TOPOLOGY, MATH-440/640
SPRING 2020

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0. Chapter 0

Question 0.4 (Prohovich)**.**

Proof. ◻

Question 0.5 (Golinski)**.**

Proof. ◻

Question 0.7 (Ewing)**.** $(A \cup B) \setminus B = A$ *is false*

Proof. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and

$$(A \cup B) - B = \{1, 2, 3\} - \{2, 3\} = \{1\} \neq A$$

◻

Question 0.12 (Macula)**.** *Let $f : X \to Y$ be a function from X onto Y, and suppose $B \subset Y$. Prove that $B = f(f^{-1}(B))$.*

Proof. Let a be any element in $f^{-1}(B)$. Then $f(a) \in B$, and so $f(f^{-1}(B)) \subset B$. Since f is surjective, $f^{-1}(B)$ contains, for all $b \in B$, at least one element a such that $f(a) = b$, making $B \subset f(f^{-1}(B))$. Thus, $B = f(f^{-1}(B))$. ◻

Question 0.32 (Hafer)**.** *QUESTION*

Proof. Let A and B be finite disjoint sets such that the cardinality of A is m and the cardinality of B is n. Then let A be the set of elements such that $Z_m = \{0, ..., m-1\}$ and B be the set of elements such that $Z_n = \{0, ..., n-1\}$. Then $Z_{n+m} = \{0, ..., n-1, n, ..., n+m-1\}$ which has cardinality of $n+m$. Now for each element $b \in B$ we can map it directly to itself in Z_{m+n}. For each $a \in A$ we can map each element to itself $+n$ in Z_{n+m}. So each element in Z_{n+m} is mapped to by either some $a \in A$ or $b \in B$. Since A and B are disjoint no element in Z_{n+m} is mapped to by both some a and b. Hence there is a one to one mapping between Z_{n+m} and $A \cup B$. Therefore $A \cup B$ has cardinality $m+n$. ◻

Question 0.34 (Ewing)**.** *If sets A, B have cardinalities n and m, respectively, then $A \times B$ has cardinality nm.*

Proof. First we show that $\mathbb{Z}_n \times \mathbb{Z}_m$ has cardinality nm by demonstrating that the function $h \colon \mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{Z}_{nm}$ defined by $h(i, j) = im + j$ is both surjective and injective. It is surjective, because for any $y \in \mathbb{Z}_{nm}$ we can find an $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m$, as follows: When $y = im + j = 0$ we must have $(i, j) = (0, 0) \in \mathbb{Z}_n \times \mathbb{Z}_m$, since both i, j are non-negative. For any other y, i.e., when $y > 0$, Euclid's division algorithm ensures that there are integers i, j s.t. $y = im + j$ with $0 \leq j < m$, and we know that $0 \leq i < n$ because $y \in \mathbb{Z}_{nm}$ means $y = im + j < nm$. So for any $y \in \mathbb{Z}_{nm}$ there is an $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m$ s.t. $h(i, j) = y$, i.e., h is surjective. It is injective since Euclid's Division algorithm ensures that those integers i, j are unique for each $y = im+j$, i.e., whenever $h(i_1, j_1) = y = h(i_2, j_2)$ we have $i_1 = i_2$ and $j_1 = j_2$. Since h is bijective, $\mathbb{Z}_n \times \mathbb{Z}_m$ is equivalent to $\mathbb{Z}_{nm}$ and so has cardinality nm. It therefore suffices to show that $A \times B$ is equivalent to $\mathbb{Z}_n \times \mathbb{Z}_m$.

Since A, B are finite with cardinalities n, m, respectively, they are equivalent to $\mathbb{Z}_n, \mathbb{Z}_m$, with bijections $f_A \colon A \to \mathbb{Z}_n$ and $f_B \colon B \to \mathbb{Z}_m$ defined by $f_A(a) = i$, $(a \in A, i \in \mathbb{Z}_n)$ and $f_B(b) = j$, $(b \in B, j \in \mathbb{Z}_m)$. Define $g \colon A \times B \to \mathbb{Z}_n \times \mathbb{Z}_m$ by $g(a, b) = (f_A(a), f_B(b))$. This is surjective, because for every $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m$ the bijectivity of f_A and f_B means we have $(i, j) = (f_A(a), f_B(b))$ for some $a \in A$ and a $b \in B$, which gives us an $(a, b) \in A \times B$. Function g is injective, because bijectivity of f_A and f_B means that whenever $g(a_1, b_1) = g(a_2, b_2)$ we have $f_A(a_1) = f_A(a_2)$ and $f_B(b_1) = f_B(b_2)$ and hence $a_1 = a_2$ and $b_1 = b_2$, i.e. $(a_1, b_1) = (a_2, b_2)$. Since g is bijective, $A \times B$ is equivalent to $\mathbb{Z}_n \times \mathbb{Z}_m$ and we are done. □

