

# MATH 368/621 Fall 2020 Homework #2

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## Problem 1

These exercises will introduce review expectation and variance and introduce covariance as well as expectation and variance of multidimensional (vector) r.v.'s.

(a) [harder] Consider a sequence of independent r.v.'s  $X_1, \dots, X_n$  and prove that

$$\mathbb{E} \left[ \prod_{i=1}^n X_i \right] = \prod_{i=1}^n \mathbb{E} [X_i].$$

$$\mathbb{E} \left[ \prod_{i=1}^n X_i \right] = \mathbb{E} [X_1, \dots, X_n] = \mathbb{E} [X_1] \cdot \dots \cdot \mathbb{E} [X_n] = \prod_{i=1}^n \mathbb{E} [X_i]$$

(b) [easy] Prove that  $\text{Cov} [X_1, X_2] = \mathbb{E} [(X_1 - \mu_1)(X_2 - \mu_2)]$ .

$$\begin{aligned} \text{Cov} [X_1, X_2] &= \mathbb{E} [(X_1 - \mu_1)(X_2 - \mu_2)] \\ &= \mathbb{E} [X_1 X_2 - \mu_1 X_1 - \mu_2 X_2 + \mu_1 \mu_2] \\ &= \mathbb{E} [X_1 X_2] - \mu_1 \mu_2 \end{aligned}$$

(c) [easy] Prove that  $\text{Cov} [X, X] = \text{Var} [X]$ .

$$\begin{aligned} \text{Cov} [X, X] &= \mathbb{E} [X X] - \mu \mu \\ &= \mathbb{E} [X] \mathbb{E} [X] - \mu \mu \\ &= \mathbb{E} [X - \mu] \mathbb{E} [X - \mu] \\ &= \mathbb{E} [(X - \mu)^2] \\ &= \text{Var} [X] \end{aligned}$$

(d) [easy] Prove that  $\mathbb{Cov}[X_1, X_2] = \mathbb{Cov}[X_2, X_1]$ .

$$\begin{aligned}\mathbb{Cov}[X_1, X_2] &= \mathbb{E}[X_1] \mathbb{E}[X_2] - \mu_1 \mu_2 \\ &= \mathbb{E}[X_2] \mathbb{E}[X_1] - \mu_2 \mu_1 \\ &= \mathbb{Cov}[X_2, X_1]\end{aligned}$$

(e) [easy] Prove that  $\mathbb{Cov}[a_1 X_1, a_2 X_2] = a_1 a_2 \mathbb{Cov}[X_1, X_2]$ .

$$\begin{aligned}\mathbb{Cov}[a_1 X_1, a_2 X_2] &= \mathbb{E}[(a_1 X_1 - a_1 \mu_1)(a_2 X_2 - a_2 \mu_2)] \\ &= \mathbb{E}[a_1 a_2 (X_1 - \mu_1)(X_2 - \mu_2)] \\ &= a_1 a_2 \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)] \\ &= a_1 a_2 \mathbb{Cov}[X_1, X_2]\end{aligned}$$

(f) [easy] Prove that  $\mathbb{Cov}[X_1 + X_3, X_2] = \mathbb{Cov}[X_1, X_2] + \mathbb{Cov}[X_3, X_2]$ .

$$\begin{aligned}\mathbb{Cov}[X_1 + X_3, X_2] &= \mathbb{E}[(X_1 + X_3 - \mu_1 - \mu_3)(X_2 - \mu_2)] \\ &= \mathbb{E}[X_1 X_2 - \mu_1 X_1 + X_3 X_2 - \mu_2 X_3] \\ &= \mathbb{E}[X_1 X_2] - \mu_2 \mu_1 + \mathbb{E}[X_3 X_2] - \mu_2 \mu_3 \\ &= \mathbb{Cov}[X_1, X_2] + \mathbb{Cov}[X_3, X_2]\end{aligned}$$

