

Matched subspace detectors and distributed detection

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1 Detection Performance Monte-Carlo vs Asymptotic ROC

$$\mathcal{H}_0 : x[n] = w[n], \quad n = 0, 1, \dots, N-1 \quad (1)$$

$$\mathcal{H}_1 : x[n] = A + w[n], \quad n = 0, 1, \dots, N-1 \quad (2)$$

Known signal($A > 0$) in known noise level: MC involves creation of the random variables under each hypothesis, and the calculation of the expectation of the random variables over multiple trials. T_0 and T_1 are the test statistic under each hypothesis $A = 0$ and $A \neq 0$

$$T_0(x) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sim \mathcal{N}(0, \sigma^2/N) \quad (3)$$

$$Similarly T_1(x) \sim \mathcal{N}(A, \sigma^2/N) \quad (4)$$

$$d^2 = \frac{NA^2}{\sigma^2} \quad (5)$$

$$Threshold \ min P_e \ case(\gamma) : \frac{d}{2} \quad (6)$$

Asymptotic Performance :

$$P_{fa} = Pr(T_0(x) > \gamma) = Q\left(\frac{\gamma}{\sqrt{\sigma^2/N}}\right) \quad (7)$$

$$P_D = Pr(T_1(x) > \gamma) = Q(Q^{-1}(P_{fa}) - \sqrt{d^2}) \quad (8)$$

Known signal($-\infty < A < \infty$) in unknown noise level: The unknown parameters are estimated using ML, and the GLRT is formed.

$$GLRT :: \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} \right)^{-N/2} \exp \left(\frac{-\sum_0^{N-1} (x(n) - A)^2}{2\hat{\sigma}_1^2} + \frac{\sum_0^{N-1} x(n)^2}{2\hat{\sigma}_0^2} \right) \quad (9)$$

$$LGLRT(unknown \sigma) : 2\ln(\hat{l}(y)) = N\ln\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right) = N\ln\left(1 + \frac{\bar{x}^2}{\hat{\sigma}_1^2}\right) \quad (10)$$

$$T_0(x) = \frac{\bar{x}_0^2}{\hat{\sigma}_0^2} \quad T_1(x) = \frac{\bar{x}_1^2}{\hat{\sigma}_1^2} \quad (11)$$

$$\hat{\sigma}_i^2 = \frac{1}{N} \sum_0^{N-1} x_i^2 - \bar{x}_i^2 \quad (12)$$

Asymptotic Performance :

$$P_{fa} = Pr(T_0(x) > \sqrt{\gamma}) + Pr(T_0(x) < -\sqrt{\gamma}) = 2Q(\sqrt{\gamma}) \quad (13)$$

$$P_D = Pr(T_1(x) > \sqrt{\gamma}) + Pr(T_1(x) < -\sqrt{\gamma}) = Q(Q^{-1}(\frac{P_{fa}}{2}) - \sqrt{d^2}) + Q(Q^{-1}(\frac{P_{fa}}{2}) + \sqrt{d^2}) \quad (14)$$

