MATH 368/621 Fall 2020 Homework #4

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Problem 1

These exercises will give you practice with the gamma function.

(a) [easy] Write the definition of $\Gamma(x)$. skip — duplicate

(b) [easy] Prove $\Gamma(x+1) = x\Gamma(x)$. skip — duplicate

(c) [easy] Write the definition of $\Gamma(x, a)$ without using the gamma function. skip — duplicate

(d) [harder] Write the definition of $Q(x, a)$ without using the gamma function. skip — duplicate

(e) [easy] For $a, c \in (0, \infty)$, prove the following:

$$\int_a^\infty t^{x-1} e^{-ct} dt = \frac{\Gamma(x, ac)}{c^x}$$

skip — duplicate

(f) [easy] Let $X \sim \text{Gamma}(\alpha, \beta)$. Show that this r.v. is equivalent to $X \sim \text{Erlang}(k, \lambda)$ and find k and λ in terms of α and β. Are there any restrictions on the values of α and β for this relationship to hold?

skip — duplicate

Problem 2

These exercises will give you practice with transformations of discrete r.v.'s.

(a) [easy] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = \ln(X+1)$.

$$g(x) = \ln(x+1) \to g^{-1}(y) = \exp(y) - 1$$

$$P_x(g^{-1}(y)) = P_x(\exp(y) - 1) = \binom{n}{\exp(y)-1} p^{\exp(y)-1}(1-p)^{n-\exp(y)-1}$$

(b) [harder] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

$g(x)$ is not monotonic but it does not matter since the $Supp[X] = \{0, n\}$

$$g(x) = x^2 \rightarrow g^{-1}(y) = \sqrt{y}$$
$$P_x(g^{-1}(y)) = P_x(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}}(1-p)^{n-\sqrt{y}}$$

(c) [difficult] Let $X \sim \text{Binomial}(n, p)$ where n is an even number. Find the PMF of $Y = g(X) = \text{mod}(X, 2)$ where "mod" denotes modulus division of the first argument by the second argument.

Since n is even then,

$$g(X) = \text{mod}(X, 2) = 0$$

$$P_X(g^{-1}(y)) = P_X(0) = (1-p)^n$$

(d) [difficult] [MA] Let $X \sim \text{NegBin}(k, p)$. Find the PMF of $Y = g(X) = \text{mod}(X, n)$ where $n \in \mathbb{N}$.

Problem 3

These exercises will give you practice with transformations of continuous r.v.'s and the quantile function.

(a) [harder] Let $X \sim U(0, 1)$. Find the PDF of $Y = g(X) = aX + c$. Make sure you're careful with the indicator function that specifies the support. There are two cases.

$$g(x) = aX + c \Leftrightarrow g^{-1}(y) = \frac{y-c}{a}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \left|\frac{1}{a}\right|$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{y-c}{a^2}$$

$$= \begin{cases} \frac{y-c}{a^2} & a > 0 \\ \frac{y-c}{|-a^2|} & a < 0 \end{cases}$$

(b) [harder] Let $X \sim \mathrm{Exp}\,(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.

$$g(x) = \ln(x) \Leftrightarrow g^{-1}(y) = \exp(y)$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = |\exp(y)|$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \lambda\exp(-\lambda\exp(y))\exp(y)$$

(c) [E.C.] Let $X \sim \mathrm{Exp}\,(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$.