1. Chapter 1

Question 1.4 (Fasano)**.** *Let X be the plane. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, let $\sigma(x, y) = |x_1 - y_1|$.*

1. *Show that σ is a pseudometric.*
2. *Describe the σ-cell of radius r centered at the point (a, b).*
3. *Find the σ-closure of $S = \{x \in X : x_1^2 + x_2^2 < 1\}$*

Proof. Let X be the plane. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, let $\sigma(x, y) = |x_1 - y_1|$.

1. We start by noting that clearly $\sigma(x, y) \geq 0$ for all $x, y \in X$ because of the absolute value.

   We next take an arbitrary point $q = (q_1, q_2)$. Then $\sigma(q, q) = |q_1 - q_1| = 0$. Thus, the σ-distance between any point and itself is always 0.

   Next, consider the point $r = (r_1, r_2)$. Then by definition $\sigma(q, r) = |q_1 - r_1| = |r_1 - q_1| = \sigma(r, q)$. Thus, σ is commutative.

   Finally, we wish to show that the triangle inequality holds for σ. That is for any points $q = (q_1, q_2)$, $r = (r_1, r_2)$, and $s = (s_1, s_2)$ we wish to show $\sigma(q, s) \leq \sigma(q, r) + \sigma(r, s)$. However, we want to note here that σ takes two points in X projects them down onto the horizontal axis and then calculates the distance. Thus, it will suffice to show that the triangle inequality holds for the real numbers: i.e $|x + y| \leq |x| + |y|$. To do this we use:
   $$-|x| \leq x \leq |x| \text{ and } -|y| \leq y \leq |y|$$
   Adding these together we get:
   $$-|x| - |y| \leq x + y \leq |x| + |y|$$
   which gives us
   $$|x + y| \leq ||x| + |y|| = |x| + |y|$$
   Since σ satisfies the above properties, it is thus a pseudometric.
2. By definition, the σ-cell of radius r centered at the point $p = (a, b)$ is the set $C(p; r) = \{x : \sigma(p, x) < r\}$. Note that our pseudometric σ only looks at the first coordinate. That is the difference between (a, b) and (c, d) or (c, e) is the same. That tells us that our cell will have an infinite height, so we must only find the horizontal bounds. Therefore, $C = \{(x, y) : a - r < x < a + r, y \in \mathbf{R}\}$
3. To find the closure of S, we use $cl(S) = S \cup S'$. That is we need to find the limit points of S. We first show that the point $p_1 = (1, h)$ where $h \in \mathbf{R}$ is a limit point. Let $\epsilon > 0$. Then choose $s_1 = (1 - \frac{\epsilon}{2}, 0) \in S$. It follows that $\sigma(p_1, s_1) = |1 - (1 - \frac{\epsilon}{2})| = \frac{\epsilon}{2} < \epsilon$. Thus $p_1 = (1, h)$ is a limit point.

   We can similarly show that $p_2 = (-1, h)$ is a limit point choosing $s_2 = (-1 + \frac{\epsilon}{2}, 0)$.

We now consider any point $p_3 = (x_1, h)$ where $-1 < x_1 < 1$. It quite easily follows that p_3 is a limit point since we can find an infinite number of distinct points within S that are exactly 0 σ-distance away from p_3 (i.e. any point in S with x_1 as it's first coordinate). Finally, consider $p_4 = (x_2, h)$ where $-1 > x_2$ or $x_2 > 1$. Without loss of generality, assume $x_2 > 1$ (the other case is similar). Then $x_2 = 1 + \alpha$ where $\alpha > 0$. But then for $\epsilon = \frac{\alpha}{2} > 0$ there doesn't exist any $s \in S$ such that $\sigma(p_4, s) < \epsilon$. Thus p_4 cannot be a limit point. From these cases, we have shown that $cl(S) = S \cup S' = \{(x_1, x_2) \in X : |x_1| < 1\}$.

□

Question 1.6 (Zheng).

Proof. □

Question 1.8 (Moore).