1.1 Matlab Code

```
%Written by K.Narayanan 05-25-05
% Asymptotic vs monte-carlo
%

figure(1)
clear,clf
sig2=1;N=30;
nreal=5000;

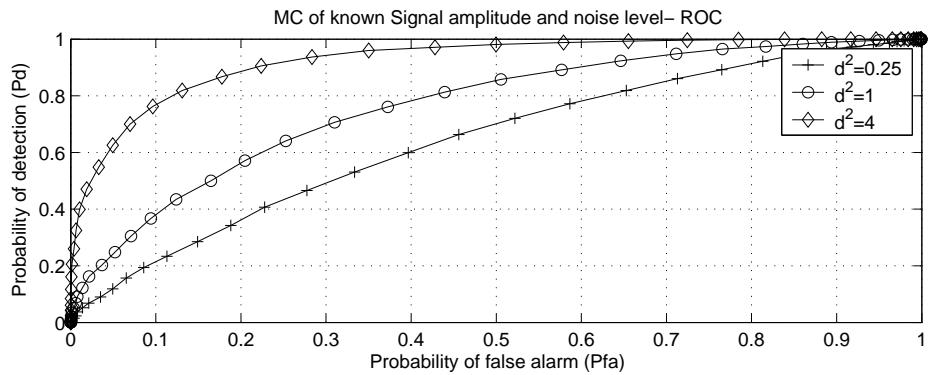
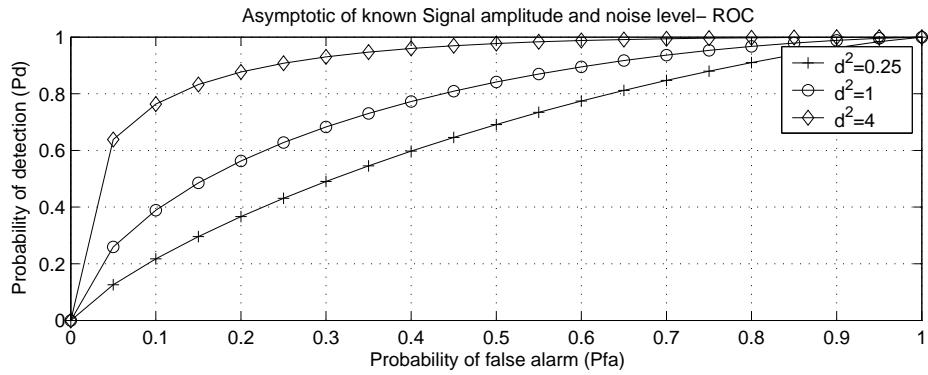
d2 =[0.25 , 1.0, 4.0];    ltype=[ '-k+'; '-ko' ;'-kd' ];
pfa = [0:.05:1];
for j = 1: length(d2)
    for i = 1:length(pfa)
        %      ss=sprintf('j:%s i:%s ',num2str(j),num2str(i));
        %      disp(ss)

        threshold(j,i) = qinv(pfa(i));
        pd(j,i)= q(threshold(j,i) - sqrt(d2(j)));
    end

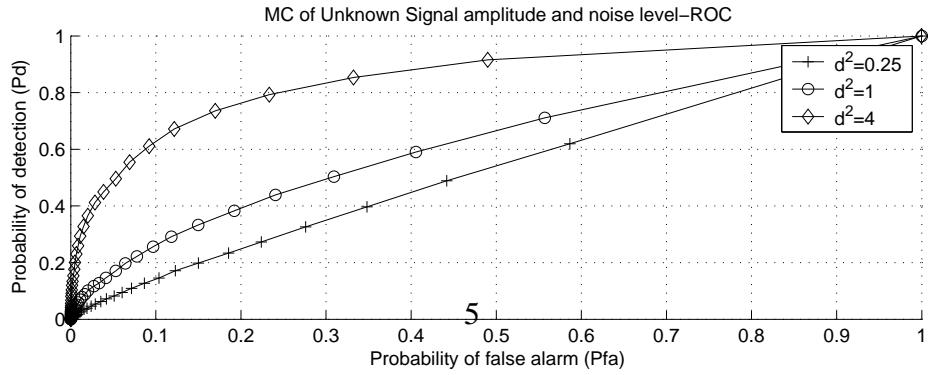
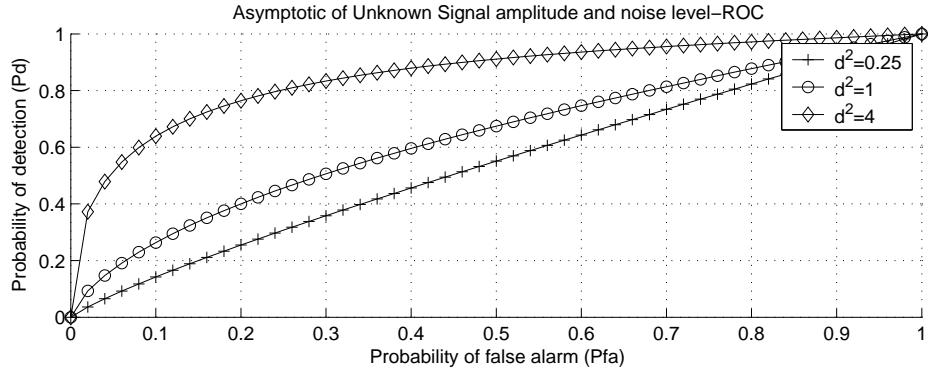
    subplot(211)
    plot(pfa(:),pd(j,:),ltype(j,:)),
    hold on

    % code for MC
    randn('seed',0)
    A=sqrt(d2(j)*sig2/N);
    for i=1:nreal
        x0=sqrt(sig2)*randn(N,1);x1=x0+A;
        T0(i,1)=mean(x0);
        T1(i,1)=mean(x1);
    end

```



(a)



(b)

Figure 1: Asymptotic and Monte-Carlo ROC (a) Known Noise Level, (b) Unknown Noise Level

```

[Pfa,Pd]=roccurve(T0,T1,51);

subplot(212)
plot(Pfa,Pd,ltype(j,:))
hold on
end
subplot(211)
grid on
xlabel('Probability of false alarm (Pfa)')
ylabel('Probability of detection (Pd)')
title('Asymptotic of known Signal amplitude and noise level- ROC');
leg1 = sprintf('d^2=%s',num2str(d2(1)));
leg2 = sprintf('d^2=%s',num2str(d2(2)));
leg3 = sprintf('d^2=%s',num2str(d2(3)));
legend(leg1,leg2,leg3);

subplot(212)
grid on
xlabel('Probability of false alarm (Pfa)')
ylabel('Probability of detection (Pd)')
title('MC of known Signal amplitude and noise level- ROC');
leg1 = sprintf('d^2=%s',num2str(d2(1)));
leg2 = sprintf('d^2=%s',num2str(d2(2)));
leg3 = sprintf('d^2=%s',num2str(d2(3)));
legend(leg1,leg2,leg3);

figure(2)
clear,clf
d2=[0.25 , 1.0, 4.0]; ltype=['-k+';'-ko';'-kd'];

sig2=1;N=30;
nreal=5000;

for j = 1:length(d2)

    for i=1:51
        Pfaa(i,1)=(i-1)/50;
        u=Qinv(Pfaa(i)/2);
        Pda(i,1)=Q(u-sqrt(d2(j)))+Q(u+sqrt(d2(j)));
    end

    subplot(211)
    hold on
    plot(Pfaa,Pda,ltype(j,:))


