y}}\sin\phi = \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\sin\phi$. Use result in Prob. 1.14:
$= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi.$

$\frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{\partial v_y}{\partial \overline{z}}\sin\phi + \frac{\partial v_z}{\partial \overline{z}}\cos\phi = -\left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\sin\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\cos\phi$
$= -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$. So

$$\frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} = \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin^2\phi + \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi$$
$$- \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi$$
$$= \frac{\partial v_y}{\partial y}\left(\cos^2\phi + \sin^2\phi\right) + \frac{\partial v_z}{\partial z}\left(\sin^2\phi + \cos^2\phi\right) = \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \ \checkmark$$

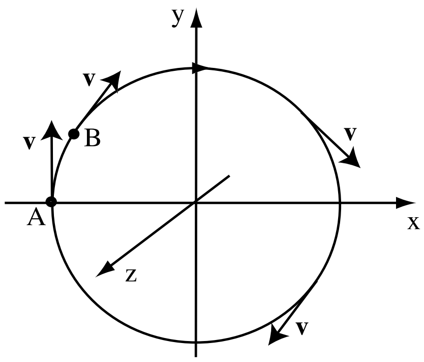
Problem 1.18

(a) $$\nabla\times\mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{\mathbf{x}}(0-6xz) + \hat{\mathbf{y}}(0+2z) + \hat{\mathbf{z}}(3z^2-0) = \boxed{-6xz\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 3z^2\,\hat{\mathbf{z}}.}$$

(b) $$\nabla\times\mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = \boxed{-2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.}$$

(c) $$\nabla\times\mathbf{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy+z^2) & 2yz \end{vmatrix} = \hat{\mathbf{x}}(2z-2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(2y-2y) = \boxed{\mathbf{0}.}$$

Problem 1.19



As we go from point A to point B (9 o'clock to 10 o'clock), x increases, y increases, v_x increases, and v_y decreases, so $\partial v_x/\partial y > 0$, while $\partial v_y/\partial y < 0$. On the circle, $v_z = 0$, and there is no dependence on z, so Eq. 1.41 says

$$\nabla\times\mathbf{v} = \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

points in the $\boxed{\text{negative } z \text{ direction}}$ (into the page), as the right hand rule would suggest. (Pick any other nearby points on the circle and you will come to the same conclusion.) [I'm sorry, but I cannot remember who suggested this cute illustration.]

Problem 1.20

$\mathbf{v} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$; or $\mathbf{v} = yz\,\hat{\mathbf{x}} + xz\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$; or $\mathbf{v} = (3x^2z - z^3)\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + (x^3 - 3xz^2)\,\hat{\mathbf{z}}$;
or $\mathbf{v} = (\sin x)(\cosh y)\,\hat{\mathbf{x}} - (\cos x)(\sinh y)\,\hat{\mathbf{y}}$; etc.

Problem 1.21

(i) $$\nabla(fg) = \frac{\partial(fg)}{\partial x}\hat{\mathbf{x}} + \frac{\partial(fg)}{\partial y}\hat{\mathbf{y}} + \frac{\partial(fg)}{\partial z}\hat{\mathbf{z}} = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{\mathbf{x}} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{\mathbf{y}} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{\mathbf{z}}$$
$$= f\left(\frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}} + \frac{\partial g}{\partial z}\hat{\mathbf{z}}\right) + g\left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}\right) = f(\nabla g) + g(\nabla f). \quad \text{qed}$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$$
$$= A_y\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_y}{\partial x} - A_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial A_z}{\partial x} + A_z\frac{\partial B_x}{\partial y} + B_x\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y}$$
$$+ A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - B_x\frac{\partial A_y}{\partial z}$$
$$= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)$$
$$- A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}). \quad \text{qed}$$

(v) $$\nabla\times(f\mathbf{A}) = \left(\frac{\partial(fA_z)}{\partial y} - \frac{\partial(fA_y)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(fA_x)}{\partial z} - \frac{\partial(fA_z)}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial(fA_y)}{\partial x} - \frac{\partial(fA_x)}{\partial y}\right)\hat{\mathbf{z}}$$

$$= \left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right)\hat{\mathbf{x}} + \left(f\frac{\partial A_x}{\partial z} + A_x\frac{\partial f}{\partial z} - f\frac{\partial A_z}{\partial x} - A_z\frac{\partial f}{\partial x}\right)\hat{\mathbf{y}}$$
$$+ \left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right)\hat{\mathbf{z}}$$
$$= f\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}\right]$$
$$- \left[\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right)\hat{\mathbf{x}} + \left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right)\hat{\mathbf{y}} + \left(A_x\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial x}\right)\hat{\mathbf{z}}\right]$$
$$= f\,(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla} f). \quad \text{qed}$$

Problem 1.22

(a) $$(\mathbf{A}\cdot\nabla)\,\mathbf{B} = \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\hat{\mathbf{x}} + \left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\hat{\mathbf{y}}$$
$$+ \left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}.$$

(b) $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$. Let's just do the x component.

$$[(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}}]_x = \frac{1}{\sqrt{\ }}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$= \frac{1}{r}\left\{x\left[\frac{1}{\sqrt{\ }} + x\left(-\tfrac{1}{2}\right)\frac{1}{(\sqrt{\ })^3}2x\right] + yx\left[-\frac{1}{2}\frac{1}{(\sqrt{\ })^3}2y\right] + zx\left[-\frac{1}{2}\frac{1}{(\sqrt{\ })^3}2z\right]\right\}$$
$$= \frac{1}{r}\left\{\frac{x}{r} - \frac{1}{r^3}\left(x^3 + xy^2 + xz^2\right)\right\} = \frac{1}{r}\left\{\frac{x}{r} - \frac{x}{r^3}\left(x^2+y^2+z^2\right)\right\} = \frac{1}{r}\left(\frac{x}{r} - \frac{x}{r}\right) = 0.$$

Same goes for the other components. Hence: $\boxed{(\hat{\mathbf{r}}\cdot\nabla)\,\hat{\mathbf{r}} = \mathbf{0}}$.

(c) $$(\mathbf{v}_a\cdot\nabla)\,\mathbf{v}_b = \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3xz\,\hat{\mathbf{z}})$$
$$= x^2\left(y\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}\right) + 3xz^2\left(x\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}\right) - 2xz\left(0\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3x\,\hat{\mathbf{z}}\right)$$
$$= \left(x^2y + 3x^2z^2\right)\hat{\mathbf{x}} + \left(6xz^3 - 4xyz\right)\hat{\mathbf{y}} + \left(3x^2z - 6x^2z\right)\hat{\mathbf{z}}$$
$$= \boxed{x^2\left(y + 3z^2\right)\hat{\mathbf{x}} + 2xz\left(3z^2 - 2y\right)\hat{\mathbf{y}} - 3x^2z\,\hat{\mathbf{z}}}$$

Problem 1.23

(ii) $$[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x = \frac{\partial}{\partial x}(A_xB_x + A_yB_y + A_zB_z) = \frac{\partial A_x}{\partial x}B_x + A_x\frac{\partial B_x}{\partial x} + \frac{\partial A_y}{\partial x}B_y + A_y\frac{\partial B_y}{\partial x} + \frac{\partial A_z}{\partial x}B_z + A_z\frac{\partial B_z}{\partial x}$$

$$[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})]_x = A_y(\boldsymbol{\nabla}\times\mathbf{B})_z - A_z(\boldsymbol{\nabla}\times\mathbf{B})_y = A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$
$$[\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})]_x = B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$
$$[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}]_x = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x = A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}$$
$$[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x = B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}$$

So $[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x$
$$= A_y\frac{\partial B_y}{\partial x} - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} + A_z\frac{\partial B_z}{\partial x} + B_y\frac{\partial A_y}{\partial x} - B_y\frac{\partial A_x}{\partial y} - B_z\frac{\partial A_x}{\partial z} + B_z\frac{\partial A_z}{\partial x}$$
$$+ A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z} + B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}$$
$$= B_x\frac{\partial A_x}{\partial x} + A_x\frac{\partial B_x}{\partial x} + B_y\left(\frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} + \cancel{\frac{\partial A_x}{\partial y}}\right) + A_y\left(\frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} + \cancel{\frac{\partial B_x}{\partial y}}\right)$$
$$+ B_z\left(-\cancel{\frac{\partial A_x}{\partial z}} + \frac{\partial A_z}{\partial x} + \cancel{\frac{\partial A_x}{\partial z}}\right) + A_z\left(-\cancel{\frac{\partial B_x}{\partial z}} + \frac{\partial B_z}{\partial x} + \cancel{\frac{\partial B_x}{\partial z}}\right)$$
$$= [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x \text{ (same for } y \text{ and } z)$$

(vi) $$[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x = \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})_z - \frac{\partial}{\partial z}(\mathbf{A}\times\mathbf{B})_y = \frac{\partial}{\partial y}(A_xB_y - A_yB_x) - \frac{\partial}{\partial z}(A_zB_x - A_xB_z)$$
$$= \frac{\partial A_x}{\partial y}B_y + A_x\frac{\partial B_y}{\partial y} - \frac{\partial A_y}{\partial y}B_x - A_y\frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial z}B_x - A_z\frac{\partial B_x}{\partial z} + \frac{\partial A_x}{\partial z}B_z + A_x\frac{\partial B_z}{\partial z}$$

$[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})]_x$
$$= B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} - A_x\frac{\partial B_x}{\partial x} - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} + A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - B_x\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= B_y \frac{\partial A_x}{\partial y} + A_x\left(-\cancel{\frac{\partial B_x}{\partial x}} + \cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) + B_x\left(\cancel{\frac{\partial A_x}{\partial x}} - \cancel{\frac{\partial A_x}{\partial x}} - \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z}\right)$$
$$+ A_y\left(-\frac{\partial B_x}{\partial y}\right) + A_z\left(-\frac{\partial B_x}{\partial z}\right) + B_z\left(\frac{\partial A_x}{\partial z}\right)$$
$$= [\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x \text{ (same for } y \text{ and } z)$$

Problem 1.24

$$\begin{aligned}
\boldsymbol{\nabla}(f/g) &= \frac{\partial}{\partial x}(f/g)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(f/g)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(f/g)\,\hat{\mathbf{z}} \\
&= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\,\hat{\mathbf{x}} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\,\hat{\mathbf{y}} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\,\hat{\mathbf{z}} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}\right) - f\left(\frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}} + \frac{\partial g}{\partial z}\hat{\mathbf{z}}\right)\right] = \frac{g\nabla f - f\nabla g}{g^2}. \quad \text{qed}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{A}/g) &= \frac{\partial}{\partial x}(A_x/g) + \frac{\partial}{\partial y}(A_y/g) + \frac{\partial}{\partial z}(A_z/g) \\
&= \frac{g\frac{\partial A_x}{\partial x} - A_x\frac{\partial g}{\partial x}}{g^2} + \frac{g\frac{\partial A_y}{\partial y} - A_y\frac{\partial g}{\partial y}}{g^2} + \frac{g\frac{\partial A_z}{\partial z} - A_z\frac{\partial g}{\partial x}}{g^2} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} + A_z\frac{\partial g}{\partial z}\right)\right] = \frac{g\nabla\cdot\mathbf{A} - \mathbf{A}\cdot\nabla g}{g^2}. \quad \text{qed}
\end{aligned}$$

$$\begin{aligned}
[\boldsymbol{\nabla}\times(\mathbf{A}/g)]_x &= \frac{\partial}{\partial y}(A_z/g) - \frac{\partial}{\partial z}(A_y/g) \\
&= \frac{g\frac{\partial A_z}{\partial y} - A_z\frac{\partial g}{\partial y}}{g^2} - \frac{g\frac{\partial A_y}{\partial z} - A_y\frac{\partial g}{\partial z}}{g^2} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \left(A_z\frac{\partial g}{\partial y} - A_y\frac{\partial g}{\partial z}\right)\right] \\
&= \frac{g(\nabla\times\mathbf{A})_x + (\mathbf{A}\times\nabla g)_x}{g^2} \text{ (same for } y \text{ and } z). \quad \text{qed}
\end{aligned}$$

Problem 1.25

(a) $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} = \hat{\mathbf{x}}(6xz) + \hat{\mathbf{y}}(9zy) + \hat{\mathbf{z}}(-2x^2 - 6y^2)$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9zy) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) = 6z + 9z + 0 = 15z$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(3z)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(x)\right) = 0;\ \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2x)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(3y) - \frac{\partial}{\partial x}(0)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y)\right) = -5\,\hat{\mathbf{z}};\ \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = -15z$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) \stackrel{?}{=} \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = 0 - (-15z) = 15z.\ \checkmark$$

(b) $\mathbf{A}\cdot\mathbf{B} = 3xy - 4xy = -xy$; $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \boldsymbol{\nabla}(-xy) = \hat{\mathbf{x}}\frac{\partial}{\partial x}(-xy) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(-xy) = -y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}$

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = \hat{\mathbf{x}}(-10y) + \hat{\mathbf{y}}(5x);\ \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{0}$$

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}) = \hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x)$$

$$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} = \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}) = \hat{\mathbf{x}}(3y) + \hat{\mathbf{y}}(-4x)$$

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}$$
$$= -10y\,\hat{\mathbf{x}} + 5x\,\hat{\mathbf{y}} + 6y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}} + 3y\,\hat{\mathbf{x}} - 4x\,\hat{\mathbf{y}} = -y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}} = \boldsymbol{\nabla}\cdot(\mathbf{A}\cdot\mathbf{B}).\ \checkmark$$

(c) $$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(-2x^2 - 6y^2) - \frac{\partial}{\partial z}(9zy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(6xz) - \frac{\partial}{\partial x}(-2x^2 - 6y^2)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(9zy) - \frac{\partial}{\partial y}(6xz)\right)$$
$$= \hat{\mathbf{x}}(-12y - 9y) + \hat{\mathbf{y}}(6x + 4x) + \hat{\mathbf{z}}(0) = -21y\,\hat{\mathbf{x}} + 10x\,\hat{\mathbf{y}}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1 + 2 + 3 = 6;\ \boldsymbol{\nabla}\cdot\mathbf{B} = \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x) = 0$$

$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})=3y\,\hat{\mathbf{x}}-4x\,\hat{\mathbf{y}}-6y\,\hat{\mathbf{x}}+2x\,\hat{\mathbf{y}}-18y\,\hat{\mathbf{x}}+12x\,\hat{\mathbf{y}}=-21y\,\hat{\mathbf{x}}+10x\,\hat{\mathbf{y}}$
$=\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}).$ ✓

Problem 1.26

(a) $\frac{\partial^2 T_a}{\partial x^2}=2;\ \frac{\partial^2 T_a}{\partial y^2}=\frac{\partial^2 T_a}{\partial z^2}=0\ \Rightarrow\ \boxed{\nabla^2 T_a=2.}$

(b) $\frac{\partial^2 T_b}{\partial x^2}=\frac{\partial^2 T_b}{\partial y^2}=\frac{\partial^2 T_b}{\partial z^2}=-T_b\ \Rightarrow\ \boxed{\nabla^2 T_b=-3T_b=-3\sin x\sin y\sin z.}$

(c) $\frac{\partial^2 T_c}{\partial x^2}=25T_c\ ;\ \frac{\partial^2 T_c}{\partial y^2}=-16T_c\ ;\ \frac{\partial^2 T_c}{\partial z^2}=-9T_c\ \Rightarrow\ \boxed{\nabla^2 T_c=0.}$

(d) $$\left.\begin{array}{l}\frac{\partial^2 v_x}{\partial x^2}=2\ ;\ \frac{\partial^2 v_x}{\partial y^2}=\frac{\partial^2 v_x}{\partial z^2}=0\ \Rightarrow\ \nabla^2 v_x=2\\ \frac{\partial^2 v_y}{\partial x^2}=\frac{\partial^2 v_y}{\partial y^2}=0\ ;\ \frac{\partial^2 v_y}{\partial z^2}=6x\ \Rightarrow\ \nabla^2 v_y=6x\\ \frac{\partial^2 v_z}{\partial x^2}=\frac{\partial^2 v_z}{\partial y^2}=\frac{\partial^2 v_z}{\partial z^2}=0\ \Rightarrow\ \nabla^2 v_z=0\end{array}\right\}\ \boxed{\nabla^2\mathbf{v}=2\,\hat{\mathbf{x}}+6x\,\hat{\mathbf{y}}.}$$

Problem 1.27

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v})=\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$
$$=\left(\frac{\partial^2 v_z}{\partial x\,\partial y}-\frac{\partial^2 v_z}{\partial y\,\partial x}\right)+\left(\frac{\partial^2 v_x}{\partial y\,\partial z}-\frac{\partial^2 v_x}{\partial z\,\partial y}\right)+\left(\frac{\partial^2 v_y}{\partial z\,\partial x}-\frac{\partial^2 v_y}{\partial x\,\partial z}\right)=0,\ \text{by equality of cross-derivatives.}$$

From Prob. 1.18: $\boldsymbol{\nabla}\times\mathbf{v}_a=-6xz\,\hat{\mathbf{x}}+2z\,\hat{\mathbf{y}}+3z^2\,\hat{\mathbf{z}}\ \Rightarrow\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}_a)=\frac{\partial}{\partial x}(-6xz)+\frac{\partial}{\partial y}(2z)+\frac{\partial}{\partial z}(3z^2)=-6z+6z=0.$

Problem 1.28

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}t)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z}\end{vmatrix}=\hat{\mathbf{x}}\left(\frac{\partial^2 t}{\partial y\,\partial z}-\frac{\partial^2 t}{\partial z\,\partial y}\right)+\hat{\mathbf{y}}\left(\frac{\partial^2 t}{\partial z\,\partial x}-\frac{\partial^2 t}{\partial x\,\partial z}\right)+\hat{\mathbf{z}}\left(\frac{\partial^2 t}{\partial x\,\partial y}-\frac{\partial^2 t}{\partial y\,\partial x}\right)$$
$=0$, by equality of cross-derivatives.

In Prob. 1.11(b), $\boldsymbol{\nabla}f=2xy^3z^4\,\hat{\mathbf{x}}+3x^2y^2z^4\,\hat{\mathbf{y}}+4x^2y^3z^3\,\hat{\mathbf{z}}$, so

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3\end{vmatrix}$$
$=\hat{\mathbf{x}}(3\cdot 4x^2y^2z^3-4\cdot 3x^2y^2z^3)+\hat{\mathbf{y}}(4\cdot 2xy^3z^3-2\cdot 4xy^3z^3)+\hat{\mathbf{z}}(2\cdot 3xy^2z^4-3\cdot 2xy^2z^4)=0.$ ✓

Problem 1.29

(a) $(0,0,0)\longrightarrow(1,0,0)$. $x:0\to 1, y=z=0; d\mathbf{l}=dx\,\hat{\mathbf{x}}; \mathbf{v}\cdot d\mathbf{l}=x^2\,dx; \int\mathbf{v}\cdot d\mathbf{l}=\int_0^1 x^2\,dx=(x^3/3)|_0^1=1/3.$
$(1,0,0)\longrightarrow(1,1,0)$. $x=1, y:0\to 1, z=0; d\mathbf{l}=dy\,\hat{\mathbf{y}}; \mathbf{v}\cdot d\mathbf{l}=2yz\,dy=0; \int\mathbf{v}\cdot d\mathbf{l}=0.$
$(1,1,0)\longrightarrow(1,1,1)$. $x=y=1, z:0\to 1; d\mathbf{l}=dz\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{l}=y^2\,dz=dz; \int\mathbf{v}\cdot d\mathbf{l}=\int_0^1 dz=z|_0^1=1.$
Total: $\int\mathbf{v}\cdot d\mathbf{l}=(1/3)+0+1=\boxed{4/3.}$

(b) $(0,0,0)\longrightarrow(0,0,1)$. $x=y=0, z:0\to 1; d\mathbf{l}=dz\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{l}=y^2\,dz=0; \int\mathbf{v}\cdot d\mathbf{l}=0.$
$(0,0,1)\longrightarrow(0,1,1)$. $x=0, y:0\to 1, z=1; d\mathbf{l}=dy\,\hat{\mathbf{y}}; \mathbf{v}\cdot d\mathbf{l}=2yz\,dy=2y\,dy; \int\mathbf{v}\cdot d\mathbf{l}=\int_0^1 2y\,dy=y^2|_0^1=1.$
$(0,1,1)\longrightarrow(1,1,1)$. $x:0\to 1, y=z=1; d\mathbf{l}=dx\,\hat{\mathbf{x}}; \mathbf{v}\cdot d\mathbf{l}=x^2\,dx; \int\mathbf{v}\cdot d\mathbf{l}=\int_0^1 x^2\,dx=(x^3/3)|_0^1=1/3.$
Total: $\int\mathbf{v}\cdot d\mathbf{l}=0+1+(1/3)=\boxed{4/3.}$

(c) $x=y=z:0\to 1; dx=dy=dz; \mathbf{v}\cdot d\mathbf{l}=x^2\,dx+2yz\,dy+y^2\,dz=x^2\,dx+2x^2\,dx+x^2\,dx=4x^2\,dx;$
$\int\mathbf{v}\cdot d\mathbf{l}=\int_0^1 4x^2\,dx=(4x^3/3)|_0^1=\boxed{4/3.}$

(d) $\oint\mathbf{v}\cdot d\mathbf{l}=(4/3)-(4/3)=\boxed{0.}$

Problem 1.30

$x, y : 0 \to 1, z = 0; d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{a} = y(z^2-3)\,dx\,dy = -3y\,dx\,dy; \int \mathbf{v}\cdot d\mathbf{a} = -3\int_0^2 dx \int_0^2 y\,dy = -3(x|_0^2)(\frac{y^2}{2}|_0^2) = -3(2)(2) = \boxed{-12.}$ In Ex. 1.7 we got 20, for the same boundary line (the square in the xy-plane), so the answer is $\boxed{\text{no:}}$ the surface integral does *not* depend only on the boundary line. The *total* flux for the cube is $20 + 12 = \boxed{32.}$

Problem 1.31

$\int T\,d\tau = \int z^2\,dx\,dy\,dz$. You can do the integrals in any order—here it is simplest to save z for last:

$$\int z^2 \left[\int \left(\int dx\right) dy\right] dz.$$

The sloping surface is $x+y+z=1$, so the x integral is $\int_0^{(1-y-z)} dx = 1-y-z$. For a given z, y ranges from 0 to $1-z$, so the y integral is $\int_0^{(1-z)}(1-y-z)\,dy = [(1-z)y - (y^2/2)]|_0^{(1-z)} = (1-z)^2 - [(1-z)^2/2] = (1-z)^2/2 = (1/2) - z + (z^2/2)$. Finally, the z integral is $\int_0^1 z^2(\frac{1}{2} - z + \frac{z^2}{2})\,dz = \int_0^1(\frac{z^2}{2} - z^3 + \frac{z^4}{2})\,dz = (\frac{z^3}{6} - \frac{z^4}{4} + \frac{z^5}{10})|_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \boxed{1/60.}$

Problem 1.32

$T(\mathbf{b}) = 1+4+2 = 7;\ T(\mathbf{a}) = 0.\ \Rightarrow \boxed{T(\mathbf{b}) - T(\mathbf{a}) = 7.}$

$\nabla T = (2x+4y)\hat{\mathbf{x}} + (4x+2z^3)\hat{\mathbf{y}} + (6yz^2)\hat{\mathbf{z}};\ \ \nabla T\cdot d\mathbf{l} = (2x+4y)dx + (4x+2z^3)dy + (6yz^2)dz$

$$\left.\begin{array}{l}\text{(a) Segment 1: } x: 0\to 1,\ y = z = dy = dz = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (2x)\,dx = x^2\big|_0^1 = 1.\\ \quad\ \text{Segment 2: } y: 0\to 1,\ x = 1,\ z = 0,\ dx = dz = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (4)\,dy = 4y\big|_0^1 = 4.\\ \quad\ \text{Segment 3: } z: 0\to 1,\ x = y = 1,\ dx = dy = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (6z^2)\,dz = 2z^3\big|_0^1 = 2.\end{array}\right\} \int_{\mathbf{a}}^{\mathbf{b}} \nabla T\cdot d\mathbf{l} = 7.\ \checkmark$$

$$\left.\begin{array}{l}\text{(b) Segment 1: } z: 0\to 1,\ x = y = dx = dy = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (0)\,dz = 0.\\ \quad\ \text{Segment 2: } y: 0\to 1,\ x = 0,\ z = 1,\ dx = dz = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (2)\,dy = 2y\big|_0^1 = 2.\\ \quad\ \text{Segment 3: } x: 0\to 1,\ y = z = 1,\ dy = dz = 0.\ \int \nabla T\cdot d\mathbf{l} = \int_0^1 (2x+4)\,dx\\ \qquad\qquad = (x^2+4x)\big|_0^1 = 1+4 = 5.\end{array}\right\} \int_{\mathbf{a}}^{\mathbf{b}} \nabla T\cdot d\mathbf{l} = 7.\ \checkmark$$

(c) $x: 0\to 1,\ y = x,\ z = x^2,\ dy = dx,\ dz = 2x\,dx.$

$\nabla T\cdot d\mathbf{l} = (2x+4x)dx + (4x+2x^6)dx + (6xx^4)2x\,dx = (10x+14x^6)dx.$

$\int_{\mathbf{a}}^{\mathbf{b}} \nabla T\cdot d\mathbf{l} = \int_0^1 (10x+14x^6)dx = (5x^2+2x^7)\big|_0^1 = 5+2 = 7.\ \checkmark$

Problem 1.33

$\nabla\cdot\mathbf{v} = y + 2z + 3x$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(y+2z+3x)\,dx\,dy\,dz = \iint\left\{\int_0^2(y+2z+3x)\,dx\right\}dy\,dz$$

$$\hookrightarrow \left[(y+2z)x + \tfrac{3}{2}x^2\right]_0^2 = 2(y+2z)+6$$

$$= \int\left\{\int_0^2(2y+4z+6)dy\right\}dz$$

$$\hookrightarrow \left[y^2 + (4z+6)y\right]_0^2 = 4 + 2(4z+6) = 8z+16$$

$$= \int_0^2(8z+16)dz = (4z^2+16z)\big|_0^2 = 16+32 = \boxed{48.}$$

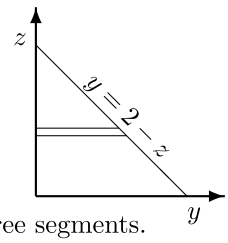
Numbering the surfaces as in Fig. 1.29:

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}, x = 2$. $\mathbf{v}\cdot d\mathbf{a} = 2y\,dy\,dz$. $\int \mathbf{v}\cdot d\mathbf{a} = \iint 2y\,dy\,dz = 2y^2\big|_0^2 = 8$.
(ii) $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}, x = 0$. $\mathbf{v}\cdot d\mathbf{a} = 0$. $\int \mathbf{v}\cdot d\mathbf{a} = 0$.
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}, y = 2$. $\mathbf{v}\cdot d\mathbf{a} = 4z\,dx\,dz$. $\int \mathbf{v}\cdot d\mathbf{a} = \iint 4z\,dx\,dz = 16$.
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}, y = 0$. $\mathbf{v}\cdot d\mathbf{a} = 0$. $\int \mathbf{v}\cdot d\mathbf{a} = 0$.
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}, z = 2$. $\mathbf{v}\cdot d\mathbf{a} = 6x\,dx\,dy$. $\int \mathbf{v}\cdot d\mathbf{a} = 24$.
(vi) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}, z = 0$. $\mathbf{v}\cdot d\mathbf{a} = 0$. $\int \mathbf{v}\cdot d\mathbf{a} = 0$.
$\Rightarrow \int \mathbf{v}\cdot d\mathbf{a} = 8 + 16 + 24 = 48$ ✓

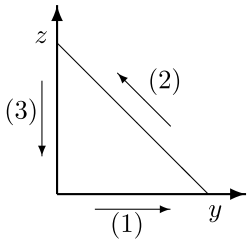
Problem 1.34

$\nabla\times\mathbf{v} = \hat{\mathbf{x}}(0 - 2y) + \hat{\mathbf{y}}(0 - 3z) + \hat{\mathbf{z}}(0 - x) = -2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}$.
$d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -2y\,dy\,dz$.

$$\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int\left\{\int_0^{2-z}(-2y)dy\right\}dz$$
$$\hookrightarrow y^2\Big|_0^{2-z} = -(2-z)^2$$
$$= -\int_0^2(4 - 4z + z^2)dz = -\left(4z - 2z^2 + \frac{z^3}{3}\right)\Big|_0^2$$
$$= -\left(8 - 8 + \frac{8}{3}\right) = \boxed{-\frac{8}{3}}$$



Meanwhile, $\mathbf{v}\cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.



(1) $x = z = 0$; $dx = dz = 0$. $y : 0 \to 2$. $\int \mathbf{v}\cdot d\mathbf{l} = 0$.
(2) $x = 0$; $z = 2 - y$; $dx = 0$, $dz = -dy$, $y : 2 \to 0$. $\mathbf{v}\cdot d\mathbf{l} = 2yz\,dy$.
$\int \mathbf{v}\cdot d\mathbf{l} = \int_2^0 2y(2-y)dy = -\int_0^2(4y - 2y^2)dy = -\left(2y^2 - \frac{2}{3}y^3\right)\Big|_0^2 = -\left(8 - \frac{2}{3}\cdot 8\right) = -\frac{8}{3}$.
(3) $x = y = 0$; $dx = dy = 0$; $z : 2 \to 0$. $\mathbf{v}\cdot d\mathbf{l} = 0$. $\int \mathbf{v}\cdot d\mathbf{l} = 0$. So $\oint \mathbf{v}\cdot d\mathbf{l} = -\frac{8}{3}$. ✓

Problem 1.35

By Corollary 1, $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$ should equal $\frac{4}{3}$. $\nabla\times\mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}}$.

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, $x = 1$; $y, z : 0 \to 1$. $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = (4z^2 - 2)dy\,dz$; $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1(4z^2 - 2)dz$ $= \left(\frac{4}{3}z^3 - 2z\right)\Big|_0^1 = \frac{4}{3} - 2 = -\frac{2}{3}$.
(ii) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}$, $z = 0$; $x, y : 0 \to 1$. $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$; $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$.
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}$, $y = 1$; $x, z : 0 \to 1$. $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$; $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$.
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}$, $y = 0$; $x, z : 0 \to 1$. $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$; $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$.
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}$, $z = 1$; $x, y : 0 \to 1$. $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2\,dx\,dy$; $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2$.

$\Rightarrow \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -\frac{2}{3} + 2 = \frac{4}{3}$. ✓

Problem 1.36

(a) Use the product rule $\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)$:

$$\int_{\mathcal{S}} f(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a} = \int_{\mathcal{S}} \boldsymbol{\nabla}\times(f\mathbf{A})\cdot d\mathbf{a} + \int_{\mathcal{S}} [\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot d\mathbf{a} = \oint_{\mathcal{P}} f\mathbf{A}\cdot d\mathbf{l} + \int_{\mathcal{S}} [\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot d\mathbf{a}. \quad \text{qed}$$

(I used Stokes' theorem in the last step.)

(b) Use the product rule $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$:

$$\int_{\mathcal{V}} \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})d\tau = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})\,d\tau + \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,d\tau = \oint_{\mathcal{S}} (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} + \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,d\tau. \quad \text{qed}$$

(I used the divergence theorem in the last step.)

Problem 1.37 $\boxed{r = \sqrt{x^2+y^2+z^2}; \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right); \quad \phi = \tan^{-1}\left(\frac{y}{x}\right).}$

Problem 1.38

There are many ways to do this one—probably the most illuminating way is to work it out by trigonometry from Fig. 1.36. The most systematic approach is to study the expression:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}.$$

If I only vary r slightly, then $d\mathbf{r} = \frac{\partial}{\partial r}(\mathbf{r})dr$ is a short vector pointing in the direction of increase in r. To make it a unit vector, I must divide by its length. Thus:

$$\hat{\mathbf{r}} = \frac{\frac{\partial\mathbf{r}}{\partial r}}{\left|\frac{\partial\mathbf{r}}{\partial r}\right|}; \quad \hat{\boldsymbol{\theta}} = \frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|}; \quad \hat{\boldsymbol{\phi}} = \frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}.$$

$\frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial\mathbf{r}}{\partial r}\right|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1.$

$\frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{\mathbf{x}} + r\cos\theta\sin\phi\,\hat{\mathbf{y}} - r\sin\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2.$

$\frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{x}} + r\sin\theta\cos\phi\,\hat{\mathbf{y}}$; $\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta.$

$$\Rightarrow \boxed{\begin{aligned}&\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}.\\ &\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}.\\ &\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.\end{aligned}}$$

Check: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1,$ ✓

$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = -\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi = 0,$ ✓ etc.

