MATH 341 / 650.3 Spring 2020 Homework #3

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Problem 1

Assume $\mathcal{F}$ = binomial with n fixed and $\mathbb{P}(\theta) = \text{Beta}(\alpha, \beta)$. To solve numerically, use `R` on your computer (or rdrr.io online).

(a) [harder] Design a prior where you believe $\mathbb{E}[\theta] = 0.5$ and you feel as if your belief represents information contained in five coin flips.

If $\text{E}(\theta) = 0.5$

$$0.5 = \frac{\alpha}{\alpha + \beta}$$
$$\alpha = \beta$$

If $a = b$ and our belief represent 5 coin flips then we can say our prior is

$$Beta(2.5, 2.5)$$

(b) [harder] You flip the same coin 100 times and you observe 39 heads. Test the hypothesis that this coin is fair given prior information from (a). Write out the hypotheses for this test, declare an α and use the credible region method. Make sure you say whether you retain or reject the null and justify why.

Let $\alpha = 5\%$ and $\delta = 0.01$

$$H_0 : \theta = 0.5$$
$$H_a : \theta \neq 0.5$$

$$CR_{\theta,95\%} = [\texttt{qbeta(0.025, 41.5, 63.5)}, \texttt{qbeta(0.975, 41.5, 63.5)}]$$
$$= [0.304, 0.490]$$

Therefore, we reject the null hypothesis and accept the alternative hypothesis since θ is not in the credible region.

(c) [harder] Let's say you wanted to test whether the coin is fair but you are indifferent to any θ which is different from fair by a margin of δ that you pick. Write out the hypotheses for this test, declare an α and calculate the Bayesian p-val for the test to determine if you should retain or reject H_0.

Let $\alpha = 5\%$ and $\delta = 0.01$

$$
\begin{aligned}
\mathbb{P}\left(\theta \in [\theta_0 \pm \delta]]\right) &= \texttt{qbeta(0.51, 41.5, 63.5)} - \texttt{qbeta(0.49, 41.5, 63.5)} \\
&= 0.002
\end{aligned}
$$

Therefore, we retain H_0 since $0.003 < \alpha$

(d) [harder] Test the hypothesis that this coin has a bias towards tails given prior information from (a) and the data from (b). Write out the hypotheses for this test, declare an α and calculate the Bayesian p-val for the test to determine if you should retain or reject H_0.

$$
\begin{aligned}
H_a &: \theta < 0.5 \\
H_0 &: \theta \geq 0.5
\end{aligned}
$$

If $\alpha = 0.05$,

$$
\begin{aligned}
\mathbb{P}\left(\theta \geq 0.5\right) &= \int_0^1 \text{Beta}\left(41.5,\, 63.5\right) d\theta - \int_0^{0.5} \text{Beta}\left(41.5,\, 63.5\right) d\theta \\
&= 1 - \texttt{pbeta}(0.5, 41.5, 63.5) \\
&= 0.015
\end{aligned}
$$

(e) [easy] Assume again the prior information from (a). What is the shrinkage proportion ρ for this prior when estimating θ via $\hat{\theta}_{\text{MMSE}}$?.

$$
\rho = \frac{\alpha + \beta}{n + \alpha + \beta} = \frac{2.5 + 2.5}{100 + 2.5 + 2.5} = 0.048
$$

(f) [difficult] Prove that $\hat{\theta}_{\text{MMSE}}$ is a biased estimator (i.e. its expectation is *not* θ).

Recall,

$$\hat{\theta}_{\text{MMSE}} = \frac{X+\alpha}{n+\alpha+\beta}$$

$$\begin{aligned}
\mathbb{E}\left[\hat{\theta}_{\text{MMSE}}\right] &= \mathbb{E}\left[\frac{X+\alpha}{n+\alpha+\beta}\right] \\
&= (\frac{1}{n+\alpha+\beta})\mathbb{E}\left[X+\alpha\right] \\
&= (\frac{1}{n+\alpha+\beta})\mathbb{E}\left[X\right]+\alpha \\
&= (\frac{1}{n+\alpha+\beta})n\theta+\alpha \neq \theta
\end{aligned}$$

(g) [easy] Prove that $\lim_{n\to\infty} \rho = 0$ and therefore this bias $\to 0$ as your dataset gets larger.

$$\lim_{n\to\infty} \rho = \lim_{n\to\infty} \frac{\alpha+\beta}{n+\alpha+\beta} = \lim_{n\to\infty} \frac{1}{n+\alpha+\beta} = 0$$

(h) [difficult] [MA] Why should anyone use shrinkage estimators if they're biased? Google it. Discuss.

- More stable estimates for true population parameters
- Reduce sampling and non-sampling errors
- Smoothed spatial fluctuations

(i) [difficult] Find the posterior predictive distribution, $X^* \mid X$. MA students — do this yourself. Other students — use my notes and justify each step. Remember, if $W \sim \text{Bernoulli}\,(p)$ then $\mathbb{P}\,(W=1)=p$. Use that trick.

$$W \sim \text{Bernoulli}\left(\hat{\theta}_{\text{MMSE}}\right) = \text{Bernoulli}\left(\frac{\alpha+n}{n+\alpha+\beta}\right) = \text{Bernoulli}\,(0.227)$$

(j) [difficult] Using the prior in (a), can $X^* \mid X$ ever be degenerate? Yes/no and explain.

Recall,

$$\mathbb{P}\left(X^* \mid X\right) = \text{Bernoulli}\left(\frac{\alpha+X}{\alpha+\beta+n}\right)$$

If we can in consideration the prior from (a) then,

$$\mathbb{P}\left(X^* \mid X\right) = \text{Bernoulli}\left(\frac{2.5 + X}{5 + n}\right)$$

Therefore NO, it can never be degenerate becasue n or X can not be negative.

(k) [difficult] Using the general prior $\mathbb{P}(\theta) = \text{Beta}(\alpha, \beta)$, can $X^* \mid X$ ever be degenerate? Yes/no and explain.

No since α and β have to be greater than 0.

(l) [harder] Given the prior information in (a) and the data in (b), what is the probability the next flip is a tail?

$$\begin{aligned}
\mathbb{P}\left(X^* = 0 \mid X\right) &= 1 - \hat{\theta}_{\text{MMSE}} \\
&= 1 - \frac{\sum x_i + \alpha}{\alpha + \beta + n} \\
&= 1 - \left(\frac{41.5 + 5}{63.5 + 41.5 + 100}\right) \\
&= 1 - 0.227 \\
&= 0.773
\end{aligned}$$