```

```

randn('seed',0)
A=sqrt(d2(j)*sig2/N);
for i=1:nreal
    x0=sqrt(sig2)*randn(N,1);x1=x0+A;
    y0=mean(x0)^2/(x0'*x0/N-mean(x0)^2);
    y1=mean(x1)^2/(x1'*x1/N-mean(x1)^2);
    T0(i,1)=N*log(1+y0);
    T1(i,1)=N*log(1+y1);
end

[Pfa,Pd]=roccurve(T0,T1,51);

subplot(212)
hold on
plot(Pfa,Pd,ltype(j,:))

end

subplot(211)
grid
xlabel('Probability of false alarm (Pfa)')
ylabel('Probability of detection (Pd)')
title('Asymptotic of Unknown Signal amplitude and noise level-ROC');

leg1 = sprintf('d^2=%s',num2str(d2(1)));
leg2 = sprintf('d^2=%s',num2str(d2(2)));
leg3 = sprintf('d^2=%s',num2str(d2(3)));
legend(leg1,leg2,leg3);

subplot(212)
grid
xlabel('Probability of false alarm (Pfa)')
ylabel('Probability of detection (Pd)')
title('MC of Unknown Signal amplitude and noise level-ROC')

leg1 = sprintf('d^2=%s',num2str(d2(1)));
leg2 = sprintf('d^2=%s',num2str(d2(2)));
leg3 = sprintf('d^2=%s',num2str(d2(3)));
legend(leg1,leg2,leg3);

```

2 Matched Subspace Detectors and Distributed Detection

A review presentation of:

1. Signal processing applications of oblique projection operators-RT Behrens, LL Scharf,1994
2. Matched subspace detectors LL Scharf and B Friedlander, 1994
3. Statistical Signal Processing, LL Scharf, Addison Wesley, 1991
4. Adaptive Subspace Detectors, SKraut, LL Scharf and T McWhorter 2001
5. Distributed Detection-Notes by Michael McCloud

2.1 Agenda

- Projections
 - Orthogonal : $P^2 = P, P^T = P$
 - Oblique : $P^T \neq P$
- Pseudo-Inverse, SVD, Whitening
- Four Detection Problems for $y = \mu\psi + \eta$
 - Coherent signal,known noise level:(i)Matched filter(MF):
$$n = \frac{\psi^T y}{\sigma \sqrt{\psi^T \psi}}$$
 - Coherent signal,unknown noise level:(ii)Constant false alarm rate (CFAR)MF:
$$\cos = \frac{\psi^T y}{\sqrt{\psi^T \psi} \sqrt{y^T y}} = \frac{t}{|t|^2+1}, \quad t = \frac{\psi^T y}{\sqrt{\psi^T \psi} \sqrt{y^T P_\psi^\perp y}}$$
 - In Coherent signal,known noise level:(iii) Matches subspace filter(MSD):
$$|n|^2 = \frac{|\psi^T y|^2}{\sigma^2 \psi^T \psi}$$
 - In Coherent signal,unknown noise level:(iv)CFAR MSD:
$$|\cos|^2 = \frac{|\psi^T y|^2}{(\psi^T \psi)(y^T y)} = \frac{F}{F+1}$$
- Application to Distributed detection

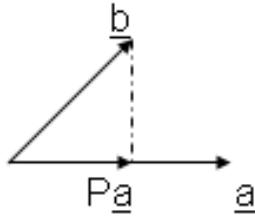


Figure 2: Projection of b on a

3 Mathematical Preliminaries

3.1 Projection, Pseudo-Inverse

- Orthogonal Projection of b on a : $P_a = \frac{aa^T b}{a^T a}$
- Projection Matrix $P_a = A(A^T A)^{-1} A^T$
 P_a is symmetric and idempotent for \perp projection
- $Ax = y$, Over determined case, $x = A^\# y$ is a least square solution
 $A^\# = (A^T A)^{-1}$ or $A^\# = A^T (AA^T)^{-1}$ or Moore – Penrose conditions
- Relationship between Projection and Pseudo-inverse operators: $P_a = AA^\#$

3.2 SVD, Whitening Transformation

- SVD: $A_{m \times n} = Q_{m \times m} \Lambda_{m \times n} V_{n \times n}^T$; $A^T A = Q \Lambda^T \Lambda Q^T$; $AA^T = V \Lambda \Lambda^T V^T$
- Rank m projection operator from eigen functions of SVD, $P_m = QQ^T$
- Pseudo-Inverse and Singular values (σ_i)
 $A^\# = V \Lambda^\# Q^T$, $\Lambda^\# = \text{diag} \left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \dots, \frac{1}{\sigma_r} \right)$
- Whitening Transformation:
 $Y = \mu X + v$; Y is distributed gaussian $\mathcal{N}(\mu x, R)$, $R = E(vv^T)$
 $Z = \mu \phi + \omega$; Z is distributed gaussian $\mathcal{N}(\mu \phi, \sigma^2 I)$

whitening or decorrelation of R is obtained by the transformation: $z = R^{\frac{-1}{2}}Y$, $\phi = R^{\frac{-1}{2}}X$, $\omega = R^{\frac{-1}{2}}v$