$\sin\theta\,\hat{\mathbf{r}} = \sin^2\theta\cos\phi\,\hat{\mathbf{x}} + \sin^2\theta\sin\phi\,\hat{\mathbf{y}} + \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

$\cos\theta\,\hat{\boldsymbol{\theta}} = \cos^2\theta\cos\phi\,\hat{\mathbf{x}} + \cos^2\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

Add these:

(1) $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} = +\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}};$

(2) $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$

Multiply (1) by $\cos\phi$, (2) by $\sin\phi$, and subtract:

$$\boxed{\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$$

Multiply (1) by $\sin\phi$, (2) by $\cos\phi$, and add:

$$\boxed{\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$$

$\cos\theta\,\hat{\mathbf{r}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \cos^2\theta\,\hat{\mathbf{z}}.$
$\sin\theta\,\hat{\boldsymbol{\theta}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin^2\theta\,\hat{\mathbf{z}}.$
Subtract these:

$$\boxed{\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.}$$

Problem 1.39

(a) $\boldsymbol{\nabla}\cdot\mathbf{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = \frac{1}{r^2}4r^3 = 4r$

$\int(\boldsymbol{\nabla}\cdot\mathbf{v}_1)d\tau = \int(4r)(r^2\sin\theta\,dr\,d\theta\,d\phi) = (4)\int_0^R r^3 dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi = (4)\left(\frac{R^4}{4}\right)(2)(2\pi) = \boxed{4\pi R^4}$

$\int\mathbf{v_1}\cdot d\mathbf{a} = \int(r^2\hat{\mathbf{r}})\cdot(r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}) = r^4\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\phi = 4\pi R^4$ ✓ (*Note:* at surface of sphere $r = R$.)

(b) $\boldsymbol{\nabla}\cdot\mathbf{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0 \;\Rightarrow \boxed{\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)d\tau = 0}$

$\int\mathbf{v}_2\cdot d\mathbf{a} = \int\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)(r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}) = \int\sin\theta\,d\theta\,d\phi = \boxed{4\pi.}$

They *don't* agree! The point is that this divergence is zero *except at the origin*, where it blows up, so our calculation of $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)$ is *incorrect.* The right answer is 4π.

Problem 1.40

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\
&= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}r\sin\theta(-\sin\phi)\\
&= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi
\end{aligned}$$

$$\int(\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr\int_0^{\frac{\theta}{2}}\left[\underbrace{\int_0^{2\pi}(5\cos\theta - \sin\phi)\,d\phi}_{\hookrightarrow 2\pi(5\cos\theta)}\right]d\theta\sin\theta$$

$$= \left(\frac{R^3}{3}\right)(10\pi)\underbrace{\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta}_{\hookrightarrow \left.\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}}$$

$$= \boxed{\frac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\; r = R;\; \phi: 0\to 2\pi,\; \theta: 0\to\frac{\pi}{2}.$

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\,d\theta\,d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3.$

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\,d\phi)(+\hat{\boldsymbol{\theta}}) = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$ (here $\theta = \frac{\pi}{2}$). $r: 0\to R,\; \phi: 0\to 2\pi.$

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}.$

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3.$ ✓

Problem 1.41 $\boxed{\boldsymbol{\nabla}t = (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} + \frac{1}{\cancel{\sin\theta}}(-\cancel{\sin\theta}\sin\phi)\hat{\boldsymbol{\phi}}}$

$$\begin{aligned}
\nabla^2 t &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}t)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\cos\theta + \sin\theta\cos\phi)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(-\sin\theta + \cos\theta\cos\phi)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi)\\
&= \frac{1}{r^2}2r(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}(-2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi) - \frac{1}{r\sin\theta}\cos\phi\\
&= \frac{1}{r\sin\theta}[2\sin\theta\cos\theta + 2\sin^2\theta\cos\phi - 2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi - \cos\phi]\\
&= \frac{1}{r\sin\theta}\left[(\sin^2\theta + \cos^2\theta)\cos\phi - \cos\phi\right] = 0.
\end{aligned}$$

$\Rightarrow \boxed{\nabla^2 t = 0}$

Check: $r\cos\theta = z,\ r\sin\theta\cos\phi = x \ \Rightarrow$ in Cartesian coordinates $t = x + z$. Obviously Laplacian is zero.

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} t \cdot d\mathbf{l} = t(\mathbf{b}) - t(\mathbf{a})$

Segment 1: $\theta = \frac{\pi}{2},\ \phi = 0,\ r : 0 \to 2.\ d\mathbf{l} = dr\,\hat{\mathbf{r}};\ \boldsymbol{\nabla} t \cdot d\mathbf{l} = (\cos\theta + \sin\theta\cos\phi)dr = (0+1)dr = dr.$

$\int \boldsymbol{\nabla} t \cdot d\mathbf{l} = \int_0^2 dr = 2.$

Segment 2: $\theta = \frac{\pi}{2},\ r = 2,\ \phi : 0 \to \frac{\pi}{2}.\quad d\mathbf{l} = r\sin\theta\, d\phi\, \hat{\boldsymbol{\phi}} = 2\, d\phi\, \hat{\boldsymbol{\phi}}.$

$\boldsymbol{\nabla} t \cdot d\mathbf{l} = (-\sin\phi)(2\,d\phi) = -2\sin\phi\, d\phi.\ \int \boldsymbol{\nabla} t \cdot d\mathbf{l} = -\int_0^{\frac{\pi}{2}} 2\sin\phi\, d\phi = \ 2\cos\phi\big|_0^{\frac{\pi}{2}} = -2.$

Segment 3: $r = 2,\ \phi = \frac{\pi}{2};\ \theta : \frac{\pi}{2} \to 0.$

$d\mathbf{l} = r\, d\theta\, \hat{\boldsymbol{\theta}} = 2\, d\theta\, \hat{\boldsymbol{\theta}};\ \boldsymbol{\nabla} t \cdot d\mathbf{l} = (-\sin\theta + \cos\theta\cos\phi)(2\, d\theta) = -2\sin\theta\, d\theta.$

$\int \boldsymbol{\nabla} t \cdot d\mathbf{l} = -\int_{\frac{\pi}{2}}^{0} 2\sin\theta\, d\theta = \ 2\cos\theta\big|_{\frac{\pi}{2}}^{0} = 2.$

Total: $\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} t \cdot d\mathbf{l} = 2 - 2 + 2 = \boxed{2}$. Meanwhile, $t(\mathbf{b}) - t(\mathbf{a}) = [2(1+0)] - [0(\ )] = 2.$ ✓

Problem 1.42 From Fig. 1.42, $\boxed{\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}};\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}};\ \hat{\mathbf{z}} = \hat{\mathbf{z}}}$

Multiply first by $\cos\phi$, second by $\sin\phi$, and subtract:

$\hat{\mathbf{s}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi = \cos^2\phi\,\hat{\mathbf{x}} + \cos\phi\sin\phi\,\hat{\mathbf{y}} + \sin^2\phi\,\hat{\mathbf{x}} - \sin\phi\cos\phi\,\hat{\mathbf{y}} = \hat{\mathbf{x}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{x}}.$

So $\boxed{\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$

Multiply first by $\sin\phi$, second by $\cos\phi$, and add:

$\hat{\mathbf{s}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi = \sin\phi\cos\phi\,\hat{\mathbf{x}} + \sin^2\phi\,\hat{\mathbf{y}} - \sin\phi\cos\phi\,\hat{\mathbf{x}} + \cos^2\phi\,\hat{\mathbf{y}} = \hat{\mathbf{y}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{y}}.$

So $\boxed{\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$ $\qquad \boxed{\hat{\mathbf{z}} = \hat{\mathbf{z}}.}$

Problem 1.43

(a)
$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\,s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}\,2s(2+\sin^2\phi) + \frac{1}{s}\,s(\cos^2\phi - \sin^2\phi) + 3\\ &= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3\\ &= 4 + \sin^2\phi + \cos^2\phi + 3 = \boxed{8.}\end{aligned}$$

(b) $\int(\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \int(8)s\,ds\,d\phi\,dz = 8\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi\int_0^5 dz = 8(2)\left(\frac{\pi}{2}\right)(5) = \boxed{40\pi.}$

Meanwhile, the surface integral has five parts:

top: $z = 5,\ d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = 3z\,s\,ds\,d\phi = 15s\,ds\,d\phi.\ \int\mathbf{v}\cdot d\mathbf{a} = 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi = 15\pi.$
bottom: $z = 0,\ d\mathbf{a} = -s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = -3z\,s\,ds\,d\phi = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$
back: $\phi = \frac{\pi}{2},\ d\mathbf{a} = ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = s\sin\phi\cos\phi\,ds\,dz = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$
left: $\phi = 0,\ d\mathbf{a} = -ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = -s\sin\phi\cos\phi\,ds\,dz = 0.\ \int\mathbf{v}\cdot d\mathbf{a} = 0.$
front: $s = 2,\ d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}};\ \mathbf{v}\cdot d\mathbf{a} = s(2+\sin^2\phi)s\,d\phi\,dz = 4(2+\sin^2\phi)d\phi\,dz.$

$\int\mathbf{v}\cdot d\mathbf{a} = 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)d\phi\int_0^5 dz = (4)(\pi + \frac{\pi}{4})(5) = 25\pi.$

So $\oint\mathbf{v}\cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi.$ ✓

(c)
$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v} &= \left(\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right)\hat{\mathbf{s}} + \left(\frac{\partial}{\partial z}\left(s(2+\sin^2\phi)\right) - \frac{\partial}{\partial s}(3z)\right)\hat{\boldsymbol{\phi}}\\ &\quad + \frac{1}{s}\left(\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}\left(s(2+\sin^2\phi)\right)\right)\hat{\mathbf{z}}\\ &= \frac{1}{s}(2s\sin\phi\cos\phi - s\,2\sin\phi\cos\phi)\,\hat{\mathbf{z}} = \boxed{\mathbf{0}.}\end{aligned}$$

Problem 1.44

(a) $3(3^2) - 2(3) - 1 = 27 - 6 - 1 = \boxed{20.}$

(b) $\cos\pi = \boxed{\text{-1.}}$

(c) $\boxed{\text{zero.}}$

(d) $\ln(-2+3) = \ln 1 = \boxed{\text{zero.}}$

Problem 1.45

(a) $\int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)\,dx = \frac{1}{3}(0+3) = \boxed{1.}$

(b) By Eq. 1.94, $\delta(1-x) = \delta(x-1)$, so $1 + 3 + 2 = \boxed{6.}$

(c) $\int_{-1}^{1} 9x^2\,\frac{1}{3}\,\delta(x+\frac{1}{3})\,dx = 9\left(-\frac{1}{3}\right)^2\frac{1}{3} = \boxed{\tfrac{1}{3}.}$

(d) $\boxed{1 \text{ (if } a > b\text{), } 0 \text{ (if } a < b\text{).}}$

Problem 1.46

(a) $\int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\delta(x)\right]dx = x\,f(x)\delta(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left(x\,f(x)\right)\delta(x)\,dx.$

The first term is zero, since $\delta(x) = 0$ at $\pm\infty$; $\frac{d}{dx}\left(x\,f(x)\right) = x\frac{df}{dx} + \frac{dx}{dx}f = x\frac{df}{dx} + f.$

So the integral is $-\int_{-\infty}^{\infty}\left(x\frac{df}{dx} + f\right)\delta(x)\,dx = 0 - f(0) = -f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$

So, $x\frac{d}{dx}\delta(x) = -\delta(x).$ qed

(b) $\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(x)\theta(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta(x)dx = f(\infty) - \int_{0}^{\infty}\frac{df}{dx}dx = f(\infty) - \left(f(\infty) - f(0)\right)$
$= f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$ So $\frac{d\theta}{dx} = \delta(x).$ qed

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Problem 1.47

(a) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}').}$ Check: $\int\rho(\mathbf{r})d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau = q.$ ✓

(b) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{a}) - q\delta^3(\mathbf{r}).}$

(c) Evidently $\rho(r) = A\delta(r - R)$. To determine the constant A, we require

$Q = \int\rho\,d\tau = \int A\delta(r-R)4\pi r^2\,dr = A\,4\pi R^2.$ So $A = \frac{Q}{4\pi R^2}.$ $\boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r - R).}$

Problem 1.48

(a) $a^2 + \mathbf{a}\cdot\mathbf{a} + a^2 = \boxed{3a^2.}$

(b) $\int(\mathbf{r} - \mathbf{b})^2\frac{1}{5^3}\delta^3(\mathbf{r})\,d\tau = \frac{1}{125}b^2 = \frac{1}{125}(4^2 + 3^2) = \boxed{\tfrac{1}{5}.}$

(c) $c^2 = 25 + 9 + 4 = 38 > 36 = 6^2$, so $\mathbf{c}$ is outside $\mathcal{V}$, so the integral is $\boxed{\text{zero.}}$

(d) $\left(\mathbf{e} - (2\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}})\right)^2 = \left(1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + (-1)\,\hat{\mathbf{z}}\right)^2 = 1 + 1 = 2 < (1.5)^2 = 2.25$, so $\mathbf{e}$ is inside $\mathcal{V}$,
and hence the integral is $\mathbf{e}\cdot(\mathbf{d} - \mathbf{e}) = (3,2,1)\cdot(-2,0,2) = -6 + 0 + 2 = \boxed{\text{-4.}}$

Problem 1.49

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\boldsymbol{\delta}^3(\mathbf{r})\right)\,d\tau = 4\pi e^{-0} = \boxed{4\pi.}$
Second method: integrating by parts (use Eq. 1.59).

$$J = -\int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot \boldsymbol{\nabla}(e^{-r})\, d\tau + \oint_{\mathcal{S}} e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}. \quad \text{But} \quad \boldsymbol{\nabla}\left(e^{-r}\right) = \left(\frac{\partial}{\partial r} e^{-r}\right) \hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.$$

$$= \int \frac{1}{r^2} e^{-r} 4\pi r^2\, dr + \int e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}} = 4\pi \int_0^R e^{-r}\, dr + e^{-R} \int \sin\theta\, d\theta\, d\phi$$

$$= 4\pi \left(-e^{-r}\right)\Big|_0^R + 4\pi e^{-R} = 4\pi\left(-e^{-R} + e^{-0}\right) + 4\pi e^{-R} = 4\pi. \checkmark \qquad (\text{Here } R = \infty,\ \text{so } e^{-R} = 0.)$$

Problem 1.50 (a) $\boldsymbol{\nabla}\cdot\mathbf{F_1} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(x^2\right) = \boxed{0}$; $\quad \boldsymbol{\nabla}\cdot\mathbf{F_2} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}$

$$\boldsymbol{\nabla}\times\mathbf{F_1} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{\mathbf{y}} \frac{\partial}{\partial x}\left(x^2\right) = \boxed{-2x\hat{\mathbf{y}}}; \quad \boldsymbol{\nabla}\times\mathbf{F_2} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{\mathbf{0}}$$

$\boxed{\mathbf{F_2} \text{ is a gradient; } \mathbf{F_1} \text{ is a curl}}$ $\quad \boxed{U_2 = \tfrac{1}{2}\left(x^3 + y^2 + z^2\right)}$ would do ($\mathbf{F_2} = \boldsymbol{\nabla} U_2$).
For $\mathbf{A_1}$, we want $\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = 0$; $\quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $\quad A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.
$\boxed{\mathbf{A_1} = \tfrac{1}{3}x^2\,\hat{\mathbf{y}}}$ $\quad (\mathbf{F_1} = \boldsymbol{\nabla}\times\mathbf{A_1})$. (But these are <u>not</u> unique.)

(b) $\boldsymbol{\nabla}\cdot\mathbf{F_3} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\quad \boldsymbol{\nabla}\times\mathbf{F_3} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\mathbf{x}}\,(x - x) + \hat{\mathbf{y}}\,(y - y) + \hat{\mathbf{z}}\,(z - z) = \mathbf{0}.$
So $\mathbf{F_3}$ can be written as the gradient of a scalar ($\mathbf{F_3} = \boldsymbol{\nabla} U_3$) and as the curl of a vector ($\mathbf{F_3} = \boldsymbol{\nabla}\times\mathbf{A_3}$). In fact, $\boxed{U_3 = xyz}$ does the job. For the vector potential, we have

$$\left\{\begin{array}{lll} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, & \text{which suggests} & A_z = \frac{1}{4}y^2 z + f(x,z);\ A_y = -\frac{1}{4}yz^2 + g(x,y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting} & A_x = \frac{1}{4}z^2 x + h(x,y);\ A_z = -\frac{1}{4}zx^2 + j(y,z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy, & \text{so} & A_y = \frac{1}{4}x^2 y + k(y,z);\ A_x = -\frac{1}{4}xy^2 + l(x,z) \end{array}\right\}$$

Putting this all together: $\boxed{\mathbf{A_3} = \tfrac{1}{4}\left\{x\left(z^2 - y^2\right)\hat{\mathbf{x}} + y\left(x^2 - z^2\right)\hat{\mathbf{y}} + z\left(y^2 - x^2\right)\hat{\mathbf{z}}\right\}}$ (again, <u>not</u> unique).

Problem 1.51
(d) $\Rightarrow$ (a): $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times(-\boldsymbol{\nabla} U) = \mathbf{0}$ (Eq. 1.44 – curl of gradient is always zero).
(a) $\Rightarrow$ (c): $\oint \mathbf{F}\cdot d\mathbf{l} = \int(\boldsymbol{\nabla}\times\mathbf{F})\cdot d\mathbf{a} = 0$ (Eq. 1.57–Stokes' theorem).
(c) $\Rightarrow$ (b): $\int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} - \int_{\mathbf{a}\ II}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} + \int_{\mathbf{b}\ II}^{\mathbf{a}} \mathbf{F}\cdot d\mathbf{l} = \oint \mathbf{F}\cdot d\mathbf{l} = 0$, so

$$\int_{\mathbf{a}\ I}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ II}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l}.$$

(b) $\Rightarrow$ (c): same as (c) $\Rightarrow$ (b), only in reverse; (c) $\Rightarrow$ (a): same as (a)$\Rightarrow$ (c).

Problem 1.52
(d) $\Rightarrow$ (a): $\boldsymbol{\nabla}\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{W}) = 0$ (Eq 1.46—divergence of curl is always zero).
(a) $\Rightarrow$ (c): $\oint \mathbf{F}\cdot d\mathbf{a} = \int(\boldsymbol{\nabla}\cdot\mathbf{F})\, d\tau = 0$ (Eq. 1.56—divergence theorem).

(c) $\Rightarrow$ (b): $\int_I \mathbf{F}\cdot d\mathbf{a} - \int_{II} \mathbf{F}\cdot d\mathbf{a} = \oint \mathbf{F}\cdot d\mathbf{a} = 0$, so

$$\int_I \mathbf{F}\cdot d\mathbf{a} = \int_{II} \mathbf{F}\cdot d\mathbf{a}.$$

(*Note:* sign change because for $\oint \mathbf{F}\cdot d\mathbf{a}$, d**a** is *outward*, whereas for surface II it is *inward*.)
(b) $\Rightarrow$ (c): same as (c) $\Rightarrow$ (b), in reverse; (c)$\Rightarrow$ (a): same as (a)$\Rightarrow$ (c) .

Problem 1.53

In Prob. 1.15 we found that $\boldsymbol{\nabla}\cdot\mathbf{v}_a = 0$; in Prob. 1.18 we found that $\boldsymbol{\nabla}\times\mathbf{v}_c = \mathbf{0}$. So

$\boxed{\mathbf{v}_c \text{ can be written as the gradient of a scalar; } \mathbf{v}_a \text{ can be written as the curl of a vector.}}$

(a) To find t:

(1) $\frac{\partial t}{\partial x} = y^2 \Rightarrow t = y^2 x + f(y,z)$

(2) $\frac{\partial t}{\partial y} = \left(2xy + z^2\right)$

(3) $\frac{\partial t}{\partial z} = 2yz$

From (1) & (3) we get $\frac{\partial f}{\partial z} = 2yz \Rightarrow f = yz^2 + g(y) \Rightarrow t = y^2x + yz^2 + g(y)$, so $\frac{\partial t}{\partial y} = 2xy + z^2 + \frac{\partial g}{\partial y} = 2xy + z^2$ (from (2)) $\Rightarrow \frac{\partial g}{\partial y} = 0$. We may as well pick $g = 0$; then $\boxed{t = xy^2 + yz^2.}$

(b) To find $\mathbf{W}$: $\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = x^2$; $\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} = 3z^2x$; $\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} = -2xz$.
Pick $W_x = 0$; then

$$\begin{aligned} \frac{\partial W_z}{\partial x} &= -3xz^2 \Rightarrow W_z = -\frac{3}{2}x^2z^2 + f(y,z) \\ \frac{\partial W_y}{\partial x} &= -2xz \Rightarrow W_y = -x^2z + g(y,z). \end{aligned}$$

$\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = \frac{\partial f}{\partial y} + x^2 - \frac{\partial g}{\partial z} = x^2 \Rightarrow \frac{\partial f}{\partial y} - \frac{\partial g}{\partial z} = 0$. May as well pick $f = g = 0$.

$\boxed{\mathbf{W} = -x^2z\,\hat{\mathbf{y}} - \frac{3}{2}x^2z^2\,\hat{\mathbf{z}}.}$

Check: $$\boldsymbol{\nabla}\times\mathbf{W} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x^2z & -\frac{3}{2}x^2z^2 \end{vmatrix} = \hat{\mathbf{x}}\left(x^2\right) + \hat{\mathbf{y}}\left(3xz^2\right) + \hat{\mathbf{z}}\left(-2xz\right).\checkmark$$

You can add any gradient ($\boldsymbol{\nabla}t$) to $\mathbf{W}$ without changing its curl, so this answer is far from unique. Some other solutions:

$\mathbf{W} = xz^3\,\hat{\mathbf{x}} - x^2z\,\hat{\mathbf{y}}$;

$\mathbf{W} = \left(2xyz + xz^3\right)\hat{\mathbf{x}} + x^2y\,\hat{\mathbf{z}}$;

$\mathbf{W} = xyz\,\hat{\mathbf{x}} - \frac{1}{2}x^2z\,\hat{\mathbf{y}} + \frac{1}{2}x^2\left(y - 3z^2\right)\hat{\mathbf{z}}$.

Problem 1.54

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\cos\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, r^2\cos\phi\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right)\\ &= \frac{1}{r^2}4r^3\cos\theta+\frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi+\frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)\\ &= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta+\cos\phi-\cos\phi\right]=4r\cos\theta.\end{aligned}$$

$$\begin{aligned}\int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau &= \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi=4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi\\ &= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)=\boxed{\tfrac{\pi R^4}{4}.}\end{aligned}$$

Surface consists of four parts:
(1) *Curved:* $d\mathbf{a}=R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\ r=R.\quad \mathbf{v}\cdot d\mathbf{a}=\left(R^2\cos\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right).$

$$\int\mathbf{v}\cdot d\mathbf{a}=R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi=R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)=\frac{\pi R^4}{4}.$$

(2) *Left:* $d\mathbf{a}=-r\,dr\,d\theta\,\hat{\boldsymbol{\phi}};\ \phi=0.\quad \mathbf{v}\cdot d\mathbf{a}=\left(r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta)=0.\quad \int\mathbf{v}\cdot d\mathbf{a}=0.$
(3) *Back:* $d\mathbf{a}=r\,dr\,d\theta\,\hat{\boldsymbol{\phi}};\ \phi=\pi/2.\quad \mathbf{v}\cdot d\mathbf{a}=\left(-r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta)=-r^3\cos\theta\,dr\,d\theta.$

$$\int\mathbf{v}\cdot d\mathbf{a}=\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\,d\theta=-\left(\frac{1}{4}R^4\right)(+1)=-\frac{1}{4}R^4.$$

(4) *Bottom:* $d\mathbf{a}=r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}};\ \theta=\pi/2.\quad \mathbf{v}\cdot d\mathbf{a}=\left(r^2\cos\phi\right)(r\,dr\,d\phi).$

$$\int\mathbf{v}\cdot d\mathbf{a}=\int_0^R r^3\,dr\int_0^{\pi/2}\cos\phi\,d\phi=\frac{1}{4}R^4.$$

Total: $\oint\mathbf{v}\cdot d\mathbf{a}=\pi R^4/4+0-\frac{1}{4}R^4+\frac{1}{4}R^4=\frac{\pi R^4}{4}.\quad\checkmark$

Problem 1.55

$\boldsymbol{\nabla}\times\mathbf{v}=\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ ay&bx&0\end{vmatrix}=\hat{\mathbf{z}}\,(b-a).\quad$ So $\quad\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a}=(b-a)\pi R^2.$

$\mathbf{v}\cdot d\mathbf{l}=(ay\,\hat{\mathbf{x}}+bx\,\hat{\mathbf{y}})\cdot(dx\,\hat{\mathbf{x}}+dy\,\hat{\mathbf{y}}+dz\,\hat{\mathbf{z}})=ay\,dx+bx\,dy;\ x^2+y^2=R^2\Rightarrow 2x\,dx+2y\,dy=0,$
so $dy=-(x/y)\,dx$. So $\quad\mathbf{v}\cdot d\mathbf{l}=ay\,dx+bx(-x/y)\,dx=\frac{1}{y}\left(ay^2-bx^2\right)\,dx.$

For the "upper" semicircle, $y = \sqrt{R^2 - x^2}$, so $\mathbf{v} \cdot d\mathbf{l} = \frac{a\left(R^2-x^2\right)-bx^2}{\sqrt{R^2-x^2}}\,dx$.

$$\begin{aligned}
\int \mathbf{v} \cdot d\mathbf{l} &= \int_R^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2-x^2}}\,dx = \left\{ aR^2 \sin^{-1}\left(\frac{x}{R}\right) - (a+b)\left[-\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]\right\}\Bigg|_{+R}^{-R} \\
&= \frac{1}{2}R^2(a-b)\sin^{-1}(x/R)\Big|_{+R}^{-R} = \frac{1}{2}R^2(a-b)\left(\sin^{-1}(-1) - \sin^{-1}(+1)\right) = \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) \\
&= \frac{1}{2}\pi R^2(b-a).
\end{aligned}$$

And the same for the lower semicircle (y changes sign, but the limits on the integral are reversed) so $\oint \mathbf{v} \cdot d\mathbf{l} = \pi R^2(b-a)$. ✓

Problem 1.56

(1) $x = z = 0;\ dx = dz = 0;\ y : 0 \to 1.\quad \mathbf{v} \cdot d\mathbf{l} = (yz^2)\,dy = 0;\quad \int \mathbf{v} \cdot d\mathbf{l} = 0.$
(2) $x = 0;\ z = 2-2y;\ dz = -2\,dy;\ y : 1 \to 0.\quad \mathbf{v} \cdot d\mathbf{l} = (yz^2)\,dy + (3y+z)\,dz = y(2-2y)^2\,dy - (3y+2-2y)2\,dy;$

$$\int \mathbf{v} \cdot d\mathbf{l} = 2\int_1^0 (2y^3 - 4y^2 + y - 2)\,dy = 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Bigg|_1^0 = \frac{14}{3}.$$

(3) $x = y = 0;\ dx = dy = 0;\ z : 2 \to 0.\quad \mathbf{v} \cdot d\mathbf{l} = (3y+z)\,dz = z\,dz;$

$$\int \mathbf{v} \cdot d\mathbf{l} = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = -2.$$

Total: $\oint \mathbf{v} \cdot d\mathbf{l} = 0 + \frac{14}{3} - 2 = \boxed{\frac{8}{3}.}$
Meanwhile, Stokes' thereom says $\oint \mathbf{v} \cdot d\mathbf{l} = \int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a}$. Here $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, so all we need is $(\boldsymbol{\nabla} \times \mathbf{v})_x = \frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2) = 3 - 2yz.$ Therefore

$$\begin{aligned}
\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a} &= \int\int (3 - 2yz)\,dy\,dz = \int_0^1 \left[\int_0^{2-2y} (3-2yz)\,dz\right] dy \\
&= \int_0^1 \left[3(2-2y) - 2y\tfrac{1}{2}(2-2y)^2\right] dy = \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy \\
&= \left(-y^4 + \tfrac{8}{3}y^3 - 5y^2 + 6y\right)\Big|_0^1 = -1 + \tfrac{8}{3} - 5 + 6 = \tfrac{8}{3}. ✓
\end{aligned}$$

Problem 1.57

Start at the origin.

(1) $\theta = \frac{\pi}{2},\ \phi = 0;\ r : 0 \to 1.\quad \mathbf{v} \cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = 0.\quad \int \mathbf{v} \cdot d\mathbf{l} = 0.$

(2) $r = 1,\ \theta = \frac{\pi}{2};\ \phi : 0 \to \pi/2.\quad \mathbf{v} \cdot d\mathbf{l} = (3r)(r\sin\theta\,d\phi) = 3\,d\phi.\quad \int \mathbf{v} \cdot d\mathbf{l} = 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}.$

(3) $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$, so $r = \frac{1}{\sin\theta}$, $dr = \frac{-1}{\sin^2\theta}\cos\theta\,d\theta$, $\theta : \frac{\pi}{2} \to \theta_0 \equiv \tan^{-1}(1/2)$.

$$\begin{aligned}\mathbf{v}\cdot d\mathbf{l} &= \left(r\cos^2\theta\right)(dr) - (r\cos\theta\sin\theta)(r\,d\theta) = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta \\ &= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta.\end{aligned}$$

Therefore

$$\int \mathbf{v}\cdot d\mathbf{l} = -\int_{\pi/2}^{\theta_0}\frac{\cos\theta}{\sin^3\theta}\,d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\theta_0} = \frac{1}{2\cdot(1/5)} - \frac{1}{2\cdot(1)} = \frac{5}{2} - \frac{1}{2} = 2.$$

(4) $\theta = \theta_0$, $\phi = \frac{\pi}{2}$; $r : \sqrt{5} \to 0$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = \frac{4}{5}r\,dr$.

$$\int \mathbf{v}\cdot d\mathbf{l} = \frac{4}{5}\int_{\sqrt{5}}^{0} r\,dr = \left.\frac{4}{5}\frac{r^2}{2}\right|_{\sqrt{5}}^{0} = -\frac{4}{5}\cdot\frac{5}{2} = -2.$$

Total:

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\tfrac{3\pi}{2}.}$$

Stokes' theorem says this should equal $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a}$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(r\cos^2\theta\right) - \frac{\partial}{\partial r}(r3r)\right]\hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(-rr\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}\left(r\cos^2\theta\right)\right]\hat{\boldsymbol{\phi}} \\ &= \frac{1}{r\sin\theta}[3r\cos\theta]\,\hat{\mathbf{r}} + \frac{1}{r}[-6r]\,\hat{\boldsymbol{\theta}} + \frac{1}{r}[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta]\,\hat{\boldsymbol{\phi}} \\ &= 3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}}.\end{aligned}$$

(1) *Back face:* $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 0$. $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 0$.

(2) *Bottom:* $d\mathbf{a} = -r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 6r\sin\theta\,dr\,d\phi$. $\theta = \frac{\pi}{2}$, so $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 6r\,dr\,d\phi$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1 6r\,dr\int_0^{\pi/2}d\phi = 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}. \quad \checkmark$$

Problem 1.58

$\mathbf{v}\cdot d\mathbf{l} = y\,dz$.

(1) *Left side:* $z = a - x$; $dz = -dx$; $y = 0$. Therefore $\int\mathbf{v}\cdot d\mathbf{l} = 0$.

(2) *Bottom:* $dz = 0$. Therefore $\int\mathbf{v}\cdot d\mathbf{l} = 0$.

(3) *Back:* $z = a - \frac{1}{2}y$; $dz = -1/2\,dy$; $y : 2a \to 0$. $\int \mathbf{v}\cdot d\mathbf{l} = \int_{2a}^{0} y\left(-\frac{1}{2}\,dy\right) = -\frac{1}{2}\frac{y^2}{2}\Big|_{2a}^{0} = \frac{4a^2}{4} = \boxed{a^2.}$

Meanwhile, $\boldsymbol{\nabla}\times\mathbf{v} = \hat{\mathbf{x}}$, so $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a}$ is the projection of this surface on the $x\,y$ plane $= \frac{1}{2}\cdot a\cdot 2a = a^2$. ✓

Problem 1.59

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, 4r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2\tan\theta\right) \\
&= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r}{\sin\theta}\left(\sin^2\theta + \cos^2\theta - \sin^2\theta\right) \\
&= 4r\frac{\cos^2\theta}{\sin\theta}.
\end{aligned}$$

$$\begin{aligned}
\int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau &= \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \int_0^R 4r^3\,dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/6} \\
&= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right).}
\end{aligned}$$

Surface coinsists of two parts:

(1) *The ice cream:* $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$; $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\sin\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right) = R^4\sin^2\theta\,d\theta\,d\phi$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} = 2\pi R^4\left(\frac{\pi}{12} - \frac{1}{4}\sin 60^\circ\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$$

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \to 2\pi$; $r : 0 \to R$; $d\mathbf{a} = r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2}r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\,r^3\,dr\,d\phi$

$$\int\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4.$$

Therefore $\int\mathbf{v}\cdot d\mathbf{a} = \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right) = \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)$. ✓.

Problem 1.60

(a) Corollary 2 says $\oint(\boldsymbol{\nabla}T)\cdot d\mathbf{l} = 0$. Stokes' theorem says $\oint(\boldsymbol{\nabla}T)\cdot d\mathbf{l} = \int[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T)]\cdot d\mathbf{a}$. So $\int[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T)]\cdot d\mathbf{a} = 0$, and since this is true for *any* surface, the integrand must vanish: $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T) = \mathbf{0}$, confirming Eq. 1.44.

(b) Corollary 2 says $\oint(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 0$. Divergence theorem says $\oint(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \int\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v})\,d\tau$. So $\int\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v})\,d\tau = 0$, and since this is true for *any* volume, the integrand must vanish: $\boldsymbol{\nabla}(\boldsymbol{\nabla}\times\mathbf{v}) = 0$, confirming Eq. 1.46.