(d) [harder] Let $X \sim \mathrm{U}\,(0, 1)$. Find the PDF of $Y = g(X) = \ln\left(\frac{X}{1-X}\right)$. If this is a brand name r.v., mark it so and include its parameter values.

$$g(x) = \ln(\frac{x}{1-x}) \Leftrightarrow g^{-1}(y) = \frac{\exp(y)}{1+\exp(y)}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \left|\frac{\exp(y)}{(1+\exp(y))^2}\right|$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{\exp(y)}{(1+\exp(y))^2}$$

$$= \frac{\exp(y)}{(1+\exp(y))^2}\frac{\exp(-2y)}{\exp(-2y)}$$

$$= \frac{\exp(-y)}{(\exp(-y)+1)^2}$$

$$\sim \text{Logistic}\,(0,\ 1)$$

(e) [easy] Find the Quantile function of X where $X \sim \text{Logistic}(0, 1)$.

$$F(x) = \frac{\exp(x)}{1 + \exp(x)}$$

$$Q[X, p] = \ln(\frac{p}{1-p})$$

(f) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Logistic}(\mu, \sigma)$ where $X \sim \text{Logistic}(0, 1)$.

$$g(x) = \sigma X + \mu \Leftrightarrow g^{-1}(y) = \frac{y-\mu}{\sigma}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{|\sigma|}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{\exp\left(\frac{y-\mu}{\sigma}\right)}{\sigma(1 + \exp\left(\frac{y-\mu}{\sigma}\right))^2}$$

$$= \frac{\exp\left(\frac{y-\mu}{\sigma}\right)}{\sigma(1 + \exp\left(\frac{y-\mu}{\sigma}\right))^2}\frac{\exp(-2y)}{\exp(-2y)}$$

$$= \frac{\exp\left(-(\frac{y-\mu}{\sigma})\right)}{\sigma(1 + \exp\left(-(\frac{y-\mu}{\sigma})\right))^2}$$

(g) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

$$g(x) = \frac{1}{1 + e^{-x}} \Leftrightarrow g^{-1}(y) = -\ln(\frac{1-y}{y})$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{y(1-y)}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{\lambda}{y(1-y)}(\frac{y}{(1-y)})^{-\lambda}$$

$$\sim \text{ParetoI}(1-y,\lambda)$$

(h) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^X$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.

$$g(x) = ke^X \Leftrightarrow g^{-1}(y) = \ln(\frac{y}{k})$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{y}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \lambda\text{exp}\left(\lambda(\ln(\frac{y}{k}))\right)\frac{1}{y}$$

$$= \frac{\lambda}{y}(\frac{y}{k})^{-\lambda}$$

$$\sim \text{ParetoI}(k,\lambda)$$

(i) [easy] Rederive the $X \sim \text{Laplace}(0,1)$ r.v. model by taking the difference of two standard exponential r.v.'s.

$$f_D(d) = \int_{\text{Supp}[X_1]} f(x)^{old} f_t^{old}(d-x)\mathbb{1}_{d-x\in\text{Supp}[X]}$$

$$= \int_0^\infty \exp(-x)\exp(d-x)\,\mathbb{1}_{x\in[d,\infty]}$$

$$= \exp(d)\begin{cases}\int_d^\infty \exp(-2x)\,dx & d \geq 0\\ \int_0^\infty \exp(-2x)\,dx & d < 0\end{cases}$$

$$= \frac{1}{2}\text{exp}(-|d|)$$

$$\sim \text{Laplace}(0,1)$$

(j) [easy] Let $X \sim \text{Laplace}(0,1)$. Prove that $\mathbb{E}[X] = 0$ without using the integral definition. There's a trick.

Since $X_1, X_2 \overset{iid}{\sim} \text{Exp}(\lambda)$

$$\mathbb{E}[X_1 - X_2] = \mathbb{E}[X_1] - \mathbb{E}[X_2] = 0$$

(k) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Laplace}(\mu, \sigma)$ where $X \sim \text{Laplace}(0,1)$.

$$f(\sigma x + \mu) = \frac{1}{2\sigma} \exp\left(-\left|\frac{y-\mu}{\sigma}\right|\right)$$

(l) [difficult] Show that $\mathcal{E} \sim \text{Laplace}(0,\sigma)$ is a reasonable error distribution.