3.3 Pseudo-inverse and oblique projection

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} H^T H & H^T S \\ S^T H & S^T S \end{pmatrix}^{-1} \begin{pmatrix} H^T \\ S^T \end{pmatrix} y$$

$$\begin{aligned} \text{Woodbury Identity: } (A \pm XBX^T)^{-1} &= A^{-1} \mp A^{-1}X(B^{-1} \pm X^TA^{-1}X)^{-1}X^TA^{-1} \\ \theta &= [[H^T H - H^T S(S^T S)^{-1}S^T H]^{-1}H^T - \\ &\quad (H^T H)^{-1}H^T S[S^T S - S^T H(H^T H)^{-1}H^T S]^{-1}S^T]y \\ \theta &= [(H^T H)^{-1}H^T + (H^T H)^{-1}H^T S[(S^T S) - \\ &\quad S^T H(H^T H)^{-1}H^T S]^{-1}S^T H(H^T H)^{-1}H^T - \\ &\quad (H^T H)^{-1}H^T S[S^T S - S^T H(H^T H)^{-1}H^T S]^{-1}S^T]y \\ \theta &= (H^T H)^{-1}H^T [I - S(S^T(I - P_H)S)^{-1}S^T(I - P_H)]y \\ H\theta &= E_{HS}y \\ E_{HS} &= P_H(I - S(S^T P_H^\perp S)^{-1}S^T P_H^\perp) \\ E_{HS} &= H(H^T P_S^\perp H)^{-1}H^T P_S^\perp \end{aligned}$$

4 Structured Noise Model and Projections

In a multi-user environment, a receiver sees more than one transmission. The competing users transmission can be modeled as structured noise, since its characteristics can be modelled. A receiver in light of the interference characteristics can nullify it to process the desired signal.

$$Y_n = Hx_n + S_n\phi + v_n, \text{ } n - \text{dimensional observations}$$

H and S are modal matrices of the signal(rank p) and interference(rank t)respectively, such that $p + t < n$ i.e. accounts for non-zero null space

$$\text{Pseudo-inverse: } A^\# = \begin{pmatrix} (H^T P_S^\perp H)^{-1}H^T P_S^\perp \\ (S^T P_H^\perp S)^{-1}S^T P_H^\perp \end{pmatrix}$$

Note: P_S^\perp is a null steering operator that produces an interference free vector, which is resolved further onto P_G^\perp and P_G for detecting hypothesis. Also given: $P_{HS}^\perp = I - P_{HS}$, $P_S^\perp = I - P_S$, the following relations between \perp and oblique projections can be deduced:

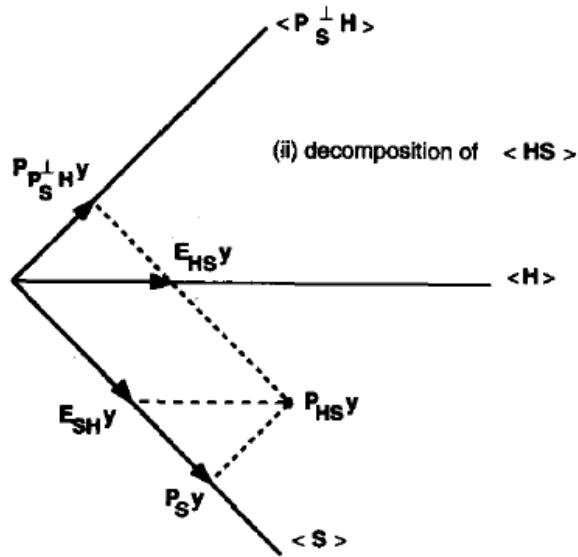


Figure 3: Signal and Interference subspace, oblique and orthogonal operators

$$\begin{aligned}
 P_{HS} &= E_{HS} + E_{SH} \\
 P_{HS} &= P_S + P_{P_S^\perp H}(P_G) \\
 P_{P_S^\perp H}(P_G) &= P_S^\perp - P_{HS}^\perp \\
 P_{P_S^\perp H} &= P_S^\perp P_G P_S^\perp
 \end{aligned}$$

5 Detection and Hypothesis Testing

- Likelihood function is used to form the GLRT
- P_{FA} is chosen for the test which determines the threshold
- Performance ROC is: P_D vs P_{FA} or P_{error}
- When noise level(σ) is unknown it is estimated using ML

$$\text{Likelihood Function: } f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{aligned}
GLRT : &= \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} \right)^{-N/2} e^{\left(\frac{-\|\hat{n}_1\|_2^2 + \|\hat{n}_0\|_2^2}{2\sigma_1^2} \right)} \\
LGLRT(knownd\sigma) : 2ln\hat{l}(y) &= \frac{1}{\sigma^2} [\|\hat{n}_0\|_2^2 - \|\hat{n}_1\|_2^2] \\
LGLRT(unknown\sigma) : \hat{l}(y)^{2/N} &= \frac{\|\hat{n}_0\|_2^2}{\|\hat{n}_1\|_2^2} \\
Performance : P_{fa} &= P[LGLRT(y) > \eta | H_0] \\
P_d &= P[LGLRT(y) > \eta | H_1] \\
\phi(\mathbf{y}) &= \begin{cases} 1 \sim H_1, & L(\mathbf{y}) > \eta \\ 0 \sim H_0, & L(\mathbf{y}) \leq \eta \end{cases}
\end{aligned}$$

6 (i) Detection of Coherent signal in known noise level

The noise estimates under each hypothesis is obtained by using the null steering operator, that removes interference and or signal

$$\begin{aligned}
\hat{n}_1 &= \begin{cases} P_S^\perp y, & \hat{\mu} \leq 0 \\ P_{HS}^\perp y, & \hat{\mu} > 0 \end{cases} \\
\hat{n}_0 &= P_S^\perp y
\end{aligned}$$

