MATH 368/621 Fall 2020 Homework #1

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Problem 1

These exercises give you practice with sums and indicator functions.

(a) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \mathbb{1}_{x=17}$.

$$\sum_{x\in\mathbb{R}} \mathbb{1}_{x=17} = 1$$

(b) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} c\mathbb{1}_{x=17}$ where $c \in \mathbb{R}$ is a constant.

$$\sum_{x\in\mathbb{R}} c\mathbb{1}_{x=17} = c$$

(c) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}}$.

$$\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}} = 3$$

(d) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,3\}}$.

$$\begin{aligned}\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,3\}} &= 1(1) + 2(1) + 3(1) + 4(0) + 5(0) + \ldots \\ &= 6\end{aligned}$$

(e) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}}$.

$$\begin{aligned}\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}} &= 1^1 + 2^1 + 3^1 + 4^0 + \dots \\ &= 6 + \infty \\ &= \infty\end{aligned}$$

(f) [easy] Expand and simplify as much as you can: $\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}}$.

$$\begin{aligned}\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} &= 1 * 1 * 1 * 0 * \dots \\ &= 0\end{aligned}$$

(g) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}}$.

$$\begin{aligned}\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}} &= (1)(0) + (1)(0) + (1)(0) + (0)(1) + (0)(1) + (0)(1) + \dots \\ &= 0\end{aligned}$$

(h) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} c\mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x \in \mathbb{R}} c\mathbb{1}_{x \in \{1,2,\dots,t\}} &= c \sum_{i=0}^{t} 1 \\ &= tc\end{aligned}$$

(i) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} t\mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x \in \mathbb{R}} t\mathbb{1}_{x \in \{1,2,\dots,t\}} &= t \sum_{i=1}^{t} i \\ &= t^2\end{aligned}$$

(j) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1,2,\ldots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1,2,\ldots,t\}} &= \sum_{i=1}^{t} i \\ &= \frac{(t-1)(t)}{2}\end{aligned}$$

(k) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \frac{1}{x!} \mathbb{1}_{x \in \mathbb{N}}$.

$$\begin{aligned}\sum_{x \in \mathbb{R}} \frac{1}{x!} \mathbb{1}_{x \in \mathbb{N}} &= \sum_{x \in \mathbb{N}} \frac{1}{x!} \\ &= \exp(1) - 1\end{aligned}$$

(l) [harder] Prove $\mathbb{E}[\mathbb{1}_{X \in A}] = \mathbb{P}(X \in A)$.

Problem 2

These exercises review convolutions.

(a) [easy] Is a JMF a type of PMF or PMF a type of JMF? Explain.

A JMF is a type of PMF because a JMF is derived from 2 or more PMF's

(b) [easy] Let $X_1, X_2 \stackrel{iid}{\sim} \text{Bernoulli}(p)$. Find the PMF of the sum of $T = X_1 + X_2$ using the appropriate discrete convolution formula that would make the problem easiest.

$$\begin{aligned}\mathbb{P}(t) &= \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t-x} \\ &= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x} \binom{1}{t-x} \\ &= p^t (1-p)^{2-t} \left(\binom{1}{t} \binom{1}{t-1} \right) \\ &= \binom{2}{t} p^t (1-p)^{2-t}\end{aligned}$$

(c) [easy] Let $X_1 \sim \text{Bernoulli}(p_1)$ independent of $X_2 \sim \text{Bernoulli}(p_2)$. Find the JMF of for X_1, X_2. Denote it using a 2×2 grid or the piecewise function notation.

$$\begin{cases} 1 & \text{w.p.} \quad (p_1)(p_2) \\ 0 & \text{w.p.} \quad (1-p_1)(1-p_2) \end{cases}$$

(d) [difficult] Let

$$X_1 \sim \begin{cases} 3 & \text{w.p.} \quad 0.3 \\ 6 & \text{w.p.} \quad 0.7 \end{cases} \qquad \text{independent of} \quad X_2 \sim \begin{cases} 4 & \text{w.p.} \quad 0.4 \\ 8 & \text{w.p.} \quad 0.6 \end{cases}$$

Find the PMF of $T = X_1 + X_2$ using a convolution. Denote it using the piecewise function notation.

