MATH 341 / 650.3 Spring 2020 Homework #5

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Sunday 29th March, 2020

Problem 1

These are questions about McGrayne's book, chapters 11–14.

(a) [easy] Did Savage like Shlaifer? Yes / No and why?

Yes because Savage describes Schlaifer as "hot as a pistol, sharp as a knife, clear as a bell, quick as a whip, and as exhausting as a marathon run"

(b) [easy] How did Neyman-Pearson approach statistical decision theory? What is the weakness to this approach? (p145)

The approach involved the test of hypotheses, confidence intervals, and unbiased estimation. Together, they could not infer what steps to take based on an observed sample outcome without thinking about the other potential sample outcomes that could have occurred.

(c) [easy] Who popularized "probability trees" (and "tree flipping") similar to exercises we did in Math 241?

Raiffa popularized "probability trees"

(d) [easy] Where are Bayesian methods taught more widely than any other discipline in academia?

They were taught more widly at wildcat drilling.

(e) [easy] Despite the popularity of his Bayesian textbook on business decision theory, why didn't Schlaifer's Bayesianism catch on in the real world of business executives making decisions?

Because Shlaifer was more of a theoretician and his inexperience on real world problems drove him away from that direction.

(f) [easy] Why did the pollsters fail (big time) to predict Harry Truman's victory in the 1948 presidential election?

Pollsters rejected randomized sampling and were relying on results that were formulated by outdated sampling designs that underrespresented the population.

(g) [easy] When does the diference between Bayesianism and Frequentism grow “immense”?

When you have an immense numbers of parameters.

(h) [easy] How did Mosteller demonstrate that Madison wrote the 12 Federalist papers of unknown authorship?

They found odds to be easier in computating as opposed to mathematical vocabulary.

(i) [easy] Write a one paragraph biography of John Tukey.

John Tukey was a professor at Princeton University were he helped to establish a cryptography think tank. Tukey was part of the CIA and the NSA. At the same time of the Cold War, Tukey used paper and pencil in order to sketch the surface-to-air missle system. With his expirience, Tukey helped to build the U-2 spy plane and helped NBC develop a network to predict and analyse elections.

(j) [easy] Why did Alfred Kinsey’s wife want to poison John Tukey?

Kinsey and Tukey did not agree with sampling methods and ways to control randomness.

(k) [easy] Tukey helped NBC with polling predictions for the presidential campaign. What was NBC’s polling algorithm based on?

The algorithm was based on empirical Bayes, were past data was used to construct the priors.

(l) [easy] Why is “objectivity an heirloom ... and ... a fallacy?”

Economist and engineers should not be expected to have similar argumens regarding the same data.

(m) [easy] Why do you think Tukey called Bayes Rule by the name “borrowing strength?”

Tukey borrowed rules from Bayes Rule.

(n) [easy] Why is it that we don’t know a lot of Bayes Rule’s modern history?

Tukey was banned from speaking and writting about Bayes rule.

(o) [easy] Generally speaking, how does Nate Silver predict elections?

Sliver combines different strengths from small samples and weights the results of other pollsets to keep track of how up to date is the data.

(p) [easy] How many Bayesians of import were there in 1979?

About 100.

(q) [easy] What advice did Chernoff give to Susan Holmes? (Note: Susan Holmes was my undergraduate advisor).

Chernoff advices Holmes to start with Bayesian thinking to get the right answer. Then to justify her answer in which ever way she prefered.

(r) [easy] How did Rasmussen's team estimate the probability of a nuclear plant core meltdown?

Using Raiffa's decision trees.

(s) [easy] How did the Three Mile Island accident vindicate Rasmussen's committee report?

Human error and the release of radioactivity.

Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	`qbeta`$(p,\ \alpha,\ \beta)$	`d-`$(x,\ \alpha,\ \beta)$	`p-`$(x,\ \alpha,\ \beta)$	`r-`$(\alpha,\ \beta)$
betabinomial	`qbetabinom`$(p,\ n,\ \alpha,\ \beta)$	`d-`$(x,\ n,\ \alpha,\ \beta)$	`p-`$(x,\ n,\ \alpha,\ \beta)$	`r-`$(n,\ \alpha,\ \beta)$
binomial	`qbinom`$(p,\ n,\ \theta)$	`d-`$(x,\ n,\ \theta)$	`p-`$(x,\ n,\ \theta)$	`r-`$(n,\ \theta)$
exponential	`qexp`$(p,\ \theta)$	`d-`$(x,\ \theta)$	`p-`$(x,\ \theta)$	`r-`(θ)
gamma	`qgamma`$(p,\ \alpha,\ \beta)$	`d-`$(x,\ \alpha,\ \beta)$	`p-`$(x,\ \alpha,\ \beta)$	`r-`$(\alpha,\ \beta)$
inversegamma	`qinvgamma`$(p,\ \alpha,\ \beta)$	`d-`$(x,\ \alpha,\ \beta)$	`p-`$(x,\ \alpha,\ \beta)$	`r-`$(\alpha,\ \beta)$
negative-binomial	`qnbinom`$(p,\ r,\ \theta)$	`d-`$(x,\ r,\ \theta)$	`p-`$(x,\ r,\ \theta)$	`r-`$(r,\ \theta)$
normal (univariate)	`qnorm`$(p,\ \theta,\ \sigma)$	`d-`$(x,\ \theta,\ \sigma)$	`p-`$(x,\ \theta,\ \sigma)$	`r-`$(\theta,\ \sigma)$
poisson	`qpois`$(p,\ \theta)$	`d-`$(x,\ \theta)$	`p-`$(x,\ \theta)$	`r-`(θ)
T (standard)	`qt`$(p,\ \nu)$	`d-`$(x,\ \nu)$	`p-`$(x,\ \nu)$	`r-`(ν)
T (nonstandard)	`qt.scaled`$(p,\ \nu,\ \mu,\ \sigma)$	`d-`$(x,\ \nu,\ \mu,\ \sigma)$	`p-`$(x,\ \nu,\ \mu,\ \sigma)$	`r-`$(\nu,\ \mu,\ \sigma)$
uniform	`qunif`$(p,\ a,\ b)$	`d-`$(x,\ a,\ b)$	`p-`$(x,\ a,\ b)$	`r-`$(a,\ b)$

Table 1: Functions from `R` (in alphabetical order) that can be used on this assignment and exams. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 2

We will ask some basic problems on the Gamma-Poisson conjugate model.

(a) [easy] Write the PDF of $\theta \sim \text{Gamma}(\alpha, \beta)$ which is the gamma distribution with the standard parameterization and notated with the hyperparameters we used in class.

$$\mathbb{P}(\theta) = \text{Gamma}(\alpha,\ \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(\beta\theta)$$

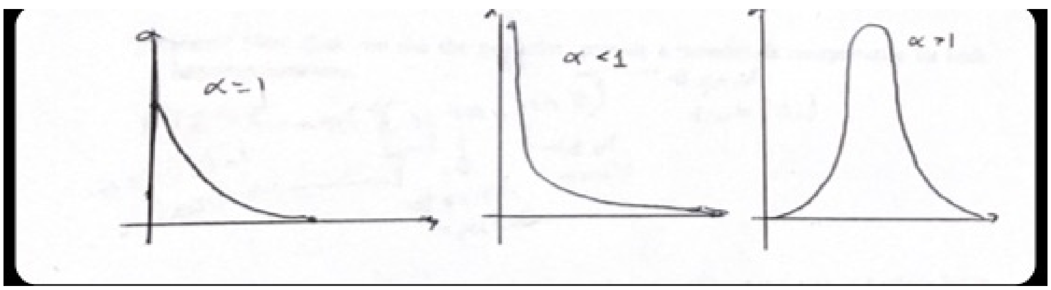
(b) [easy] What is the support and parameter space?

$$\text{Supp}\left[X\right] = (0, \infty), \alpha, \beta > 0$$

(c) [easy] What is the expectation and standard error and mode?

- $\mathbb{E}\left[Y\right] = \frac{\alpha}{\beta}$
- $\mathbb{V}\text{ar}\left[Y\right] = \frac{\alpha}{(\beta)^2}$
- $\text{Mode}\left[Y\right] = \frac{\alpha-1}{\beta}$ if $\alpha > 1$

(d) [easy] Draw four different pictures of different hyperparameter combinations to demonstrate this model's flexibility



(e) [harder] Prove that the Poisson likelihood for $n = 1$ with a gamma prior yields a gamma posterior and find its parameters.