The variance of estimate is found using CRB, as the estimate is efficient. Matched filter statistic, normalized to unit variance is :

$$\begin{aligned}
Var(\hat{\mu}(R) - \mu) &= - \left(E \left\{ \frac{d^2 \ln p(R|\mu)}{d\mu^2} \right\} \right)^{-1} = \sigma^2 (x^T P_S^\perp x)^{-1} \\
\hat{\mu}(R) &= \mathcal{N} \left[\frac{\mu}{\sigma} (x^T P_S^\perp x)^{0.5}, 1 \right], \\
\mu &= 0 \text{ under } H_0, \mu > 0 \text{ under } H_1
\end{aligned}$$

6.1 Detection of Coherent signal in known noise level (contd.)

Performance of one-sided test,i.e. $H_0 : \mu = 0, H_1 : \mu > 0$:

$$GLRT L_1(y) = \begin{cases} 0, & \hat{\mu} \leq 0 \\ \frac{1}{\sigma^2} (y^T P_S^\perp y - y^T P_{HS}^\perp y), & \hat{\mu} > 0 \end{cases}$$

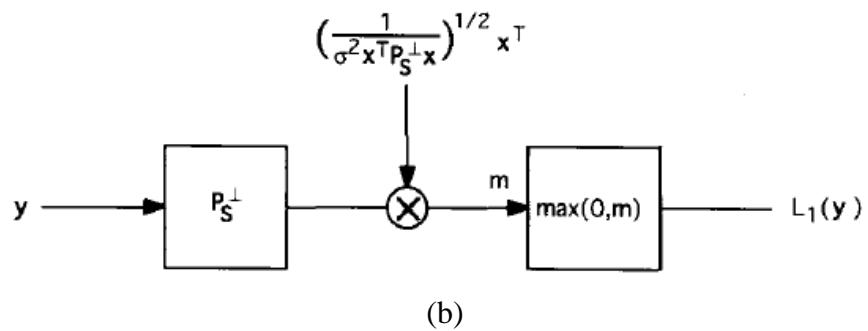
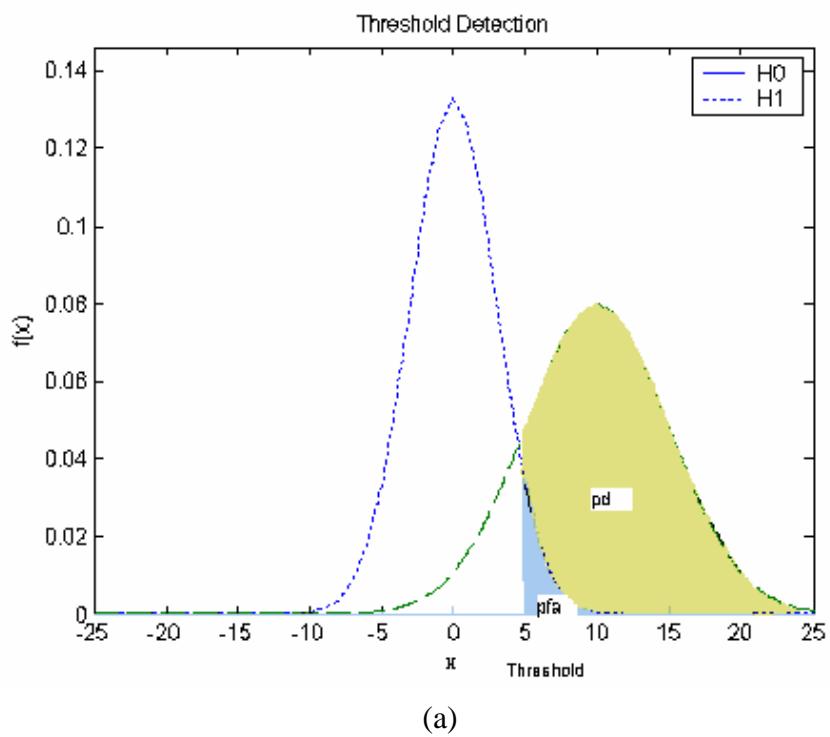


