MATH 368/621 Fall 2020 Homework #4

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Problem 1

These exercises will give you practice with the gamma function.

(a) [easy] Write the definition of $\Gamma(x)$. skip — duplicate

(b) [easy] Prove $\Gamma(x+1) = x\Gamma(x)$. skip — duplicate

(c) [easy] Write the definition of $\Gamma(x, a)$ without using the gamma function. skip — duplicate

(d) [harder] Write the definition of $Q(x, a)$ without using the gamma function. skip — duplicate

(e) [easy] For $a, c \in (0, \infty)$, prove the following:

$$\int_a^\infty t^{x-1} e^{-ct} dt = \frac{\Gamma(x, ac)}{c^x}$$

skip — duplicate

(f) [easy] Let $X \sim \text{Gamma}(\alpha, \beta)$. Show that this r.v. is equivalent to $X \sim \text{Erlang}(k, \lambda)$ and find k and λ in terms of α and β. Are there any restrictions on the values of α and β for this relationship to hold?

skip — duplicate

Problem 2

These exercises will give you practice with transformations of discrete r.v.'s.

(a) [easy] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = \ln(X+1)$.

$$g(x) = \ln(x+1) \to g^{-1}(y) = \exp(y) - 1$$

$$P_x(g^{-1}(y)) = P_x(\exp(y) - 1) = \binom{n}{\exp(y) - 1} p^{\exp(y)-1}(1-p)^{n-\exp(y)-1}$$

(b) [harder] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

$g(x)$ is not monotonic but it does not matter since the $Supp[X] = \{0, n\}$

$$g(x) = x^2 \rightarrow g^{-1}(y) = \sqrt{y}$$
$$P_x(g^{-1}(y)) = P_x(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}}(1-p)^{n-\sqrt{y}}$$

(c) [difficult] Let $X \sim \text{Binomial}(n, p)$ where n is an even number. Find the PMF of $Y = g(X) = \text{mod}(X, 2)$ where "mod" denotes modulus division of the first argument by the second argument.

(d) [difficult] [MA] Let $X \sim \text{NegBin}(k, p)$. Find the PMF of $Y = g(X) = \text{mod}(X, n)$ where $n \in \mathbb{N}$.

Problem 3

These exercises will give you practice with transformations of continuous r.v.'s and the quantile function.

(a) [harder] Let $X \sim \text{U}(0, 1)$. Find the PDF of $Y = g(X) = aX + c$. Make sure you're careful with the indicator function that specifies the support. There are two cases.

(b) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.

(c) [E.C.] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$.

(d) [harder] Let $X \sim \text{U}(0, 1)$. Find the PDF of $Y = g(X) = \ln\left(\frac{X}{1-X}\right)$. If this is a brand name r.v., mark it so and include its parameter values.

(e) [easy] Find the Quantile function of X where $X \sim \text{Logistic}(0, 1)$.

(f) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Logistic}(\mu, \sigma)$ where $X \sim \text{Logistic}(0, 1)$.

(g) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

(h) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^X$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.

(i) [easy] Rederive the $X \sim \text{Laplace}(0, 1)$ r.v. model by taking the difference of two standard exponential r.v.'s.

(j) [easy] Let $X \sim \text{Laplace}(0, 1)$. Prove that $\mathbb{E}[X] = 0$ without using the integral definition. There's a trick.

(k) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Laplace}(\mu, \sigma)$ where $X \sim \text{Laplace}(0, 1)$.

(l) [difficult] Show that $\mathcal{E} \sim \text{Laplace}(0, \sigma)$ is a reasonable error distribution.

(m) [harder] [MA] Find the Quantile function of X where $X \sim \text{Laplace}(0, 1)$.

(n) [difficult] [MA] Let $X \sim \text{ParetoI}(k, \lambda)$. Show that $Y = X \mid X > c$ where $c > k$ is also a ParetoI r.v. and find its parameter values.