$$\begin{aligned}
\mathbb{P}\left(\theta|X\right) &= \frac{\mathbb{P}\left(X|\theta\right)\mathbb{P}\left(\theta\right)}{\mathbb{P}\left(X\right)} \\
&\propto \mathbb{P}\left(X|\theta\right)\mathbb{P}\left(\theta\right) \\
&= \frac{\exp\left(-n\theta\right)\theta^{x}}{x!}\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\theta^{\alpha-1}\exp\left(\beta\theta\right) \\
&= \exp\left(-n\theta\right)\theta^{x+\alpha-1}\exp\left(-\beta\alpha\right) \\
&= \theta^{x+\alpha-1}\exp\left(-(n+\beta)\theta\right) \\
&= \text{Gamma}\left(x+\alpha,\, n+\beta\right) \\
&= \text{Gamma}\left(x+\alpha,\, 1+\beta\right)
\end{aligned}$$

(f) [harder] Prove that the Poisson likelihood for n observations, i.e. $X_1, \ldots, X_n; \theta \overset{iid}{\sim} \text{Poisson}\left(\theta\right)$, with a gamma prior yields a gamma posterior and find its parameters.

$$
\begin{aligned}
\mathbb{P}(\theta|X) &= \frac{\mathbb{P}(X|\theta)\,\mathbb{P}(\theta)}{\mathbb{P}(X)} \\
&\propto \mathbb{P}(X|\theta)\,\mathbb{P}(\theta) \\
&= \frac{\exp(-n\theta)\,\theta^{\sum x_i}}{\prod x_i!}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(\beta\theta) \\
&= \exp(-n\theta)\,\theta^{\sum x_i+\alpha-1}\exp(-\beta\alpha) \\
&= \theta^{\sum x_i+\alpha-1}\exp(-(n+\beta)\theta) \\
&= \text{Gamma}\left(\sum x_i+\alpha,\, n+\beta\right)
\end{aligned}
$$

(g) [easy] Now that you see the posterior, provide a pseudodata interpretation for both hyperparameters.

$$
\theta|X \sim \text{Gamma}\left(\sum x_i+\alpha,\, n+\beta\right)
$$

- $\sum x_i$ is the number of total successes
- α is the number of psuedo successes
- n is the number of trials
- β is the number of pseudo trials

(h) [harder] Find the Bayesian point estimates as function of the data and prior's hyperparameters (i.e. $\hat{\theta}_{\text{MMSE}}$, $\hat{\theta}_{\text{MMAE}}$ and $\hat{\theta}_{\text{MAP}}$).

- $\hat{\theta}_{\text{MMSE}} = \mathbb{E}[\theta|X] = \frac{\sum x_i+\alpha}{n+\beta}$
- $\hat{\theta}_{\text{MMAE}} = \text{median}[\theta|X] = qgamma(0.5, \sum x_i+\alpha, n+\beta)$
- $\hat{\theta}_{\text{MAP}} = \text{Mode}[\theta|X] = \frac{\sum x_i+\alpha-1}{n+\beta}$ if $\sum x_i+\alpha \geq 1$

(i) [harder] If $X_1, \ldots, X_n; \theta \overset{iid}{\sim} \text{Poisson}(\theta)$, find $\hat{\theta}_{\text{MLE}}$.

$$
\begin{aligned}
\mathcal{L}(\theta; X) &= \frac{\exp(-n\theta)\,\theta^{\sum x_i}}{\prod x_i!} \\
\ell'(\theta; X) &= -n + \frac{\sum x_i}{\theta} = 0 \\
\hat{\theta}_{\text{MLE}} &= \frac{1}{n}\sum x_i = \bar{x}
\end{aligned}
$$

(j) [harder] Demonstrate that $\hat{\theta}_{\text{MMSE}}$ is a shrinkage estimator and find ρ.

$$
\begin{aligned}
\hat{\theta}_{\text{MMSE}} &= \frac{\sum x_i}{n+\beta}\frac{n}{n} + \frac{\alpha}{n+\beta}\frac{\beta}{\beta} \\
&= \frac{n}{n+\beta}\bar{x} + \frac{\beta}{n+\beta}\mathbb{E}\left[\theta\right] \\
&= (1-\rho)\bar{x} + \rho\mathbb{E}\left[\theta\right] \\
\rho &= \frac{\bar{x}}{\bar{x}+\mathbb{E}\left[\theta\right]}
\end{aligned}
$$

(k) [harder] Demonstrate that $\mathbb{P}\left(\theta\right) \propto 1$ is improper.

If $\theta \sim U(0,\infty)$ then, $\mathbb{P}\left(\theta\right) \propto 1$ but $\mathbb{P}\left(\theta\right) = \frac{1}{\infty}$ therefore is not a valid pdf.