Problem 1.61

(a) Divergence theorem: $\oint\mathbf{v}\cdot d\mathbf{a} = \int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau$. Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector. Using product rule #5 in front cover: $\boldsymbol{\nabla}\cdot\mathbf{v} = \boldsymbol{\nabla}\cdot(\mathbf{c}T) = T(\boldsymbol{\nabla}\cdot\mathbf{c}) + \mathbf{c}\cdot(\boldsymbol{\nabla}T)$. But $\mathbf{c}$ is constant so $\boldsymbol{\nabla}\cdot\mathbf{c} = 0$. Therefore we have: $\int\mathbf{c}\cdot(\boldsymbol{\nabla}T)\,d\tau = \int T\mathbf{c}\cdot d\mathbf{a}$. Since $\mathbf{c}$ is constant, take it outside the integrals: $\mathbf{c}\cdot\int\boldsymbol{\nabla}T\,d\tau = \mathbf{c}\cdot\int T\,d\mathbf{a}$. But $\mathbf{c}$

is *any* constant vector—in particular, it could be be $\hat{\mathbf{x}}$, or $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$—so each *component* of the integral on left equals corresponding component on the right, and hence

$$\int \boldsymbol{\nabla} T \, d\tau = \int T \, d\mathbf{a}. \qquad \text{qed}$$

(b) Let $\mathbf{v} \to (\mathbf{v} \times \mathbf{c})$ in divergence theorem. Then $\int \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v} \times \mathbf{c}) d\tau = \int (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a}$. Product rule #6 $\Rightarrow$ $\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v} \times \mathbf{c}) = \mathbf{c} \cdot (\boldsymbol{\nabla}\times\mathbf{v}) - \mathbf{v} \cdot (\boldsymbol{\nabla}\times\mathbf{c}) = \mathbf{c} \cdot (\boldsymbol{\nabla}\times\mathbf{v})$. (Note: $\boldsymbol{\nabla}\times\mathbf{c} = \mathbf{0}$, since $\mathbf{c}$ is constant.) Meanwhile vector indentity (1) says $d\mathbf{a} \cdot (\mathbf{v} \times \mathbf{c}) = \mathbf{c} \cdot (d\mathbf{a} \times \mathbf{v}) = -\mathbf{c} \cdot (\mathbf{v} \times d\mathbf{a})$. Thus $\int \mathbf{c} \cdot (\boldsymbol{\nabla}\times\mathbf{v}) \, d\tau = -\int \mathbf{c} \cdot (\mathbf{v} \times d\mathbf{a})$. Take $\mathbf{c}$ outside, and again let $\mathbf{c}$ be $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ then:

$$\int (\boldsymbol{\nabla}\times\mathbf{v}) \, d\tau = -\int \mathbf{v} \times d\mathbf{a}. \quad \text{qed}$$

(c) Let $\mathbf{v} = T\boldsymbol{\nabla} U$ in divergence theorem: $\int \boldsymbol{\nabla}\boldsymbol{\cdot}(T\boldsymbol{\nabla} U) \, d\tau = \int T\boldsymbol{\nabla} U \cdot d\mathbf{a}$. Product rule #(5) $\Rightarrow$ $\boldsymbol{\nabla}\boldsymbol{\cdot}(T\boldsymbol{\nabla} U) = T\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\nabla} U) + (\boldsymbol{\nabla} U) \cdot (\boldsymbol{\nabla} T) = T\nabla^2 U + (\boldsymbol{\nabla} U) \cdot (\boldsymbol{\nabla} T)$. Therefore

$$\int \left(T\nabla^2 U + (\boldsymbol{\nabla} U) \cdot (\boldsymbol{\nabla} T) \right) \, d\tau = \int (T\boldsymbol{\nabla} U) \cdot d\mathbf{a}. \quad \text{qed}$$

(d) Rewrite (c) with $T \leftrightarrow U$: $\int \left(U\nabla^2 T + (\boldsymbol{\nabla} T) \cdot (\boldsymbol{\nabla} U) \right) \, d\tau = \int (U\boldsymbol{\nabla} T) \cdot d\mathbf{a}$. Subtract this from (c), noting that the $(\boldsymbol{\nabla} U) \cdot (\boldsymbol{\nabla} T)$ terms cancel:

$$\int \left(T\nabla^2 U - U\nabla^2 T \right) \, d\tau = \int (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T) \cdot d\mathbf{a}. \quad \text{qed}$$

(e) Stokes' theorem: $\int (\boldsymbol{\nabla}\times\mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l}$. Let $\mathbf{v} = \mathbf{c}T$. By Product Rule #(7): $\boldsymbol{\nabla}\times(\mathbf{c}T) = T(\boldsymbol{\nabla}\times\mathbf{c}) - \mathbf{c} \times (\boldsymbol{\nabla} T) = -\mathbf{c} \times (\boldsymbol{\nabla} T)$ (since $\mathbf{c}$ is constant). Therefore, $-\int (\mathbf{c} \times (\boldsymbol{\nabla} T)) \cdot d\mathbf{a} = \oint T\mathbf{c} \cdot d\mathbf{l}$. Use vector indentity #1 to rewrite the first term $(\mathbf{c} \times (\boldsymbol{\nabla} T)) \cdot d\mathbf{a} = \mathbf{c} \cdot (\boldsymbol{\nabla} T \times d\mathbf{a})$. So $-\int \mathbf{c} \cdot (\boldsymbol{\nabla} T \times d\mathbf{a}) = \oint \mathbf{c} \cdot T \, d\mathbf{l}$. Pull $\mathbf{c}$ outside, and let $\mathbf{c} \to \hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ to prove:

$$\int \boldsymbol{\nabla} T \times d\mathbf{a} = -\oint T \, d\mathbf{l}. \quad \text{qed}$$

Problem 1.62

(a) $d\mathbf{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$. Let the surface be the northern hemisphere. The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components clearly integrate to zero, and the $\hat{\mathbf{z}}$ component of $\hat{\mathbf{r}}$ is $\cos\theta$, so

$$\mathbf{a} = \int R^2 \sin\theta \cos\theta \, d\theta \, d\phi \, \hat{\mathbf{z}} = 2\pi R^2 \, \hat{\mathbf{z}} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = 2\pi R^2 \, \hat{\mathbf{z}} \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} = \boxed{\pi R^2 \, \hat{\mathbf{z}}.}$$

(b) Let $T = 1$ in Prob. 1.61(a). Then $\boldsymbol{\nabla} T = 0$, so $\oint d\mathbf{a} = 0$. qed

(c) This follows from (b). For suppose $\mathbf{a}_1 \neq \mathbf{a}_2$; then if you put them together to make a closed surface, $\oint d\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \neq 0$.

(d) For one such triangle, $d\mathbf{a} = \frac{1}{2}(\mathbf{r} \times d\mathbf{l})$ (since $\mathbf{r} \times d\mathbf{l}$ is the area of the parallelogram, and the direction is perpendicular to the surface), so for the entire conical surface, $\mathbf{a} = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{l}$.

(e) Let $T = \mathbf{c} \cdot \mathbf{r}$, and use product rule #4: $\boldsymbol{\nabla} T = \boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c} \times (\boldsymbol{\nabla}\times\mathbf{r}) + (\mathbf{c} \cdot \boldsymbol{\nabla})\mathbf{r}$. But $\boldsymbol{\nabla}\times\mathbf{r} = 0$, and $(\mathbf{c} \cdot \boldsymbol{\nabla})\mathbf{r} = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})(x \, \hat{\mathbf{x}} + y \, \hat{\mathbf{y}} + z \, \hat{\mathbf{z}}) = c_x \, \hat{\mathbf{x}} + c_y \, \hat{\mathbf{y}} + c_z \, \hat{\mathbf{z}} = \mathbf{c}$. So Prob. 1.61(e) says

$$\oint T \, d\mathbf{l} = \oint (\mathbf{c} \cdot \mathbf{r}) \, d\mathbf{l} = -\int (\boldsymbol{\nabla} T) \times d\mathbf{a} = -\int \mathbf{c} \times d\mathbf{a} = -\mathbf{c} \times \int d\mathbf{a} = -\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}. \qquad \text{qed}$$

Problem 1.63

(1)

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \boxed{\tfrac{1}{r^2}.}$$

For a sphere of radius R:

$$\left.\begin{aligned} \textstyle\int \mathbf{v}\cdot d\mathbf{a} &= \textstyle\int \left(\frac{1}{R}\,\hat{\mathbf{r}}\right)\cdot\left(R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}\right) = R\int\sin\theta\,d\theta\,d\phi = 4\pi R. \\ \textstyle\int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau &= \textstyle\int\left(\frac{1}{r^2}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \left(\int_0^R dr\right)\left(\int\sin\theta\,d\theta\,d\phi\right) = 4\pi R. \end{aligned}\right\}\ \begin{array}{l}\text{So divergence}\\ \text{theorem checks.}\end{array}$$

Evidently there is *no* delta function at the origin.

$$\boldsymbol{\nabla}\times(r^n\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^n\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = \frac{1}{r^2}(n+2)r^{n+1} = \boxed{(n+2)r^{n-1}}$$

(except for $n = -2$, for which we already know (Eq. 1.99) that the divergence is $4\pi\boldsymbol{\delta}^3(\mathbf{r})$).

(2) *Geometrically,* it should be zero. Likewise, the curl in the spherical coordinates obviously gives $\boxed{\text{zero.}}$ To be certain there is no lurking delta function here, we integrate over a sphere of radius R, using Prob. 1.61(b): If $\boldsymbol{\nabla}\times(r^n\,\hat{\mathbf{r}}) = \mathbf{0}$, then $\int(\boldsymbol{\nabla}\times\mathbf{v})\,d\tau = \mathbf{0} \stackrel{?}{=} -\oint \mathbf{v}\times d\mathbf{a}$. But $\mathbf{v} = r^n\,\hat{\mathbf{r}}$ and $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$ are both in the $\hat{\mathbf{r}}$ directions, so $\mathbf{v}\times d\mathbf{a} = \mathbf{0}$. ✓

Problem 1.64

(a) Since the argument is not a function of angle, Eq. 1.73 says

$$\begin{aligned} D &= -\frac{1}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{2}\right)\frac{2r}{(r^2+\epsilon^2)^{3/2}}\right] = \frac{1}{4\pi r^2}\frac{d}{dr}\left[\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right] \\ &= \frac{1}{4\pi r^2}\left[\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3}{2}\frac{r^3\,2r}{(r^2+\epsilon^2)^{5/3}}\right] = \frac{1}{4\pi r^2}\frac{3r^2}{(r^2+\epsilon^2)^{5/2}}\left(r^2+\epsilon^2-r^2\right) = \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}.\ ✓ \end{aligned}$$

(b) Setting $r \to 0$:

$$D(0,\epsilon) = \frac{3\epsilon^2}{4\pi\epsilon^5} = \frac{3}{4\pi\epsilon^3},$$

which goes to infinity as $\epsilon \to 0$. ✓

(c) From (a) it is clear that $D(r,0) = 0$ for $r \neq 0$. ✓

(d)

$$\int D(r,\epsilon)\,4\pi r^2\,dr = 3\epsilon^2\int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\,dr = 3\epsilon^2\left(\frac{1}{3\epsilon^2}\right) = 1.\ ✓$$

(I looked up the integral.) Note that (b), (c), and (d) are the defining conditions for $\delta^3(\mathbf{r})$.

Chapter 2

Electrostatics

Problem 2.1

(a) $\boxed{\text{Zero.}}$

(b) $\boxed{F = \dfrac{1}{4\pi\epsilon_0}\dfrac{qQ}{r^2},}$ where r is the distance from center to each numeral. $\mathbf{F}$ points *toward* the missing q. *Explanation:* by superposition, this is equivalent to (a), with an extra $-q$ at 6 o'clock—since the force of all twelve is zero, the net force is that of $-q$ only.

(c) $\boxed{\text{Zero.}}$

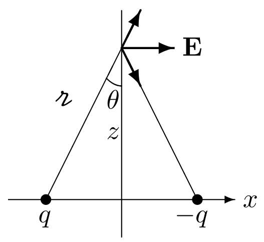
(d) $\boxed{\dfrac{1}{4\pi\epsilon_0}\dfrac{qQ}{r^2},}$ pointing toward the missing q. Same reason as (b). Note, however, that if you explained (b) as a cancellation in pairs of opposite charges (1 o'clock against 7 o'clock; 2 against 8, etc.), with one unpaired q doing the job, then you'll need a *different* explanation for (d).

Problem 2.2

This time the "vertical" components cancel, leaving

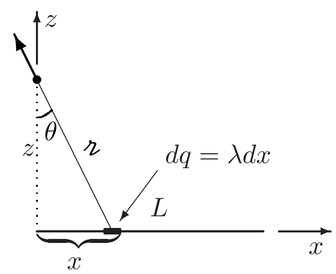
$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \sin\theta\,\hat{\mathbf{x}}$, or

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{x}}.}$$



From far away, ($z \gg d$), the field goes like $\mathbf{E} \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\,\hat{\mathbf{z}}$, which, as we shall see, is the field of a *dipole.* (If we set $d \to 0$, we get $\mathbf{E} = \mathbf{0}$, as is appropriate; to the extent that this configuration looks like a single point charge from far away, the net charge is zero, so $\mathbf{E} \to \mathbf{0}$.)

Problem 2.3



$$
\begin{aligned}
E_z &= \tfrac{1}{4\pi\epsilon_0}\int_0^L \tfrac{\lambda\,dx}{\mathcal{r}^2}\cos\theta;\ (\mathcal{r}^2 = z^2+x^2;\ \cos\theta = \tfrac{z}{\mathcal{r}})\\
&= \tfrac{1}{4\pi\epsilon_0}\lambda z\int_0^L \tfrac{1}{(z^2+x^2)^{3/2}}dx\\
&= \tfrac{1}{4\pi\epsilon_0}\lambda z\left[\tfrac{1}{z^2}\tfrac{x}{\sqrt{z^2+x^2}}\right]\Big|_0^L = \tfrac{1}{4\pi\epsilon_0}\tfrac{\lambda}{z}\tfrac{L}{\sqrt{z^2+L^2}}.\\
E_x &= -\tfrac{1}{4\pi\epsilon_0}\int_0^L \tfrac{\lambda\,dx}{\mathcal{r}^2}\sin\theta = -\tfrac{1}{4\pi\epsilon_0}\lambda\int\tfrac{x\,dx}{(x^2+z^2)^{3/2}}\\
&= -\tfrac{1}{4\pi\epsilon_0}\lambda\left[-\tfrac{1}{\sqrt{x^2+z^2}}\right]\Big|_0^L = -\tfrac{1}{4\pi\epsilon_0}\lambda\left[\tfrac{1}{z}-\tfrac{1}{\sqrt{z^2+L^2}}\right].
\end{aligned}
$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\left[\left(-1+\frac{z}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{x}} + \left(\frac{L}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{z}}\right].}$$

For $z \gg L$ you expect it to look like a point charge $q = \lambda L$: $\mathbf{E} \to \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{z}}$. It checks, for with $z \gg L$ the $\hat{\mathbf{x}}$ term $\to 0$, and the $\hat{\mathbf{z}}$ term $\to \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\frac{L}{z}\hat{\mathbf{z}}$.

Problem 2.4

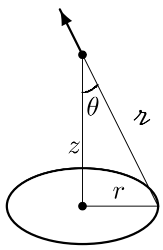
From Ex. 2.2, with $L \to \frac{a}{2}$ and $z \to \sqrt{z^2+\left(\frac{a}{2}\right)^2}$ (distance from center of edge to P), field of *one* edge is:

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{\lambda a}{\sqrt{z^2+\frac{a^2}{4}}\sqrt{z^2+\frac{a^2}{4}+\frac{a^2}{4}}}.$$

There are 4 sides, and we want vertical components only, so multiply by $4\cos\theta = 4\frac{z}{\sqrt{z^2+\frac{a^2}{4}}}$:

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\hat{\mathbf{z}}.}$$

Problem 2.5



"Horizontal" components cancel, leaving: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{\int\frac{\lambda dl}{\mathcal{r}^2}\cos\theta\right\}\hat{\mathbf{z}}$.
Here, $\mathcal{r}^2 = r^2+z^2$, $\cos\theta = \frac{z}{\mathcal{r}}$ (both constants), while $\int dl = 2\pi r$. So

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)z}{\left(r^2+z^2\right)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 2.6

Break it into rings of radius r, and thickness dr, and use Prob. 2.5 to express the field of each ring. Total charge of a ring is $\sigma\cdot 2\pi r\cdot dr = \lambda\cdot 2\pi r$, so $\lambda = \sigma dr$ is the "line charge" of each ring.

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{\left(r^2+z^2\right)^{3/2}};\quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R\frac{r}{\left(r^2+z^2\right)^{3/2}}dr.$$

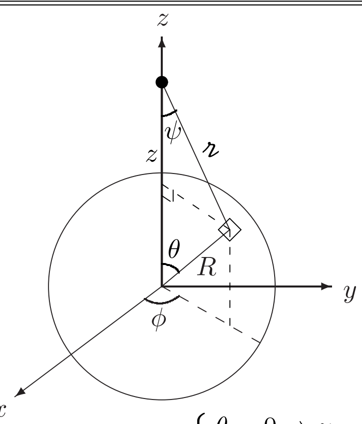
$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.}$$

For $R \gg z$ the second term $\to 0$, so $\mathbf{E}_{\text{plane}} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma\hat{\mathbf{z}} = \boxed{\dfrac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.}$

For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)$, so $[\;] \approx \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} = \frac{R^2}{2z^3}$,
and $E = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, where $Q = \pi R^2\sigma$. ✓

Problem 2.7

$\mathbf{E}$ is clearly in the z direction. From the diagram,
$dq = \sigma da = \sigma R^2 \sin\theta\, d\theta\, d\phi$,
$\mathscr{r}^{\,2} = R^2 + z^2 - 2Rz\cos\theta$,
$\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$.



So

$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z-R\cos\theta)\sin\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\,d\theta. \qquad \text{Let } u = \cos\theta;\ du = -\sin\theta\, d\theta; \left\{\begin{array}{l}\theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1\end{array}\right\}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^1 \frac{z - Ru}{(R^2+z^2-2Rzu)^{3/2}}du. \quad \text{Integral can be done by partial fractions—or look it up.}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu-R}{\sqrt{R^2+z^2-2Rzu}}\right]_{-1}^1 = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

Problem 2.8

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller r) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\hat{\mathbf{r}},$$

where Q_{int} is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.}$$

Inside the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \quad \text{so} \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

Problem 2.9

(a) $\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0\frac{1}{r^2}k(5r^4) = \boxed{5\epsilon_0 k r^2.}$

Problem 2.15

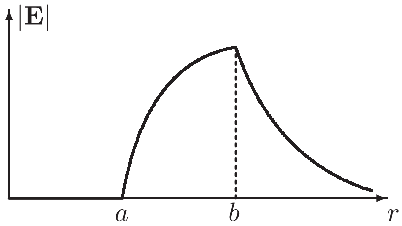
(i) $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = \mathbf{0}}$.

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int \frac{k}{\bar{r}^2}\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi$

$= \frac{4\pi k}{\epsilon_0}\int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r-a)\ \therefore \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$

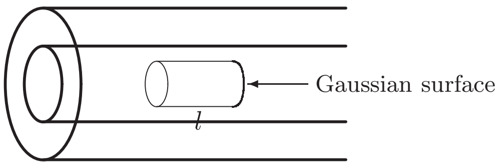
(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0}\int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b-a)$, so

$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$



Problem 2.16

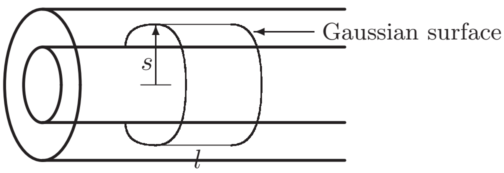
(i)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\rho\pi s^2 l;$

$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}.}$

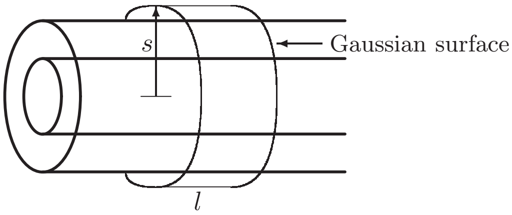
(ii)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\rho\pi a^2 l;$

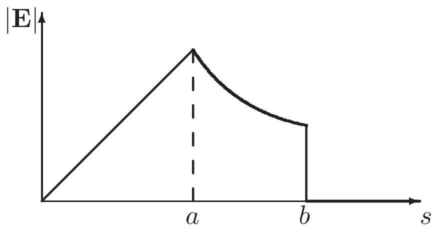
$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}.}$

(iii)

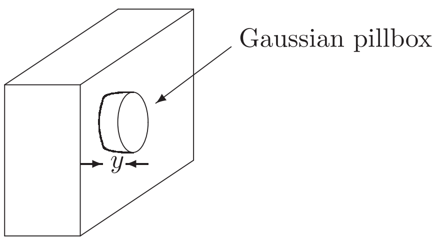


$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = 0;$

$\boxed{\mathbf{E} = \mathbf{0}.}$



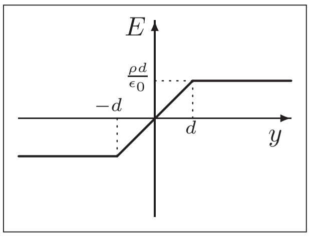
Problem 2.17 On the $x\,z$ plane $E = 0$ by symmetry. Set up a Gaussian "pillbox" with one face in this plane and the other at y.



$\int \mathbf{E}\cdot d\mathbf{a} = E\cdot A = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Ay\rho;$

$\boxed{\mathbf{E} = \frac{\rho}{\epsilon_0}y\,\hat{\mathbf{y}}}$ (for $|y| < d$).

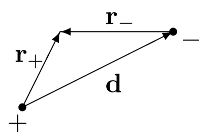
$Q_{\text{enc}} = \frac{1}{\epsilon_0} A d\rho \Rightarrow \boxed{\mathbf{E} = \frac{\rho}{\epsilon_0} d\,\hat{\mathbf{y}}}$ (for $y > d$).



Problem 2.18

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$



But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d}.}$

Problem 2.19

$$\nabla\times\mathbf{E} = \frac{1}{4\pi\epsilon_0}\nabla\times\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho\,d\tau = \frac{1}{4\pi\epsilon_0}\int\left[\nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\rho\,d\tau \quad \text{(since } \rho \text{ depends on } \mathbf{r}'\text{, not } \mathbf{r}\text{)}$$

$$= \mathbf{0} \quad \left(\text{since } \nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \mathbf{0}\text{, from Prob. 1.63}\right).$$

Problem 2.20

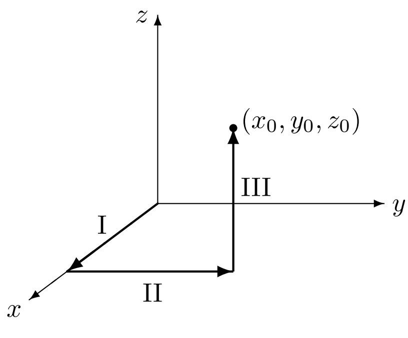
$$(1)\ \nabla\times\mathbf{E}_1 = k\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = k\left[\hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x)\right] \neq \mathbf{0},$$

so $\mathbf{E}_1$ is an *impossible* electrostatic field.

$$(2)\ \nabla\times\mathbf{E}_2 = k\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix} = k\left[\hat{\mathbf{x}}(2z-2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(2y-2y)\right] = \mathbf{0},$$

so $\mathbf{E}_2$ is a *possible* electrostatic field.

Let's go by the indicated path:



$$\mathbf{E}\cdot d\mathbf{l} = (y^2\,dx + (2xy+z^2)dy + 2yz\,dz)k$$

Step I: $y = z = 0$; $dy = dz = 0$. $\mathbf{E}\cdot d\mathbf{l} = ky^2\,dx = 0$.

Step II: $x = x_0$, $y: 0 \to y_0$, $z = 0$. $dx = dz = 0$.
$\mathbf{E}\cdot d\mathbf{l} = k(2xy+z^2)dy = 2kx_0y\,dy$.
$\int_{II}\mathbf{E}\cdot d\mathbf{l} = 2kx_0\int_0^{y_0} y\,dy = kx_0y_0^2$.

Step III: $x = x_0$, $y = y_0$, $z: 0 \to z_0$; $dx = dy = 0$.
$\mathbf{E}\cdot d\mathbf{l} = 2kyz\,dz = 2ky_0z\,dz$.

$\int_{III} \mathbf{E}\cdot d\mathbf{l} = 2y_0 k \int_0^{z_0} z\,dz = k y_0 z_0^2.$

$$V(x_0, y_0, z_0) = -\int_0^{(x_0,y_0,z_0)} \mathbf{E}\cdot d\mathbf{l} = -k(x_0 y_0^2 + y_0 z_0^2), \text{ or } \boxed{V(x,y,z) = -k(xy^2 + yz^2).}$$

Check: $-\boldsymbol{\nabla}V = k\left[\frac{\partial}{\partial x}(xy^2+yz^2)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(xy^2+yz^2)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(xy^2+yz^2)\,\hat{\mathbf{z}}\right] = k[y^2\,\hat{\mathbf{x}} + (2xy+z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}] = \mathbf{E}.$ ✓

Problem 2.21

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}. \quad \begin{cases} \text{Outside the sphere } (r > R): \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \\ \text{Inside the sphere } (r < R): \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}. \end{cases}$$

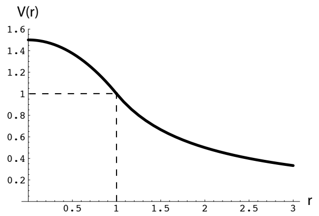
So for $r > R$: $V(r) = -\int_\infty^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right) d\bar{r} = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\bar{r}}\right)\Big|_\infty^r = \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{r},}$

and for $r < R$: $V(r) = -\int_\infty^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right) d\bar{r} - \int_R^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar{r}\right) d\bar{r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$

$$= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right).}$$

When $r > R$, $\boldsymbol{\nabla}V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = -\boldsymbol{\nabla}V = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}.$ ✓

When $r < R$, $\boldsymbol{\nabla}V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$; so $\mathbf{E} = -\boldsymbol{\nabla}V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}.$✓



(In the figure, r is in units of R, and $V(r)$ is in units of $q/4\pi\epsilon_0 R$.)

Problem 2.22

$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$ (Prob. 2.13). In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at $s = a$. Then

$V(s) = -\int_a^s \left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\bar{s}}\right) d\bar{s} = \boxed{-\frac{1}{4\pi\epsilon_0}2\lambda\ln\left(\frac{s}{a}\right).}$

(In this form it is clear why $a = \infty$ would be no good—likewise the other "natural" point, $a = 0$.)

$\boldsymbol{\nabla}V = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{\partial}{\partial s}\left(\ln\left(\frac{s}{a}\right)\right)\hat{\mathbf{s}} = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{1}{s}\hat{\mathbf{s}} = -\mathbf{E}.$ ✓

Problem 2.23

$V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{k}{\epsilon_0}\frac{(b-a)}{r^2}\right)dr - \int_b^a \left(\frac{k}{\epsilon_0}\frac{(r-a)}{r^2}\right)dr - \int_a^0 (0)dr = \frac{k}{\epsilon_0}\frac{(b-a)}{b} - \frac{k}{\epsilon_0}\left(\ln\left(\frac{a}{b}\right) + a\left(\frac{1}{a} - \frac{1}{b}\right)\right)$

$= \frac{k}{\epsilon_0}\left\{1 - \frac{a}{b} - \ln\left(\frac{a}{b}\right) - 1 + \frac{a}{b}\right\} = \boxed{\frac{k}{\epsilon_0}\ln\left(\frac{b}{a}\right).}$

Problem 2.24

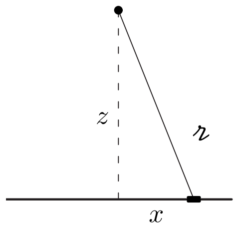
Using Eq. 2.22 and the fields from Prob. 2.16:

$V(b) - V(0) = -\int_0^b \mathbf{E}\cdot d\mathbf{l} = -\int_0^a \mathbf{E}\cdot d\mathbf{l} - \int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{\rho}{2\epsilon_0}\int_0^a s\,ds - \frac{\rho a^2}{2\epsilon_0}\int_a^b \frac{1}{s}ds$

$$= -\left(\frac{\rho}{2\epsilon_0}\right) \left.\frac{s^2}{2}\right|_0^a + \frac{\rho a^2}{2\epsilon_0} \ln s\Big|_a^b = \boxed{-\frac{\rho a^2}{4\epsilon_0}\left(1 + 2\ln\left(\frac{b}{a}\right)\right).}$$

Problem 2.25

(a) $\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2 + \left(\frac{d}{2}\right)^2}}.}$



(b) $V = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda\,dx}{\sqrt{z^2+x^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln(x + \sqrt{z^2+x^2})\Big|_{-L}^{L}$

$$= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right]} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{L+\sqrt{z^2+L^2}}{z}\right).$$

(c) $V = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma\,2\pi r\,dr}{\sqrt{r^2+z^2}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma\left(\sqrt{r^2+z^2}\right)\Big|_0^R = \boxed{\frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - z\right).}$

In each case, by symmetry $\frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} = 0$. $\quad\therefore \mathbf{E} = -\frac{\partial V}{\partial z}\hat{\mathbf{z}}$.

(a) $\mathbf{E} = -\frac{1}{4\pi\epsilon_0}2q\left(-\frac{1}{2}\right)\frac{2z}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}}}$ (agrees with Ex. 2.1).

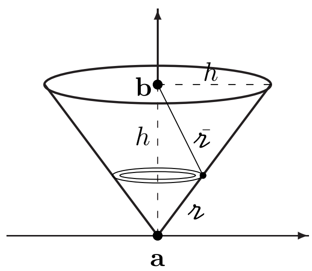
(b) $\mathbf{E} = -\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{(L+\sqrt{z^2+L^2})}\frac{1}{2}\frac{1}{\sqrt{z^2+L^2}}2z - \frac{1}{(-L+\sqrt{z^2+L^2})}\frac{1}{2}\frac{1}{\sqrt{z^2+L^2}}2z\right\}\hat{\mathbf{z}}$

$= -\frac{\lambda}{4\pi\epsilon_0}\frac{z}{\sqrt{z^2+L^2}}\left\{\frac{-L+\sqrt{z^2+L^2}-L-\sqrt{z^2+L^2}}{(z^2+L^2)-L^2}\right\}\hat{\mathbf{z}} = \boxed{\frac{2L\lambda}{4\pi\epsilon_0}\frac{1}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}}$ (agrees with Ex. 2.2).

(c) $\mathbf{E} = -\frac{\sigma}{2\epsilon_0}\left\{\frac{1}{2}\frac{1}{\sqrt{R^2+z^2}}2z - 1\right\}\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}}$ (agrees with Prob. 2.6).

If the right-hand charge in (a) is $-q$, then $\boxed{V=0}$, which, naively, suggests $\mathbf{E} = -\nabla V = \mathbf{0}$, in contradiction with the answer to Prob. 2.2. The point is that we only know V *on the z axis*, and from this we cannot hope to compute $E_x = -\frac{\partial V}{\partial x}$ or $E_y = -\frac{\partial V}{\partial y}$. That was OK in part (a), because we knew from symmetry that $E_x = E_y = 0$. But now $\mathbf{E}$ points in the x direction, so knowing V on the z axis is insufficient to determine $\mathbf{E}$.