$\mathcal{E}$ is a reasonable error distribution because it has thin tails at it extremes. Meaning that it is less mass at the ends and its symmetric

$$P(X) = \frac{1}{2\sigma} \exp\left(-\frac{|y|}{\sigma}\right)$$

(m) [harder] [MA] Find the Quantile function of X where $X \sim \text{Laplace}(0,1)$.

$$\begin{cases} \frac{1}{2}\exp(x) & x \leq \mu \\ \\ 1 - \frac{1}{2}\exp(-x) & x \geq \mu \end{cases} = \begin{cases} q = \frac{1}{2}\exp(x) & q \leq \frac{1}{2} \\ \\ 1 = 1 - \frac{1}{2}\exp(-x) & q \geq \frac{1}{2} \end{cases}$$

$$Q[X,q] = \begin{cases} \ln(2q) = x & q \leq \frac{1}{2} \\ \\ \ln(-2q+2) = x & q \geq \frac{1}{2} \end{cases}$$

(n) [difficult] [MA] Let $X \sim \text{ParetoI}(k,\lambda)$. Show that $Y = X \mid X > c$ where $c > k$ is also a ParetoI r.v. and find its parameter values.

$$P(X > y | X > c) = \frac{1 - F(y)}{1 - F(c)}$$

$$= \frac{((\frac{\lambda}{k})(\frac{k}{y})^{\lambda})}{(\frac{\lambda}{k})(\frac{k}{c})^{\lambda}}$$

$$= (\frac{c}{y})^{\lambda}$$

$$\sim \text{ParetoI}(c, \lambda)$$

Problem 4

We will now explore a couple of extreme distributions.

(a) [harder] Let $X \sim \text{Exp}(1)$ and $Y = -\ln(X) \sim \text{Gumbel}(0, 1)$. Find the PDF of this standard Gumbel distribution. Make sure you include the indicator function throughout your proof.

$$g(x) = -\ln(X) \;\Leftrightarrow\; g^{-1}(y) = \exp(-Y)$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = |-\exp(-Y)|$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \exp(-\exp(-y) - y)$$

(b) [easy] Find the CDF of Y.

$$\int \exp(-\exp(-x) - x)\, dx = \exp(-\exp(-x))$$

(c) [easy] Let $G = \beta Y + \mu \sim \text{Gumbel}(\mu, \beta)$. Find the PDF of G, the general Gumbel distribution.

$$g(x) = \beta Y + \mu \;\Leftrightarrow\; g^{-1}(y) = \frac{x - \mu}{\beta}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{\beta}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{\beta}\exp\left(-(\frac{x-\mu}{\beta})\right)$$

(d) [easy] [MA] Show that for any r.v. X, if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

$$F_Y(y) = P(Y \leq y)$$

$$= P(F_X(X) \leq y)$$

$$= P(X \leq F_X(y^{-1}))$$

$$= F_X(y^{-1}) = F_X(\frac{y-b}{a})$$

(e) [easy] Using the answer in the previous question, find the CDF of $G \sim \text{Gumbel}(\mu, \beta)$.

$$\int \frac{1}{\beta}\exp\left(-(\frac{x-\mu}{\beta})\right) dx = \exp\left(-(\frac{x-\mu}{\beta})\right)$$

Problem 5

These exercises will give you practice with the Weibull distribution.

(a) [easy] If $X \sim \text{Exp}\,(1)$ then show that $Y = \frac{1}{\lambda}X^{\frac{1}{k}} \sim \text{Weibull}\,(k, \lambda)$ where $k, \lambda > 0$.

$$g(x) = \frac{1}{\lambda}x^{\frac{1}{k}} \;\Leftrightarrow\; g^{-1}(y) = \lambda^k y^k$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = k\lambda^k y^{k-1}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = k\lambda(\lambda y)^{k-1}\exp\left(-(\lambda^k y^k)\right)$$

$$\sim \text{Weibull}\,(k, \lambda)$$

(b) [harder] Find Med $[Y]$.