(l) [easy] [MA] Demonstrate that $\mathbb{P}\left(\theta\right) \propto 1$ can be created by using an improper Gamma distribution (i.e. a Gamma distribution with parameters that are not technically in its parameter space and thereby does not admit a distribution function).

(m) [harder] Find Jeffrey's prior for the Poisson likelihood model. Try to do it yourself.

$$
\begin{aligned}
\ell'(\theta; X) &= -n + \frac{\sum x_i}{\theta} \\
\ell''(\theta; X) &= \frac{\sum x_i}{\theta} \\
\boldsymbol{I}(\theta) &= \mathbb{E}\left[\ell''(\theta; X)\right] \\
&= \frac{1}{(\theta)^2}\sum \mathbb{E}\left[x_i\right] \\
&= \frac{n}{\theta}
\end{aligned}
$$

(n) [easy] What is the equivalent of the Haldane prior in the Binomial likelihood model for the Poisson likelihood model? Use an interpretation of pseudocounts to explain.

$\mathbb{P}\left(\theta|X\right) = \text{Gamma}\left(\sum x_i + \alpha,\, n+\beta\right)$ where α and β are 0. Therefore, $\text{Gamma}\left(\sum x_i,\, n\right)$

(o) [harder] Prove that posterior predictive distribution for the next Poisson realization (i.e. $n^* = 1$) given n observed Poisson realizations is negative binomially distributed and show its parameters are $p = \beta/(\beta+1)$ and $r = \alpha$ for $\alpha \in \mathbb{N}$.

$$
\mathbb{P}\left(X_*|X\right) = \int \mathbb{P}\left(X_*|\theta\right)\mathbb{P}\left(\theta|X\right)\, d\theta
$$

$$
\begin{aligned}
&= p^r(1-p)^{x_*}\frac{\Gamma\left(\sum x_i + x_* + \alpha\right)}{x_*!\Gamma\left(\sum x_i + \alpha\right)} \\
&= p^r(1-p)^{x_*}\frac{\Gamma\left(x_* + r\right)}{x_*!\Gamma\left(r\right)} \\
&= \binom{x_* + r - 1}{x_*} p^r(1-p)^{x_*} \\
&= \text{NegBin}\left(r,\, p\right)
\end{aligned}
$$

(p) [harder] If $\alpha \notin \mathbb{N}$, create an "extended negative binomial" r.v. and find its PMF. You can copy from Wikipedia.

$$
\mathbb{P}(X; n, r, p) = \frac{\binom{x+r-1}{x} p^x}{(1-p)^{-r} - \sum_{i=0}^{n-1} \binom{i+r-1}{x} p^i}
$$

(q) [harder] Why is the extended negative binomial r.v. also known as the gamma-Poisson mixture distribution? Why is it also called the "overdispersed Poisson"?

It is the posterior predictive result of the Poisson distribution given a prior Gamma. It is called ther overdisperse Poisson because it has a greater variability.

(r) [harder] If you observe $0, 3, 2, 4, 2, 6, 1, 0, 5$, give a 90% CR for θ. Pick an principled objective (uninformative) prior.

$$
\begin{aligned}
CR_{\theta,90\%} &= [qgamma(0.05, 24, 9), qgamma(0.95, 24, 9)] \\
&= [1.839, 3.621]
\end{aligned}
$$

(s) [harder] Using the data and the prior from (s), test if $\theta < 2$.

$$
\begin{aligned}
pval = \mathbb{P}\left(\theta < 2 | X\right) &= pgamma(2, 24, 9) - pgamma(0, 24, 9) \\
&= 0.101
\end{aligned}
$$

(t) [harder] Using the data and the prior from (s), find the probability the next observation will be a 7. Leave in exact form then use a calculator to compute it to the nearest two significat digits.

$$
\mathbb{P}\left(X_* = 7 | X\right) = \binom{30}{7}(0.9)^{24}(0.1)^7 = 0.02
$$

(u) [difficult] [MA] We talked about that the negative binomial is an "overdispersed" Poisson. Show that the negative binomial converges to a Poisson.

(v) [E.C.] [MA] Find the joint posterior predictive distribution for m future observations. I couldn't find the answer to this myself nor compute the integral.

Problem 3

We now discuss the theory of the normal-normal conjugate model. Assume

$$X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} \mathcal{N}\left(\theta,\, \sigma^2\right)$$

but where you see only "X", this is shorthand for all n samples.