(e) [difficult] Prove the PMF of a binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$. You will need to use Pascal's Triangle combinatorial identity we employed in class.

$$\text{Let } T_n = X_1 + X_2 + \ldots + X_n \text{ and } T_n = X_n + T_{n-1}$$

$$\begin{aligned}
\mathbb{P}(t) &= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{n-1}{t-x} p^{t-x} (1-p)^{n-1-t+x} \\
&= \sum_{x \in \{0,1\}} p^t (1-p)^{n-t} \binom{n-1}{t-x} \\
&= p^t (1-p)^{n-t} \sum_{x \in \{0,1\}} \binom{n-1}{t-x} \\
&= p^t (1-p)^{n-t} \left(\binom{n-1}{t} + \binom{n-1}{t-1} \right) \\
&= p^t (1-p)^{n-t} \binom{n}{t}
\end{aligned}$$

(f) [difficult] [MA] Prove the PMF of a negative binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim} \text{Geometric}(p)$. You will need to use the "hockey stick identity" [click here].

$$\text{Let} T_n = X_n + T_{n-1}$$

$$\begin{aligned}
\mathbb{P}(t) &= \sum \mathbb{P}(X_n)\,\mathbb{P}(T_n - X_n) \\
&= \sum_{x\in\{0,1,\ldots\}} (1-p)^x \binom{(t-x)+n-2}{n-2}(1-p)^{t-x}p^{n-1}\mathbb{1}_{t-x\in\{0,1,\ldots\}} \\
&= (1-p)^t p^n \sum_{x\in\{0,1,\ldots\}} \binom{(t-x)+n-2}{n-2} \\
&= (1-p)^t p^n \sum x = 0^t \binom{t+n-x-2}{n-2} \\
&= (1-p)^t p^n \binom{t+n-1}{n-1}
\end{aligned}$$

(g) [difficult] Let $X_1 \sim \text{Binomial}(n_1, p)$ independent of $X_2 \sim \text{Binomial}(n_2, p)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

$$\begin{aligned}
\mathbb{P}(t) &= \sum_{x\in\mathbb{R}} \binom{n_1}{x} p^x (1-p)^{n_1-x} \binom{n_2}{t-x} p^{t-x}(1-p)^{n_2-t+x}\mathbb{1}_{t-x\in\text{Supp}[X_1]} \\
&= p^t(1-p)^{n_1+n_2-t} \sum_{x\in\{0,\ldots,t\}} \binom{n_1}{x}\binom{n_2}{t-x} \\
&= p^t(1-p)^{n_1+n_2-t}\binom{n_1+n_2}{t}
\end{aligned}$$

(h) [easy] Prove the PMF of $X \sim \text{Poisson}(\lambda)$ using the limit as $n \to \infty$ and let $p = \frac{\lambda}{n}$.

$$\text{Let } X \sim \text{Binomial}(n, p) \text{ where } n \to \infty \text{ and } p = \frac{\lambda}{n}$$

$$\begin{aligned}
\mathbb{P}(x) &= \lim_{n\to\infty} \frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x}\mathbb{1}_{x\in 0,\ldots,n} \\
&= \lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{1-x}\mathbb{1}_{x\in 0,\ldots,n} \\
&= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)n^x}\lim_{n\to\infty}(1-\frac{\lambda}{n})^n\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-x}\mathbb{1}_{x\in 0,\ldots,n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n(n-1)\dots(n-x+1)}{n\cdot n \dots \cdot n} \exp\left(-\lambda\right) \mathbb{1}_{x\in\{0,1,\dots\}} \\
&= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\{0,1,\dots\}} \\
&= \text{Poisson}\left(\lambda\right)
\end{aligned}
$$