Figure 4: (a) $H_0 = 0, H_1 = A > 0$, one-sided Hypothesis Test, (b)MF-detector

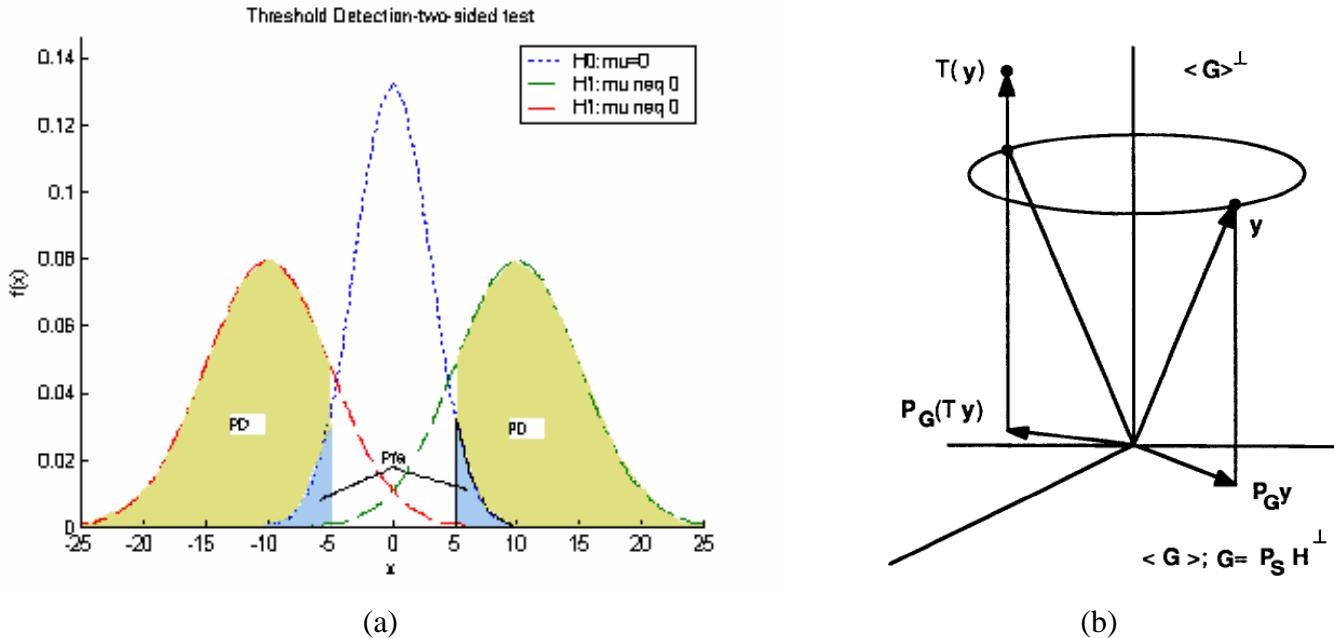


Figure 5: (a)two-sided hypothesis test (b)invariances of the GLRT $L_1(y)$

$$\begin{aligned} GLRT L_1(y) &= \begin{cases} 0, & \hat{\mu} \leq 0 \\ \frac{1}{\sigma^2} (y^T P_{P_S^\perp H} y), & \hat{\mu} > 0 \end{cases} \\ P_{fa} &= 1 - P[\mathcal{N}(0, 1) \leq \eta], \eta > 0 \\ P_D &= 1 - P[\mathcal{N}(\lambda, 1) \leq \eta], \lambda : \frac{\mu}{\sigma} (x^T P_S^\perp x)^{0.5} \end{aligned}$$

Performance of two-sided test,i.e. $H_0 : \mu = 0, H_1 : \mu \neq 0$ is given below. $L_1(y)$ exhibits Invariance to translation and rotations in $\langle P_S^\perp x \rangle^\perp$ plane by construction, shown in figure[5].

$$\begin{aligned} L_1(y) &= \frac{1}{\sigma^2} y^T P_{P_S^\perp x} y : \chi_1^2(\lambda^2), \lambda^2 = \frac{\mu^2}{\sigma^2} x^T P_S^\perp x \\ P_{fa} &= 1 - P[\chi_1^2(0) \leq \eta] \\ P_D &= 1 - P[\chi_1^2(\lambda^2) \leq \eta] \end{aligned}$$

7 (ii) Detection of Coherent signal in unknown noise level

The LGLRT for the unknown noise level is found taking the N/2-root :

$$GLRT L_2(y) = \begin{cases} 1, & \hat{\mu} \leq 0 \\ \frac{y^T P_S^\perp y}{y^T P_{HS}^\perp y}, & \hat{\mu} > 0 \end{cases}$$

L2 -1 is also a monotonic use this as the LGLRT L2:

$$\begin{aligned} GLRT L_2(y) - 1 &= \begin{cases} 0, & \hat{\mu} \leq 0 \\ \frac{y^T (P_S^\perp - P_{HS}^\perp) y}{y^T P_{HS}^\perp y}, & \hat{\mu} > 0 \end{cases} \\ &= \begin{cases} 0, & \hat{\mu} \leq 0 \\ \frac{y^T P_S^\perp P_G P_S^\perp y}{y^T P_S^\perp P_G^\perp P_S^\perp y}, & \hat{\mu} > 0 \end{cases} \end{aligned}$$

Detector structure and invariances are shown in figure[6]. Noise is estimated by using the null steering operator it is the received sample variance. The CFAR matched filter statistic is 'T' distributed.

$$\begin{aligned} \text{Noise; level ;} \hat{\sigma}^2 &= \frac{y^T P_\Psi^\perp y}{N-1} \\ \text{Test statistic : } t &= \frac{\Psi^T y}{\sqrt{\Psi^T \Psi} \sqrt{y^T P_\Psi^\perp y}} \sim \frac{\mathcal{N}(\frac{\mu}{\sigma} (x^T P_S^\perp x)^{0.5}, 1)}{\chi_{N-t-1}^2(\lambda^2)} \end{aligned}$$

Performance of detector :

$$P_{fa} = 1 - P[t_{1,N-t-1}(0) \leq \eta]$$

$$P_D = 1 - P[t_{1,N-t-1}(\lambda^2) \leq \eta], \lambda^2 : \frac{\mu^2}{\sigma^2} (x^T P_S^\perp x)$$