(a) [easy] What is the kernel of $\theta \mid X,\ \sigma^2$?

$$\exp\left(\theta\frac{n\bar{x}}{\sigma^2} - \theta^2\frac{n}{2\sigma^2}\right) \propto \mathcal{N}\left(n\bar{x},\, \frac{\sigma^2}{n}\right)$$

(b) [difficult] Show that posterior of $\theta \mid X,\ \sigma^2$ is normal if $\theta \sim \mathcal{N}\left(\mu_0,\, \tau^2\right)$. Try to do it yourself and only copy from the notes if you have to.

$$\begin{aligned}
\mathbb{P}\left(\theta \mid X,\ \sigma^2\right) &\propto \mathbb{P}\left(X \mid \sigma^2\right)\mathbb{P}\left(\theta \mid \sigma^2\right) \\
&= e^{a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2} \\
&= e^{(a+\alpha)\theta - (b+\beta)\theta^2} \\
&\propto \mathcal{N}\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right)
\end{aligned}$$

Let $\mu_0 = \frac{\alpha}{2\beta}$ and $\tau^2 = \frac{1}{2\beta}$, if $a = \frac{n\bar{x}}{\sigma^2}$ and $b = \frac{n\theta^2}{2\sigma^2}$

$$\begin{aligned}
\mathbb{P}\left(\theta \mid X,\ \sigma^2\right) &\propto \exp\left(\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)\theta - \left(\frac{n\theta^2}{2\sigma^2} + \frac{1}{2\tau^2}\right)\theta^2\right) \\
&\propto \mathcal{N}\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n\theta^2}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n\theta^2}{\sigma^2} + \frac{1}{\tau^2}}\right)
\end{aligned}$$

(c) [easy] Find the Bayesian point estimates as function of the data and prior's hyperparameters (i.e. $\hat{\theta}_{\text{MMSE}}$, $\hat{\theta}_{\text{MMAE}}$ and $\hat{\theta}_{\text{MAP}}$).

- $\hat{\theta}_{\text{MMSE}} = \mathbb{E}\left[\theta|X\right] = \frac{\frac{n\bar{x}}{(\sigma)^2}+\frac{\mu_0}{(\tau)^2}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}}$
- $\hat{\theta}_{\text{MMAE}} = \text{median}\left[\theta|X\right] = \frac{\frac{n\bar{x}}{(\sigma)^2}+\frac{\mu_0}{(\tau)^2}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}}$
- $\hat{\theta}_{\text{MAP}} = \text{Mode}\left[\theta|X\right] = \frac{\frac{n\bar{x}}{(\sigma)^2}+\frac{\mu_0}{(\tau)^2}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}}$

(d) [harder] On a previous homework we showed that if $X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} \mathcal{N}(\theta,\, \sigma^2)$ then $\hat{\theta}_{\text{MLE}} = \bar{x}$. Show that $\hat{\theta}_{\text{MMSE}}$ is a shrinkage estimator and find ρ.

$$\begin{aligned}
\hat{\theta}_{\text{MMSE}} &= \frac{\frac{n\bar{x}}{(\sigma)^2}+\frac{\mu_0}{(\tau)^2}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}} \\
&= \frac{\frac{n\bar{x}}{(\sigma)^2}+\hat{\theta}_{\text{MLE}}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}} \\
&= \frac{\frac{\mathbb{E}[\theta]}{(\tau)^2}}{\frac{n}{(\sigma)^2}+\frac{1}{(\tau)^2}} \\
&= (1-\rho)\hat{\theta}_{\text{MLE}} + \rho\mathbb{E}\left[\theta\right] \\
\rho &= \frac{\hat{\theta}_{\text{MMSE}} - \hat{\theta}_{\text{MLE}}}{\mathbb{E}\left[\theta\right] - \hat{\theta}_{\text{MLE}}}
\end{aligned}$$

(e) [harder] Setup the integral to find $\mathbb{P}\left(X_* \mid X\right)$ where $n_* = 1$ but don't solve.

$$\begin{aligned}
\mathbb{P}\left(X_*|X\right) &= \int \mathbb{P}\left(X_*|\theta, (\sigma)^2\right)\mathbb{P}\left(\theta|X, (\sigma)^2\right)\, d\theta \\
&= \int \mathcal{N}\left(\theta,\, \sigma^2\right)\mathcal{N}\left(\theta_p,\, \sigma_p^2\right)\, d\theta \\
&= \int \exp\left(-\frac{1}{2\sigma^2}\left(x_* - \theta\right)^2\right)\exp\left(-\frac{1}{2\sigma_p^2}\left(x - \theta_p\right)^2\right)\, d\theta
\end{aligned}$$