Proof. □

Question 1.9 (Kelly).

Proof. □

Question 1.10 (Fernandez). *Let X be the set of real numbers, and let d be the pseudometric given by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ (Example 1.1(e)).*

(1) *Describe the closure of the cell $C(0; 1)$.*
(2) *Describe the set $B(0; 1) = \{y \in X : d(0, y) \leq 1\}$*

Proof. (1) The cell $C(0; 1) = \{x : d(0, x) < 1\} = \{0\}$. If $d(0, x) < 1$ then $x = 0$ by definition of the pseudometric. The d-derived set $C'(0; 1) = \emptyset$ since there are no d-limit points of the set. By definition of closure $cl(C(0; 1)) = C(0; 1) \cup C'(0; 1) = \{0\}$.
(2) The set $B(0; 1) = \{y \in X : d(0, y) \leq 1\} = X$. If $d(0, y) < 1$ then $y = 0$ and if $d(0, y) = 1$ then y is all other real numbers. There fore $B(0; 1) = X$

□

Question 1.11 (Prohovich).

Proof. □

Question 1.12 (Ewing). *If (X, d) is the space of real numbers with the usual pseudometric and for each $n \in \mathbb{Z}^+$, $A_n = (1/n, 1]$, find $cl(\cup\{A_n\})$ and $\cup\{cl(A_n)\}$.*

(a) $\mathrm{cl}(\cup\{A_n\}) = [0, 1]$

Proof. We must show that all and only points in $[0, 1]$ are either in $\cup\{A_n\}$ or limit points of $\cup\{A_n\}$. First note that $\cup\{A_n\}$ contains all $1/n$ with $n \in \mathbb{Z}^+$ and all real numbers x greater than some $1/n$ and less than or equal to 1. Because we can always find an $n > 1/x$ and hence $1/n < x$ for any real number $x \in (0, 1]$, this means $\cup\{A_n\} = (0, 1]$. Now consider four cases:

- Points $x > 1$ are neither in $(0, 1] = \cup\{A_n\}$ nor limit points for $\cup\{A_n\}$, since for $\epsilon < x - 1$, there are no points $y \in (0, 1]$ s.t. $|x - y| < \epsilon$.
- All points $x \in (0, 1] = \cup\{A_n\}$ are in $\mathrm{cl}(\cup\{A_n\})$ by definition of closure.
- The point $x = 0$ is a limit point of $\cup\{A_n\}$ and hence in $\mathrm{cl}(\cup\{A_n\})$, since all points $1/n$ are in $\cup\{A_n\}$ and hence for any $\epsilon > 0$ we can find a $1/n < \epsilon$ s.t. there is a point y with $1/n < y < \epsilon \leq 1$, i.e., y in $\cup\{A_n\} \subset \mathrm{cl}(\cup\{A_n\})$.
- All points $x < 0$ are neither in $(0, 1] = \cup\{A_n\}$ nor limit points for $\cup\{A_n\}$, since for $\epsilon < |x|$ there are no points $y \in (0, 1]$, s.t. $|y - x| < \epsilon$.

Therefore all and only points in $[0, 1]$ are either elements or limit points of $\mathrm{cl}(\cup\{A_n\})$, i.e., $\mathrm{cl}(\cup\{A_n\}) = [0, 1]$. □

(b) $\cup\{\mathrm{cl}(A_n)\} = (0,1]$

Proof. Notice that for any $A_n = (1/n, 1]$,

- No points $x > 1$ are elements of $\mathrm{cl}(A_n)$ or limit points, since for $\epsilon < x - 1$, there are no points $y \in (1/n, 1]$ s.t. $|x - y| < \epsilon$.
- All points $x \in (1/n, 1]$ are in that A_n and hence in that $\mathrm{cl}(A_n)$.
- The point $x = 1/n$ is a limit point of that A_n, since for all $\epsilon > 0$ all points y s.t. $1/n < y < \epsilon$ are in that A_n.
- All points $x < 1/n$ are neither elements nor limit points of A_n, since every element in A_n must be greater than $1/n$ and given any $\epsilon < |x - 1/n|$ there can be no $y \in (1/n, 1]$ s.t. $|x - y| < \epsilon$.