Problem 2.26



$$V(\mathbf{a}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\mathscr{r}}\right)d\mathscr{r} = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}(\sqrt{2}h) = \frac{\sigma h}{2\epsilon_0}$$

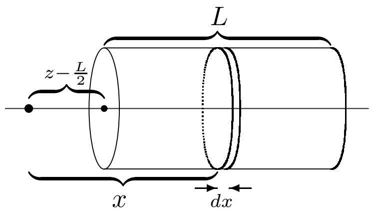
(where $r = \mathscr{r}/\sqrt{2}$)

$$
\begin{aligned}
V(\mathbf{b}) &= \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\bar{\mathscr{r}}}\right)d\mathscr{r} \qquad (\text{where } \bar{\mathscr{r}} = \sqrt{h^2+\mathscr{r}^2-\sqrt{2}h\mathscr{r}}\;) \\
&= \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\frac{\mathscr{r}}{\sqrt{h^2+\mathscr{r}^2-\sqrt{2}h\mathscr{r}}}\,d\mathscr{r} \\
&= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[\sqrt{h^2+\mathscr{r}^2-\sqrt{2}h\mathscr{r}}+\frac{h}{\sqrt{2}}\ln(2\sqrt{h^2+\mathscr{r}^2-\sqrt{2}h\mathscr{r}}+2\mathscr{r}-\sqrt{2}h)\right]\Bigg|_0^{\sqrt{2}h}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[h+\frac{h}{\sqrt{2}}\ln(2h+2\sqrt{2}h-\sqrt{2}h)-h-\frac{h}{\sqrt{2}}\ln(2h-\sqrt{2}h)\right] \\
&= \frac{\sigma}{2\sqrt{2}\epsilon_0}\frac{h}{\sqrt{2}}\left[\ln(2h+\sqrt{2}h)-\ln(2h-\sqrt{2}h)\right] = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{(2+\sqrt{2})^2}{2}\right) \\
&= \frac{\sigma h}{2\epsilon_0}\ln(1+\sqrt{2}). \qquad \therefore \boxed{V(\mathbf{a})-V(\mathbf{b}) = \frac{\sigma h}{2\epsilon_0}\left[1-\ln(1+\sqrt{2})\right]}.
\end{aligned}
$$

Problem 2.27

Cut the cylinder into slabs, as shown in the figure, and use result of Prob. 2.25c, with $z \to x$ and $\sigma \to \rho\,dx$:



$$
\begin{aligned}
V &= \frac{\rho}{2\epsilon_0}\int_{z-L/2}^{z+L/2}\left(\sqrt{R^2+x^2}-x\right)dx \\
&= \frac{\rho}{2\epsilon_0}\frac{1}{2}\left[x\sqrt{R^2+x^2}+R^2\ln(x+\sqrt{R^2+x^2})-x^2\right]\Big|_{z-L/2}^{z+L/2} \\
&= \boxed{\frac{\rho}{4\epsilon_0}\left\{\left(z+\frac{L}{2}\right)\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}-\left(z-\frac{L}{2}\right)\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}+R^2\ln\left[\frac{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right]-2zL\right\}}.
\end{aligned}
$$

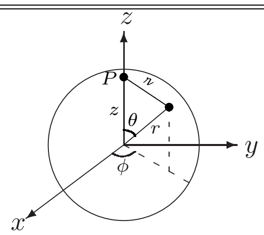
(*Note:* $-\left(z+\frac{L}{2}\right)^2+\left(z-\frac{L}{2}\right)^2 = -z^2-zL-\frac{L^2}{4}+z^2-zL+\frac{L^2}{4} = -2zL.$)

$$
\begin{aligned}
\mathbf{E} = -\boldsymbol{\nabla}V = -\hat{\mathbf{z}}\frac{\partial V}{\partial z} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\Bigg\{&\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}+\frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}-\frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}} \\
&+R^2\underbrace{\left[\frac{1+\frac{z+\frac{L}{2}}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}}{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\frac{1+\frac{z-\frac{L}{2}}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right]}_{\frac{1}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\frac{1}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}-2L\Bigg\}
\end{aligned}
$$

$$
\mathbf{E} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{2\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}-2\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}-2L\right\}
$$

$$= \boxed{\frac{\rho}{2\epsilon_0}\left[L - \sqrt{R^2 + \left(z + \frac{L}{2}\right)^2} + \sqrt{R^2 + \left(z - \frac{L}{2}\right)^2}\right]\hat{\mathbf{z}}.}$$

Problem 2.28



Orient axes so P is on z axis.

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\,d\tau.\quad \begin{cases} \text{Here } \rho \text{ is constant, } d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi, \\ \mathscr{r} \;=\sqrt{z^2 + r^2 - 2rz\cos\theta}. \end{cases}$

$V = \frac{\rho}{4\pi\epsilon_0}\int \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{z^2+r^2-2rz\cos\theta}}\ ;\ \int_0^{2\pi} d\phi = 2\pi.$

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{z^2+r^2-2rz\cos\theta}}\,d\theta = \frac{1}{rz}\left(\sqrt{r^2+z^2-2rz\cos\theta}\right)\Big|_0^{\pi} = \frac{1}{rz}\left(\sqrt{r^2+z^2+2rz} - \sqrt{r^2+z^2-2rz}\right)$$

$$= \frac{1}{rz}\left(r + z - |r - z|\right) = \begin{Bmatrix} 2/z\ ,\ \text{if } r < z, \\ 2/r\ ,\ \text{if } r > z. \end{Bmatrix}$$

$$\therefore V = \frac{\rho}{4\pi\epsilon_0}\cdot 2\pi \cdot 2\left\{\int_0^z \frac{1}{z}r^2 dr + \int_z^R \frac{1}{r}r^2 dr\right\} = \frac{\rho}{\epsilon_0}\left\{\frac{1}{z}\frac{z^3}{3} + \frac{R^2-z^2}{2}\right\} = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{z^2}{3}\right).$$

But $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, so $V(z) = \frac{1}{2\epsilon_0}\frac{3q}{4\pi R^3}\left(R^2 - \frac{z^2}{3}\right) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{z^2}{R^2}\right)$; $\boxed{V(r) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right).}$ ✓

Problem 2.29

$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\nabla^2 \int\left(\frac{\rho}{\mathscr{r}}\right)d\tau = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left(\nabla^2\frac{1}{\mathscr{r}}\right)d\tau$ (since ρ is a function of $\mathbf{r}'$, not $\mathbf{r}$)

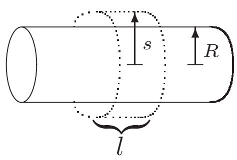
$= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')[-4\pi\delta^3(\mathbf{r}-\mathbf{r}')]\,d\tau = -\frac{1}{\epsilon_0}\rho(\mathbf{r}).$ ✓

Problem 2.30.

(a) Ex. 2.5: $\mathbf{E}_{\text{above}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$; $\mathbf{E}_{\text{below}} = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ always pointing *up*); $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$. ✓

Ex. 2.6: At each surface, $E = 0$ one side and $E = \frac{\sigma}{\epsilon_0}$ other side, so $\Delta E = \frac{\sigma}{\epsilon_0}$. ✓

Prob. 2.11: $\mathbf{E}_{\text{out}} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}$; $\mathbf{E}_{\text{in}} = 0$; so $\Delta\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}$. ✓

(b) 

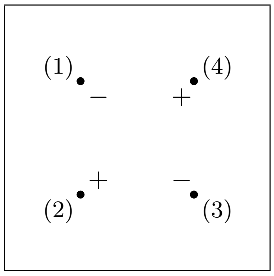
Outside: $\oint \mathbf{E}\cdot d\mathbf{a} = E(2\pi s)l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{\sigma}{\epsilon_0}(2\pi R)l \Rightarrow \mathbf{E} = \frac{\sigma}{\epsilon_0}\frac{R}{s}\hat{\mathbf{s}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{s}}$ (at surface).

Inside: $Q_{\text{enc}} = 0$, so $\mathbf{E} = 0$. $\therefore \Delta\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{s}}$. ✓

(c) $V_{\text{out}} = \frac{R^2\sigma}{\epsilon_0 r} = \frac{R\sigma}{\epsilon_0}$ (at surface); $V_{\text{in}} = \frac{R\sigma}{\epsilon_0}$; so $V_{\text{out}} = V_{\text{in}}$. ✓

$\frac{\partial V_{\text{out}}}{\partial r} = -\frac{R^2\sigma}{\epsilon_0 r^2} = -\frac{\sigma}{\epsilon_0}$ (at surface); $\frac{\partial V_{\text{in}}}{\partial r} = 0$; so $\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} = -\frac{\sigma}{\epsilon_0}$. ✓

Problem 2.31



(a) $V = \frac{1}{4\pi\epsilon_0}\sum \frac{q_i}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q}{a} + \frac{q}{\sqrt{2}a} + \frac{-q}{a}\right\} = \frac{q}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$.

$\therefore W_4 = qV = \boxed{\frac{q^2}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right).}$

(b) $W_1 = 0$, $W_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{a}\right)$; $W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{\sqrt{2}\,a} - \frac{q^2}{a}\right)$; $W_4 =$ (see (a)).

$W_{\text{tot}} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\left\{-1 + \frac{1}{\sqrt{2}} - 1 - 2 + \frac{1}{\sqrt{2}}\right\} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}\left(-2 + \frac{1}{\sqrt{2}}\right).}$

Problem 2.32

Conservation of energy (kinetic plus potential):

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 + \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{r} = E.$$

At release $v_A = v_B = 0,\ r = a$, so

$$E = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a}.$$

When they are very far apart ($r \to \infty$) the potential energy is zero, so

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a}.$$

Meanwhile, conservation of momentum says $m_A v_A = m_B v_B$, or $v_B = (m_A/m_B)v_A$. So

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B\left(\frac{m_A}{m_B}\right)^2 v_A^2 = \frac{1}{2}\left(\frac{m_A}{m_B}\right)(m_A + m_B)v_A^2 = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a}.$$

$$\boxed{v_A = \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\left(\frac{m_A}{m_B}\right)};\quad v_B = \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\left(\frac{m_B}{m_A}\right)}.}$$

Problem 2.33

From Eq. 2.42, the energy of one charge is

$$W = \frac{1}{2}qV = \frac{1}{2}(2)\sum_{n=1}^{\infty}\frac{1}{4\pi\epsilon_0}\frac{(-1)^n q^2}{na} = \frac{q^2}{4\pi\epsilon_0 a}\sum_{1}^{\infty}\frac{(-1)^n}{n}.$$

(The factor of 2 out front counts the charges to the left as well as to the right of q.) The sum is $-\ln 2$ (you can get it from the Taylor expansion of $\ln(1+x)$:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

with $x = 1$. Evidently $\boxed{\alpha = \ln 2}$.

Problem 2.34

(a) $W = \frac{1}{2}\int \rho V\,d\tau$. From Prob. 2.21 (or Prob. 2.28): $V = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{r^2}{3}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$

$$W = \frac{1}{2}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^R \left(3 - \frac{r^2}{R^2}\right)4\pi r^2 dr = \frac{q\rho}{4\epsilon_0 R}\left[3\frac{r^3}{3} - \frac{1}{R^2}\frac{r^5}{5}\right]\Bigg|_0^R = \frac{q\rho}{4\epsilon_0 R}\left(R^3 - \frac{R^3}{5}\right)$$
$$= \frac{q\rho}{5\epsilon_0}R^2 = \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3} = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. *Outside* $(r > R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$; *Inside* $(r < R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$.

$$\therefore W = \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}q^2\left\{\int_R^\infty \frac{1}{r^4}(r^2 4\pi\,dr) + \int_0^R \left(\frac{r}{R^3}\right)^2(4\pi r^2 dr)\right\}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\left(-\frac{1}{r}\right)\Bigg|_R^\infty + \frac{1}{R^6}\left(\frac{r^5}{5}\right)\Bigg|_0^R\right\} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

(c) $W = \frac{\epsilon_0}{2}\left\{\oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a} + \int_{\mathcal{V}} E^2 d\tau\right\}$, where $\mathcal{V}$ is large enough to enclose all the charge, but otherwise arbitrary. Let's use a sphere of radius $a > R$. Here $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

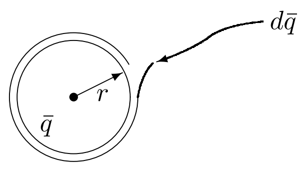
$$W = \frac{\epsilon_0}{2}\left\{\int_{r=a}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)r^2\sin\theta\,d\theta\,d\phi + \int_0^R E^2 d\tau + \int_R^a\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2(4\pi r^2 dr)\right\}$$
$$= \frac{\epsilon_0}{2}\left\{\frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{a}4\pi + \frac{q^2}{(4\pi\epsilon_0)^2}\frac{4\pi}{5R} + \frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left(-\frac{1}{r}\right)\Bigg|_R^a\right\}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\frac{1}{a} + \frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right\} = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

As $a \to \infty$, the contribution from the surface integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\right)$ goes to zero, while the volume integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\left(\frac{6a}{5R} - 1\right)\right)$ picks up the slack.

Problem 2.35

$$dW = d\bar{q}\,V = d\bar{q}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\bar{q}}{r}, \quad (\bar{q} = \text{charge on sphere of radius } r).$$

$$\bar{q} = \frac{4}{3}\pi r^3\rho = q\frac{r^3}{R^3} \quad (q = \text{total charge on sphere}).$$



$$d\bar{q} = 4\pi r^2 dr\,\rho = \frac{4\pi r^2}{\frac{4}{3}\pi R^3}q\,dr = \frac{3q}{R^3}r^2 dr.$$

$$dW = \frac{1}{4\pi\epsilon_0}\left(\frac{qr^3}{R^3}\right)\frac{1}{r}\left(\frac{3q}{R^3}r^2 dr\right) = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}r^4 dr$$

$$W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\int_0^R r^4 dr = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\frac{R^5}{5} = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).\checkmark$$

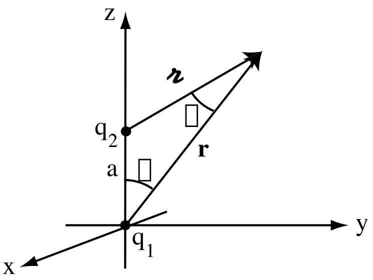
Problem 2.36

(a) $W = \frac{\epsilon_0}{2}\int E^2\,d\tau.$ $\quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $(a < r < b)$, zero elsewhere.

$$W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} = \boxed{\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).}$$

(b) $W_1 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{a},\ W_2 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{b},\quad \mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\ (r > a),\ \mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\,\hat{\mathbf{r}}\ (r > b).$ So $\mathbf{E}_1\cdot\mathbf{E}_2 = \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{-q^2}{r^4},\ (r > b)$, and hence $\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2\int_b^\infty \frac{1}{r^4}4\pi r^2 dr = -\frac{q^2}{4\pi\epsilon_0 b}.$
$W_{\text{tot}} = W_1 + W_2 + \epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau = \frac{1}{8\pi\epsilon_0}q^2\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{b}\right) = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).\checkmark$

Problem 2.37



$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\,\hat{\mathbf{r}};\quad \mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\ ;\quad W_i = \epsilon_0\frac{q_1q_2}{(4\pi\epsilon_0)^2}\int\frac{1}{r^2\mathscr{r}^2}\cos\beta\,r^2\sin\theta\,dr\,d\theta\,d\phi,$$

where (from the figure)

$$\mathscr{r} = \sqrt{r^2+a^2-2ra\cos\theta},\quad \cos\beta = \frac{(r-a\cos\theta)}{\mathscr{r}}.$$

Therefore

$$W_i = \frac{q_1q_2}{(4\pi)^2\epsilon_0}2\pi\int\frac{(r-a\cos\theta)}{\mathscr{r}^3}\sin\theta\,dr\,d\theta.$$

It's simplest to do the r integral first, changing variables to $\mathscr{r}$:

$$2\mathscr{r}\,d\mathscr{r} = (2r-2a\cos\theta)\,dr \quad\Rightarrow\quad (r-a\cos\theta)\,dr = \mathscr{r}\,d\mathscr{r}\ .$$

As $r: 0\to\infty$, $\mathscr{r}: a\to\infty$, so

$$W_i = \frac{q_1q_2}{8\pi\epsilon_0}\int_0^\pi\left(\int_a^\infty\frac{1}{\mathscr{r}^2}\,d\mathscr{r}\right)\sin\theta\,d\theta.$$

The $\mathscr{r}$ integral is $1/a$, so

$$W_i = \frac{q_1q_2}{8\pi\epsilon_0 a}\int_0^\pi\sin\theta\,d\theta = \boxed{\frac{q_1q_2}{4\pi\epsilon_0 a}.}$$

Of course, this is precisely the interaction energy of two point charges.

Problem 2.38

(a) $\boxed{\sigma_R = \frac{q}{4\pi R^2}\ ;\ \sigma_a = \frac{-q}{4\pi a^2}\ ;\ \sigma_b = \frac{q}{4\pi b^2}.}$

(b) $V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_b^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{b} + \frac{q}{R} - \frac{q}{a}\right).}$

(c) $\boxed{\sigma_b \to 0}$ (the charge "drains off"); $V(0) = -\int_\infty^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right).}$

Problem 2.39

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2};}$ $\boxed{\sigma_b = -\frac{q_b}{4\pi b^2};}$ $\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}.}$

(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\,\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\,\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ $(\mathbf{r}_b)$ is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

Problem 2.40

(a) $\boxed{\text{No.}}$ For example, if it is very close to the wall, it will induce charge of the opposite sign on the wall, and it will be attracted.

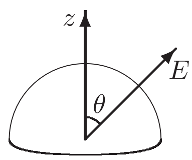
(b) $\boxed{\text{No.}}$ *Typically* it will be attractive, but see footnote 12 for a extraordinary counterexample.

Problem 2.41

Between the plates, $E = 0$; outside the plates $E = \sigma/\epsilon_0 = Q/\epsilon_0 A$. So

$$P = \frac{\epsilon_0}{2}E^2 = \frac{\epsilon_0}{2}\frac{Q^2}{\epsilon_0^2 A^2} = \boxed{\frac{Q^2}{2\epsilon_0 A^2}.}$$

Problem 2.42



Inside, $\mathbf{E} = \mathbf{0}$; outside, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$; so

$\mathbf{E}_{\text{ave}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\,\hat{\mathbf{r}}$; $f_z = \sigma(E_{\text{ave}})_z$; $\sigma = \frac{Q}{4\pi R^2}$.

$$F_z = \int f_z da = \int \left(\frac{Q}{4\pi R^2}\right)\frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\right)\cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{2\epsilon_0}\left(\frac{Q}{4\pi R}\right)^2 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{\pi\epsilon_0}\left(\frac{Q}{4R}\right)^2 \left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2} = \frac{1}{2\pi\epsilon_0}\left(\frac{Q}{4R}\right)^2 = \boxed{\frac{Q^2}{32\pi R^2\epsilon_0}.}$$

Problem 2.43

Say the charge on the inner cylinder is Q, for a length L. The field is given by Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = E \cdot 2\pi s \cdot L = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Q \Rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\,\hat{\mathbf{s}}$. Potential difference between the cylinders is

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right).$$

As set up here, a is at the higher potential, so $V = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

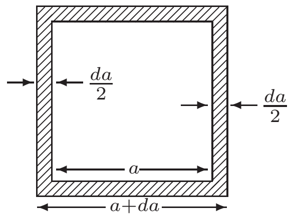
$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance *per unit length* is $\boxed{\dfrac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}.}$

Problem 2.44

(a) $W = (\text{force})\times(\text{distance}) = (\text{pressure})\times(\text{area})\times(\text{distance}) = \boxed{\dfrac{\epsilon_0}{2}E^2 A\epsilon.}$

(b) $W = (\text{energy per unit volume})\times(\text{decrease in volume}) = \left(\epsilon_0 \frac{E^2}{2}\right)(A\epsilon)$. Same as (a), confirming that the energy lost is equal to the work done.

Problem 2.45



From Prob. 2.4, the field at height z above the center of a square loop (side a) is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\,\hat{\mathbf{z}}.$$

Here $\lambda \to \sigma\frac{da}{2}$ (see figure), and we integrate over a from 0 to $\bar{a}$:

$$E = \frac{1}{4\pi\epsilon_0}2\sigma z\int_0^{\bar{a}}\frac{a\,da}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\text{. Let } u = \frac{a^2}{4}\text{, so } a\,da = 2\,du.$$

$$= \frac{1}{4\pi\epsilon_0}4\sigma z\int_0^{\bar{a}^2/4}\frac{du}{(u+z^2)\sqrt{2u+z^2}} = \frac{\sigma z}{\pi\epsilon_0}\left[\frac{2}{z}\tan^{-1}\left(\frac{\sqrt{2u+z^2}}{z}\right)\right]_0^{\bar{a}^2/4}$$

$$= \frac{2\sigma}{\pi\epsilon_0}\left\{\tan^{-1}\left(\frac{\sqrt{\frac{\bar{a}^2}{2}+z^2}}{z}\right) - \tan^{-1}(1)\right\};$$

$$\boxed{\mathbf{E} = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\sqrt{1+\frac{a^2}{2z^2}} - \frac{\pi}{4}\right]\hat{\mathbf{z}}} = \frac{\sigma}{\pi\epsilon_0}\tan^{-1}\left(\frac{a^2}{4z\sqrt{z^2+(a^2/2)}}\right)\hat{\mathbf{z}}.$$

$a \to \infty$ (infinite plane): $E = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}(\infty) - \frac{\pi}{4}\right] = \frac{2\sigma}{\pi\epsilon_0}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\sigma}{2\epsilon_0}$. ✓

$z \gg a$ (point charge): Let $f(x) = \tan^{-1}\sqrt{1+x} - \frac{\pi}{4}$, and expand as a Taylor series:

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \cdots$$

Here $f(0) = \tan^{-1}(1) - \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{4} = 0$; $f'(x) = \frac{1}{1+(1+x)}\frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2(2+x)\sqrt{1+x}}$, so $f'(0) = \frac{1}{4}$, so

$$f(x) = \frac{1}{4}x + (\ )x^2 + (\ )x^3 + \cdots$$

Thus (since $\frac{a^2}{2z^2} = x \ll 1$), $E \approx \frac{2\sigma}{\pi\epsilon_0}\left(\frac{1}{4}\frac{a^2}{2z^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{\sigma a^2}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$. ✓

The (double) integral is a pure number; Mathematica says it is 2. So

$$\boxed{V = \frac{\sigma R}{\pi\epsilon_0}.}$$

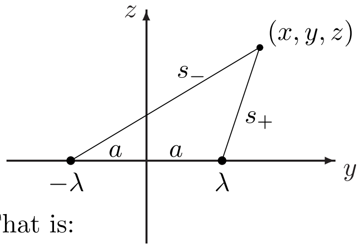
Problem 2.52

(a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_-.

$\therefore$ Total $\boxed{V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right).}$

Now $s_+ = \sqrt{(y-a)^2+z^2}$, and $s_- = \sqrt{(y+a)^2+z^2}$, so

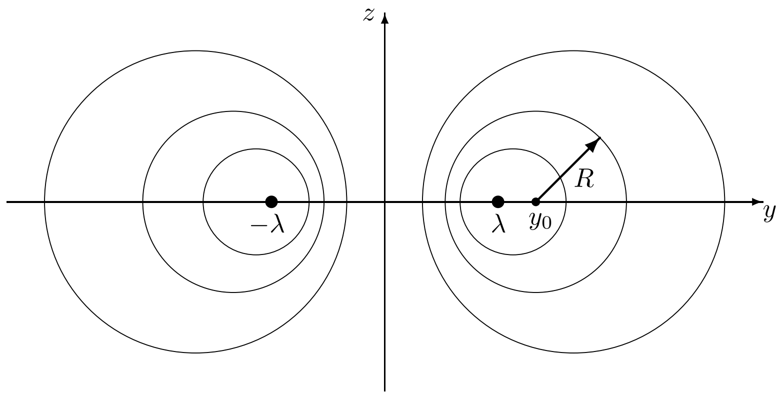
$$V(x,y,z) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right) = \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right].}$$



(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k = $ constant. That is:
$y^2+2ay+a^2+z^2 = k(y^2-2ay+a^2+z^2) \Rightarrow y^2(k-1)+z^2(k-1)+a^2(k-1)-2ay(k+1)=0$, or
$y^2+z^2+a^2-2ay\left(\frac{k+1}{k-1}\right)=0$. The equation for a *circle*, with center at $(y_0,0)$ and radius R, is
$(y-y_0)^2+z^2=R^2$, or $y^2+z^2+(y_0^2-R^2)-2yy_0=0$.
Evidently the equipotentials *are* circles, with $y_0 = a\left(\frac{k+1}{k-1}\right)$ and
$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - a^2 = a^2\frac{(k^2+2k+1-k^2+2k-1)}{(k-1)^2} = a^2\frac{4k}{(k-1)^2}$, or
$R = \frac{2a\sqrt{k}}{|k-1|}$; or, in terms of V_0:

$$y_0 = a\frac{e^{4\pi\epsilon_0 V_0/\lambda}+1}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}} = \boxed{a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$

$$R = 2a\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{2}{(e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda})} = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} = \boxed{a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$



Problem 2.53

(a) $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Eq. 2.24), so $\boxed{\dfrac{d^2V}{dx^2} = -\dfrac{1}{\epsilon_0}\rho.}$

(b) $qV = \frac{1}{2}mv^2 \rightarrow \boxed{v = \sqrt{\dfrac{2qV}{m}}.}$

(c) $dq = A\rho\, dx$; $\frac{dq}{dt} = a\rho \frac{dx}{dt} = \boxed{A\rho v = I}$ (constant). (*Note*: ρ, hence also I, is *negative*.)

(d) $\frac{d^2V}{dx^2} = -\frac{1}{\epsilon_0}\rho = -\frac{1}{\epsilon_0}\frac{I}{Av} = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2qV}} \Rightarrow \boxed{\dfrac{d^2V}{dx^2} = \beta V^{-1/2}}$, where $\beta = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2q}}$.
(*Note: I* is *negative*, so β is positive; q is *positive*.)

(e) Multiply by $V' = \frac{dV}{dx}$:

$$V'\frac{dV'}{dx} = \beta V^{-1/2}\frac{dV}{dx} \Rightarrow \int V'\,dV' = \beta\int V^{-1/2}\,dV \Rightarrow \frac{1}{2}V'^2 = 2\beta V^{1/2} + \text{constant}.$$

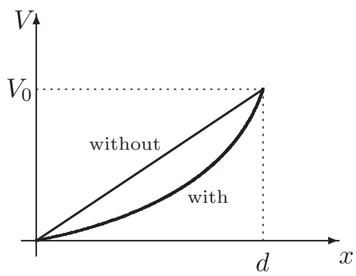
But $V(0) = V'(0) = 0$ (cathode is at potential zero, and field at cathode is zero), so the constant is zero, and

$$V'^2 = 4\beta V^{1/2} \Rightarrow \frac{dV}{dx} = 2\sqrt{\beta}\,V^{1/4} \Rightarrow V^{-1/4}dV = 2\sqrt{\beta}\,dx;$$
$$\int V^{-1/4}\,dV = 2\sqrt{\beta}\int dx \Rightarrow \frac{4}{3}V^{3/4} = 2\sqrt{\beta}\,x + \text{constant}.$$

But $V(0) = 0$, so this constant is also zero.

$$V^{3/4} = \frac{3}{2}\sqrt{\beta}\,x, \text{ so } V(x) = \left(\frac{3}{2}\sqrt{\beta}\right)^{4/3} x^{4/3}, \text{ or } V(x) = \left(\frac{9}{4}\beta\right)^{2/3} x^{4/3} = \left(\frac{81I^2m}{32\epsilon_0^2A^2q}\right)^{1/3} x^{4/3}.$$

Interms of V_0 (instead of I): $\boxed{V(x) = V_0\left(\dfrac{x}{d}\right)^{4/3}}$ (see graph).
Without space-charge, V would increase linearly: $V(x) = V_0\left(\frac{x}{d}\right)$.



$$\rho = -\epsilon_0\frac{d^2V}{dx^2} = -\epsilon_0 V_0\frac{1}{d^{4/3}}\frac{4}{3}\cdot\frac{1}{3}x^{-2/3} = \boxed{-\frac{4\epsilon_0 V_0}{9(d^2x)^{2/3}}.}$$
$$v = \sqrt{\frac{2q}{m}}\sqrt{V} = \boxed{\sqrt{2qV_0/m}\left(\frac{x}{d}\right)^{2/3}.}$$

(f) $V(d) = V_0 = \left(\frac{81I^2m}{32\epsilon_0^2A^2q}\right)^{1/3} d^{4/3} \Rightarrow V_0^3 = \frac{81md^4}{32\epsilon_0^2A^2q}I^2$; $I^2 = \frac{32\epsilon_0^2A^2q}{81md^4}V_0^3$;
$I = \frac{4\sqrt{2}\,\epsilon_0 A\sqrt{q}}{9\sqrt{m}\,d^2}V_0^{3/2} = KV_0^{3/2}$, where $\boxed{K = \dfrac{4\epsilon_0 A}{9d^2}\sqrt{\dfrac{2q}{m}}.}$

Problem 2.54

(a) $\boxed{\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int\frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\mathscr{r}/\lambda}d\tau.}$

(b) $\boxed{\text{Yes.}}$ The field of a point charge at the origin is radial and symmetric, so $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$, and hence this is also true (by superposition) for any *collection* of charges.

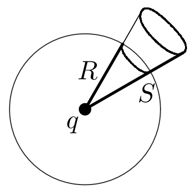
(c) $$V = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{4\pi\epsilon_0} q \int_\infty^r \frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}dr$$
$$= \frac{1}{4\pi\epsilon_0} q \int_r^\infty \frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}dr = \frac{q}{4\pi\epsilon_0}\left\{\int_r^\infty \frac{1}{r^2}e^{-r/\lambda}dr + \frac{1}{\lambda}\int_r^\infty \frac{1}{r}e^{-r/\lambda}dr\right\}.$$

Now $\int \frac{1}{r^2}e^{-r/\lambda}dr = -\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda}\int \frac{e^{-r/\lambda}}{r}dr \longleftarrow$ exactly right to kill the last term. Therefore

$$V(r) = \frac{q}{4\pi\epsilon_0}\left\{-\frac{e^{-r/\lambda}}{r}\Big|_r^\infty\right\} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}.}$$

(d) $$\oint_\mathcal{S} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\cancel{4\pi}\epsilon_0} q \frac{1}{\cancel{R^2}}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\,\cancel{4\pi}\,\cancel{R^2} = \frac{q}{\epsilon_0}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}.$$
$$\int_\mathcal{V} V\,d\tau = \frac{q}{\cancel{4\pi}\epsilon_0}\int_0^R \frac{e^{-r/\lambda}}{r}r^2\,\cancel{4\pi}\,dr = \frac{q}{\epsilon_0}\int_0^R re^{-r/\lambda}dr = \frac{q}{\epsilon_0}\left[\frac{e^{-r/\lambda}}{(1/\lambda)^2}\left(-\frac{r}{\lambda}-1\right)\right]_0^R$$
$$= \lambda^2\frac{q}{\epsilon_0}\left\{-e^{-R/\lambda}\left(1+\frac{R}{\lambda}\right)+1\right\}.$$
$$\therefore \oint_\mathcal{S}\mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int_\mathcal{V} V\,d\tau = \frac{q}{\epsilon_0}\left\{\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}+1\right\} = \frac{q}{\epsilon_0}.\quad \text{qed}$$

(e) Does the result in (d) hold for a *non*spherical surface? Suppose we make a "dent" in the sphere—pushing a patch (area $R^2\sin\theta\,d\theta\,d\phi$) from radius R out to radius S (area $S^2\sin\theta\,d\theta\,d\phi$).

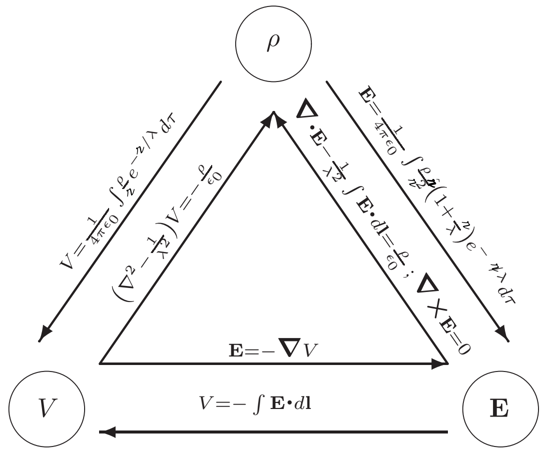


$$\Delta\oint\mathbf{E}\cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{S^2}\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda}(S^2\sin\theta\,d\theta\,d\phi) - \frac{1}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}(R^2\sin\theta\,d\theta\,d\phi)\right\}$$
$$= \frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

$$\Delta\frac{1}{\lambda^2}\int V\,d\tau = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\int\frac{e^{-r/\lambda}}{r}r^2\sin\theta\,dr\,d\theta\,d\phi = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\int_R^S re^{-r/\lambda}dr$$
$$= -\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\Big|_R^S$$
$$= -\frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

So the change in $\frac{1}{\lambda^2}\int V\,d\tau$ exactly compensates for the change in $\oint\mathbf{E}\cdot d\mathbf{a}$, and we get $\frac{1}{\epsilon_0}q$ for the total using the dented sphere, just as we did with the perfect sphere. Any closed surface can be built up by successive distortions of the sphere, so the result holds for all shapes. By superposition, if there are many charges inside, the total is $\frac{1}{\epsilon_0}Q_{\text{enc}}$. Charges *outside* do not contribute (in the argument above we found that  for this volume $\oint\mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int V\,d\tau = 0$—and, again, the sum is not changed by distortions of the surface, as long as q remains outside). So the new "Gauss's Law" holds for *any* charge configuration.

(f) In differential form, "Gauss's law" reads: $\boxed{\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho,}$ or, putting it all in terms of $\mathbf{E}$:
$\boldsymbol{\nabla}\cdot\mathbf{E} - \frac{1}{\lambda^2}\int \mathbf{E}\cdot d\mathbf{l} = \frac{1}{\epsilon_0}\rho$. Since $\mathbf{E} = -\boldsymbol{\nabla}V$, this also yields "Poisson's equation": $-\nabla^2 V + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho$.



(g) Refer to "Gauss's law" in differential form (f). Since $\mathbf{E}$ is zero, inside a conductor (otherwise charge would move, and in such a direction as to cancel the field), V is constant (inside), and hence ρ is uniform, throughout the volume. Any "extra" charge must reside on the surface. (The fraction at the surface depends on λ, and on the shape of the conductor.)

Problem 2.55

$\rho = \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \epsilon_0\frac{\partial}{\partial x}(ax) = \boxed{\epsilon_0 a}$ (constant everywhere).

The same charge density would be compatible (as far as Gauss's law is concerned) with $\mathbf{E} = ay\hat{\mathbf{y}}$, for instance, or $\mathbf{E} = (\frac{a}{3})\mathbf{r}$, etc. The point is that Gauss's law (and $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$) by themselves *do not determine the field*—like any differential equations, they must be supplemented by appropriate *boundary conditions*. Ordinarily, these are so "obvious" that we impose them almost subconsciously ("E must go to zero far from the source charges")—or we appeal to symmetry to resolve the ambiguity ("the field must be the same—in magnitude—on both sides of an infinite plane of surface charge"). But in this case there are *no* natural boundary conditions, and no persuasive symmetry conditions, to fix the answer. The question "What is the electric field produced by a uniform charge density filling all of space?" is simply *ill-posed*: it does not give us sufficient information to determine the answer. (Incidentally, it won't help to appeal to Coulomb's law $\left(\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau\right)$—the integral is hopelessly indefinite, in this case.)