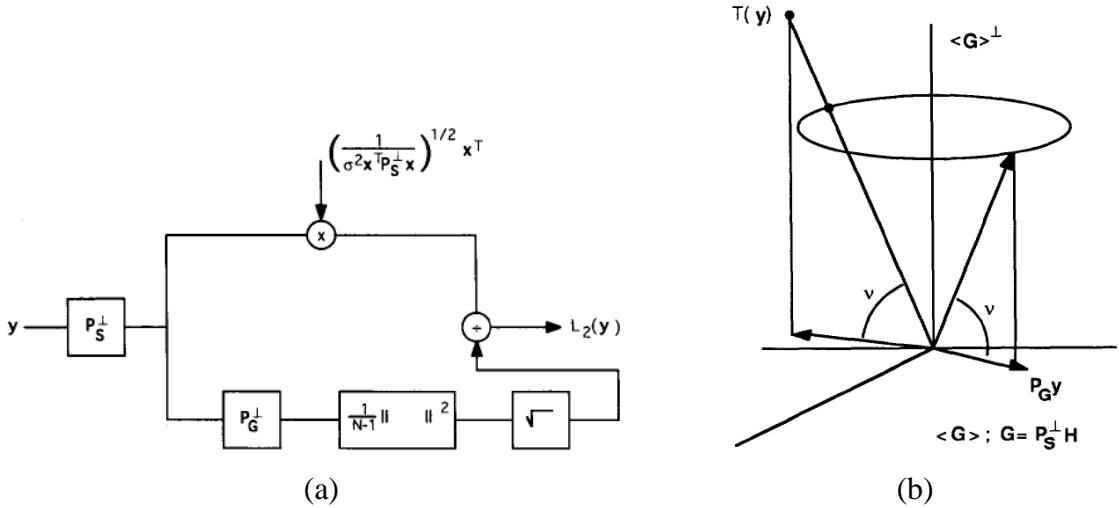


Figure 6: (a)CFAR MF detector structure (b)invariances of the GLRT $L_2(y)$

8 (iii) and (iv) Incoherent Detection: MSD,CFAR-MSD

When the signal phase is not known, the statistics obtained in the coherent case is multiplied by its conjugate to obtain energy and eliminate the phase terms. Detector structure shown in figure[7]

Matched Subspace Detectors

Description	MSD
Test Stat:	$ n ^2 = \frac{ \psi^T y ^2}{\sigma^2 \psi^T \psi}$
Threshold:	$P_{fa} = 1 - P[\chi_p^2(0) \leq \eta]$
Performance:	$P_D = 1 - P[\chi_p^2(\lambda^2) \leq \eta], \lambda^2 : \frac{\mu^2}{\sigma^2} (x^T P_S^\perp x)$

Matched Subspace Detectors

Description	CFAR-MSD
Test Stat:	$ \cos ^2 = \frac{ \psi^T y ^2}{(\psi^T \psi)(y^T y)}$
Threshold:	$P_{fa} = 1 - P[F_{p,s-p}(0) \leq \eta]$
Performance:	$P_D = 1 - P[F_{p,s-p}(\lambda^2) \leq \eta], \lambda^2 : \frac{\mu^2}{\sigma^2} (x^T P_S^\perp x)$

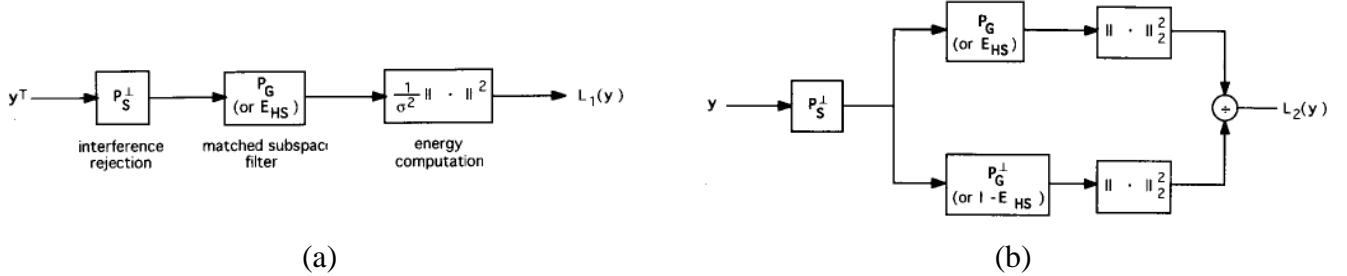


Figure 7: (a)MSD detector structure (b)CFAR MSD detector structure

9 Distributed Detection- Strong target,known noise level[5]

- Measurement in the k^{th} sensor in the absence of interference is: $y_k = \mu_k s_k + n_k$
- ML estimate of μ_k is obtained by minimizing the quadratic in the exponent of the likelihood function, i.e $\nabla_{\mu_k} f_1(y/\mu_k) = 0$

$$\begin{aligned}
 \text{Likelihood function under } H_1 : f_1(y|\mu_k) &= \frac{1}{\sigma^{2NK}} \exp \left[-\frac{1}{\sigma^2} \sum_k \|y_k - \mu_k S_k\|^2 \right] \\
 \nabla_{\mu_k} \frac{1}{\sigma^2} \sum_k (y_k - \mu_k S_k)^* (y_k - \mu_k S_k) &= 0 \\
 \nabla_{\mu_k} \frac{1}{\sigma^2} \sum_k (y_k^* y_k - 2\mu_k S_k^* y_k + \mu_k^2 S_k^* S_k) &= 0 \\
 \nabla_{\mu_k} \frac{1}{\sigma^2} \sum_k (y_k^* y_k + \left(\mu_k - \frac{S_k^* y_k}{S_k^* S_k} \right)^2 S_k^* S_k - \frac{(S_k^* y_k)^2}{S_k^* S_k}) &= 0 \\
 \hat{\mu}_k &= \frac{S_k^* y_k}{S_k^* S_k} \\
 \text{Likelihood function: } f_1(y|\hat{\mu}_k) &= \frac{1}{\sigma^{2NK}} \exp \left[-\frac{1}{\sigma^2} \sum_k \left(\|y_k\|^2 - \frac{y_k^* S_k S_k^* y_k}{S_k^* S_k} \right) \right] \\
 f_0(y) &= \frac{1}{\sigma^{2NK}} \exp \frac{1}{\sigma^2} \sum_k \|y_k\|^2 \\
 \text{LLRT: } \log \frac{f_1(y|\hat{\mu}_k)}{f_0(y)} &= \sum_k \left(\frac{y_k^* S_k S_k^* y_k}{S_k^* S_k} \right)
 \end{aligned}$$