$$\frac{1}{2} = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^k\right)$$

$$\text{Med}\,[Y] = \lambda \ln(2)^{\frac{1}{k}}$$

(c) [difficult] [MA] Prove that if $k > 1$ then $\mathbb{P}\left(Y \geq y + c \mid Y \geq c\right) < \mathbb{P}\left(Y \geq y\right)$ for $c > 0$.

$$\frac{P(Y \geq y + c \;\cap\; Y \geq c)}{P(Y \geq c)} = \exp\left(\left(-\lambda^2(y+c)(c)\right)^k\right)\left(\frac{y+c}{c}\right)^{k+1}$$

$$P(Y \geq y) = k\lambda(\lambda y)^{k-1}\exp\left(-(\lambda y)^k\right)$$

$$\exp\left(\left(-\lambda^2(y+c)(c)\right)^k\right)\left(\frac{y+c}{c}\right)^{k+1} < k\lambda(\lambda y)^{k-1}\exp\left(-(\lambda y)^k\right)$$

(d) [difficult] If $X \sim \text{Exp}\,(\lambda)$ then show that $Y = X^\beta \sim$ Weibull where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class (i.e. your answer in part a).

$$g(x) = X^\beta \Leftrightarrow \; g^{-1}(y) = y^{\frac{1}{\beta}}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{\beta}y^{\frac{1}{\beta}-1}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \lambda\exp\left(\lambda(y^{\frac{1}{\beta}})\right)\frac{1}{\beta}y^{\frac{1}{\beta}-1} = \frac{\lambda}{\beta}y^{\frac{1}{\beta}-1}\exp\left(\lambda(y^{\frac{1}{\beta}})\right)$$

Let $k = \frac{1}{\beta}$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = k\lambda y^{k-1}\exp\left(-\lambda(y^k)\right)$$

(e) [easy] Using Y, the Weibull in terms of the parameterization we learned in class (i.e. your answer in part a), find the PDF of $W = Y + c \sim \text{Weibull}\,(k, \lambda, c)$ which is known as the "translated Weibull" or "3-parameter Weibull model".

$$k\lambda(\lambda(y+c))^{k-1}\text{exp}\left(-(\lambda^k(y+c)^k)\right)$$

Problem 6

We will practice finding kernels and relating them to known distributions. The gamma function and the beta function will come up as well.

(a) [easy] Find the kernel of the negative binomial PMF.

$$P(X=k) = \binom{k+r-1}{k}(1-p)^r p^k \mathbb{1}_{k\in\{0,1,\ldots\}}$$

$$= \frac{(k+r-1)!}{k!(k+r-1-k)!}(1-p)^r p^k \mathbb{1}_{k\in\{0,1,\ldots\}}$$

$$= \frac{(k+r-1)!}{k!(r-1)!}(1-p)^r p^k \mathbb{1}_{k\in\{0,1,\ldots\}}$$

$$\propto \frac{(k+r-1)!}{k!}p^k \mathbb{1}_{k\in\{0,1,\ldots\}}$$

(b) [easy] Find the kernel of the beta PDF.

$$P(X=x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]}$$

$$\propto x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]}$$

(c) [easy] If $k(x) = e^{-\lambda x}x^{k-1}\mathbb{1}_{x>0}$ how would you know if the r.v. X was an $\text{Erlang}\,(k,\,\lambda)$ or a $\text{Gamma}\,(k,\lambda)$?

We know that X will be $\text{Gamma}\,(k,\lambda)$ since the indicator functions tell us about their supports. We know that:

$$\text{Erlang}\,(k,\,\lambda) \to \text{Supp}\,[X] = [0,\infty)$$

$$\text{Gamma}\,(k, \lambda) \rightarrow \text{Supp}\,[X] = (0, \infty)$$

(d) [harder] If $k(x) = xe^{-x^2}\mathbb{1}_{x>0}$, how is X distributed?

$X \sim \text{Weibull}(2, 1)$

$$c = \left(\int_{-\infty}^{\infty} f(x)dx\right)^{-1}$$

$$= 2$$

Problem 7

We will now practice using order statistics concepts.