Problem 2.56

Compare Newton's law of universal gravitation to Coulomb's law:

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}; \qquad \mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}.$$

Evidently $\frac{1}{4\pi\epsilon_0} \to G$ and $q \to m$. The gravitational energy of a sphere (translating Prob. 2.34) is therefore

$$\boxed{W_{\text{grav}} = \frac{3}{5}G\frac{M^2}{R}.}$$

Now, $G = 6.67 \times 10^{-11}$ N m^2/kg^2, and for the sun $M = 1.99 \times 10^{30}$ kg, $R = 6.96 \times 10^8$ m, so the sun's gravitational energy is $W = 2.28 \times 10^{41}$ J. At the current rate this energy would be dissipated in a time

$$t = \frac{W}{P} = \frac{2.28 \times 10^{41}}{3.86 \times 10^{26}} = 5.90 \times 10^{14}\,\mathrm{s} = \boxed{1.87 \times 10^7 \text{ years.}}$$

Problem 2.57

First eliminate z, using the formula for the ellipsoid:

$$\sigma(x,y) = \frac{Q}{4\pi ab}\frac{1}{\sqrt{c^2(x^2/a^4) + c^2(y^2/b^4) + 1 - (x^2/a^2) - (y^2/b^2)}}.$$

Now (for parts (a) and (b)) set $c \to 0$, "squashing" the ellipsoid down to an ellipse in the $x\,y$ plane:

$$\sigma(x,y) = \frac{Q}{2\pi ab}\frac{1}{\sqrt{1 - (x/a)^2 - (y/b)^2}}.$$

(I multiplied by 2 to count both surfaces.)

(a) For the circular disk, set $a = b = R$ and let $r \equiv \sqrt{x^2 + y^2}$. $\boxed{\sigma(r) = \frac{Q}{2\pi R}\frac{1}{\sqrt{R^2 - r^2}}.}$

(b) For the ribbon, let $Q/b \equiv \Lambda$, and then take the limit $b \to \infty$: $\boxed{\sigma(x) = \frac{\Lambda}{2\pi}\frac{1}{\sqrt{a^2 - x^2}}.}$

(c) Let $b = c$, $r \equiv \sqrt{y^2 + z^2}$, making an ellipsoid of revolution:

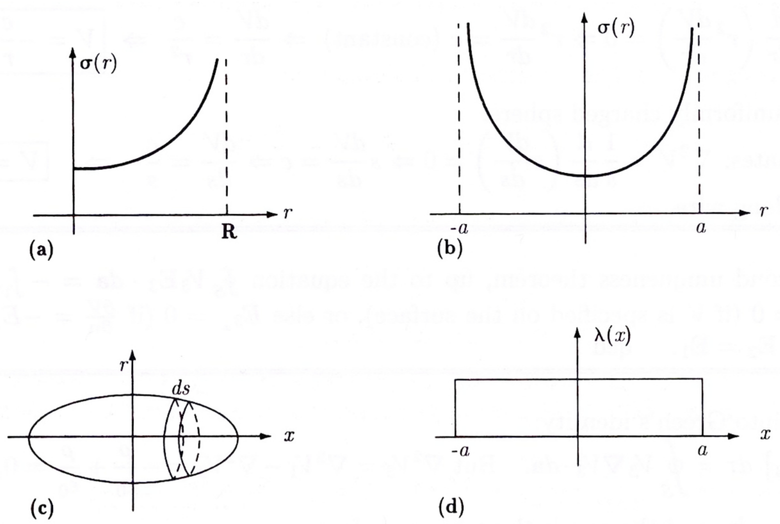
$$\frac{x^2}{a^2} + \frac{r^2}{c^2} = 1, \quad \text{with } \sigma = \frac{Q}{4\pi ac^2}\frac{1}{\sqrt{x^2/a^4 + r^2/c^4}}.$$

The charge on a ring of width dx is

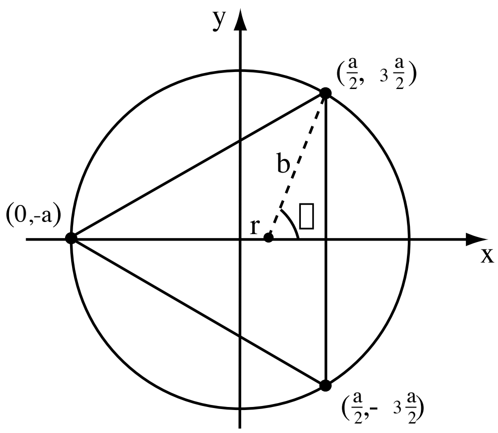
$$dq = \sigma 2\pi r\, ds, \quad \text{where } ds = \sqrt{dx^2 + dr^2} = dx\sqrt{1 + (dr/dx)^2}.$$

Now $\frac{2x\,dx}{a^2} + \frac{2r\,dr}{c^2} = 0 \Rightarrow \frac{dr}{dx} = -\frac{c^2 x}{a^2 r}$, so $ds = dx\sqrt{1 + \frac{c^4 x^2}{a^4 r^2}} = dx\frac{c^2}{r}\sqrt{x^2/a^4 + r^2/c^4}$. Thus

$$\lambda(x) = \frac{dq}{dx} = 2\pi r \frac{Q}{4\pi ac^2}\frac{1}{\sqrt{x^2/a^4 + r^2/c^4}}\frac{c^2}{r}\sqrt{x^2/a^4 + r^2/c^4} = \boxed{\frac{Q}{2a}. \quad \textit{(Constant!)}}$$



Problem 2.58



(a) One such point is on the x axis (see diagram) at $x = r$. Here the field is

$$E_x = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(a+r)^2} - 2\frac{\cos\theta}{b^2}\right] = 0, \quad \text{or} \quad \frac{2\cos\theta}{b^2} = \frac{1}{(a+r)^2}.$$

Now,

$$\cos\theta = \frac{(a/2)-r}{b}; \quad b^2 = \left(\frac{a}{2} - r\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2 = (a^2 - ar + r^2).$$

Therefore

$$\frac{2[(a/2)-r]}{(a^2 - ar + r^2)^{3/2}} = \frac{1}{(a+r)^2}. \quad \text{To simplify, let} \quad \frac{r}{a} \equiv u:$$

$$\frac{(1-2u)}{(1-u+u^2)^{3/2}} = \frac{1}{(1+u)^2}, \quad \text{or} \quad (1-2u)^2(1+u)^4 = (1-u+u^2)^3.$$

Multiplying out each side:

$$1 - 6u^2 - 4u^3 + 9u^4 + 12u^5 + 4u^6 = 1 - 3u + 6u^2 - 7u^3 + 6u^4 - 3u^5 + u^6,$$

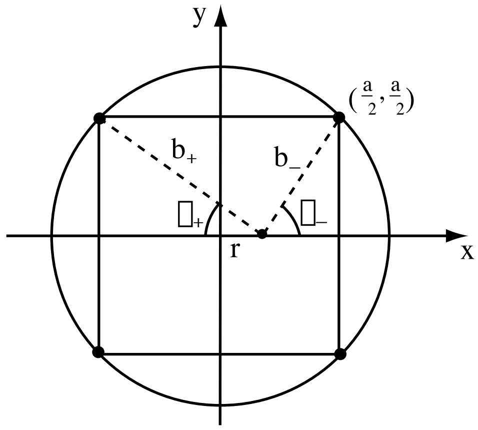
or

$$3u - 12u^2 + 3u^3 + 3u^4 + 15u^5 + 3u^6 = 0.$$

$u = 0$ is a solution (of course—the center of the triangle); factoring out $3u$ we are left with a quintic equation:

$$1 - 4u + u^2 + u^3 + 5u^4 + u^5 = 0.$$

According to Mathematica, this has two complex roots, and one negative root. The two remaining solutions are $u = 0.284718$ and $u = 0.626691$. The latter is outside the triangle, and clearly spurious. So $\boxed{r = 0.284718\,a.}$ (The other two places where $\mathbf{E} = \mathbf{0}$ are at the symmetrically located points, of course.)



(b) For the square:

$$E_x = \frac{q}{4\pi\epsilon_0}\left(2\frac{\cos\theta_+}{b_+^2} - 2\frac{\cos\theta_-}{b_-^2}\right) = 0 \quad\Rightarrow\quad \frac{\cos\theta_+}{b_+^2} = \frac{\cos\theta_-}{b_-^2},$$

where

$$\cos\theta_\pm = \frac{(a/\sqrt{2}) \pm r}{b_\pm}; \quad b_\pm^2 = \left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}} \pm r\right)^2 = a^2 \pm \sqrt{2}\,ar + r^2.$$

Thus

$$\frac{(a/\sqrt{2}) + r}{(a^2 + \sqrt{2}\,ar + r^2)^{3/2}} = \frac{(a/\sqrt{2}) - r}{(a^2 - \sqrt{2}\,ar + r^2)^{3/2}}.$$

To simplify, let $w \equiv \sqrt{2}\,r/a$; then

$$\frac{1+w}{(2 + 2w + w^2)^{3/2}} = \frac{1-w}{(2 - 2w + w^2)^{3/2}}, \quad\text{or}\quad (1+w)^2(2 - 2w + w^2)^3 = (1-w)^2(2 + 2w + w^2)^3.$$

Multiplying out the left side:

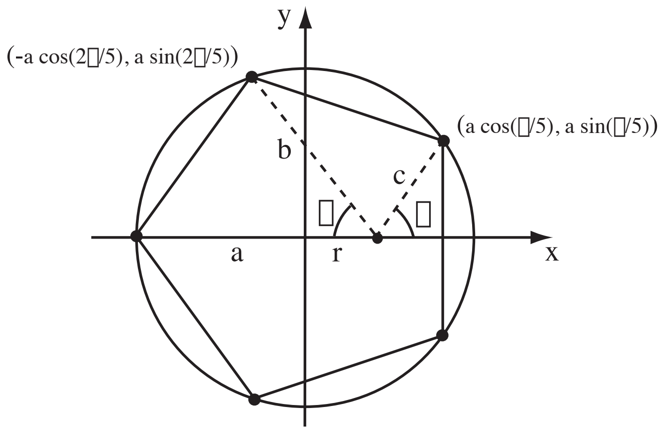
$$8 - 8w - 4w^2 + 16w^3 - 10w^4 - 2w^5 + 7w^6 - 4w^7 + w^8 = (\text{same thing with } w \to -w).$$

The even powers cancel, leaving

$$8w - 16w^3 + 2w^5 + 4w^7 = 0, \quad \text{or} \quad 4 - 8v + v^2 + 2v^3 = 0,$$

where $v \equiv w^2$. According to Mathematica, this cubic equation has one negative root, one root that is spurious (the point lies outside the square), and $v = 0.598279$, which yields

$$r = \sqrt{\frac{v}{2}}\, a = \boxed{0.546936\, a}.$$



For the pentagon:

$$E_x = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(a+r)^2} + 2\frac{\cos\theta}{b^2} - 2\frac{\cos\phi}{c^2}\right) = 0,$$

where

$$\cos\theta = \frac{a\cos(2\pi/5) + r}{b}, \quad \cos\phi = \frac{a\cos(\pi/5) - r}{c};$$

$$\begin{aligned} b^2 &= [a\cos(2\pi/5) + r]^2 + [a\sin(2\pi/5)]^2 = a^2 + r^2 + 2ar\cos(2\pi/5), \\ c^2 &= [a\cos(\pi/5) - r]^2 + [a\sin(\pi/5)]^2 = a^2 + r^2 - 2ar\cos(\pi/5). \end{aligned}$$

$$\frac{1}{(a+r)^2} + 2\frac{r + a\cos(2\pi/5)}{[a^2 + r^2 + 2ar\cos(2\pi/5)]^{3/2}} + 2\frac{r - a\cos(\pi/5)}{[a^2 + r^2 - 2ar\cos(\pi/5)]^{3/2}} = 0.$$

Mathematica gives the solution $\boxed{r = 0.688917\, a.}$

For an n-sided regular polygon there are evidently n such points, lying on the radial spokes that bisect the sides; their distance from the center appears to grow monotonically with n: $r(3) = 0.285$, $r(4) = 0.547$, $r(5) = 0.689, \ldots$. As $n \to \infty$ they fill out a circle that (in the limit) coincides with the ring of charge itself.

Problem 2.59 The theorem is *false*. For example, suppose the conductor is a neutral sphere and the external field is due to a nearby positive point charge q. A negative charge will be induced on the near side of the sphere (and a positive charge on the far side), so the force will be *attractive* (toward q). If we now reverse the sign of q, the induced charges will also reverse, but the force will still be attractive.

If the external field is *uniform*, then the net force on the induced charges is zero, and the total force on the conductor is $Q\mathbf{E}_e$, which *does* switch signs if $\mathbf{E}_e$ is reversed. So the "theorem" is valid in this very special case.

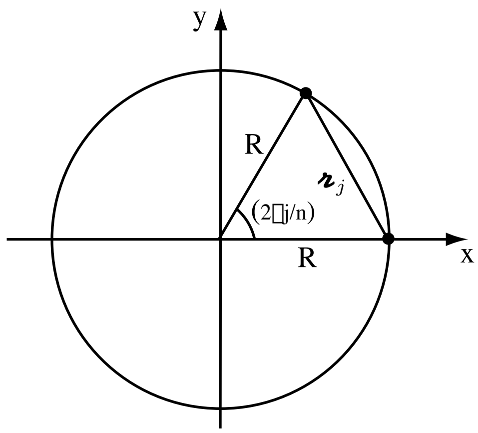
Problem 2.60 The initial configuration consists of a point charge q at the center, $-q$ induced on the inner surface, and $+q$ on the outer surface. What is the energy of this configuration? Imagine assembling it piece-by-piece. First bring in q and place it at the origin—this takes no work. Now bring in $-q$ and spread it over the surface at a—this takes $-qV = -q(1/4\pi\epsilon_0)(q/a)$. Finally, bring in $+q$ and spread it over the surface at b—this costs nothing, since the net field of the other two charges is zero out there. Thus the energy of the initial configuration is

$$W_i = -\frac{q^2}{4\pi\epsilon_0 a}.$$

The final configuration is neutral shell and a distant point charge—the energy is zero. Thus the work necessary to go from the initial to the final state is

$$W = \boxed{\frac{q^2}{4\pi\epsilon_0 a}.}$$

Problem 2.61



Suppose the n point charges are evenly spaced around the circle, with the jth particle at angle $j(2\pi/n)$. According to Eq. 2.42, the energy of the configuration is

$$W_n = n\frac{1}{2}qV,$$

where V is the potential due to the $(n-1)$ other charges, at charge # n (on the x axis).

$$V = \frac{1}{4\pi\epsilon_0}q\sum_{j=1}^{n-1}\frac{1}{\mathscr{r}_j}, \quad \mathscr{r}_j = 2R\sin\left(\frac{j\pi}{n}\right)$$

(see the figure). So

$$W_n = \frac{q^2}{4\pi\epsilon_0 R}\frac{n}{4}\sum_{j=1}^{n-1}\frac{1}{\sin(j\pi/n)} = \frac{q^2}{4\pi\epsilon_0 R}\Omega_n.$$

Mathematica says

$$\begin{aligned}
\Omega_{10} &= \frac{10}{4}\sum_{j=1}^{9}\frac{1}{\sin(j\pi/10)} = 38.6245 \\
\Omega_{11} &= \frac{11}{4}\sum_{j=1}^{10}\frac{1}{\sin(j\pi/11)} = \boxed{48.5757} \\
\Omega_{12} &= \frac{12}{4}\sum_{j=1}^{11}\frac{1}{\sin(j\pi/12)} = \boxed{59.8074}
\end{aligned}$$

If $(n-1)$ charges are on the circle (energy $\Omega_{n-1}q^2/4\pi\epsilon_0 R$), and the nth is at the center, the total energy is

$$W_n = [\Omega_{n-1} + (n-1)]\,\frac{q^2}{4\pi\epsilon_0 R}.$$

For

$$\begin{aligned}
n = 11: &\qquad \Omega_{10} + 10 = 38.6245 + 10 = \boxed{48.6245} > \Omega_{11} \\
n = 12: &\qquad \Omega_{11} + 11 = 48.5757 + 11 = \boxed{59.5757} < \Omega_{12}
\end{aligned}$$

Thus a lower energy is achieved for 11 charges if they are all at the rim, but for 12 it is better to put one at the center.

Chapter 3

Potential

Problem 3.1

The argument is exactly the same as in Sect. 3.1.4, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = (R - z)$, instead of $(z - R)$. Hence $V_{\text{ave}} = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{2zR}\left[(z+R) - (R-z)\right] = \boxed{\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R}.}$ If there is more than one charge inside the sphere, the average potential due to interior charges is $\dfrac{1}{4\pi\epsilon_0}\dfrac{Q_{\text{enc}}}{R}$, and the average due to exterior charges is V_{center}, so $V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. ✓

Problem 3.2

A stable equilibrium is a point of local minimum in the potential energy. Here the potential energy is qV. But we know that Laplace's equation allows no local minima for V. What *looks* like a minimum, in the figure, must in fact be a saddle point, and the box "leaks" through the center of each face.

Problem 3.3

Laplace's equation in *spherical* coordinates, for V dependent only on r, reads:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \Rightarrow r^2\frac{dV}{dr} = c \text{ (constant)} \;\Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \;\Rightarrow\; \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

In *cylindrical* coordinates: $\nabla^2 V = \dfrac{1}{s}\dfrac{d}{ds}\left(s\dfrac{dV}{ds}\right) = 0 \Rightarrow s\dfrac{dV}{ds} = c \Rightarrow \dfrac{dV}{ds} = \dfrac{c}{s} \quad\Rightarrow\quad \boxed{V = c\ln s + k.}$

Example: potential of a long wire.

Problem 3.4

Refer to Fig. 3.3, letting α be the angle between $\mathbf{\mathscr{r}}$ and the z axis. Obviously, $\mathbf{E}_{\text{ave}}$ points in the $-\hat{\mathbf{z}}$ direction, so

$$\mathbf{E}_{\text{ave}} = \frac{1}{4\pi R^2}\oint \mathbf{E}\,da = -\hat{\mathbf{z}}\frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\cos\alpha\,da.$$

By the law of cosines,

$$R^2 = z^2 + \mathscr{r}^2 - 2\mathscr{r}\,z\cos\alpha \quad\Rightarrow\quad \cos\alpha = \frac{z^2 + \mathscr{r}^2 - R^2}{2\mathscr{r}\,z},$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta \quad\Rightarrow\quad \frac{\cos\alpha}{\mathscr{r}^2} = \frac{z^2 + \mathscr{r}^2 - R^2}{2z\mathscr{r}^3} = \frac{z - R\cos\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}.$$

$$\mathbf{E}_{\text{ave}} = -\hat{\mathbf{z}}\frac{q}{16\pi^2R^2\epsilon_0}\int \frac{z-R\cos\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}R^2\sin\theta\,d\theta\,d\phi$$
$$= -\frac{q\hat{\mathbf{z}}}{8\pi\epsilon_0}\int_0^\pi \frac{z-R\cos\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\sin\theta\,d\theta = -\frac{q\hat{\mathbf{z}}}{8\pi\epsilon_0}\int_{-1}^1 \frac{z-Ru}{(R^2+z^2-2Rzu)^{3/2}}\,du$$

(where $u \equiv \cos\theta$). The integral is

$$I = \frac{1}{R\sqrt{R^2+z^2-2Rzu}}\bigg|_{-1}^{1} - \frac{1}{2Rz^2}\left(\sqrt{R^2+z^2-2Rzu}+\frac{R^2+z^2}{\sqrt{R^2+z^2-2Rzu}}\right)\bigg|_{-1}^{1}$$
$$= \frac{1}{R}\left(\frac{1}{|z-R|}-\frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[|z-R|-(z+R)+(R^2+z^2)\left(\frac{1}{|z-R|}-\frac{1}{z+R}\right)\right].$$

(a) If $z > R$,

$$I = \frac{1}{R}\left(\frac{1}{z-R}-\frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[(z-R)-(z+R)+(R^2+z^2)\left(\frac{1}{z-R}-\frac{1}{z+R}\right)\right]$$
$$= \frac{1}{R}\left(\frac{2R}{z^2-R^2}\right) - \frac{1}{2Rz^2}\left[-2R+(R^2+z^2)\frac{2R}{z^2-R^2}\right] = \frac{2}{z^2}.$$

So

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\,\hat{\mathbf{z}},$$

the same as the field at the center. By superposition the same holds for any *collection* of charges outside the sphere.

(b) If $z < R$,

$$I = \frac{1}{R}\left(\frac{1}{R-z}-\frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[(R-z)-(z+R)+(R^2+z^2)\left(\frac{1}{R-z}-\frac{1}{z+R}\right)\right]$$
$$= \frac{1}{R}\left(\frac{2z}{R^2-z^2}\right) - \frac{1}{2Rz^2}\left[-2z+(R^2+z^2)\frac{2z}{R^2-z^2}\right] = 0.$$

So

$$\mathbf{E}_{\text{ave}} = \mathbf{0}.$$

By superposition the same holds for any *collection* of charges inside the sphere.

Problem 3.5

Same as proof of second uniqueness theorem, up to the equation $\oint_{\mathcal{S}} V_3\mathbf{E}_3\cdot d\mathbf{a} = -\int_{\mathcal{V}}(E_3)^2\,d\tau$. But on each surface, either $V_3 = 0$ (if V is specified on the surface), or else $E_{3_\perp} = 0$ (if $\frac{\partial V}{\partial n} = -E_\perp$ is specified). So $\int_{\mathcal{V}}(E_3)^2 = 0$, and hence $\mathbf{E}_2 = \mathbf{E}_1$. qed

Problem 3.6

Putting $U = T = V_3$ into Green's identity:

$\int_{\mathcal{V}}\left[V_3\nabla^2V_3 + \boldsymbol{\nabla}V_3\cdot\boldsymbol{\nabla}V_3\right]d\tau = \oint_{\mathcal{S}} V_3\boldsymbol{\nabla}V_3\cdot d\mathbf{a}$. But $\nabla^2V_3 = \nabla^2V_1 - \nabla^2V_2 = -\frac{\rho}{\epsilon_0}+\frac{\rho}{\epsilon_0} = 0$, and $\boldsymbol{\nabla}V_3 = -\mathbf{E}_3$.

So $\int_{\mathcal{V}} E_3^2\,d\tau = -\oint_{\mathcal{S}} V_3\mathbf{E}_3\cdot d\mathbf{a}$, and the rest is the same as before.

Problem 3.7

Place image charges $+2q$ at $z = -d$ and $-q$ at $z = -3d$. Total force on $+q$ is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2}+\frac{2q}{(4d)^2}+\frac{-q}{(6d)^2}\right]\hat{\mathbf{z}} = \frac{q^2}{4\pi\epsilon_0 d^2}\left(-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right)\hat{\mathbf{z}} = \boxed{-\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.}$$

Problem 3.8

(a) From Fig. 3.13: $\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta}$; $\mathscr{r}\,' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$. Therefore:

$$\begin{aligned}\frac{q'}{\mathscr{r}\,'} &= -\frac{R}{a}\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} \quad \text{(Eq. 3.15), while } b = \frac{R^2}{a} \text{ (Eq. 3.16).}\\ &= -\frac{q}{\left(\frac{a}{R}\right)\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}} = -\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2+R^2-2ra\cos\theta}}.\end{aligned}$$

Therefore:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}\,'}\right) = \boxed{\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right\}}.$$

Clearly, when $r = R$, $V \to 0$.

(b) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$. Therefore,

$$\begin{aligned}\sigma(\theta) &= -\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left\{-\frac{1}{2}(r^2+a^2-2ra\cos\theta)^{-3/2}(2r-2a\cos\theta)\right.\\ &\quad\left.\left.+\ \frac{1}{2}\left(R^2+(ra/R)^2-2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r-2a\cos\theta\right)\right\}\right|_{r=R}\\ &= -\frac{q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta) + \left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}\\ &= \frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]\\ &= \boxed{\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}.}\end{aligned}$$

$$\begin{aligned}q_{\text{induced}} &= \int\sigma\,da = \frac{q}{4\pi R}(R^2-a^2)\int(R^2+a^2-2Ra\cos\theta)^{-3/2}R^2\sin\theta\,d\theta\,d\phi\\ &= \frac{q}{4\pi R}(R^2-a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2}\right]\Big|_0^\pi\\ &= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right].\end{aligned}$$

But $a > R$ (else q would be *inside*), so $\sqrt{R^2+a^2-2Ra} = a - R$.

$$\begin{aligned}&= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{(a+R)}-\frac{1}{(a-R)}\right] = \frac{q}{2a}\left[(a-R)-(a+R)\right] = \frac{q}{2a}(-2R)\\ &= \boxed{-\frac{qR}{a} = q'.}\end{aligned}$$

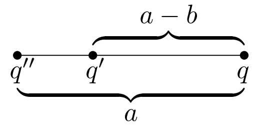
(c) The force on q, due to the sphere, is the same as the force of the image charge q', to wit:

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}.$$

To bring q in from infinity to a, then, we do work

$$W = \frac{q^2 R}{4\pi\epsilon_0}\int_{\infty}^{a} \frac{\bar{a}}{(\bar{a}^2 - R^2)^2}\, d\bar{a} = \frac{q^2 R}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2 - R^2)}\right]\Bigg|_{\infty}^{a} = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2 R}{2(a^2 - R^2)}.}$$

Problem 3.9



Place a second image charge, q'', at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$;

$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$

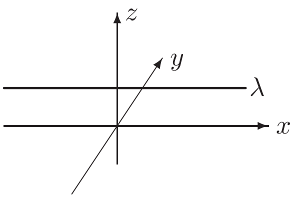
For a *neutral* sphere, $q' + q'' = 0$.

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) = \frac{qq'}{4\pi\epsilon_0}\left(-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right)\\
&= \frac{qq'}{4\pi\epsilon_0}\frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)}{4\pi\epsilon_0}\frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2}\\
&= -\boxed{\frac{q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^2}.}
\end{aligned}$$

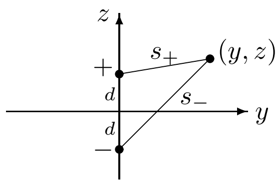
(Drop the minus sign, because the problem asks for the force of *attraction*.)

Problem 3.10

(a) Image problem: λ above, $-\lambda$ below. Potential was found in Prob. 2.52:



$$\begin{aligned}
V(y,z) &= \frac{2\lambda}{4\pi\epsilon_0}\ln(s_-/s_+) = \frac{\lambda}{4\pi\epsilon_0}\ln(s_-^2/s_+^2)\\
&= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}\right\}}
\end{aligned}$$



(b) $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$. Here $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}$, evaluated at $z = 0$.

$$\begin{aligned}
\sigma(y) &= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{y^2 + (z+d)^2}2(z+d) - \frac{1}{y^2 + (z-d)^2}2(z-d)\right\}\Bigg|_{z=0}\\
&= -\frac{2\lambda}{4\pi}\left\{\frac{d}{y^2 + d^2} - \frac{-d}{y^2 + d^2}\right\} = \boxed{-\frac{\lambda d}{\pi(y^2 + d^2)}.}
\end{aligned}$$

Check: Total charge induced on a strip of width l parallel to the y axis:

$$\begin{aligned}
q_{\text{ind}} &= -\frac{l\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{1}{y^2 + d^2}\, dy = -\frac{l\lambda d}{\pi}\left[\frac{1}{d}\tan^{-1}\left(\frac{y}{d}\right)\right]\Bigg|_{-\infty}^{\infty} = -\frac{l\lambda d}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]\\
&= -\lambda l. \quad \text{Therefore } \lambda_{\text{ind}} = -\lambda, \text{ as it should be.}
\end{aligned}$$

In this case $V_0(y) = \left\{ \begin{array}{l} +V_0, \text{ for } 0 < y < a/2 \\ -V_0, \text{ for } a/2 < y < a \end{array} \right\}$. Therefore,

$$\begin{aligned} C_n &= \frac{2}{a} V_0 \left\{ \int_0^{a/2} \sin(n\pi y/a)\, dy - \int_{a/2}^{a} \sin(n\pi y/a)\, dy \right\} = \frac{2V_0}{a} \left\{ - \left. \frac{\cos(n\pi y/a)}{(n\pi/a)} \right|_0^{a/2} + \left. \frac{\cos(n\pi y/a)}{(n\pi/a)} \right|_{a/2}^{a} \right\} \\ &= \frac{2V_0}{n\pi} \left\{ -\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right\} = \frac{2V_0}{n\pi} \left\{ 1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right) \right\}. \end{aligned}$$

The term in curly brackets is:

$$\left\{ \begin{array}{ll} n = 1 & : 1 - 1 - 2\cos(\pi/2) = 0, \\ n = 2 & : 1 + 1 - 2\cos(\pi) = 4, \\ n = 3 & : 1 - 1 - 2\cos(3\pi/2) = 0, \\ n = 4 & : 1 + 1 - 2\cos(2\pi) = 0, \end{array} \right\} \text{etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

Therefore

$$C_n = \left\{ \begin{array}{ll} 8V_0/n\pi, & n = 2, 6, 10, 14, \text{etc. (in general, } 4j+2, \text{ for } j = 0, 1, 2, ...), \\ 0, & \text{otherwise.} \end{array} \right.$$

So

$$\boxed{V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,...} \frac{e^{-n\pi x/a} \sin(n\pi y/a)}{n}} = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a} \sin[(4j+2)\pi y/a]}{(4j+2)}.$$

Problem 3.14

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,...} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a) \quad \text{(Eq. 3.36);} \quad \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad \text{(Eq. 2.49).}$$

So

$$\begin{aligned} \sigma(y) &= -\epsilon_0 \frac{\partial}{\partial x} \left. \left\{ \frac{4V_0}{\pi} \sum \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a) \right\} \right|_{x=0} = -\epsilon_0 \frac{4V_0}{\pi} \sum \frac{1}{n} (-\frac{n\pi}{a}) e^{-n\pi x/a} \sin(n\pi y/a) \Big|_{x=0} \\ &= \boxed{\frac{4\epsilon_0 V_0}{a} \sum_{n=1,3,5,...} \sin(n\pi y/a).} \end{aligned}$$

Or, using the closed form 3.37:

$$\begin{aligned} V(x,y) &= \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)} \right) \Rightarrow \sigma = -\epsilon_0 \frac{2V_0}{\pi} \frac{1}{1 + \frac{\sin^2(\pi y/a)}{\sinh^2(\pi x/a)}} \left(\frac{-\sin(\pi y/a)}{\sinh^2(\pi x/a)} \right) \frac{\pi}{a} \cosh(\pi x/a) \Big|_{x=0} \\ &= \frac{2\epsilon_0 V_0}{a} \left. \frac{\sin(\pi y/a)\cosh(\pi x/a)}{\sin^2(\pi y/a) + \sinh^2(\pi x/a)} \right|_{x=0} = \boxed{\frac{2\epsilon_0 V_0}{a} \frac{1}{\sin(\pi y/a)}.} \end{aligned}$$

[*Comment:* Technically, the series solution for σ is defective, since term-by-term differentiation has produced a (naively) non-convergent sum. More sophisticated definitions of convergence permit one to work with series of this form, but it is better to sum the series *first* and *then* differentiate (the second method.)]