Problem 4

We will now explore a couple of extreme distributions.

(a) [harder] Let $X \sim \text{Exp}(1)$ and $Y = -\ln(X) \sim \text{Gumbel}(0, 1)$. Find the PDF of this standard Gumbel distribution. Make sure you include the indicator function throughout your proof.

$$g(x) = -\ln(X) \;\Leftrightarrow\; g^{-1}(y) = \exp(-Y)$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = |-\exp(-Y)|$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \exp(-\exp(-y) - y)$$

(b) [easy] Find the CDF of Y.

$$\int \exp(-\exp(-x) - x)\,dx = \exp(-\exp(-x))$$

(c) [easy] Let $G = \beta Y + \mu \sim \text{Gumbel}(\mu, \beta)$. Find the PDF of G, the general Gumbel distribution.

$$g(x) = \beta Y + \mu \iff g^{-1}(y) = \frac{x - \mu}{\beta}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{\beta}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = \frac{1}{\beta}\text{exp}\left(-(\frac{x-\mu}{\beta})\right)$$

(d) [easy] [MA] Show that for any r.v. X, if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

(e) [easy] Using the answer in the previous question, find the CDF of $G \sim \text{Gumbel}(\mu, \beta)$.

$$\int \frac{1}{\beta}\text{exp}\left(-(\frac{x-\mu}{\beta})\right)dx = \text{exp}\left(-(\frac{x-\mu}{\beta})\right)$$

Problem 5

These exercises will give you practice with the Weibull distribution.

(a) [easy] If $X \sim \text{Exp}\,(1)$ then show that $Y = \frac{1}{\lambda}X^{\frac{1}{k}} \sim \text{Weibull}\,(k, \lambda)$ where $k, \lambda > 0$.

$$g(x) = \frac{1}{\lambda}x^{\frac{1}{k}} \iff g^{-1}(y) = \lambda^k y^k$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = k\lambda^k y^{k-1}$$

$$f_x(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}\left[g^{-1}(y)\right]\right| = k\lambda(\lambda y)^{k-1}\text{exp}\left(-(\lambda^k y^k)\right)$$

$$\sim \text{Weibull}\,(k, \lambda)$$

(b) [harder] Find Med $[Y]$.

$$\frac{1}{2} = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^k\right)$$

$$\text{Med}[Y] = \lambda \ln(2)^{\frac{1}{k}}$$

(c) [difficult] [MA] Prove that if $k > 1$ then $\mathbb{P}(Y \geq y + c \mid Y \geq c) < \mathbb{P}(Y \geq y)$ for $c > 0$.

(d) [difficult] If $X \sim \text{Exp}(\lambda)$ then show that $Y = X^\beta \sim$ Weibull where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class (i.e. your answer in part a).

(e) [easy] Using Y, the Weibull in terms of the parameterization we learned in class (i.e. your answer in part a), find the PDF of $W = Y + c \sim \text{Weibull}(k, \lambda, c)$ which is known as the "translated Weibull" or "3-parameter Weibull model".

Problem 6

We will practice finding kernels and relating them to known distributions. The gamma function and the beta function will come up as well.

(a) [easy] Find the kernel of the negative binomial PMF.

$$P(X = k) = \binom{k+r-1}{k}(1-p)^r p^k \mathbb{1}_{k \in \{0,1,\ldots\}}$$

$$= \frac{(k+r-1)!}{k!(k+r-1-k)!}(1-p)^r p^k \mathbb{1}_{k \in \{0,1,\ldots\}}$$

$$= \frac{(k+r-1)!}{k!(r-1)!}(1-p)^r p^k \mathbb{1}_{k \in \{0,1,\ldots\}}$$

$$\propto \frac{(k+r-1)!}{k!} p^k \mathbb{1}_{k \in \{0,1,\ldots\}}$$

(b) [easy] Find the kernel of the beta PDF.

$$P(X = x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]}$$

$$\propto x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]}$$

(c) [easy] If $k(x) = e^{-\lambda x}x^{k-1}\mathbb{1}_{x>0}$ how would you know if the r.v. X was an Erlang $(k,\ \lambda)$ or a Gamma (k, λ)?