10 Distributed Detection- Strong target, unknown noise level[5]

- Estimates of the Noise from the observations under H0 and H1:

$$\begin{aligned}
\hat{\sigma}_1^2 &= \frac{\sum_k \|P_{S_k}^\perp y_k\|^2}{NK} \\
\hat{\sigma}_0^2 &= \frac{\sum_k \|y_k\|^2}{NK} \\
N/2 \text{ root LGLRT} : \hat{l} &= \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \\
\hat{l} &= \frac{\sum_k \|P_{S_k}^\perp y_k\|^2 + \sum_k \|P_{S_k} y_k\|^2}{\sum_k \|P_{S_k}^\perp y_k\|^2} \\
\hat{L} = \hat{l} - 1 &= \frac{\sum_k \|P_{S_k} y_k\|^2}{\sum_k \|P_{S_k}^\perp y_k\|^2}
\end{aligned}$$

11 Distributed Detection- Weak target, known noise level[5]

- Weak target case only one of the k sensors detect the target, \hat{k} is chosen to minimize the exponent and $\mu_{\hat{k}}$ is the resulting ML estimate:

$$\begin{aligned}
\text{Likelihood function: } f_1(y|k, \mu_k) &= \frac{1}{\sigma^{2NK}} \exp \left[-\frac{1}{\sigma^2} \left(\sum_{l \neq k} \|y_l\|^2 + \|y_k - \mu_k S_k\|^2 \right) \right] \\
\nabla_{\mu_k, k} \frac{1}{\sigma^2} \left(\sum_{l \neq k} \|y_l\|^2 + (y_k - \mu_k S_k)^* (y_k - \mu_k S_k) \right) &= 0 \\
\nabla_{\mu_k, k} \frac{1}{\sigma^2} \left(\sum_{l \neq k} \|y_l\|^2 + (y_k^* y_k - 2\mu_k S_k^* y_k + \mu_k^2 S_k^* S_k) \right) &= 0, \quad \hat{k} = \underset{k}{\operatorname{argmax}} \left(\frac{y_k^* S_k S_k^* y_k}{S_k^* S_k} \right) = \underset{k}{\operatorname{argmax}} \|P_{S_k} y_k\|^2 \\
\nabla_{\mu_k, k} \frac{1}{\sigma^2} \left(\sum_{l \neq k} \|y_l\|^2 + \left[y_k^* y_k + \left(\mu_k - \frac{S_k^* y_k}{S_k^* S_k} \right)^2 S_k^* S_k - \frac{(S_k^* y_k)^2}{S_k^* S_k} \right] \right) &= 0, \quad \hat{\mu}_{\hat{k}} = \frac{S_{\hat{k}}^* y_{\hat{k}}}{S_{\hat{k}}^* S_{\hat{k}}} \\
\text{Likelihood function: } f_1(y|k, \hat{\mu}_k) &= \frac{1}{\sigma^{2NK}} \exp \left[-\frac{1}{\sigma^2} \left(\sum_k \|y_k\|^2 - \underset{k}{\operatorname{argmax}} \frac{y_k^* S_k S_k^* y_k}{S_k^* S_k} \right) \right]
\end{aligned}$$

$$f_0(y) = \frac{1}{\sigma^{2NK}} \exp \frac{1}{\sigma^2} \sum_k \|y_k\|^2$$

$$LLRT : \log \frac{f_1(y|k, \hat{\mu}_k)}{f_0(y)} = \underset{k}{argmax} \left(\frac{y_k^* S_k S_k^* y_k}{S_k^* S_k} \right)$$

Note: This is a order-statistic, whos pdf may be found.

12 Distributed Detection- Weak target, unknown noise level[5]

$$\begin{aligned}\hat{\sigma}_1^2 &= \frac{1}{NK} \left[\sum_{l \neq m} y_l^2 + \|P_{S_m}^\perp y_m\|^2 \right], m = \underset{k}{argmax} \|P_{S_k} y_k\|^2 \\ &= \frac{1}{NK} [\sum y_k^2 - \|P_{S_m} y_m\|^2], P_{S_m}^\perp = 1 - P_{S_m} \\ \hat{\sigma}_0^2 &= \frac{\sum_k \|y_k\|^2}{NK} \\ N/2 \text{ root LGLRT} : \hat{l} &= \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \\ \hat{l} &= \frac{1}{1 - \frac{\|P_{S_m} y_m\|^2}{\sum y_k^2}}, m = \underset{k}{argmax} \|P_{S_k} y_k\|^2 \\ \hat{L} = 1 - \frac{1}{\hat{l}} &= \frac{\|P_{S_m} y_m\|^2}{\sum y_k^2}\end{aligned}$$

Note: This is a order-statistic, whos pdf may be found as a maximisation of a F statistic if Numerator and Denominator are independent χ^2 random-variables.