Summation of series Eq. 3.36

$$V(x,y) = \frac{4V_0}{\pi} I, \text{ where } I \equiv \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).$$

Now $\sin w = \mathcal{I}m\left(e^{iw}\right)$, so

$$I = \mathcal{I}m \sum \frac{1}{n} e^{-n\pi x/a} e^{in\pi y/a} = \mathcal{I}m \sum \frac{1}{n} \mathcal{Z}^n,$$

where $\mathcal{Z} \equiv e^{-\pi(x-iy)/a}$. Now

$$\sum_{1,3,5,\ldots} \frac{1}{n} \mathcal{Z}^n = \sum_{j=0}^{\infty} \frac{1}{(2j+1)} \mathcal{Z}^{(2j+1)} = \int_0^{\mathcal{Z}} \left\{ \sum_{j=0}^{\infty} u^{2j} \right\} du$$
$$= \int_0^{\mathcal{Z}} \frac{1}{1-u^2}\, du = \frac{1}{2} \ln\left(\frac{1+\mathcal{Z}}{1-\mathcal{Z}}\right) = \frac{1}{2} \ln\left(Re^{i\,\theta}\right) = \frac{1}{2}\left(\ln R + i\,\theta\right),$$

where $Re^{i\,\theta} = \frac{1+\mathcal{Z}}{1-\mathcal{Z}}$. Therefore

$$I = \mathcal{I}m \left\{ \frac{1}{2}(\ln R + i\,\theta) \right\} = \frac{1}{2}\theta. \quad \text{But } \frac{1+\mathcal{Z}}{1-\mathcal{Z}} = \frac{1+e^{-\pi(x-iy)/a}}{1-e^{-\pi(x-iy)/a}} = \frac{\left(1+e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)}{\left(1-e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)}$$
$$= \frac{1+e^{-\pi x/a}\left(e^{i\pi y/a}-e^{-i\pi y/a}\right)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2} = \frac{1+2ie^{-\pi x/a}\sin(\pi y/a)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2},$$

so

$$\tan\theta = \frac{2e^{-\pi x/a}\sin(\pi y/a)}{1-e^{-2\pi x/a}} = \frac{2\sin(\pi y/a)}{e^{\pi x/a}-e^{-\pi x/a}} = \frac{\sin(\pi y/a)}{\sinh(\pi x/a)}.$$

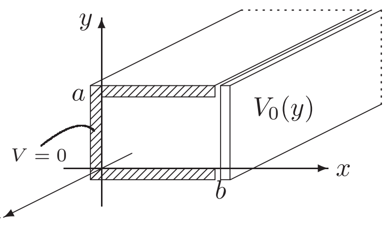
Therefore

$$I = \frac{1}{2}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right), \text{ and } \boxed{V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right).}$$

Problem 3.15

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions

$$\left\{\begin{array}{ll} \text{(i)} & V(x,0)=0, \\ \text{(ii)} & V(x,a)=0, \\ \text{(iii)} & V(0,y)=0, \\ \text{(iv)} & V(b,y)=V_0(y). \end{array}\right\}$$



As in Ex. 3.4, separation of variables yields

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right).$$

Here (i)$\Rightarrow D=0$, (iii)$\Rightarrow B=-A$, (ii)$\Rightarrow ka$ is an integer multiple of π:

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a).}$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y). \text{ Fourier's trick} \Rightarrow C_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

Therefore

$$\boxed{C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.}$$

$$\text{(b) } C_n = \frac{2}{a\sinh(n\pi b/a)}V_0\int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{a\sinh(n\pi b/a)} \times \left\{ \begin{array}{l} 0, \text{ if } n \text{ is even,} \\ \frac{2a}{n\pi}, \text{ if } n \text{ is odd.} \end{array} \right\}$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.}$$

Problem 3.16

Same format as Ex. 3.5, only the boundary conditions are:

$$\left\{ \begin{array}{lll} \text{(i)} & V = 0 & \text{when } x = 0, \\ \text{(ii)} & V = 0 & \text{when } x = a, \\ \text{(iii)} & V = 0 & \text{when } y = 0, \\ \text{(iv)} & V = 0 & \text{when } y = a, \\ \text{(v)} & V = 0 & \text{when } z = 0, \\ \text{(vi)} & V = V_0 & \text{when } z = a. \end{array} \right\}$$

This time we want sinusoidal fuctions in x and y, exponential in z:

$$X(x) = A\sin(kx) + B\cos(kx), \quad Y(y) = C\sin(ly) + D\cos(ly), \quad Z(z) = Ee^{\sqrt{k^2+l^2}z} + Ge^{-\sqrt{k^2+l^2}z}.$$

(i)$\Rightarrow B = 0$; (ii)$\Rightarrow k = n\pi/a$; (iii)$\Rightarrow D = 0$; (iv)$\Rightarrow l = m\pi/a$; (v)$\Rightarrow E + G = 0$. Therefore

$$Z(z) = 2E\sinh(\pi\sqrt{n^2+m^2}z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh(\pi\sqrt{n^2+m^2}z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum\sum\left[C_{n,m}\sinh(\pi\sqrt{n^2+m^2})\right]\sin(n\pi x/a)\sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin(n\pi x/a)\sin(m\pi y/a)\,dx\,dy = \left\{\begin{array}{ll} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if both are odd.}\end{array}\right\}$$

Therefore

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\dots}\sum_{m=1,3,5,\dots}\frac{1}{nm}\sin(n\pi x/a)\sin(m\pi y/a)\frac{\sinh\left(\pi\sqrt{n^2+m^2}z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.}$$

Consider the superposition of *six* such cubes, one with V_0 on each of the six faces. The result is a cube with V_0 on its entire surface, so the potential at the center is V_0. Evidently the potential at the center of the original cube (with V_0 on just one face) is one sixth of this: $\boxed{V_0/6.}$ To check it, put in $x = y = z = a/2$:

$$V(a/2,a/2,a/2) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\dots}\sum_{m=1,3,5,\dots}\frac{1}{nm}\sin(n\pi/2)\sin(m\pi/2)\frac{\sinh\left(\pi\sqrt{n^2+m^2}/2\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.$$

Let $n \equiv 2i+1$, $m \equiv 2j+1$, and note that $\sinh(2u) = 2\sinh(u)\cosh(u)$. The double sum is then

$$S = \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{(2i+1)(2j+1)}\operatorname{sech}\left[\pi\sqrt{(2i+1)^2+(2j+1)^2}/2\right].$$

Setting the upper limits at $i = 3$, $j = 3$ (or above) Mathematica returns $S = 0.102808$, which (to 6 digits) is equal to $\pi^2/96$, confirming (at least, numerically) that $V(a/2,a/2,a/2) = V_0/6$.

Problem 3.17

$$\begin{aligned}
P_3(x) &= \frac{1}{8\cdot 6}\frac{d^3}{dx^3}\left(x^2-1\right)^3 = \frac{1}{48}\frac{d^2}{dx^2}3\left(x^2-1\right)^2 2x = \frac{1}{8}\frac{d^2}{dx^2}x\left(x^2-1\right)^2 \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2 + 2x\left(x^2-1\right)2x\right] = \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(x^2-1+4x^2\right)\right] \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(5x^2-1\right)\right] = \frac{1}{8}\left[2x\left(5x^2-1\right)+\left(x^2-1\right)10x\right] \\
&= \frac{1}{4}\left(5x^3-x+5x^3-5x\right) = \frac{1}{4}\left(10x^3-6x\right) = \boxed{\frac{5}{2}x^3-\frac{3}{2}x.}
\end{aligned}$$

We need to show that $P_3(\cos\theta)$ satisfies

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -l(l+1)P, \text{ with } l = 3,$$

where $P_3(\cos\theta) = \frac{1}{2}\cos\theta\left(5\cos^2\theta - 3\right)$.

$$\begin{aligned}
\frac{dP_3}{d\theta} &= \frac{1}{2}\left[-\sin\theta\left(5\cos^2\theta-3\right)+\cos\theta(10\cos\theta(-\sin\theta)\right] = -\frac{1}{2}\sin\theta\left(5\cos^2\theta-3+10\cos^2\theta\right) \\
&= -\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right).
\end{aligned}$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dP_3}{d\theta}\right) = -\frac{3}{2}\frac{d}{d\theta}\left[\sin^2\theta\left(5\cos^2\theta-1\right)\right] = -\frac{3}{2}\left[2\sin\theta\cos\theta\left(5\cos^2\theta-1\right)+\sin^2\theta\left(-10\cos\theta\sin\theta\right)\right]$$
$$= -3\sin\theta\cos\theta\left[5\cos^2\theta-1-5\sin^2\theta\right].$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -3\cos\theta\left[5\cos^2-1-5\left(1-\cos^2\theta\right)\right] = -3\cos\theta\left(10\cos^2\theta-6\right)$$
$$= -3\cdot 4\cdot\frac{1}{2}\cos\theta\left(5\cos^2\theta-3\right) = -l(l+1)P_3.\quad \text{qed}$$

$$\int_{-1}^{1} P_1(x)P_3(x)\,dx = \int_{-1}^{1}(x)\frac{1}{2}\left(5x^3-3x\right)\,dx = \frac{1}{2}\left(x^5-x^3\right)\Big|_{-1}^{1} = \frac{1}{2}(1-1+1-1) = 0.\ \checkmark$$

Problem 3.18

(a) *Inside:* $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (Eq. 3.66) where

$$A_l = \frac{(2l+1)}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69).}$$

In this case $V_0(\theta) = V_0$ comes outside the integral, so

$$A_l = \frac{(2l+1)V_0}{2R^l}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta.$$

But $P_0(\cos\theta) = 1$, so the integral can be written

$$\int_0^\pi P_0(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{l} 0,\ \text{if } l\neq 0 \\ 2,\ \text{if } l=0\end{array}\right\} \quad \text{(Eq. 3.68).}$$

Therefore

$$A_l = \left\{\begin{array}{ll} 0, & \text{if } l\neq 0 \\ V_0, & \text{if } l=0\end{array}\right\}.$$

Plugging this into the general form:

$$V(r,\theta) = A_0\,r^0 P_0(\cos\theta) = \boxed{V_0.}$$

The potential is *constant throughout the sphere.*

Outside: $V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$ (Eq. 3.72), where

$$B_l = \frac{(2l+1)}{2}R^{l+1}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.73).}$$
$$= \frac{(2l+1)}{2}R^{l+1}V_0\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l\neq 0 \\ RV_0, & \text{if } l=0\end{array}\right\}.$$

Therefore $\boxed{V(r,\theta) = V_0\frac{R}{r}}$ (i.e. equals V_0 at $r = R$, then falls off like $\frac{1}{r}$).

(b)

$$V(r,\theta) = \left\{ \begin{array}{l} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \text{ for } r \le R \text{ (Eq. 3.78)} \\ \sum\limits_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \text{ for } r \ge R \text{ (Eq. 3.79)} \end{array} \right\},$$

where

$$B_l = R^{2l+1} A_l \quad \text{(Eq. 3.81)}$$

and

$$\begin{aligned} A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \quad \text{(Eq. 3.84)} \\ &= \frac{1}{2\epsilon_0 R^{l-1}} \sigma_0 \int_0^\pi P_l(\cos\theta) \sin\theta \, d\theta = \left\{ \begin{array}{ll} 0, & \text{if } l \ne 0 \\ R\sigma_0/\epsilon_0, & \text{if } l = 0 \end{array} \right\}. \end{aligned}$$

Therefore

$$\boxed{V(r,\theta) = \left\{ \begin{array}{ll} \dfrac{R\sigma_0}{\epsilon_0}, & \text{for } r \le R \\ \dfrac{R^2\sigma_0}{\epsilon_0}\dfrac{1}{r}, & \text{for } r \ge R \end{array} \right\}}.$$

Note: in terms of the total charge $Q = 4\pi R^2 \sigma_0$,

$$V(r,\theta) = \left\{ \begin{array}{ll} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R}, & \text{for } r \le R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}, & \text{for } r \ge R \end{array} \right\}.$$

Problem 3.19

$$V_0(\theta) = k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right].$$

(I know that any 3^{rd} order polynomial can be expressed as a linear combination of the first four Legendre polynomials; in this case, since the polynomial is *odd,* I only need P_1 and P_3.)

$$4\cos^3\theta - 3\cos\theta = \alpha\left[\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta + \left(\beta - \frac{3}{2}\alpha\right)\cos\theta,$$

so

$$4 = \frac{5\alpha}{2} \Rightarrow \alpha = \frac{8}{5}; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{3}{2}\cdot\frac{8}{5} = \beta - \frac{12}{5} \Rightarrow \beta = \frac{12}{5} - 3 = -\frac{3}{5}.$$

Therefore

$$V_0(\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right].$$

Now

$$V(r,\theta)=\left\{\begin{array}{l}\sum\limits_{l=0}^{\infty}A_l r^l P_l(\cos\theta),\ \text{ for } r\le R\ \text{ (Eq. 3.66)}\\ \sum\limits_{l=0}^{\infty}\dfrac{B_l}{r^{l+1}}P_l(\cos\theta),\ \text{ for } r\ge R\ \text{ (Eq. 3.72)}\end{array}\right\},$$

where

$$\begin{aligned}A_l &= \frac{(2l+1)}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)}\\ &= \frac{(2l+1)}{2R^l}\frac{k}{5}\left\{8\int_0^\pi P_3(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta-3\int_0^\pi P_1(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta\right\}\\ &= \frac{k}{5}\frac{(2l+1)}{2R^l}\left\{8\frac{2}{(2l+1)}\,\delta_{l3}-3\frac{2}{(2l+1)}\,\delta_{l1}\right\}=\frac{k}{5}\frac{1}{R^l}\left[8\,\delta_{l3}-3\,\delta_{l1}\right]\\ &= \left\{\begin{array}{ll}8k/5R^3, & \text{if } l=3\\ -3k/5R, & \text{if } l=1\end{array}\right\}\text{(zero otherwise).}\end{aligned}$$

Therefore

$$V(r,\theta)=-\frac{3k}{5R}rP_1(\cos\theta)+\frac{8k}{5R^3}r^3P_3(\cos\theta)=\boxed{\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3P_3(\cos\theta)-3\left(\frac{r}{R}\right)P_1(\cos\theta)\right]},$$

or

$$\frac{k}{5}\left\{8\left(\frac{r}{R}\right)^3\frac{1}{2}\left[5\cos^3\theta-3\cos\theta\right]-3\left(\frac{r}{R}\right)\cos\theta\right\}\Rightarrow\boxed{V(r,\theta)=\frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2\left[5\cos^2\theta-3\right]-3\right\}}$$

(for $r\le R$). Meanwhile, $B_l=A_lR^{2l+1}$ (Eq. 3.81—this follows from the continuity of V at R). Therefore

$$B_l=\left\{\begin{array}{ll}8kR^4/5, & \text{if } l=3\\ -3kR^2/5, & \text{if } l=1\end{array}\right\}\quad\text{(zero otherwise).}$$

So

$$V(r,\theta)=\frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta)+\frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta)=\boxed{\frac{k}{5}\left[8\left(\frac{R}{r}\right)^4P_3(\cos\theta)-3\left(\frac{R}{r}\right)^2P_1(\cos\theta)\right]},$$

or

$$\boxed{V(r,\theta)=\frac{k}{5}\left(\frac{R}{r}\right)^2\cos\theta\left\{4\left(\frac{R}{r}\right)^2\left[5\cos^2\theta-3\right]-3\right\}}$$

(for $r \geq R$). Finally, using Eq. 3.83:

$$\begin{aligned}
\sigma(\theta) &= \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) = \epsilon_0\left[3A_1P_1 + 7A_3R^2P_3\right] \\
&= \epsilon_0\left[3\left(-\frac{3k}{5R}\right)P_1 + 7\left(\frac{8k}{5R^3}\right)R^2P_3\right] = \boxed{\frac{\epsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56P_3(\cos\theta)\right]} \\
&= \frac{\epsilon_0 k}{5R}\left[-9\cos\theta + \frac{56}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] = \frac{\epsilon_0 k}{5R}\cos\theta[-9 + 28\cdot 5\cos^2\theta - 28\cdot 3] \\
&= \boxed{\frac{\epsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right].}
\end{aligned}$$

Problem 3.20

Use Eq. 3.83: $\sigma(\theta) = \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)$. But Eq. 3.69 says: $A_l = \dfrac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta$.

Putting them together:

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \quad \text{with } C_l = \int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta. \quad \text{qed}$$

Problem 3.21

Set $V = 0$ on the equatorial plane, far from the sphere. Then the potential is the same as Ex. 3.8 *plus* the potential of a uniformly charged spherical shell:

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.}$$

Problem 3.22

(a) $V(r,\theta) = \sum_{l=0}^{\infty}\dfrac{B_l}{r^{l+1}}P_l(\cos\theta)$ $(r > R)$, so $V(r,0) = \sum_{l=0}^{\infty}\dfrac{B_l}{r^{l+1}}P_l(1) = \sum_{l=0}^{\infty}\dfrac{B_l}{r^{l+1}} = \dfrac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$.

Since $r > R$ in this region, $\sqrt{r^2+R^2} = r\sqrt{1+(R/r)^2} = r\left[1 + \frac{1}{2}(R/r)^2 - \frac{1}{8}(R/r)^4 + \dots\right]$, so

$$\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0}r\left[1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \dots - 1\right] = \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots\right).$$

Comparing like powers of r, I see that $B_0 = \dfrac{\sigma R^2}{4\epsilon_0}$, $B_1 = 0$, $B_2 = -\dfrac{\sigma R^4}{16\epsilon_0}, \dots$. Therefore

$$\boxed{\begin{aligned}
V(r,\theta) &= \frac{\sigma R^2}{4\epsilon_0}\left[\frac{1}{r} - \frac{R^2}{4r^3}P_2(\cos\theta) + \dots\right], \\
&= \frac{\sigma R^2}{4\epsilon_0 r}\left[1 - \frac{1}{8}\left(\frac{R}{r}\right)^2\left(3\cos^2\theta - 1\right) + \dots\right],
\end{aligned}} \quad (\text{for } r > R).$$

(b) $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad (r < R)$. In the northern hemispere, $0 \le \theta \le \pi/2$,

$$V(r,0) = \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

Since $r < R$ in this region, $\sqrt{r^2+R^2} = R\sqrt{1+(r/R)^2} = R\left[1 + \frac{1}{2}(r/R)^2 - \frac{1}{8}(r/R)^4 + \dots\right]$. Therefore

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \dots - r\right].$$

Comparing like powers: $A_0 = \frac{\sigma}{2\epsilon_0}R,\ A_1 = -\frac{\sigma}{2\epsilon_0},\ A_2 = \frac{\sigma}{4\epsilon_0 R}, \dots$, so

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R - rP_1(\cos\theta) + \frac{1}{2R}r^2P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 - \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \dots\right], \end{aligned}} \quad \text{(for } r < R, \text{ northern hemisphere).}$$

In the southern hemisphere we'll have to go for $\theta = \pi$, using $P_l(-1) = (-1)^l$.

$$V(r,\pi) = \sum_{l=0}^{\infty}(-1)^l \overline{A}_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

(I put an overbar on $\overline{A}_l$ to distinguish it from the northern A_l). The only difference is the sign of $\overline{A}_1$: $\overline{A}_1 = +(\sigma/2\epsilon_0),\ \overline{A}_0 = A_0,\ \overline{A}_2 = A_2$. So:

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R + rP_1(\cos\theta) + \frac{1}{2R}r^2P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 + \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \dots\right], \end{aligned}} \quad \text{(for } r < R, \text{ southern hemisphere).}$$

Problem 3.23

$$V(r,\theta) = \left\{\begin{array}{l} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \ (r \le R) \text{ (Eq. 3.78)}, \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \ (r \ge R) \text{ (Eq. 3.79)}, \end{array}\right\}$$

where $B_l = A_l R^{2l+1}$ (Eq. 3.81) and

$$\begin{aligned}
A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta \quad \text{(Eq. 3.84)} \\
&= \frac{1}{2\epsilon_0 R^{l-1}} \sigma_0 \left\{ \int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right\} \quad (\text{let } x = \cos\theta) \\
&= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left\{ \int_0^1 P_l(x)\, dx - \int_{-1}^0 P_l(x)\, dx \right\}.
\end{aligned}$$

Now $P_l(-x) = (-1)^l P_l(x)$, since $P_l(x)$ is even, for even l, and odd, for odd l. Therefore

$$\int_{-1}^0 P_l(x)\, dx = \int_1^0 P_l(-x)\, d(-x) = (-1)^l \int_0^1 P_l(x)\, dx,$$

and hence

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left[1 - (-1)^l\right] \int_0^1 P_l(x)\, dx = \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ \dfrac{\sigma_0}{\epsilon_0 R^{l-1}} \displaystyle\int_0^1 P_l(x)\, dx, & \text{if } l \text{ is odd} \end{array} \right\}.$$

So $A_0 = A_2 = A_4 = A_6 = 0$, and all we need are A_1, A_3, and A_5.

$$\begin{aligned}
\int_0^1 P_1(x)\, dx &= \int_0^1 x\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}. \\
\int_0^1 P_3(x)\, dx &= \frac{1}{2}\int_0^1 \left(5x^3 - 3x\right) dx = \frac{1}{2}\left.\left(5\frac{x^4}{4} - 3\frac{x^2}{2}\right)\right|_0^1 = \frac{1}{2}\left(\frac{5}{4} - \frac{3}{2}\right) = -\frac{1}{8}. \\
\int_0^1 P_5(x)\, dx &= \frac{1}{8}\int_0^1 \left(63x^5 - 70x^3 + 15x\right) dx = \frac{1}{8}\left.\left(63\frac{x^6}{6} - 70\frac{x^4}{4} + 15\frac{x^2}{2}\right)\right|_0^1 \\
&= \frac{1}{8}\left(\frac{21}{2} - \frac{35}{2} + \frac{15}{2}\right) = \frac{1}{16}(36 - 35) = \frac{1}{16}.
\end{aligned}$$

Therefore

$$\boxed{A_1 = \frac{\sigma_0}{\epsilon_0}\left(\frac{1}{2}\right);\ A_3 = \frac{\sigma_0}{\epsilon_0 R^2}\left(-\frac{1}{8}\right);\ A_5 = \frac{\sigma_0}{\epsilon_0 R^4}\left(\frac{1}{16}\right);\ \text{etc.}}$$

and

$$\boxed{B_1 = \frac{\sigma_0}{\epsilon_0}R^3\left(\frac{1}{2}\right);\ B_3 = \frac{\sigma_0}{\epsilon_0}R^5\left(-\frac{1}{8}\right);\ B_5 = \frac{\sigma_0}{\epsilon_0}R^7\left(\frac{1}{16}\right);\ \text{etc.}}$$

Thus

$$\boxed{V(r,\theta)=\left\{\begin{array}{ll}\dfrac{\sigma_0 r}{2\epsilon_0}\left[P_1(\cos\theta)-\dfrac{1}{4}\left(\dfrac{r}{R}\right)^2 P_3(\cos\theta)+\dfrac{1}{8}\left(\dfrac{r}{R}\right)^4 P_5(\cos\theta)+...\right], & (r\le R),\\ \dfrac{\sigma_0 R^3}{2\epsilon_0 r^2}\left[P_1(\cos\theta)-\dfrac{1}{4}\left(\dfrac{R}{r}\right)^2 P_3(\cos\theta)+\dfrac{1}{8}\left(\dfrac{R}{r}\right)^4 P_5(\cos\theta)+...\right], & (r\ge R).\end{array}\right\}}$$

Problem 3.24

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2}=0.$$

Look for solutions of the form $V(s,\phi)=S(s)\Phi(\phi)$:

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2}=0.$$

Multiply by s^2 and divide by $V=S\Phi$:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0.$$

Since the first term involves s only, and the second ϕ only, each is a constant:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)=C_1,\quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=C_2,\quad \text{with } C_1+C_2=0.$$

Now C_2 must be negative (else we get exponentials for Φ, which do not return to their original value—as geometrically they *must*— when ϕ is increased by 2π).

$$C_2=-k^2.\quad \text{Then } \frac{d^2\Phi}{d\phi^2}=-k^2\Phi\Rightarrow\Phi=A\cos k\phi+B\sin k\phi.$$

Moreover, since $\Phi(\phi+2\pi)=\Phi(\phi)$, *$k$ must be an integer:* $k=0,1,2,3,\dots$ (negative integers are just repeats, but $k=0$ must be included, since $\Phi=A$ (a constant) is OK).

$s\dfrac{d}{ds}\left(s\dfrac{dS}{ds}\right)=k^2S$ can be solved by $S=s^n$, provided n is chosen right:

$$s\frac{d}{ds}\left(sns^{n-1}\right)=ns\frac{d}{ds}\left(s^n\right)=n^2ss^{n-1}=n^2s^n=k^2S\Rightarrow n=\pm k.$$

Evidently the general solution is $S(s)=Cs^k+Ds^{-k}$, *unless* $k=0$, in which case we have only *one* solution to a *second-order* equation—namely, $S=$ constant. So we must treat $k=0$ separately. *One* solution is a constant—but what's the other? Go back to the diferential equation for S, and put in $k=0$:

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=0\Rightarrow s\frac{dS}{ds}=\text{constant}=C\Rightarrow\frac{dS}{ds}=\frac{C}{s}\Rightarrow dS=C\frac{ds}{s}\Rightarrow S=C\ln s+D\ \text{(another constant)}.$$

So the second solution in this case is $\ln s$. [How about Φ? That *too* reduces to a single solution, $\Phi=A$, in the case $k=0$. What's the second solution here? Well, putting $k=0$ into the Φ equation:

$$\frac{d^2\Phi}{d\phi^2}=0\Rightarrow\frac{d\Phi}{d\phi}=\text{constant}=B\Rightarrow\Phi=B\phi+A.$$

But a term of the form $B\phi$ is unacceptable, since it does not return to its initial value when ϕ is augmented by 2π.] *Conclusion:* The general solution with cylindrical symmetry is

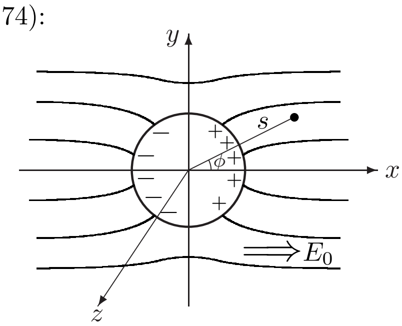
$$\boxed{V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi + b_k \sin k\phi) + s^{-k}(c_k \cos k\phi + d_k \sin k\phi)\right].}$$

Yes: the potential of a line charge goes like $\ln s$, which *is* included.

Problem 3.25

Picking $V = 0$ on the yz plane, with $\mathbf{E_0}$ in the x direction, we have (Eq. 3.74):

$$\left\{\begin{array}{lll} \text{(i)} & V = 0, & \text{when } s = R, \\ \text{(ii)} & V \to -E_0 x = -E_0 s\cos\phi, & \text{for } s \gg R. \end{array}\right\}$$



Evidently $a_0 = b_0 = b_k = d_k = 0$, and $a_k = c_k = 0$ except for $k = 1$:

$$V(s,\phi) = \left(a_1 s + \frac{c_1}{s}\right)\cos\phi.$$

(i)$\Rightarrow$ $c_1 = -a_1 R^2$; (ii)$\rightarrow$ $a_1 = -E_0$. Therefore

$$V(s,\phi) = \left(-E_0 s + \frac{E_0 R^2}{s}\right)\cos\phi, \quad \text{or} \quad \boxed{V(s,\phi) = -E_0 s\left[\left(\frac{R}{s}\right)^2 - 1\right]\cos\phi.}$$

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial s}\right|_{s=R} = -\epsilon_0 E_0 \left(-\frac{R^2}{s^2} - 1\right)\cos\phi\Bigg|_{s=R} = \boxed{2\epsilon_0 E_0 \cos\phi.}$$

Problem 3.26

Inside: $V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k\left(a_k \cos k\phi + b_k \sin k\phi\right)$. (In this region $\ln s$ and s^{-k} are no good—they blow up at $s = 0$.)

Outside: $V(s,\phi) = \overline{a}_0 + \sum_{k=1}^{\infty} \frac{1}{s^k}\left(c_k \cos k\phi + d_k \sin k\phi\right)$. (Here $\ln s$ and s^k are no good at $s \to \infty$).

$$\sigma = -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\Bigg|_{s=R} \quad \text{(Eq. 2.36)}.$$

Thus

$$a\sin 5\phi = -\epsilon_0 \sum_{k=1}^{\infty}\left\{-\frac{k}{R^{k+1}}\left(c_k\cos k\phi + d_k \sin k\phi\right) - kR^{k-1}\left(a_k \cos k\phi + b_k \sin k\phi\right)\right\}.$$

Evidently $a_k = c_k = 0$; $b_k = d_k = 0$ except $k = 5$; $a = 5\epsilon_0\left(\frac{1}{R^6}d_5 + R^4 b_5\right)$. Also, V is continuous at $s = R$: $a_0 + R^5 b_5 \sin 5\phi = \overline{a}_0 + \frac{1}{R^5} d_5 \sin 5\phi$. So $a_0 = \overline{a}_0$ (might as well choose both zero); $R^5 b_5 = R^{-5} d_5$, or $d_5 = R^{10} b_5$.

Combining these results: $a = 5\epsilon_0\left(R^4 b_5 + R^4 b_5\right) = 10\epsilon_0 R^4 b_5$; $b_5 = \dfrac{a}{10\epsilon_0 R^4}$; $d_5 = \dfrac{aR^6}{10\epsilon_0}$. Therefore

$$\boxed{V(s,\phi) = \frac{a\sin 5\phi}{10\epsilon_0}\left\{\begin{array}{l} s^5/R^4, \text{ for } s < R, \\ R^6/s^5, \text{ for } s > R. \end{array}\right\}}$$

Problem 3.27 Since $\mathbf{r}$ is on the z axis, the angle α is just the polar angle θ (I'll drop the primes, for simplicity).

Monopole term:

$$\int \rho\, d\tau = kR\int\left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the r integral is

$$\int_0^R (R-2r)\,dr = \left(Rr - r^2\right)\Big|_0^R = R^2 - R^2 = 0.$$

So the monopole term is zero.

Dipole term:

$$\int r\cos\theta \rho\, d\tau = kR\int (r\cos\theta)\left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2\sin\theta\, dr\, d\theta\, d\phi.$$

But the θ integral is

$$\int_0^\pi \sin^2\theta\cos\theta\, d\theta = \left.\frac{\sin^3\theta}{3}\right|_0^\pi = \frac{1}{3}(0-0) = 0.$$

So the dipole contribution is likewise zero.

Quadrupole term:

$$\int r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho\, d\tau = \frac{1}{2}kR\int r^2\left(3\cos^2\theta - 1\right)\left[\frac{1}{r^2}(R-2r)\sin\theta\right] r^2\sin\theta\, dr\, d\theta\, d\phi.$$

r integral:

$$\int_0^R r^2(R-2r)\,dr = \left.\left(\frac{r^3}{3}R - \frac{r^4}{2}\right)\right|_0^R = \frac{R^4}{3} - \frac{R^4}{2} = -\frac{R^4}{6}.$$

θ integral:

$$\int_0^\pi \underbrace{\left(3\cos^2\theta - 1\right)}_{3(1-\sin^2\theta)-1=2-3\sin^2\theta} \sin^2\theta\, d\theta = 2\int_0^\pi \sin^2\theta\, d\theta - 3\int_0^\pi \sin^4\theta\, d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right) = \pi\left(1 - \frac{9}{8}\right) = -\frac{\pi}{8}.$$

φ integral:

$$\int_0^{2\pi} d\phi = 2\pi.$$

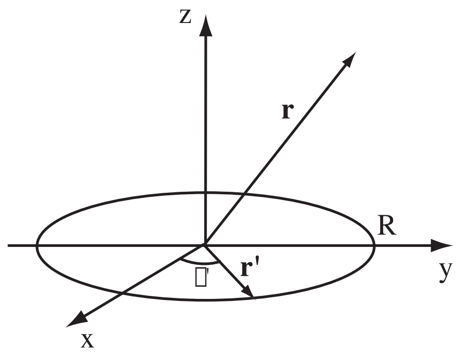
The whole integral is:

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi) = \frac{k\pi^2R^5}{48}.$$

For point P on the z axis ($r \to z$ in Eq. 3.95) the approximate potential is

$$\boxed{V(z) \cong \frac{1}{4\pi\epsilon_0}\frac{k\pi^2 R^5}{48 z^3}.} \quad \text{(Quadrupole.)}$$

Problem 3.28



For a line charge, $\rho(\mathbf{r}')\,d\tau' \to \lambda(\mathbf{r}')\,dl'$, which in this case becomes $\lambda R\,d\phi'$.

$$\begin{aligned}
\mathbf{r} &= r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}},\\
\mathbf{r}' &= R\cos\phi' + R\sin\phi', \quad \text{so}\\
\mathbf{r}\cdot\mathbf{r}' &= rR\sin\theta\cos\phi\cos\phi' + rR\sin\theta\sin\phi\sin\phi' = rR\cos\alpha,\\
\cos\alpha &= \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi').
\end{aligned}$$

$n = 0:$

$$\int \rho(\mathbf{r}')\,d\tau' \to \lambda R\int_0^{2\pi} d\phi' = 2\pi R\lambda; \quad V_0 = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda}{r} = \boxed{\frac{\lambda}{2\epsilon_0}\frac{R}{r}.}$$

$n = 1:$

$$\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' \to \int R\cos\alpha\,\lambda R\,d\phi' = \lambda R^2\sin\theta\int_0^{2\pi}(\cos\phi\cos\phi' + \sin\phi\sin\phi')d\phi' = 0; V_1 = \boxed{0.}$$

$n = 2:$

$$\int (r')^2 P_2(\cos\alpha)\,\rho(\mathbf{r}')d\tau' \to \int R^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\lambda R d\phi' = \frac{\lambda R^3}{2}\int\left[3\sin^2\theta\,(\cos\phi\cos\phi' + \sin\phi\sin\phi')^2 - 1\right]d\phi'$$

$$= \frac{\lambda R^3}{2}\left[3\sin^2\theta\left(\cos^2\phi\int_0^{2\pi}\cos^2\phi'\,d\phi' + \sin^2\phi\int_0^{2\pi}\sin^2\phi'\,d\phi' + 2\sin\phi\cos\phi\int_0^{2\pi}\sin\phi'\cos\phi'\,d\phi'\right) - \int_0^{2\pi}d\phi'\right]$$

$$= \frac{\lambda R^3}{2}\left[3\sin^2\theta\left(\pi\cos^2\phi + \pi\sin^2\phi + 0\right) - 2\pi\right] = \frac{\pi\lambda R^3}{2}\left(3\sin^2\theta - 2\right) = -\pi\lambda R^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

So

$$V_2 = \boxed{-\frac{\lambda}{8\epsilon_0}\frac{R^3}{r^3}\left(3\cos^2\theta - 1\right) = -\frac{\lambda}{4\epsilon_0}\frac{R^3}{r^3}P_2(\cos\theta).}$$

Problem 3.29

$\mathbf{p} = (3qa - qa)\,\hat{\mathbf{z}} + (-2qa - 2q(-a))\,\hat{\mathbf{y}} = 2qa\,\hat{\mathbf{z}}$. Therefore

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

and $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, so

$$V \cong \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

Problem 3.30

(a) By symmetry, $\mathbf{p}$ is clearly in the z direction: $\mathbf{p} = p\,\hat{\mathbf{z}}$; $p = \int z\rho\,d\tau \Rightarrow \int z\sigma\,da$.

$$p = \int (R\cos\theta)(k\cos\theta)R^2\sin\theta\,d\theta\,d\phi = 2\pi R^3 k\int_0^\pi \cos^2\theta\sin\theta\,d\theta = 2\pi R^3 k\left(-\frac{\cos^3\theta}{3}\right)\Big|_0^\pi$$

$$= \frac{2}{3}\pi R^3 k[1-(-1)] = \frac{4\pi R^3 k}{3}; \quad \boxed{\mathbf{p} = \frac{4\pi R^3 k}{3}\,\hat{\mathbf{z}}.}$$

(b)

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k}{3}\frac{\cos\theta}{r^2} = \boxed{\frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

This is *also* the *exact* potential. *Conclusion:* all multiple moments of this distribution (except the dipole) are exactly zero.