(a) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the CDF of the maximum X_i and express the CDF of the minimum X_i.

$$\max \leftarrow 1 - (1 - F(x))^n$$

$$\min \leftarrow F(x)^n$$

(b) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF of the maximum X_i and express the PDF of the minimum X_i.

$$\max \leftarrow nf(x)F(x)^{n-1}$$

$$\min \leftarrow nf(x)(1 - F(x))^{n-1}$$

(c) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF and the CDF of $X_{(k)}$ i.e. the kth smallest X_i.

$$f_x(x) = \frac{n!}{(k-1)!(n-k)!} f(x)F(x)^{k-1}(1 - F(x))^{n-k}$$

$$F_x(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}$$

(d) [difficult] [MA] If discrete $X_1, \ldots, X_n \overset{iid}{\sim} p(x)$, why would the formulas in (a-c) not be accurate?

The formulas above strictly assume that random vaariables are continous. Applying these formulas to the descrete case would not be accurate because in the continous case we assign mass to a region. In the discrete case, we assign mass to single point.

(e) [harder] If $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{U}(0,\, 1)$, show that $X_{(k)} \sim \mathrm{Beta}(k,\, n-k+1)$.

$$f_{x_k}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in[0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k-1)} x^{k-1}(1-x)^{n+k+1-1} \mathbb{1}_{x\in[0,1]}$$

$$= \mathrm{Beta}(k,\, n-k+1)$$

(f) [harder] Express $\binom{n}{k}$ in terms of the beta function.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k+1)\,\Gamma(n-k+1)}$$

$$= \frac{1}{n+1} \frac{\Gamma(n+2)}{\Gamma(k+1)\,\Gamma(n-k+1)}$$

$$= \frac{1}{(n+2)B(k+1, n-k+1)}$$

(g) [E.C.] If $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{U}(a,\, b)$, show that $X_{(k)}$ is a linear transformation of the beta distribution and find its parameters.

(h) [harder] [MA] Show that $I_x(\alpha, \beta+1) = \dfrac{\beta I_x(\alpha,\beta) + x^\alpha(1-x)^\beta}{\alpha+\beta}$.

Problem 8

We will now practice multivariate change of variables where $\boldsymbol{Y} = \boldsymbol{g}(\boldsymbol{X})$ where $\boldsymbol{X}$ denotes a vector of k continuous r.v.'s and $\boldsymbol{g} : \mathbb{R}^k \to \mathbb{R}^k$ and is 1:1.

(a) [easy] State the formula for the PDF of $\boldsymbol{Y}$.

$$f_{\vec{y}}(h(\vec{y})) = f_{\vec{x}}(h(\vec{y}))\,|J_n(\vec{y})|$$

(b) [harder] Demonstrate that the formula for the PDF of $\boldsymbol{Y}$ reduces to the univariate change of variables formula if the dimensions of $\boldsymbol{Y}$ and $\boldsymbol{X}$ are 1.

Recall that:

$$\begin{aligned}\boldsymbol{Y} &= [g_i(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)]^T \\ \boldsymbol{X} &= [h_1(y_1, \ldots, y_n), \ldots, h_n(y_1, \ldots, y_n)]^T\end{aligned}$$

If the dimension of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are 1 then $n = 1$ and

$$\begin{aligned}y &= g(x) \\ x &= h(y)\end{aligned}$$

This simply becomes the univariate formula

$$f_y(g^{-1}(y)) = f_x(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right] \right|$$

(c) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$.

$$f_{\vec{x}}(y_1, y_2)\,|y_2|$$

(d) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent.

$$\int_{\mathbb{R}} f_{x1}^{old}(ru)\mathbb{1}_{ru \in \text{Supp}[X_1]} f_{x2}^{old}(u)\, |u|\, du$$

(e) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent and have positive supports.