Thus all and only points in $[1/n, 1]$ are either elements or limit points of the corresponding $A_n = (1/n, 1]$ and hence $\mathrm{cl}(A_n) = [1/n, 1]$. Taking the union of all such $\mathrm{cl}(A_n)$, however, we observe that $n \in \mathbb{Z}^+$ implies $0 \notin \cup\{\mathrm{cl}(A_n)\}$ and, of course, neither $x < 0$ nor $x > 1$ are either, while for every $x \in (0, 1]$ we can find $n > 1/x$ and hence a $\mathrm{cl}(A_n)$ of which it is an element, making that x also an element of $\cup\{\mathrm{cl}(A_n)\}$. Thus $\cup(\{\mathrm{cl}(A_n)\}) = (0, 1]$. □

Question 1.13 (Golinski)**.** *In a pseudometric space, is the intersection of a collection of open sets necessarily an open set?*

Proof. There are two cases we can deal with for this problem, a finite collection of open sets and an infinite collection of open sets.

First, let's say we are given a finite collection of open sets. We know that a set is open if it is a neighborhood (or open ball) of all its points.

Let $\{O_i\}$ be a finite collection of open sets and S = $\bigcap_{i=1}^{n} O_i$. If S is the empty set then it is open by theorem 1.10a. If S doesn't equal the empty set, then we want to prove that it is open.

Suppose x $\in$ S, then x $\in O_i$ for all i $\in$ I. for each i there exists so $r_i > 0$ so there is some $B(x, r_i) \subset O_i$. We can let r = $min\{r_i\}$ so B(x,r) $\subset B(x, r_i) \subset O_i$ for every i. Therefore B(x,r) $\subset \cap O_i = S$.

This proves that a finite collection of open sets is an open set because we have shows that there is an open ball centered at each point of S and included in S.

Next we want to see if an infinite collection of open sets is open.

An infinite collection of open sets is closed, so I will provide a counterexample.

Take the set S = $\bigcap_{i=1}^{\infty}(-1/i, 1/i)$. In this case, S = $\{0\}$ which is closed.

Therefore, we have proved that for a finite collection of open sets the intersection is open, but that this is not necessarily the case for an infinite collection. □

Question 1.14 (Berkowitz)**.**

Proof. □

Question 1.15 (Macula)**.** *(a) In a pseudometric space (X, d), can it ever happen that a cell $C(x; r)$ is a closed set? Explain.*

(b) Can it ever happen that $B(x; r) = \{y \in X : d(x, y) \leq r\}$ is an open set?

Proof. (a) Yes. Let X be a set and let d be the trivial pseudometric. Then for any $r > 0$, $C(x; r) = X$, hence is closed.

(b) Yes. Let X be a set and let d be the trivial pseudometric. Then for any $r > 0$, $B(x;r) = X$, hence is open.

□

Question 1.17 (Hafer)**.**

Proof. Let x $\in cl(A)$
$\iff \forall \epsilon > 0,\ B(x,\epsilon) \cap A \neq \emptyset$
$\iff \forall \epsilon > 0 \exists a \in A s.t. d(x,a) < \epsilon$
$\iff inf\{d(x,a) : a \in A\} = 0$
$\iff d(x,A) = 0$

□

2. Chapter 2

Question 2.3 (Zheng)**.**

Proof. □

Question 2.4 (Zhang)**.**

Proof. □

Question 2.5 (Wang)**.**

Proof. □

Question 2.6 (Prohovich)**.**

Proof. □

Question 2.7 (Macula)**.**

Proof. □

Question 2.8 (Kelly)**.**

Proof. □

Question 2.9 (Hafer)**.**

Proof. □

Question 2.10 (Golinski)**.**

Proof. □

Question 2.11 (Fernandez)**.**

Proof. □

Question 2.12 (Fasano)**.**

Proof. □

Question 2.13 (Ewing)**.**

Proof. □

Question 2.14 (Berkowitz)**.**

Proof. □

Question 2.15 (Kelly)**.**

Proof. ☐

Question 2.16 (Wang).

Proof. ☐

Question 2.17 (Berkowitz).

Proof. ☐

Question 2.18 (Golinski).

Proof. ☐

Question 2.19 (Zheng).

Proof. ☐

Question 2.20 (Zhang).

Proof. ☐

Question 2.21 (Prohovich).

Proof. ☐

Question 2.22 (Fasano).

Proof. ☐

Question 2.23 (Ewing).

Proof. ☐

Question 2.24 (Hafer).

Proof. ☐

Question 2.25 (Macula).

Proof. ☐

Question 2.26 (Fernandez).

Proof. ☐

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