(g) [harder] [MA] Prove that

$$\mathbb{Cov}\left[\sum_{i \in A} X_i, \sum_{j \in B} Y_j\right] = \sum_{i \in A} \sum_{j \in B} \mathbb{Cov}[X_i, Y_j]$$

.

$$\begin{aligned}\mathbb{Cov}\left[\sum_{i \in A} X_i, \sum_{j \in B} Y_j\right] &= \mathbb{E}\left[\left(\sum_{i \in A} X_i - \sum_{i \in A} \mu_i\right)\left(\sum_{j \in B} Y_j - \sum_{j \in B} \mu_j\right)\right] \\ &= \mathbb{E}\left[\sum_{i \in A} X_i \sum_{j \in B} Y_j\right] - \sum_{j \in B} \mu_j \sum_{i \in A} \mu_i \\ &= \sum_{i \in A} \sum_{j \in B} \mathbb{E}[X_i Y_j] - \mu_i \mu_j \\ &= \sum_{i \in A} \sum_{j \in B} \mathbb{Cov}[X_i, Y_j]\end{aligned}$$

(h) [difficult] Prove that

$$\mathbb{V}\text{ar} \left[ \sum_{i=1}^n X_i \right] = \sum_{i=1}^n \sum_{j=1}^n \mathbb{C}\text{ov} [X_i, X_j]$$

without using the vector formulas.

$$\begin{aligned} \mathbb{V}\text{ar} \left[ \sum_{i=1}^n X_i \right] &= \mathbb{E} \left[ \left( \sum_{i=1}^n X_i \right)^2 \right] - \left[ \mathbb{E} \left[ \sum_{i=1}^n X_i \right] \right]^2 \\ &= \mathbb{E} \left[ \sum_{i=1}^n \sum_{j=1}^n X_i X_j \right] - \mathbb{E} \left[ \sum_{i=1}^n X_i \right] \mathbb{E} \left[ \sum_{j=1}^n X_j \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E} [X_i X_j] - \mu_i \mu_j \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{C}\text{ov} [X_i, X_j] \end{aligned}$$

(i) [easy] Prove  $\mathbb{E} [a\mathbf{X} + \mathbf{c}] = a\boldsymbol{\mu} + \mathbf{c}$  where the following are constants:  $a \in \mathbb{R}, \mathbf{c} \in \mathbb{R}^K$ .

$$\begin{aligned} \mathbb{E} [a\mathbf{X} + \mathbf{c}] &= [a\mathbb{E} [X_1] + c_1, \dots, a\mathbb{E} [X_n] + c_n]^T \\ &= a[\mathbb{E} [X_1], \dots, \mathbb{E} [X_n]]^T + [c_1, \dots, c_n]^T \\ &= a\boldsymbol{\mu} + \mathbf{c} \end{aligned}$$

(j) [easy] Prove  $\mathbb{V}\text{ar} [\mathbf{c}^\top \mathbf{X}] = \mathbf{c}^\top \Sigma \mathbf{c}$  where  $\mathbf{c} \in \mathbb{R}^K$ , a constant and  $\Sigma := \mathbb{V}\text{ar} [\mathbf{X}]$ , the variance-covariance matrix of the vector r.v.  $\mathbf{X}$ . This is marked easy since it's in the notes.

$$\begin{aligned} \mathbb{V}\text{ar} [\mathbf{c}^\top \mathbf{X}] &= \mathbb{V}\text{ar} [c_1 X_1 + \dots + c_k X_k] \\ &= \mathbb{V}\text{ar} [X_1 + \dots + X_k] \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{C}\text{ov} [c_i X_i, c_j X_j] \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i c_j \sigma_{ij} \\ &= \mathbf{c}^\top \Sigma \mathbf{c} \end{aligned}$$

- (k) [easy] Why is  $\mathbf{c}^\top \Sigma \mathbf{c}$  called a “quadratic form?” Read about it on wikipedia.

## Problem 2

These exercises are about the Multinomial distribution.