Problem 3.31

Using Eq. 3.94 with $r' = d/2$ and $\alpha = \theta$ (Fig. 3.26):

$$\frac{1}{\mathscr{r}_+} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(\cos\theta);$$

for $\mathscr{r}_-$, we let $\theta \to 180^\circ + \theta$, so $\cos\theta \to -\cos\theta$:

$$\frac{1}{\mathscr{r}_-} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(-\cos\theta).$$

But $P_n(-x) = (-1)^n P_n(x)$, so

$$V = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) = \frac{1}{4\pi\epsilon_0}q\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n[P_n(\cos\theta) - P_n(-\cos\theta)] = \frac{2q}{4\pi\epsilon_0 r}\sum_{n=1,3,5,\dots}\left(\frac{d}{2r}\right)^n P_n(\cos\theta).$$

Therefore

$$V_{\text{dip}} = \frac{2q}{4\pi\epsilon_0}\frac{1}{r}\frac{d}{2r}P_1(\cos\theta) = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}, \quad \text{while} \quad \boxed{V_{\text{quad}} = 0.}$$

$$V_{\text{oct}} = \frac{2q}{4\pi\epsilon_0 r}\left(\frac{d}{2r}\right)^3 P_3(\cos\theta) = \frac{2q}{4\pi\epsilon_0}\frac{d^3}{8r^4}\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \boxed{\frac{qd^3}{4\pi\epsilon_0}\frac{1}{8r^4}\left(5\cos^3\theta - 3\cos\theta\right).}$$

Problem 3.32

(a) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right] = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right].}$

(b) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right].}$

(c) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{y}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$ (from Eq. 1.64, $\hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = \sin\theta\sin\phi$).

Problem 3.33

(a) This point is at $r = a$, $\theta = \frac{\pi}{2}$, $\phi = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\,\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{\mathbf{z}})$; $\mathbf{F} = q\mathbf{E} = \boxed{-\frac{pq}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}.}$

(b) Here $r = a$, $\theta = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{\mathbf{r}}) = \frac{2p}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}$. $\boxed{\mathbf{F} = \frac{2pq}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}.}$

(c) $W = q\,[V(0,0,a) - V(a,0,0)] = \frac{qp}{4\pi\epsilon_0 a^2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right] = \boxed{\frac{pq}{4\pi\epsilon_0 a^2}.}$

Problem 3.34

$Q = -q$, so $V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$; $\mathbf{p} = qa\,\hat{\mathbf{z}}$, so $V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$. Therefore

$$\boxed{V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right).} \quad \boxed{\mathbf{E}(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\,\hat{\mathbf{r}} + \frac{a}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right].}$$

Problem 3.35 The total charge is zero, so the dominant term is the dipole. We need the dipole moment of this configuration. It obviously points in the z direction, and for the southern hemisphere ($\theta : \frac{\pi}{2} \to \pi$) ρ switches sign but so does z, so

$$\begin{aligned} p &= \int z\rho\,d\tau = 2\rho_0 \int_{\theta=0}^{\pi/2} r\cos\theta\, r^2\sin\theta\,dr\,d\theta\,d\phi = 2\rho_0(2\pi)\int_0^R r^3\,dr \int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \\ &= 4\pi\rho_0 \frac{R^4}{4}\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{\pi\rho_0 R^4}{2}. \end{aligned}$$

Therefore (Eq. 3.103)

$$\mathbf{E} \approx \boxed{\frac{\pi\rho_0 R^4}{8\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right).}$$

Problem 3.36

$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 3.36). So $3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}} = 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$. So Eq. 3.104 $\equiv$ Eq. 3.103. ✓

Problem 3.37

$V_{\text{ave}}(R) = \frac{1}{4\pi R^2}\int V(\mathbf{r})\,da$, where the integral is over the surface of a sphere of radius R. Now $da =$

$R^2 \sin\theta \, d\theta \, d\phi$, so $V_{\text{ave}}(R) = \dfrac{1}{4\pi} \displaystyle\int V(R,\theta,\phi) \sin\theta \, d\theta \, d\phi$.

$$\begin{aligned}
\frac{dV_{\text{ave}}}{dR} &= \frac{1}{4\pi} \int \frac{\partial V}{\partial R} \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi} \int (\nabla V \cdot \hat{\mathbf{r}}) \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi R^2} \int (\nabla V) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) \\
&= \frac{1}{4\pi R^2} \int (\nabla V) \cdot d\mathbf{a} = \frac{1}{4\pi R^2} \int (\nabla^2 V) \, d\tau = 0.
\end{aligned}$$

(The final integral, from the divergence theorem, is over the *volume* of the sphere, where by assumption the Laplacian of V is zero.) So V_{ave} is independent of R—the same for all spheres, regardless of their radius—and hence (taking the limit as $R \to 0$), $V_{\text{ave}}(R) = V(0)$. qed

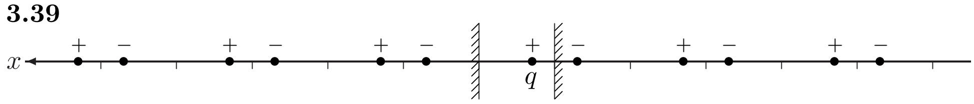
Problem 3.38 At a point (x, y) on the plane the field of q is

$$\mathbf{E}_q = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^3} \hat{\mathscr{r}}, \quad \text{and} \quad \mathscr{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} - d\,\hat{\mathbf{z}},$$

so its z component is $-\dfrac{q}{4\pi\epsilon_0} \dfrac{d}{(x^2+y^2+d^2)^{3/2}}$. Meanwhile, the field of σ (just below the surface) is $-\dfrac{\sigma}{2\epsilon_0}$, (Eq. 2.17). (Of course, this is for a *uniform* surface charge, but as long as we are infinitesimally far away σ is *effectively* uniform.) The total field inside the conductor is zero, so

$$-\frac{q}{4\pi\epsilon_0} \frac{d}{(x^2+y^2+d^2)^{3/2}} - \frac{\sigma}{2\epsilon_0} = 0 \quad \Rightarrow \quad \sigma(x,y) = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \quad \checkmark$$

Problem 3.39



The image configuration is shown in the figure; the positive image charge forces cancel in pairs. The net force of the negative image charges is:

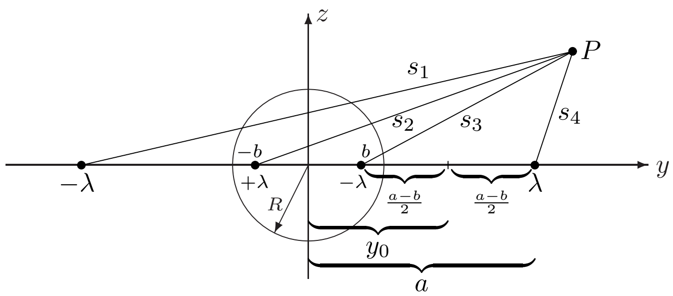
$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q^2 \left\{ \frac{1}{[2(a-x)]^2} + \frac{1}{[2a+2(a-x)]^2} + \frac{1}{[4a+2(a-x)]^2} + \dots \right. \\
&\quad \left. - \frac{1}{(2x)^2} - \frac{1}{(2a+2x)^2} - \frac{1}{(4a+2x)^2} - \dots \right\} \\
&= \boxed{\frac{1}{4\pi\epsilon_0} \frac{q^2}{4} \left\{ \left[\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \dots \right] - \left[\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \dots \right] \right\}}.
\end{aligned}$$

When $a \to \infty$ (i.e. $a \gg x$) only the $\dfrac{1}{x^2}$ term survives: $F = -\dfrac{1}{4\pi\epsilon_0} \dfrac{q^2}{(2x)^2}$ ✓ (same as for only *one* plane—Eq. 3.12). When $x = a/2$,

$$F = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4} \left\{ \left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots \right] - \left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots \right] \right\} = 0. \ \checkmark$$

Problem 3.40

Following Prob. 2.52, we place image line charges $-\lambda$ at $y = b$ and $+\lambda$ at $y = -b$ (here y is the horizontal axis, z vertical).



In the solution to Prob. 2.52 substitute:

$$a \to \frac{a-b}{2},\ y_0 \to \frac{a+b}{2} \text{ so } \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - R^2 \Rightarrow b = \frac{R^2}{a}.$$

$$\begin{aligned} V &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{s_3^2}{s_4^2}\right) + \ln\left(\frac{s_1^2}{s_2^2}\right)\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_1^2 s_3^2}{s_4^2 s_2^2}\right) \\ &= \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{[(y+a)^2+z^2][(y-b)^2+z^2]}{[(y-a)^2+z^2][(y+b)^2+z^2]}\right\}, \quad \text{or, using } y = s\cos\phi,\ z = s\sin\phi, \\ &= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(s^2+a^2+2as\cos\phi)[(as/R)^2+R^2-2as\cos\phi]}{(a^2+a^2-2as\cos\phi)[(as/R)^2+R^2+2as\cos\phi]}\right\}.} \end{aligned}$$

Problem 3.41 Same as Problem 3.9, only this time we want $q' + q'' = q$, so $q'' = q - q'$:

$$F = \frac{q}{4\pi\epsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) = \frac{q^2}{4\pi\epsilon_0 a^2} + \frac{qq'}{4\pi\epsilon_0}\left(-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right).$$

The second term is identical to Problem 3.9, and I'll just quote the answer from there:

$$F = \frac{q^2}{4\pi\epsilon_0 a^3}\left[a - R^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}\right].$$

(a) $F = 0 \Rightarrow a(a^2-R^2)^2 = R^3(2a^2-R^2)$, or (letting $x \equiv a/R$), $x(x^2-1)^2 - 2x^2 + 1 = 0$. We want a real root greater than 1; Mathematica delivers $x = (1+\sqrt{5})/2 = 1.61803$, so $a = 1.61803R = \boxed{5.66311\,\text{Å}.}$

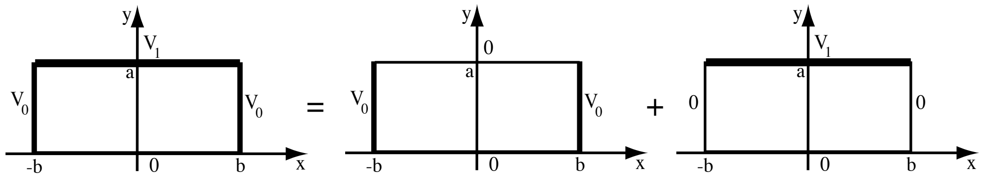
(b) Let $a_0 = x_0 R$ be the minimum value of a. The work necessary is

$$\begin{aligned} W &= -\int_\infty^{a_0} F\,da = \frac{q^2}{4\pi\epsilon_0}\int_{a_0}^\infty \frac{1}{a^3}\left[a - R^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}\right]da = \frac{q^2}{4\pi\epsilon_0 R}\int_{x_0}^\infty\left[\frac{1}{x^2} - \frac{(2x^2-1)}{x^3(x^2-1)^2}\right]dx \\ &= \frac{q^2}{4\pi\epsilon_0 R}\left[\frac{1+2x_0-2x_0^3}{2x_0^2(1-x_0^2)}\right]. \end{aligned}$$

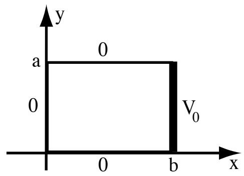
Putting in $x_0 = (1+\sqrt{5})/2$, Mathematica says the term in square brackets is 1/2 (this is not an accident; see footnote 6 on page 127), so $W = \dfrac{q^2}{8\pi\epsilon_0 R}$. Numerically,

$$W = \frac{(1.60\times10^{-19})^2}{8\pi(8.85\times10^{-12})(5.66\times10^{-10})}\,\text{J} = 2.03\times10^{-19}\,\text{J} = \boxed{1.27\,\text{eV}.}$$

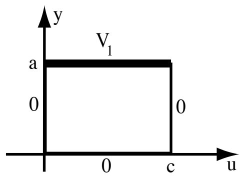
Problem 3.42



The first configuration on the right is precisely Example 3.4, but unfortunately the second configuration is not the same as Problem 3.15:



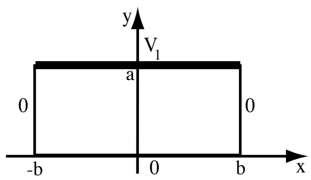
We could reconstruct Problem 3.15 with the modified boundaries, but let's see if we can't twist it around by an astute change of variables. Suppose we let $x \to y$, $y \to u$, $a \to c$, $b \to a$, and $V_0 \to V_1$:



This is closer; making the changes in the solution to Problem 3.15 we have (for this configuration)

$$V(u, y) = \frac{4V_1}{\pi} \sum_{n=1,3,5\ldots} \frac{\sinh(n\pi y/c)\sin(n\pi u/c)}{n\sinh(n\pi a/c)}.$$

Now let $c \to 2b$ and $u \to x + b$, and the configuration is just what we want:



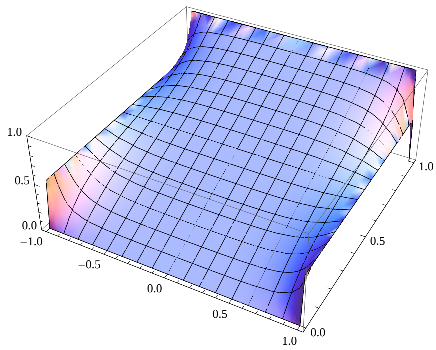
The potential for this configuration is

$$V(x, y) = \frac{4V_1}{\pi} \sum_{n=1,3,5\ldots} \frac{\sinh(n\pi y/2b)\sin(n\pi(x + b)/2b)}{n\sinh(n\pi a/2b)}.$$

(If you like, write $\sin(n\pi(x+b)/2b)$ as $(-1)^{(n-1)/2}\cos(n\pi x/2b)$.) Combining this with Eq. 3.42,

$$V(x,y) = \boxed{\frac{4}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}\left[V_0\frac{\cosh(n\pi x/a)\sin(n\pi y/a)}{\cosh(n\pi b/a)} + V_1\frac{\sinh(n\pi y/2b)\sin(n\pi(x+b)/2b)}{\sinh(n\pi a/2b)}\right]}.$$

Here's a plot of this function, for the case $a = b = 1,\ V_0 = 1/2,\ V_1 = 1$:



Problem 3.43

Since the configuration is azimuthally symmetric, $V(r,\theta) = \sum\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$.

(a) $r > b$: $\quad A_l = 0$ for all l, since $V \to 0$ at ∞. Therefore $V(r,\theta) = \sum \frac{B_l}{r^{l+1}}P_l(\cos\theta)$.

$a < r < b:\quad V(r,\theta) = \sum\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta).\qquad r < a:\quad V(r,\theta) = V_0.$

We need to determine B_l, C_l, D_l, and V_0. To do this, invoke boundary conditions as follows: (i) V is continuous at a, (ii) V is continuous at b, (iii) $\triangle\left(\frac{\partial V}{\partial r}\right) = -\frac{1}{\epsilon_0}\sigma(\theta)$ at b.

(ii) $\Rightarrow \sum\frac{B_l}{b^{l+1}}P_l(\cos\theta) = \sum\left(C_l b^l + \frac{D_l}{b^{l+1}}\right)P_l(\cos\theta);\quad \frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}} \Rightarrow \boxed{B_l = b^{2l+1}C_l + D_l.}$ (1)

(i) $\Rightarrow \sum\left(C_l a^l + \frac{D_l}{a^{l+1}}\right)P_l(\cos\theta) = V_0;\ \left\{\begin{array}{l} C_l a^l + \frac{D_l}{a^{l+1}} = 0,\ \text{if } l \neq 0, \\ C_0 a^0 + \frac{D_0}{a^1} = V_0,\ \text{if } l = 0; \end{array}\right\}\ \boxed{\begin{array}{l} D_l = -a^{2l+1}C_l,\ l \neq 0, \\ D_0 = aV_0 - aC_0. \end{array}}$ (2)

Putting (2) into (1) gives $B_l = b^{2l+1}C_l - a^{2l+1}C_l,\ l \neq 0,\quad B_0 = bC_0 + aV_0 - aC_0$. Therefore

$$\boxed{\begin{array}{l} B_l = \left(b^{2l+1} - a^{2l+1}\right)C_l,\ l \neq 0, \\ B_0 = (b-a)C_0 + aV_0. \end{array}}\ (1')$$

(iii) $\Rightarrow \sum B_l[-(l+1)]\frac{1}{b^{l+2}}P_l(\cos\theta) - \sum\left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right)P_l(\cos\theta) = \frac{-k}{\epsilon_0}P_1(\cos\theta)$. So

$$-\frac{(l+1)}{b^{l+2}}B_l - \left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right) = 0,\ \text{if } l \neq 1;$$

or

$$-(l+1)B_l - lC_l b^{2l+1} + (l+1)D_l = 0;\quad (l+1)(B_l - D_l) = -lb^{2l+1}C_l.$$

$$B_1(+2)\frac{1}{b^2} + \left(C_1 + D_1\frac{-2}{b^2}\right) = \frac{k}{\epsilon_0}, \text{ for } l = 1; \quad C_1 + \frac{2}{b^3}(B_1 - D_1) = k.$$

Therefore

$$\boxed{\begin{array}{l}(l+1)(B_l - D_l) + lb^{2l+1}C_l = 0, \text{for } l \neq 1, \\ C_1 + \dfrac{2}{b^3}(B_1 - D_1) = \dfrac{k}{\epsilon_0}.\end{array}} \quad (3)$$

Plug (2) and (1′) into (3):

For $l \neq 0$ or 1:

$(l+1)\left[\left(b^{2l+1} - a^{2l+1}\right)C_l + a^{2l+1}C_l\right] + lb^{2l+1}C_l = 0; \quad (l+1)b^{2l+1}C_l + lb^{2l+1}C_l = 0;\ (2l+1)C_l = 0 \Rightarrow C_l = 0.$

Therefore (1′) and (2) $\Rightarrow$ $\boxed{B_l = C_l = D_l = 0 \text{ for } l > 1.}$

For $l = 1$: $C_1 + \dfrac{2}{b^3}\left[\left(b^3 - a^3\right)C_1 + a^3C_1\right] = k$; $C_1 + 2C_1 = k \Rightarrow \boxed{C_1 = k/3\epsilon_0;}$ $D_1 = -a^3C_1 \Rightarrow$ $\boxed{D_1 = -a^3k/3\epsilon_0;}$ $B_1 = \left(b^3 - a^3\right)C_1 \Rightarrow \boxed{B_1 = \left(b^3 - a^3\right)k/3\epsilon_0.}$

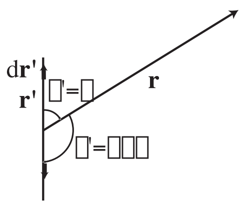
For $l = 0$: $B_0 - D_0 = 0 \Rightarrow B_0 = D_0 \Rightarrow (b-a)C_0 + aV_0 = aV_0 - aC_0$, so $bC_0 = 0 \Rightarrow$ $\boxed{C_0 = 0;\ D_0 = aV_0 = B_0.}$

Conclusion: $\boxed{V(r,\theta) = \dfrac{aV_0}{r} + \dfrac{\left(b^3 - a^3\right)k}{3r^2\epsilon_0}\cos\theta,}$ $r \geq b$. $\boxed{V(r,\theta) = \dfrac{aV_0}{r} + \dfrac{k}{3\epsilon_0}\left(r - \dfrac{a^3}{r^2}\right)\cos\theta,}$ $a \leq r \leq b$.

(b)$\sigma_i(\theta) = -\epsilon_0 \left.\dfrac{\partial V}{\partial r}\right|_a = -\epsilon_0\left[-\dfrac{aV_0}{a^2} + \dfrac{k}{3\epsilon_0}\left(1 + 2\dfrac{a^3}{a^3}\right)\cos\theta\right] = -\epsilon_0\left(-\dfrac{V_0}{a} + \dfrac{k}{\epsilon_0}\cos\theta\right) = \boxed{-k\cos\theta + V_0\dfrac{\epsilon_0}{a}.}$

(c)$q_i = \displaystyle\int \sigma_i\, da = \dfrac{V_0\epsilon_0}{a}4\pi a^2 = \boxed{4\pi a\epsilon_0 V_0 = Q_{\text{tot}}.}$ At large r: $V \approx \dfrac{aV_0}{r} \overset{?}{=} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r} = \dfrac{1}{4\pi\epsilon_0}\dfrac{4\pi a\epsilon_0 V_0}{r} = \dfrac{aV_0}{r}$. ✓

Problem 3.44



Use multipole expansion (Eq. 3.95): $\rho\, d\tau' \to \lambda\, dr'$; $\lambda = \dfrac{Q}{2a}$; the r' integral breaks into two pieces:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\left[\int_0^a (r')^n P_n(\cos\theta')\lambda\, dr' + \int_0^a (r')^n P_n(\cos\theta')\lambda\, dr'\right].$$

In the first integral $\theta' = \theta$ (see diagram); in the second integral $\theta' = \pi - \theta$, so $\cos\theta' = -\cos\theta$. But $P_n(-z) = (-1)^n P_n(z)$, so the integrals cancel when n is odd, and add when n is even.

$$V(\mathbf{r}) = 2\frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\sum_{n=0,2,4,\ldots}^{\infty}\frac{1}{r^{n+1}}P_n(\cos\theta)\int_0^a x^n\, dx.$$

The integral is $\dfrac{a^{n+1}}{n+1}$, so

$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0,2,4,\ldots}\left[\frac{1}{n+1}\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right].$$

Problem 3.45

Use separation of variables in cylindrical coordinates (Prob. 3.24):

$$V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} \left[s^k (a_k \cos k\phi + b_k \sin k\phi) + s^{-k}(c_k \cos k\phi + d_k \sin k\phi) \right].$$

$s < R:\ V(s,\phi) = \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi)$ ($\ln s$ and s^{-k} blow up at $s = 0$);
$s > R:\ V(s,\phi) = \sum_{k=1}^{\infty} s^{-k} (c_k \cos k\phi + d_k \sin k\phi)$ ($\ln s$ and s^{k} blow up as $s \to \infty$).

(We may as well pick constants so $V \to 0$ as $s \to \infty$, and hence $a_0 = 0$.) Continuity at $s = R \Rightarrow$ $\sum R^k (a_k \cos k\phi + b_k \sin k\phi) = \sum R^{-k} (c_k \cos k\phi + d_k \sin k\phi)$, so $c_k = R^{2k} a_k,\ d_k = R^{2k} b_k$. Eq. 2.36 says: $\left.\frac{\partial V}{\partial s}\right|_{R^+} - \left.\frac{\partial V}{\partial s}\right|_{R^-} = -\frac{1}{\epsilon_0}\sigma$. Therefore

$$\sum \frac{-k}{R^{k+1}} (c_k \cos k\phi + d_k \sin k\phi) - \sum k R^{k-1} (a_k \cos k\phi + b_k \sin k\phi) = -\frac{1}{\epsilon_0}\sigma,$$

or:

$$\sum 2k R^{k-1} (a_k \cos k\phi + b_k \sin k\phi) = \left\{ \begin{array}{ll} \sigma_0/\epsilon_0 & (0 < \phi < \pi) \\ -\sigma_0/\epsilon_0 & (\pi < \phi < 2\pi) \end{array} \right\}.$$

Fourier's trick: multiply by $(\cos l\phi)\, d\phi$ and integrate from 0 to 2π, using

$$\int_0^{2\pi} \sin k\phi \cos l\phi \, d\phi = 0; \quad \int_0^{2\pi} \cos\ k\phi \cos l\phi \, d\phi = \left\{ \begin{array}{l} 0,\ k \neq l \\ \pi,\ k = l \end{array} \right\}.$$

Then

$$2lR^{l-1}\pi a_l = \frac{\sigma_0}{\epsilon_0} \left[\int_0^{\pi} \cos l\phi \, d\phi - \int_{\pi}^{2\pi} \cos l\phi \, d\phi \right] = \frac{\sigma_0}{\epsilon_0} \left\{ \left.\frac{\sin l\phi}{l}\right|_0^{\pi} - \left.\frac{\sin l\phi}{l}\right|_{\pi}^{2\pi} \right\} = 0; \quad a_l = 0.$$

Multiply by $(\sin l\phi)\, d\phi$ and integrate, using $\int_0^{2\pi} \sin\ k\phi \sin l\phi \, d\phi = \left\{ \begin{array}{l} 0,\ k \neq l \\ \pi,\ k = l \end{array} \right\}$:

$$\begin{aligned} 2lR^{l-1}\pi b_l &= \frac{\sigma_0}{\epsilon_0} \left[\int_0^{\pi} \sin l\phi \, d\phi - \int_{\pi}^{2\pi} \sin l\phi \, d\phi \right] = \frac{\sigma_0}{\epsilon_0} \left\{ -\left.\frac{\cos l\phi}{l}\right|_0^{\pi} + \left.\frac{\cos l\phi}{l}\right|_{\pi}^{2\pi} \right\} = \frac{\sigma_0}{l\epsilon_0}(2 - 2\cos l\pi) \\ &= \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ 4\sigma_0/l\epsilon_0, & \text{if } l \text{ is odd} \end{array} \right\} \Rightarrow b_l = \left\{ \begin{array}{ll} 0, & \text{if } l \text{ is even} \\ 2\sigma_0/\pi\epsilon_0 l^2 R^{l-1}, & \text{if } l \text{ is odd} \end{array} \right\}. \end{aligned}$$

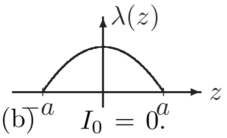
Conclusion:

$$\boxed{V(s,\phi) = \frac{2\sigma_0 R}{\pi\epsilon_0} \sum_{k=1,3,5,\dots} \frac{1}{k^2} \sin k\phi \left\{ \begin{array}{ll} (s/R)^k & (s < R) \\ (R/s)^k & (s > R) \end{array} \right\}.}$$

Problem 3.46

Use Eq. 3.95, in the form $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{r^{n+1}} I_n; \quad I_n = \int_{-a}^{a} z^n \lambda(z)\, dz.$

(a) $I_0 = k \int\limits_{-a}^{a} \cos\left(\frac{\pi z}{2a}\right) dz = k\left[\frac{2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)\right]\Bigg|_{-a}^{a} = \frac{2ak}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{4ak}{\pi}$. Therefore:

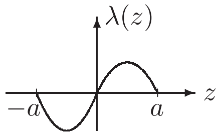


$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(\frac{4ak}{\pi}\right)\frac{1}{r}.}\quad \text{(Monopole.)}$$

(b) $I_0 = 0.$

$$I_1 = k\int\limits_{-a}^{a} z\sin(\pi z/a)\,dz = k\left\{\left(\frac{a}{\pi}\right)^2 \sin\left(\frac{\pi z}{a}\right) - \frac{az}{\pi}\cos\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a}$$

$$= k\left\{\left(\frac{a}{\pi}\right)^2[\sin(\pi) - \sin(-\pi)] - \frac{a^2}{\pi}\cos(\pi) - \frac{a^2}{\pi}\cos(-\pi)\right\} = k\frac{2a^2}{\pi};$$

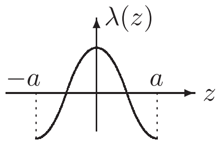


$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(\frac{2a^2k}{\pi}\right)\frac{1}{r^2}\cos\theta.}\quad \text{(Dipole.)}$$

(c) $I_0 = I_1 = 0.$

$$I_2 = k\int\limits_{-a}^{a} z^2\cos\left(\frac{\pi z}{a}\right) dz = k\left\{\frac{2z\cos(\pi z/a)}{(\pi/a)^2} + \frac{(\pi z/a)^2 - 2}{(\pi/a)^3}\sin\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a}$$

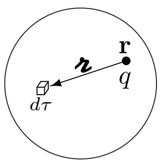
$$= 2k\left(\frac{a}{\pi}\right)^2[a\cos(\pi) + a\cos(-\pi)] = -\frac{4a^3k}{\pi^2}.$$



$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(-\frac{4a^3k}{\pi^2}\right)\frac{1}{2r^3}\left(3\cos^2\theta - 1\right).}\quad \text{(Quadrupole.)}$$

Problem 3.47

(a) The average field due to a point charge q at $\mathbf{r}$ is



$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{E}\,d\tau, \quad \text{where } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}},$$

$$\text{so}\quad \mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\frac{1}{4\pi\epsilon_0}\int q\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau.$$

(Here $\mathbf{r}$ is the source point, $d\tau$ is the field point, so $\mathscr{r}$ goes from $\mathbf{r}$ to $d\tau$.) The field at $\mathbf{r}$ due to uniform charge ρ over the sphere is $\mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau$. This time $d\tau$ is the source point and $\mathbf{r}$ is the field point, so $\mathscr{r}$ goes from $d\tau$ to $\mathbf{r}$, and hence carries the opposite sign. So with $\rho = -q/\left(\frac{4}{3}\pi R^3\right)$, the two expressions agree: $\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho$.

(b) From Prob. 2.12:

$$\mathbf{E}_\rho = \frac{1}{3\epsilon_0}\rho\mathbf{r} = -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}.$$

(c) If there are many charges inside the sphere, $\mathbf{E}_{\text{ave}}$ is the sum of the individual averages, and $\mathbf{p}_{\text{tot}}$ is the sum of the individual dipole moments. So $\mathbf{E}_{\text{ave}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$. qed

(d) The same argument, only with q placed at $\mathbf{r}$ *outside* the sphere, gives

$$\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{4}{3}\pi R^3\rho\right)}{r^2}\,\hat{\mathbf{r}} \quad \text{(field at } \mathbf{r} \text{ due to uniformly charged sphere)} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\,\hat{\mathbf{r}}.$$

But this is precisely the field produced by q (at $\mathbf{r}$) at the *center* of the sphere. So the average field (over the sphere) due to a point charge *outside* the sphere is the same as the field that same charge produces at the center. And by superposition, this holds for any *collection* of exterior charges.

Problem 3.48

(a)

$$\begin{aligned}
\mathbf{E}_{\text{dip}} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})\right. \\
&\quad \left. + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})\right] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[3\sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \underbrace{\left(2\cos^2\theta - \sin^2\theta\right)}_{=3\cos^2\theta-1}\,\hat{\mathbf{z}}\right]. \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{E}_{\text{dip}}\,d\tau \\
&= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)\int \frac{1}{r^3}\left[3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \left(3\cos^2\theta - 1\right)\,\hat{\mathbf{z}}\right] r^2\sin\theta\,dr\,d\theta\,d\phi.
\end{aligned}$$

But $\int_0^{2\pi}\cos\phi\,d\phi = \int_0^{2\pi}\sin\phi\,d\phi = 0$, so the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms drop out, and $\int_0^{2\pi} d\phi = 2\pi$, so

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)2\pi\int_0^R \frac{1}{r}\,dr \underbrace{\int_0^\pi \left(3\cos^2\theta - 1\right)\sin\theta\,d\theta}_{\left(-\cos^3\theta+\cos\theta\right)\big|_0^\pi=1-1+1-1=0}.$$

Evidently $\boxed{\mathbf{E}_{\text{ave}} = \mathbf{0},}$ which contradicts the result of Prob. 3.47. [Note, however, that the r integral, $\int_0^R \frac{1}{r}\,dr$, blows up, since $\ln r \to -\infty$ as $r \to 0$. If, as suggested, we truncate the r integral at $r = \epsilon$, then it is finite, and the θ integral gives $\mathbf{E}_{\text{ave}} = \mathbf{0}$.]