$$\int_0^\infty f_{x1}^{old}(ru) f_{x2}^{old}(u)\, |u|\, du$$

(f) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$.

$$\int f_{x_1}(ru) f_{x_2}(u - ru)\, |u|\, du$$

(g) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent.

$$\int_{\mathbb{R}} f_{x1}^{old}(ru)\mathbb{1}_{ru \in \text{Supp}[X_1]} f_{x2}^{old}(u - ru)\mathbb{1}_{u - ru \in \text{Supp}[X_2]}\, |u|\, du$$

(h) [harder] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent and have positive supports. This should be a simpler expression than the previous.

$$\int_0^\infty f_{x1}^{old}(ru) f_{x2}^{old}(u - ru)\, |u|\, du$$

(i) [difficult] Find a formula for the PDF of $E = X_1^{X_2}$ where $X_1, X_2 \overset{iid}{\sim} f(x)$.

$$\begin{cases} X_1 = Y_1^{\frac{1}{Y_2}} \\ X_2 = Y_2 \end{cases} \Leftrightarrow \begin{cases} Y_1 = X_1^{X_2} \\ Y_2 = X_2 \end{cases}$$

$$|J_n(\vec{y})| = det \begin{vmatrix} \frac{Y_1^{\frac{1-Y_2}{Y_2}}}{Y_2} & -\exp\left(\frac{\ln(Y_1)}{Y_2}\right)\frac{\ln Y_2}{Y_2^2} \\ 0 & 1 \end{vmatrix} = \left| \frac{Y_1^{\frac{1-y_2}{y_2}}}{Y_2} \right|$$

$$\int f_x(y_1, y_2) \left| \frac{Y_1^{\frac{1-y_2}{y_2}}}{Y_2} \right| dY_2$$

Let $r = Y_1$ and $u = Y_2$, then we have

$$\int_{\mathbb{R}} f_x(r^{\frac{1}{u}}, u) \left| \frac{r^{\frac{1-u}{u}}}{u} \right| du$$

(j) [difficult] Find the simplest formula you can for the PDF of $Q = \frac{X_1}{X_2} e^{X_3}$ where X_1, X_2, X_3 are dependent r.v.'s.

$$\begin{cases} X_1 = \frac{Y_1 Y_2}{\exp(Y_3)} \\ X_2 = Y_2 \\ X_3 = \ln(Y_3) \end{cases} \Leftrightarrow \begin{cases} Y_1 = \frac{X_1}{X_2} \exp(X_3) \\ Y_2 = X_2 \\ Y_3 = \exp(X_3) \end{cases}$$

$$|J_n(\vec{y})| = \frac{Y_2}{\exp(Y_3)\, Y_3}$$

$$f_Q(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f_y(y_1, y_2, y_3)\ dy_2 dy_3$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(\frac{uv}{\exp(w)}, v, \ln(w))\ dv dw$$

(k) [difficult] Show that $R = \frac{X_1}{X_2} \sim \beta'(\alpha, \beta)$, the beta prime distribution, if $X_1 \sim \text{Gamma}(\alpha, 1)$ independent of $X_2 \sim \text{Gamma}(\beta, 1)$.

$$f_R(r) = \int_0^\infty \frac{1^\alpha}{\Gamma(\alpha)} (ru)^{\alpha-1} \exp(-ru)\, \mathbb{1}_{ru \in [0,\infty]} \frac{(1)^\beta u^{\beta-1}}{\Gamma(\beta)} \exp(-u)\, |u|\, du$$

$$= \frac{1^{\alpha+\beta}}{\Gamma(\alpha)} r^{\alpha-1} \mathbb{1}_{r>0} \int_0^\infty u^{\alpha+\beta-1} \exp(r+1+u)\, du$$

$$= \frac{r^{\alpha-1}}{B(\alpha,\beta)(r+1)^{\alpha+\beta}}$$

$$= \mathrm{BetaPrime}(\alpha, \beta)$$