(i) [difficult] Let $X_1 \sim \text{Poisson}\left(\lambda_1\right)$ independent of $X_2 \sim \text{Poisson}\left(\lambda_2\right)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

$$
\begin{aligned}
\mathbb{P}\left(t\right) = \sum_{x_1\in\{0,\dots\}} &= \sum \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1}}{x_1!} \frac{\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{(t-x_1)!} \mathbb{1}_{t-x\in\text{Supp}[X_1]} \\
&= \sum_{x_1\in\{0,\dots,t\}} \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1}}{x_1!} \frac{\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{(t-x_1)!} \\
&= \sum_{x_1\in\{0,\dots,t\}} \left(\frac{1}{x_1!(t-x_1)!} \exp\left(-\lambda_1\right)\lambda_1^{x_1} \exp\left(-\lambda_2\right)\lambda_2^{t-x_1}\right) \frac{t!}{t!} \\
&= \sum_{x_1\in\{0,\dots,t\}} \binom{t}{x_1} \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1} \exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{t!} \\
&= \frac{\exp\left(-\lambda_1-\lambda_2\right)}{t!} \sum_{x_1\in\{0,\dots,t\}} \binom{t}{x_1} \lambda_1^{x_1}\lambda_2^{t-x_1} \\
&= \frac{\exp\left(-\lambda_1-\lambda_2\right)\left(\lambda_2+\lambda_1\right)^t}{t!}
\end{aligned}
$$

Problem 3

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

(a) [difficult] Let $X \sim \text{Geometric}\left(p_x\right)$ independent of $Y \sim \text{Geometric}\left(p_y\right)$. Find $\mathbb{P}\left(X > Y\right)$ using the method we did in class.

$$
\begin{aligned}
\mathbb{P}\left(X > Y\right) &= \sum_{x\in\mathbb{N}_0} \sum_{y\in\mathbb{N}_0} (1-p_x)^x p_x (1-p_y)^y p_y \mathbb{1}_{x>y} \\
&= p_x p_y \sum_{y\in\mathbb{N}_0} (1-p_y)^y \sum_{x\in\mathbb{N}_0} (1-p_x)^x \mathbb{1}_{x>y} \\
&= p_x p_y \sum_{y\in\mathbb{N}_0} (1-p_y)^y \sum_{x\in\{y+1,\dots\}} (1-p_x)^x \\
&= p_x p_y \sum_{y\in\mathbb{N}_0} (1-p_y)^y \sum_{x\prime\in\mathbb{N}_0} (1-p_x)^{x\prime+y+1}
\end{aligned}
$$

$$
\begin{aligned}
&= p_x p_y (1-p_x) \sum_{y \in \mathbb{N}_0} (1-p_y)^y \sum_{x\prime \in \mathbb{N}} (1-p_x)^{x\prime - 1} \\
&= p_y (1-p_x) \sum_{y \in \mathbb{N}_0} (1-p_y)^y (1-p_x)^y \\
&= p_y (1-p_x) \sum_{y \in \mathbb{N}_0} [(1-p_y)(1-p_x)]^y \\
&= p_y (1-p_x) \sum_{y \in \mathbb{N}} [(1-p_y)(1-p_x)]^{y-1} \\
&= \frac{p_y - p_x p_y}{p_x + p_y + p_x p_y}
\end{aligned}
$$

(b) [easy] [MA] Prove this a different way by finding $\mathbb{P}(X = Y)$ and then using the law of total probability.

(c) [easy] [MA] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)?

Let $p_x = r p_y$

$$
\begin{aligned}
\mathbb{P}(X > Y) &= \frac{p_y - p_x p_y}{p_x + p_y + p_x p_y} \\
&= \frac{1 - r p_y}{r + 1 - r p_y} \\
&= \lim_{p_y \to 0} \frac{1 - r p_y}{r + 1 - r p_y} \\
&= \frac{1}{1 + r} \\
&= \frac{p_x}{p_y + p_x}
\end{aligned}
$$

(d) [difficult] Let $X \sim \text{Poisson}(\lambda)$ independent of $Y \sim \text{Poisson}(\lambda)$. Find an expression for $\mathbb{P}(X > Y)$ *as best as you are able to answer.* Part of this exercise is identifying where you cannot go any further.