(b) We want $\mathbf{E}$ within the ϵ-sphere to be a delta function: $\mathbf{E} = \mathbf{A}\delta^3(\mathbf{r})$, with $\mathbf{A}$ selected so that the *average* field is consistent with the general theorem in Prob. 3.47:

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{A}\delta^3(\mathbf{r})\,d\tau = \frac{\mathbf{A}}{\left(\frac{4}{3}\pi R^3\right)} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} \Rightarrow \mathbf{A} = -\frac{\mathbf{p}}{3\epsilon_0}, \text{ and hence } \boxed{\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}).}$$

Problem 3.49 We need to show that the field inside the sphere approaches a delta-function with the right coefficient (Eq. 3.106) in the limit as $R \to 0$. From Eq. 3.86, the potential inside is

$$V = \frac{k}{3\epsilon_0} r\cos\theta = \frac{k}{3\epsilon_0} z, \quad \text{so} \quad \mathbf{E} = -\boldsymbol{\nabla} V = -\frac{k}{3\epsilon_0}\,\hat{\mathbf{z}}.$$

From Prob. 3.30, the dipole moment of this configuration is $\mathbf{p} = (4\pi R^3 k/3)\,\hat{\mathbf{z}}$, so $k\,\hat{\mathbf{z}} = 3\mathbf{p}/(4\pi R^3)$, and hence the field inside is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0 R^3}\mathbf{p}.$$

Clearly $E \to \infty$ as $R \to 0$ (if $\mathbf{p}$ is held constant); its volume integral is

$$\int \mathbf{E}\,d\tau = -\frac{1}{4\pi\epsilon_0 R^3}\mathbf{p}\frac{4}{3}\pi R^3 = -\frac{1}{3\epsilon_0}\mathbf{p},$$

which matches the delta-function term in Eq. 3.106. ✓

Problem 3.50

(a) $I = \int (\boldsymbol{\nabla} V_1)\cdot(\boldsymbol{\nabla} V_2)\,d\tau$. But $\boldsymbol{\nabla}\cdot(V_1\boldsymbol{\nabla} V_2) = (\boldsymbol{\nabla} V_1)\cdot(\boldsymbol{\nabla} V_2) + V_1(\nabla^2 V_2)$, so

$$I = \int \boldsymbol{\nabla}\cdot(V_1\boldsymbol{\nabla} V_2)\,d\tau - \int V_1(\nabla^2 V_2) = \oint_{\mathcal{S}} V_1(\boldsymbol{\nabla} V_2)\cdot d\mathbf{a} + \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau.$$

But the surface integral is over a huge sphere "at infinity", where V_1 and $V_2 \to 0$. So $I = \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau$. By the same argument, with 1 and 2 reversed, $I = \frac{1}{\epsilon_0}\int V_2\rho_1\,d\tau$. So $\int V_1\rho_2\,d\tau = \int V_2\rho_1\,d\tau$. qed

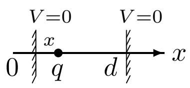
(b) $$\left\{\begin{array}{l} \textit{Situation (1)}: Q_a = \int_a \rho_1\,d\tau = Q;\ Q_b = \int_b \rho_1\,d\tau = 0;\ V_{1b} \equiv V_{ab}. \\ \textit{Situation (2)}: Q_a = \int_a \rho_2\,d\tau = 0;\ Q_b = \int_b \rho_2\,d\tau = Q;\ V_{2a} \equiv V_{ba}. \end{array}\right\}$$

$$\left\{\begin{array}{l} \int V_1\rho_2\,d\tau = V_{1a}\int_a \rho_2\,d\tau + V_{1b}\int_b \rho_2\,d\tau = V_{ab}Q. \\ \int V_2\rho_1\,d\tau = V_{2a}\int_a \rho_1\,d\tau + V_{2b}\int_b \rho_1\,d\tau = V_{ba}Q. \end{array}\right\}$$

Green's reciprocity theorem says $QV_{ab} = QV_{ba}$, so $V_{ab} = V_{ba}$. qed

Problem 3.51

(a) *Situation (1):* actual. *Situation (2):* right plate at V_0, left plate at $V = 0$, no charge at x.



$$\int V_1\rho_2\,d\tau = V_{l_1}Q_{l_2} + V_{x_1}Q_{x_2} + V_{r_1}Q_{r_2}.$$

But $V_{l_1} = V_{r_1} = 0$ and $Q_{x_2} = 0$, so $\int V_1\rho_2\,d\tau = 0$.

$$\int V_2\rho_1\,d\tau = V_{l_2}Q_{l_1} + V_{x_2}Q_{x_1} + V_{r_2}Q_{r_1}.$$

But $V_{l_2} = 0$ $Q_{x_1} = q$, $V_{r_2} = V_0$, $Q_{r_1} = Q_2$, and $V_{x_2} = V_0(x/d)$. So $0 = V_0(x/d)q + V_0 Q_2$, and hence

$$\boxed{Q_2 = -qx/d.}$$

Situation (1): actual. *Situation (2):* left plate at V_0, right plate at $V=0$, no charge at x.

$$\int V_1\rho_2\,d\tau = 0 = \int V_2\rho_1\,d\tau = V_{l_2}Q_{l_1} + V_{x_2}Q_{x_1} + V_{r_2}Q_{r_1} = V_0Q_1 + qV_{x_2} + 0.$$

But $V_{x_2} = V_0\left(1-\dfrac{x}{d}\right)$, so

$$\boxed{Q_1 = -q(1-x/d).}$$

(b) *Situation (1):* actual. *Situation (2):* inner sphere at V_0, outer sphere at zero, no charge at r.

$$\int V_1\rho_2\,d\tau = V_{a_1}Q_{a_2} + V_{r_1}Q_{r_2} + V_{b_1}Q_{b_2}.$$

But $V_{a_1} = V_{b_1} = 0,\ Q_{r_2} = 0$. So $\int V_1\rho_2\,d\tau = 0$.

$$\int V_2\rho_1\,d\tau = V_{a_2}Q_{a_1} + V_{r_2}Q_{r_1} + V_{b_2}Q_{b_1} = Q_aV_0 + qV_{r_2} + 0.$$

But V_{r_2} is the potential at r in configuration 2: $V(r) = A + B/r$, with $V(a) = V_0 \Rightarrow A + B/a = V_0$, or $aA + B = aV_0$, and $V(b) = 0 \Rightarrow A + B/b = 0$, or $bA + B = 0$. Subtract: $(b-a)A = -aV_0 \Rightarrow A = -aV_0/(b-a)$; $B\left(\frac{1}{a}-\frac{1}{b}\right) = V_0 = B\frac{(b-a)}{ab} \Rightarrow B = abV_0/(b-a)$. So $V(r) = \frac{aV_0}{(b-a)}\left(\frac{b}{r}-1\right)$. Therefore

$$Q_aV_0 + q\frac{aV_0}{(b-a)}\left(\frac{b}{r}-1\right) = 0;\quad \boxed{Q_a = -\frac{qa}{(b-a)}\left(\frac{b}{r}-1\right).}$$

Now let *Situation (2)* be: inner sphere at zero, outer at V_0, no charge at r.

$$\int V_1\rho_2\,d\tau = 0 = \int V_2\rho_1\,d\tau = V_{a_2}Q_{a_1} + V_{r_2}Q_{r_1} + V_{b_2}Q_{b_1} = 0 + qV_{r_2} + Q_bV_0.$$

This time $V(r) = A + \dfrac{B}{r}$ with $V(a) = 0 \Rightarrow A + B/a = 0$; $V(b) = V_0 \Rightarrow A + B/b = V_0$, so $V(r) = \dfrac{bV_0}{(b-a)}\left(1-\dfrac{a}{r}\right)$. Therefore $q\dfrac{bV_0}{(b-a)}\left(1-\dfrac{a}{r}\right) + Q_bV_0 = 0$; $\boxed{Q_b = -\dfrac{qb}{(b-a)}\left(1-\dfrac{a}{r}\right).}$

Problem 3.52

(a) $$\sum_{i,j=1}^{3}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_jQ_{ij} = \frac{1}{2}\int\left\{3\sum_{i=1}^{3}\hat{\mathbf{r}}_ir'_i\sum_{j=1}^{3}\hat{\mathbf{r}}_jr'_j - (r')^2\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij}\right\}\rho\,d\tau'$$

But $\displaystyle\sum_{i=1}^{3}\hat{\mathbf{r}}_ir'_i = \hat{\mathbf{r}}\cdot\mathbf{r}' = r'\cos\alpha = \sum_{j=1}^{3}\hat{\mathbf{r}}_jr'_j$; $\displaystyle\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij} = \sum\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1.$ So

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int\frac{1}{2}\left(3r'^2\cos^2\alpha - r'^2\right)\rho\,d\tau' = \text{the third term in Eq. 3.96.}\quad ✓$$

(b) Because $x^2 = y^2 = (a/2)^2$ for all four charges, $Q_{xx} = Q_{yy} = \frac{1}{2}\left[3(a/2)^2 - (\sqrt{2}a/2)^2\right](q-q-q+q) = 0$. Because $z = 0$ for all four charges, $Q_{zz} = \frac{1}{2}[-(\sqrt{2}a/2)^2](q-q-q+q) = 0$ and $Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = 0$. This leaves only

$$Q_{xy} = Q_{yx} = \frac{3}{2}\left[\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)q + \left(\frac{a}{2}\right)\left(-\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)q\right] = \boxed{\frac{3}{2}a^2q.}$$

(c)

$$
\begin{aligned}
2\overline{Q}_{ij} &= \int \left[3(r_i - d_i)(r_j - d_j) - (\mathbf{r} - \mathbf{d})^2\delta_{ij}\right]\rho\, d\tau \quad \text{(I'll drop the primes, for simplicity.)} \\
&= \int \left[3r_i r_j - r^2\delta_{ij}\right]\rho\, d\tau - 3d_i \int r_j \rho\, d\tau - 3d_j \int r_i \rho\, d\tau + 3d_i d_j \int \rho\, d\tau + 2\mathbf{d}\cdot\int \mathbf{r}\rho\, d\tau\, \delta_{ij} \\
&\quad - d^2\delta_{ij}\int \rho\, d\tau = Q_{ij} - 3(d_i p_j + d_j p_i) + 3d_i d_j Q + 2\delta_{ij}\mathbf{d}\cdot\mathbf{p} - d^2\delta_{ij}Q.
\end{aligned}
$$

So if $\mathbf{p} = 0$ and $Q = 0$ then $\overline{Q}_{ij} = Q_{ij}$. qed

(d) Eq. 3.95 with $n = 3$:

$$
V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int (r')^3 P_3(\cos\alpha)\rho\, d\tau'; \quad P_3(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right).
$$

$$
\boxed{V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\sum_{i,j,k}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\hat{\mathbf{r}}_k Q_{ijk}.}
$$

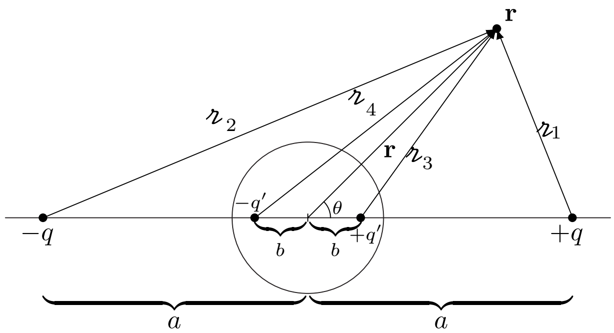
Define the "octopole moment" as

$$
\boxed{Q_{ijk} \equiv \frac{1}{2}\int \left[5r_i' r_j' r_k' - (r')^2(r_i'\delta_{jk} + r_j'\delta_{ik} + r_k'\delta_{ij})\right]\rho(\mathbf{r}')\, d\tau'.}
$$

Problem 3.53

$$
V = \frac{1}{4\pi\epsilon_0}\left\{q\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) + q'\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right)\right\}
$$

$$
\begin{aligned}
\mathscr{r}_1 &= \sqrt{r^2 + a^2 - 2ra\cos\theta}, \\
\mathscr{r}_2 &= \sqrt{r^2 + a^2 + 2ra\cos\theta}, \\
\mathscr{r}_3 &= \sqrt{r^2 + b^2 - 2rb\cos\theta}, \\
\mathscr{r}_1 &= \sqrt{r^2 + b^2 + 2rb\cos\theta}.
\end{aligned}
$$



Expanding as in Ex. 3.10: $\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) \cong \frac{2r}{a^2}\cos\theta$ (we want $a \gg r$, not $r \gg a$, this time).

$$
\begin{aligned}
\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right) &\cong \frac{2b}{r^2}\cos\theta \text{ (here we want } b \ll r, \text{ because } b = R^2/a, \text{ Eq. 3.16)} \\
&= \frac{2}{a}\frac{R^2}{r^2}\cos\theta.
\end{aligned}
$$

But $q' = -\frac{R}{a}q$ (Eq. 3.15), so

$$
V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left[q\frac{2r}{a^2}\cos\theta - \frac{R}{a}q\frac{2}{a}\frac{R^2}{r^2}\cos\theta\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)\left(r - \frac{R^3}{r^2}\right)\cos\theta.
$$

Set $E_0 = -\frac{1}{4\pi\epsilon_0}\frac{2q}{a^2}$ (field in the vicinity of the sphere produced by $\pm q$):

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta} \quad \text{(agrees with Eq. 3.76).}$$

Problem 3.54

The boundary conditions are

$$\left.\begin{array}{rl} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = V_0 \text{ when } y = a, \\ \text{(iii)} & V = 0 \text{ when } x = b, \\ \text{(iv)} & V = 0 \text{ when } x = -b. \end{array}\right\}$$

Go back to Eq. 3.26 and examine the case $k = 0$: $d^2X/dx^2 = d^2Y/dy^2 = 0$, so $X(x) = Ax + B$, $Y(y) = Cy + D$. But this configuration is symmetric in x, so $A = 0$, and hence the $k = 0$ solution is $V(x,y) = Cy + D$. Pick $D = 0$, $C = V_0/a$, and subtract off this part:

$$V(x,y) = V_0\frac{y}{a} + \bar{V}(x,y).$$

The *remainder* ($\bar{V}(x,y)$) satisfies boundary conditions similar to Ex. 3.4:

$$\left.\begin{array}{rl} \text{(i)} & \bar{V} = 0 \text{ when } y = 0, \\ \text{(ii)} & \bar{V} = 0 \text{ when } y = a, \\ \text{(iii)} & \bar{V} = -V_0(y/a) \text{ when } x = b, \\ \text{(iv)} & \bar{V} = -V_0(y/a) \text{ when } x = -b. \end{array}\right\}$$

(The point of peeling off $V_0(y/a)$ was to recover (ii), on which the constraint $k = n\pi/a$ depends.)

The solution (following Ex. 3.4) is

$$\bar{V}(x,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\sin(n\pi y/a),$$

and it remains to fit condition (iii):

$$\bar{V}(b,y) = \sum C_n \cosh(n\pi b/a)\sin(n\pi y/a) = -V_0(y/a).$$

Invoke Fourier's trick:

$$\sum C_n \cosh(n\pi b/a)\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = -\frac{V_0}{a}\int_0^a y\sin(n'\pi y/a)\,dy,$$

$$\frac{a}{2}C_n\cosh(n\pi b/a) = -\frac{V_0}{a}\int_0^a y\sin(n\pi y/a)\,dy.$$

$$\begin{aligned} C_n &= -\frac{2V_0}{a^2\cosh(n\pi b/a)}\left[\left(\frac{a}{n\pi}\right)^2\sin(n\pi y/a) - \left(\frac{ay}{n\pi}\right)\cos(n\pi y/a)\right]\Bigg|_0^a \\ &= \frac{2V_0}{a^2\cosh(n\pi b/a)}\left(\frac{a^2}{n\pi}\right)\cos(n\pi) = \frac{2V_0}{n\pi}\frac{(-1)^n}{\cosh(n\pi b/a)}. \end{aligned}$$

$$V(x,y) = \boxed{V_0\left[\frac{y}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)\right].}$$

Alternatively, start with the separable solution

$$V(x,y) = (C\sin kx + D\cos kx)\left(Ae^{ky} + Be^{-ky}\right).$$

Note that the configuration is symmetric in x, so $C = 0$, and $V(x,0) = 0 \Rightarrow B = -A$, so (combining the constants)

$$V(x,y) = A\cos kx\sinh ky.$$

But $V(b,y) = 0$, so $\cos kb = 0$, which means that $kb = \pm\pi/2, \pm 3\pi/2, \ldots$, or $k = (2n-1)\pi/2b \equiv \alpha_n$, with $n = 1, 2, 3, \ldots$ (negative k does not yield a different solution—the sign can be absorbed into A). The general linear combination is

$$V(x,y) = \sum_{n=1}^{\infty} A_n \cos\alpha_n x\sinh\alpha_n y,$$

and it remains to fit the final boundary condition:

$$V(x,a) = V_0 = \sum_{n=1}^{\infty} A_n \cos\alpha_n x\sinh\alpha_n a.$$

Use Fourier's trick, multiplying by $\cos\alpha_{n'}x$ and integrating:

$$V_0\int_{-b}^{b}\cos\alpha_{n'}x\,dx = \sum_{n=1}^{\infty} A_n\sinh\alpha_n a\int_{-b}^{b}\cos\alpha_{n'}x\cos\alpha_n x\,dx,$$

$$V_0\frac{2\sin\alpha_{n'}b}{\alpha_{n'}} = \sum_{n=1}^{\infty} A_n\sinh\alpha_n a(b\delta_{n'n}) = bA_{n'}\sinh\alpha_{n'}a.$$

So $A_n = \dfrac{2V_0}{b}\dfrac{\sin\alpha_n b}{\alpha_n\sinh\alpha_n a}$. But $\sin\alpha_n b = \sin\left(\dfrac{2n-1}{2}\pi\right) = -(-1)^n$, so

$$V(x,y) = \boxed{-\frac{2V_0}{b}\sum_{n=1}^{\infty}(-1)^n\frac{\sinh\alpha_n y}{\alpha_n\sinh\alpha_n a}\cos\alpha_n x.}$$

Problem 3.55

(a) Using Prob. 3.15b (with $b = a$):

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n\text{ odd}}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi)}.$$

$$\sigma(y) = -\epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0} = -\epsilon_0 \frac{4V_0}{\pi} \sum_{n \text{ odd}} \left(\frac{n\pi}{a}\right) \frac{\cosh(n\pi x/a)\sin(n\pi y/a)}{n \sinh(n\pi)}\Bigg|_{x=0}$$
$$= -\frac{4\epsilon_0 V_0}{a} \sum_{n \text{ odd}} \frac{\sin(n\pi y/a)}{\sinh(n\pi)}.$$
$$\lambda = \int_0^a \sigma(y)\, dy = -\frac{4\epsilon_0 V_0}{a} \sum_{n \text{ odd}} \frac{1}{\sinh(n\pi)} \int_0^a \sin(n\pi y/a)\, dy.$$

But $\displaystyle\int_0^a \sin(n\pi y/a)\, dy = -\frac{a}{n\pi}\cos(n\pi y/a)\big|_0^a = \frac{a}{n\pi}[1-\cos(n\pi)] = \frac{2a}{n\pi}$(since n is odd).

$$= -\frac{8\epsilon_0 V_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n \sinh(n\pi)} = \boxed{-\frac{\epsilon_0 V_0}{\pi} \ln 2.}$$

[Summing the series numerically (using Mathematica) gives 0.0866434, which agrees precisely with $\ln 2/8$. The series can be summed analytically, by manipulation of elliptic integrals—see "Integrals and Series, Vol. I: Elementary Functions," by A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (Gordon and Breach, New York, 1986), p. 721. I thank Ram Valluri for calling this to my attention.]

Using Prob. 3.54 (with $b = a/2$):

$$V(x,y) = V_0 \left[\frac{y}{a} + \frac{2}{\pi} \sum_n \frac{(-1)^n \cosh(n\pi x/a)\sin(n\pi y/a)}{n \cosh(n\pi/2)}\right].$$

$$\sigma(x) = -\epsilon_0 \frac{\partial V}{\partial y}\Big|_{y=0} = -\epsilon_0 V_0 \left[\frac{1}{a} + \frac{2}{\pi} \sum_n \left(\frac{n\pi}{a}\right) \frac{(-1)^n \cosh(n\pi x/a)\cos(n\pi y/a)}{n \cosh(n\pi/2)}\right]\Bigg|_{y=0}$$
$$= -\epsilon_0 V_0 \left[\frac{1}{a} + \frac{2}{a} \sum_n \frac{(-1)^n \cosh(n\pi x/a)}{\cosh(n\pi/2)}\right] = -\frac{\epsilon_0 V_0}{a} \left[1 + 2\sum_n \frac{(-1)^n \cosh(n\pi x/a)}{\cosh(n\pi/2)}\right].$$
$$\lambda = \int_{-a/2}^{a/2} \sigma(x)\, dx = -\frac{\epsilon_0 V_0}{a} \left[a + 2\sum_n \frac{(-1)^n}{\cosh(n\pi/2)} \int_{-a/2}^{a/2} \cosh(n\pi x/a)\, dx\right].$$

But $\displaystyle\int_{-a/2}^{a/2} \cosh(n\pi x/a)\, dx = \frac{a}{n\pi}\sinh(n\pi x/a)\Big|_{-a/2}^{a/2} = \frac{2a}{n\pi}\sinh(n\pi/2).$

$$= -\frac{\epsilon_0 V_0}{a} \left[a + \frac{4a}{\pi} \sum_n \frac{(-1)^n \tanh(n\pi/2)}{n}\right] = -\epsilon_0 V_0 \left[1 + \frac{4}{\pi} \sum_n \frac{(-1)^n \tanh(n\pi/2)}{n}\right]$$
$$= \boxed{-\frac{\epsilon_0 V_0}{\pi} \ln 2.}$$

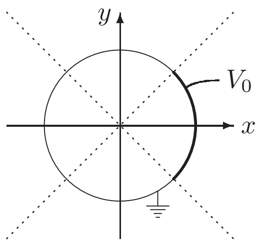
[The numerical value is -0.612111, which agrees with the expected value $(\ln 2 - \pi)/4$.]

(b) From Prob. 3.24:

$$V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} \left(a_k s^k + b_k \frac{1}{s^k}\right)[c_k \cos(k\phi) + d_k \sin(k\phi)].$$

In the interior ($s < R$) b_0 and b_k must be zero ($\ln s$ and $1/s$ blow up at the origin). Symmetry $\Rightarrow d_k = 0$. So

$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} a_k s^k \cos(k\phi).$$



At the surface:

$$V(R,\phi) = \sum_{k=0} a_k R^k \cos(k\phi) = \begin{cases} V_0, \text{ if } -\pi/4 < \phi < \pi/4, \\ 0, \quad \text{otherwise.} \end{cases}$$

Fourier's trick: multiply by $\cos(k'\phi)$ and integrate from $-\pi$ to π:

$$\sum_{k=0}^{\infty} a_k R^k \int_{-\pi}^{\pi} \cos(k\phi)\cos(k'\phi)\,d\phi = V_0 \int_{-\pi/4}^{\pi/4} \cos(k'\phi)\,d\phi = \begin{cases} V_0 \sin(k'\phi)/k' \Big|_{-\pi/4}^{\pi/4} = (V_0/k')\sin(k'\pi/4), \text{ if } k' \neq 0, \\ V_0\pi/2, \text{ if } k' = 0. \end{cases}$$

But

$$\int_{-\pi}^{\pi} \cos(k\phi)\cos(k'\phi)\,d\phi = \begin{cases} 0, & \text{if } k \neq k' \\ 2\pi, & \text{if } k = k' = 0, \\ \pi, & \text{if } k = k' \neq 0. \end{cases}$$

So $2\pi a_0 = V_0\pi/2 \Rightarrow a_0 = V_0/4$; $\pi a_k R^k = (2V_0/k)\sin(k\pi/4) \Rightarrow a_k = (2V_0/\pi k R^k)\sin(k\pi/4)$ $(k \neq 0)$; hence

$$V(s,\phi) = V_0 \left[\frac{1}{4} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(k\pi/4)}{k} \left(\frac{s}{R}\right)^k \cos(k\phi)\right].$$

Using Eq. 2.49, and noting that in this case $\hat{\mathbf{n}} = -\hat{\mathbf{s}}$:

$$\sigma(\phi) = \epsilon_0 \frac{\partial V}{\partial s}\Big|_{s=R} = \epsilon_0 V_0 \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi/4)}{kR^k} k s^{k-1} \cos(k\phi)\Big|_{s=R} = \frac{2\epsilon_0 V_0}{\pi R} \sum_{k=1}^{\infty} \sin(k\pi/4)\cos(k\phi).$$

We want the net (line) charge on the segment opposite to V_0 ($-\pi < \phi < -3\pi/4$ and $3\pi/4 < \phi < \pi$):

$$\begin{aligned} \lambda &= \int \sigma(\phi) R\,d\phi = 2R \int_{3\pi/4}^{\pi} \sigma(\phi)\,d\phi = \frac{4\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \sin(k\pi/4) \int_{3\pi/4}^{\pi} \cos(k\phi)\,d\phi \\ &= \frac{4\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \sin(k\pi/4) \left[\frac{\sin(k\phi)}{k}\Big|_{3\pi/4}^{\pi}\right] = -\frac{4\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi/4)\sin(3k\pi/4)}{k}. \end{aligned}$$

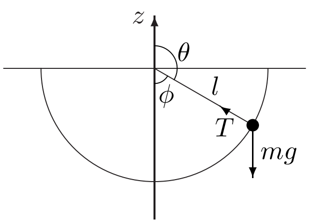
k	$\sin(k\pi/4)$	$\sin(3k\pi/4)$	product
1	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
2	1	-1	-1
3	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
4	0	0	0
5	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
6	-1	1	-1
7	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
8	0	0	0

$$\lambda = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \sum_{2,6,10,\ldots}\frac{1}{k}\right] = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \frac{1}{2}\sum_{1,3,5,\ldots}\frac{1}{k}\right] = 0.$$

Ouch! What went wrong? The problem is that the series $\sum(1/k)$ is divergent, so the "subtraction" $\infty - \infty$ is suspect. One way to avoid this is to go back to $V(s,\phi)$, calculate $\epsilon_0(\partial V/\partial s)$ at $s \neq R$, and save the limit $s \to R$ until the end:

$$\begin{aligned}
\sigma(\phi, s) &\equiv \epsilon_0\frac{\partial V}{\partial s} = \frac{2\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)}{k}\frac{ks^{k-1}}{R^k}\cos(k\phi)\\
&= \frac{2\epsilon_0 V_0}{\pi R}\sum_{k=1}^{\infty}x^{k-1}\sin(k\pi/4)\cos(k\phi) \quad \text{(where } x \equiv s/R \to 1 \text{ at the end).}\\
\lambda(x) &\equiv \sigma(\phi, s)R\,d\phi = -\frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}x^{k-1}\sin(k\pi/4)\sin(3k\pi/4)\\
&= -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2x}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^6}{6} + \frac{x^{10}}{10} + \cdots\right)\right]\\
&= -\frac{2\epsilon_0 V_0}{\pi x}\left[\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - \left(x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \cdots\right)\right].\\
&\quad \text{But (see math tables)} : \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).\\
&= -\frac{2\epsilon_0 V_0}{\pi x}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)\right] = -\frac{\epsilon_0 V_0}{\pi x}\ln\left[\left(\frac{1+x}{1-x}\right)\left(\frac{1+x^2}{1-x^2}\right)\right]\\
&= -\frac{\epsilon_0 V_0}{\pi x}\ln\left[\frac{(1+x)^2}{1+x^2}\right]; \quad \lambda = \lim_{x\to 1}\lambda(x) = \boxed{\frac{-\epsilon_0 V_0}{\pi}\ln 2.}
\end{aligned}$$

Problem 3.56



$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

Now consider the pendulum: $\mathbf{F} = -mg\,\hat{\mathbf{z}} - T\,\hat{\mathbf{r}}$, where $T - mg\cos\phi = mv^2/l$ and (by conservation of energy) $mgl\cos\phi = (1/2)mv^2 \Rightarrow v^2 = 2gl\cos\phi$ (assuming it started from rest at $\phi = 90°$, as stipulated). But $\cos\phi = -\cos\theta$, so $T = mg(-\cos\theta) + (m/l)(-2gl\cos\theta) = -3mg\cos\theta$, and hence

$$\mathbf{F} = -mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) + 3mg\cos\theta\,\hat{\mathbf{r}} = mg(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

This total force is such as to keep the pendulum on a circular arc, and it is identical to the force on q in the field of a dipole, with $mg \leftrightarrow qp/4\pi\epsilon_0 l^3$. Evidently q also executes semicircular motion, as though it were on a tether of fixed length l.

Problem 3.57 Symmetry suggests that the plane of the orbit is perpendicular to the z axis, and since we need a centripetal force, pointing in toward the axis, the orbit must lie at the bottom of the field loops (Fig. 3.37a), where the z component of the field is zero. Referring to Eq. 3.104,

$\mathbf{E}\cdot\hat{\mathbf{z}} = 0 \Rightarrow 3(\mathbf{p}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}) - \mathbf{p}\cdot\hat{\mathbf{z}} = 0$, or $3\cos^2\theta - 1 = 0$. So $\cos^2\theta = 1/3$, $\cos\theta = -1/\sqrt{3}$, $\sin\theta = \sqrt{2/3}$, $z/s = \tan\theta \Rightarrow \boxed{z = -\sqrt{2}\,s.}$ The field at the orbit is (Eq. 3.103)

$$\begin{aligned}
\mathbf{E} &= \frac{p}{4\pi\epsilon_0 r^3}\left(-2\frac{1}{\sqrt{3}}\,\hat{\mathbf{r}} + \sqrt{\frac{2}{3}}\,\hat{\boldsymbol{\theta}}\right) \\
&= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{\frac{2}{3}}\left[-\sqrt{2}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) + (\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})\right] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{\frac{2}{3}}\left[\left(-\sqrt{2}\sin\theta + \cos\theta\right)\cos\phi\,\hat{\mathbf{x}} + \left(-\sqrt{2}\sin\theta + \cos\theta\right)\sin\phi\,\hat{\mathbf{y}} + \left(-\sqrt{2}\cos\theta - \sin\theta\right)\,\hat{\mathbf{z}}\right] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{\frac{2}{3}}\left[\left(-\sqrt{2}\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}}\right)(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \left(\sqrt{2}\frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\right)\,\hat{\mathbf{z}}\right] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{\frac{2}{3}}\left[-\sqrt{3}\,(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})\right] = -\frac{p}{4\pi\epsilon_0 r^3}\sqrt{2}\,\hat{\mathbf{s}} = -\frac{p}{3\sqrt{3}\pi\epsilon_0 s^3}\,\hat{\mathbf{s}}.
\end{aligned}$$

(I used $s = r\sin\theta = r\sqrt{2/3}$, in the last step.)

The centripetal force is

$$F = qE = -\frac{qp}{3\sqrt{3}\pi\epsilon_0 s^3} = -\frac{mv^2}{s} \quad\Rightarrow\quad v^2 = \frac{qp}{3\sqrt{3}\pi\epsilon_0 m s^2} \quad\Rightarrow\quad v = \boxed{\frac{1}{s}\sqrt{\frac{qp}{3\sqrt{3}\pi\epsilon_0 m}}.}$$

The angular momentum is

$$L = smv = \boxed{\sqrt{\frac{qpm}{3\sqrt{3}\pi\epsilon_0}},}$$

the same for all orbits, regardless of their radius (!), and the energy is

$$W = \frac{1}{2}mv^2 + qV = \frac{1}{2}\frac{qp}{3\sqrt{3}\pi\epsilon_0 s^2} + \frac{q}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \frac{qp}{6\sqrt{3}\pi\epsilon_0 s^2} - \frac{qp}{4\pi\epsilon_0\sqrt{3}(3/2)s^2} = \boxed{0.}$$

Problem 3.58

Potential of q: $V_q(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$, where $\mathscr{r}^2 = a^2 + r^2 - 2ar\cos\theta$.

Equation 3.94, with $r' \to a$ and $\alpha \to \theta$: $\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\cos\theta)$. So

$$V_q(r,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\cos\theta).$$

Meanwhile, the potential of σ is (Eq. 3.79) $V_\sigma(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$.

Comparing the two ($V_q = V_\sigma$) we see that $B_l = (q/4\pi\epsilon_0)a^l$, and hence (Eq. 3.81) $A_l = (q/4\pi\epsilon_0)a^l/R^{2l+1}$. Then (Eq. 3.83)

$$\sigma(\theta) = \frac{q}{4\pi R^2}\sum_{l=0}^{\infty}(2l+1)\left(\frac{a}{R}\right)^l P_l(\cos\theta) = \frac{q}{4\pi R^2}\left[2\sum_{l=0}^{\infty} l\left(\frac{a}{R}\right)^l P_l(\cos\theta) + \sum_{l=0}^{\infty}\left(\frac{a}{R}\right)^l P_l(\cos\theta)\right].$$

Now (second line above, with $r \to R$)

$$\frac{1}{\sqrt{a^2+R^2-2aR\cos\theta}} = \frac{1}{R}\sum_{l=0}^{\infty}\left(\frac{a}{R}\right)^l P_l(\cos\theta).$$

Differentiating with respect to a:

$$\frac{d}{da}\left(\frac{1}{\sqrt{a^2+R^2-2aR\cos\theta}}\right) = -\frac{(a-R\cos\theta)}{(a^2+R^2-2aR\cos\theta)^{3/2}} = \frac{1}{aR}\sum_{l=0}^{\infty} l\left(\frac{a}{R}\right)^l P_l(\cos\theta).$$

Thus

$$\begin{aligned}
\sigma(\theta) &= \frac{q}{4\pi R^2}\left[-2aR\frac{(a-R\cos\theta)}{(a^2+R^2-2aR\cos\theta)^{3/2}} + \frac{R}{(a^2+R^2-2aR\cos\theta)^{1/2}}\right] \\
&= \frac{q}{4\pi R}\frac{\left[-2a(a-R\cos\theta)+(a^2+R^2-2aR\cos\theta)\right]}{(a^2+R^2-2aR\cos\theta)^{3/2}} = \boxed{\frac{q}{4\pi R}\frac{(R^2-a^2)}{(a^2+R^2-2aR\cos\theta)^{3/2}}}.
\end{aligned}$$