In a Pseudometric Space (X,d), is the set $B(x;r) = \{y \in X: d(x,y) \leq r\}$ necessarily a closed set?

Suppose you had the point set 'S' centered around point 'a' and radius 'r'. A second point 'b' such that $d(a,b) > r$ denotes a point from the complement of S. Let a third point 'c' denote some point within a cell centered at 'b', such that $d(b,c) \leq \varepsilon$

By the triangle inequality:
$r < d(a,b) \leq d(a,c) + d(b,c) \leq d(a,c) + \varepsilon$

If $d(a,c) > r$, the point 'c' will not meet S
If $d(a,c) \leq r$,
Subtracting $d(a,c)$ from the inequality
$0 \leq r - d(a,c) < d(a,b) - d(a,c) \leq \varepsilon$
since $d(a,b) - d(a,c) > 0$, a new ε can always be generated, for example $\varepsilon_2=(d(a,b) - d(a,c))/2$, removing point 'c' from the cell

The complement of S must then be open, as any limit point captured in the cell of a point from this region could always be removed from the cell by reducing ε. Since the complement of S is open, S must be closed.