- (a) [easy] Explain in English why  $\mathbf{B} \sim \text{Multinomial}(1, \mathbf{p})$  is the multidimensional generalization of the Bernoulli r.v.
- (b) [easy] Explain in English why the following should be true. Remember how the sampling from the bag works.

$$\binom{n}{x_1, x_2, \dots, x_K} = \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-(x_1+x_2)}{x_3} \dots \binom{n-(x_1+x_2+\dots+x_{K-1})}{x_K}$$

- (c) [harder] Prove the combinatorial identity in (b).
- (d) [easy] Consider the following bag of marbles.



Draw from replacement 37 times. What is the probability of getting 10 red, 17 green, 6 blue and 4 yellow? Compute explicitly to the nearest two significant digits.

- (e) [difficult] [MA] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ , prove that its JMF sums to one, i.e.  $\sum_{\mathbf{x} \in \text{Supp}[\mathbf{X}]} p_{\mathbf{X}}(\mathbf{x}) = 1$ .
- (f) [difficult] [MA] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ , prove that any marginal distribution is binomial with  $n$  and  $p_j$  as parameters i.e.

$$p_{X_j}(x_j) = \text{Binomial}(n, p_j)$$

We only assumed this in class because it makes sense conceptual given balls being sampled from an urn, but it was never explicitly proven.

- (g) [E.C.] [MA] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ , find the JMF of any subset of  $X_1, \dots, X_k$ . Is it technically multinomial? This is not much harder than the previous problem if formulated carefully.
- (h) [harder] Explain in English why the following should be true. Remember how the sampling from the bag works.

$$\mathbf{B}_1, \dots, \mathbf{B}_n \stackrel{iid}{\sim} \text{Multinomial}(1, \mathbf{p}) \quad \text{then} \quad \mathbf{X} := \sum_{i=1}^n \mathbf{B}_i \sim \text{Multinomial}(n, \mathbf{p})$$

- (i) [harder] Find the answer by reasoning in English. No need to prove mathematically.

$$\mathbf{X}_1, \dots, \mathbf{X}_r \stackrel{iid}{\sim} \text{Multinomial}(n, \mathbf{p}) \quad \text{then} \quad \mathbf{T} := \sum_{i=1}^r \mathbf{X}_i \sim ?$$

- (j) [easy] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ , find  $p_{\mathbf{X}_{-j}|X_j}(\mathbf{x}_{-j}, x_j)$ . This is marked easy since it's in the notes.
- (k) [E.C.] [MA] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ , find a proof for  $\text{Cov}[X_i, X_j] = -np_i p_j$  that is qualitatively different than the one we did in class.
- (l) [harder] If  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$  where  $\dim[\mathbf{X}] = K$  and  $\mathbf{p} = \frac{1}{K} \mathbf{1}_K$ . What is the limit of  $\text{Cov}[X_i, X_j]$  as  $K$  gets large but  $n$  is fixed. Why does this make sense?
- (m) [easy] Correlation  $\rho$  is a unitless measure bounded between  $[-1, 1]$  and is a type of normalized covariance metric. It is defined for two r.v.'s as

$$\rho_{1,2} := \text{Corr}[X_1, X_2] := \frac{\sigma_{1,2}}{\sigma_1 \sigma_2} = \frac{\text{Cov}[X_1, X_2]}{\text{SD}[X_1] \text{SD}[X_2]} = \frac{\text{Cov}[X_1, X_2]}{\sqrt{\text{Var}[X_1] \text{Var}[X_2]}}$$

where  $\text{SD}[\cdot]$  denotes the standard deviation of a r.v., the square root of its variance. Find  $\text{Corr}[X_i, X_j]$  for two arbitrary elements in the r.v. vector  $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ .

- (n) [easy] If  $\mathbf{c} = [1 \ 2 \ 3 \ 4]^\top$ , compute the inner product  $\mathbf{c}^\top \mathbf{c}$  and the outer product  $\mathbf{c} \mathbf{c}^\top$ .