$$
\begin{aligned}
\mathbb{P}(X > Y) &= \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} \mathbb{P}(x)\, \mathbb{P}(y)\, \mathbb{1}_{x > y} \\
&= \sum_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} \frac{e^{-\lambda} \lambda^x}{x!} \frac{e^{-\lambda} \lambda^y}{y!} \mathbb{1}_{x > y} \\
&= e^{-2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^y}{y!} \sum_{x \in \{y+1, \dots\}} \frac{\lambda^x}{x!}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^y}{y!} \sum_{x\prime \in \mathbb{N}_0} \frac{\lambda^{x\prime + y + 1}}{x\prime!} \\
&= \lambda e^{2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^{2y}}{y!} \sum_{x\prime \in \mathbb{N}_0} \frac{\lambda^{x\prime}}{x\prime!} \\
&= \lambda e^{-\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^{2y}}{y!}
\end{aligned}
$$

Problem 4

These exercises will introduce the Multinomial distribution.

(a) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the parameter space for both n and $\boldsymbol{p}$?

$$
n \in \mathbb{N}, \quad \boldsymbol{p} \in \{\vec{v} : \vec{v} \cdot 1, v_1 \in (0,1), \ldots, v_k \in (0,1)\}
$$

(b) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the $\text{Supp}\left[\boldsymbol{X}\right]$?

$$
\text{Supp}\left[\boldsymbol{X}\right] = \{\vec{x} : \vec{x} \cdot 1 = n, x_1 \in \{0, \ldots, n\}, \ldots, x_k \in \{0, \ldots, n\}\}
$$

(c) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is $\dim\left[\boldsymbol{p}\right]$?

$$
\dim\left[\boldsymbol{p}\right] = k
$$

(d) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = 2$, express p_2 as a function of p_1.

$$
p_1 = 1 - p_2
$$

(e) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = 2$, how are both X_1 and X_2 distributed?

$$
X_1 \sim \text{Binomial}\left(n,\, p_1\right), \quad X_2 \sim \text{Binomial}\left(n,\, p_2\right)
$$

(f) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and $n = 10$ and $\dim[\boldsymbol{X}] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$.

$$\boldsymbol{x} = [1, 1, 1, 1, 1, 2, 3]^T$$

(g) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(9,\, [0.1\ 0.2\ 0.7]^\top\right)$, find $\mathbb{P}\left(\boldsymbol{X} = [3\ 2\ 4]^\top\right)$ to the nearest two decimal places.

$$\begin{aligned}
\mathbb{P}\left(X = [3, 2, 4]^T\right) &= \binom{9}{3, 2, 4} \cdot (0.1) \cdot (0.2) \cdot (0.7) \\
&= \frac{9!}{3!2!4!} \cdot (0.1) \cdot (0.2) \cdot (0.7) \\
&= 17.64
\end{aligned}$$

(h) [difficult] [MA] If $\boldsymbol{X}_1 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and independently $\boldsymbol{X}_2 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}_1] = \dim[\boldsymbol{X}_2] = k$. Find the JMF of $\boldsymbol{T}_2 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of $x_1, \ldots, x_K$. Finally, use Theorem 1 in this paper: [click here] for the summation.

$$\begin{aligned}
\mathbb{P}\,(t_1, \ldots, t_k) &= \sum_{x_1, \ldots, x_k} \binom{n}{x_1, \ldots, x_k} \prod_{k=1}^{k} p_k^{x_k} \binom{n}{(t_1 - x_1), \ldots, (t_k - x_k)} \prod_{k=1}^{k} p_k^{t_k - x_k} \\
&= \prod_{i=1}^{k} p_i^{t_i} \sum_{x_1, \ldots, x_k} \binom{n}{x_1, \ldots, x_k} \binom{n}{(t_1 - x_1), \ldots, (t_k - x_k)} \\
&= \prod_{i=1}^{k} p_i^{t_i} \binom{2n}{t_1, t_2, \ldots, t_k}
\end{aligned}$$