We know that X will be Gamma (k, λ) since the indicator functions tell us about their supports. We know that:

$$\text{Erlang}\,(k,\ \lambda) \to \text{Supp}\,[X] = [0, \infty)$$

$$\text{Gamma}\,(k, \lambda) \to \text{Supp}\,[X] = (0, \infty)$$

(d) [harder] If $k(x) = xe^{-x^2}\mathbb{1}_{x>0}$, how is X distributed?

$X \sim \text{Weibull}(2, 1)$

$$c = \left(\int_{-\infty}^{\infty} f(x)dx\right)^{-1}$$

$$= 2$$

Problem 7

We will now practice using order statistics concepts.

(a) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the CDF of the maximum X_i and express the CDF of the minimum X_i.

$$\max \leftarrow 1 - (1 - F(x))^n$$

$$\min \leftarrow F(x)^n$$

(b) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF of the maximum X_i and express the PDF of the minimum X_i.

$$\max \leftarrow nf(x)F(x)^{n-1}$$

$$\min \leftarrow nf(x)(1-F(x))^{n-1}$$

(c) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF and the CDF of $X_{(k)}$ i.e. the kth smallest X_i.

(d) [difficult] [MA] If discrete $X_1, \ldots, X_n \overset{iid}{\sim} p(x)$, why would the formulas in (a-c) not be accurate?

$$f_{x_k}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in[0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k-1)} x^{k-1}(1-x)^{n+k+1-1} \mathbb{1}_{x\in[0,1]}$$

$$= \text{Beta}(k,\, n-k+1)$$

(e) [harder] If $X_1, \ldots, X_n \overset{iid}{\sim} \text{U}(0,\, 1)$, show that $X_{(k)} \sim \text{Beta}(k,\, n-k+1)$.

(f) [harder] Express $\binom{n}{k}$ in terms of the beta function.

(g) [E.C.] If $X_1, \ldots, X_n \overset{iid}{\sim} \text{U}(a,\, b)$, show that $X_{(k)}$ is a linear transformation of the beta distribution and find its parameters.

(h) [harder] [MA] Show that $I_x(\alpha, \beta+1) = \dfrac{\beta I_x(\alpha,\beta) + x^{\alpha}(1-x)^{\beta}}{\alpha+\beta}$.

Problem 8

We will now practice multivariate change of variables where $\boldsymbol{Y} = \boldsymbol{g}(\boldsymbol{X})$ where $\boldsymbol{X}$ denotes a vector of k continuous r.v.'s and $\boldsymbol{g} : \mathbb{R}^k \to \mathbb{R}^k$ and is 1:1.

(a) [easy] State the formula for the PDF of $\boldsymbol{Y}$.

$$f_{\vec{y}}(h(\vec{y})) = f_{\vec{x}}(h(\vec{y}))\,|J_n(\vec{y})|$$

(b) [harder] Demonstrate that the formula for the PDF of $\boldsymbol{Y}$ reduces to the univariate change of variables formula if the dimensions of $\boldsymbol{Y}$ and $\boldsymbol{X}$ are 1.

Recall that:

$$
\begin{aligned}
\boldsymbol{Y} &= [g_i(x_1,\dots,x_n),\dots,g_n(x_1,\dots,x_n)]^T \\
\boldsymbol{X} &= [h_1(y_1,\dots,y_n),\dots,h_n(y_1,\dots,y_n)]^T
\end{aligned}
$$

If the dimension of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are 1 then $n = 1$ and

$$
\begin{aligned}
y &= g(x) \\
x &= h(y)
\end{aligned}
$$

This simply becomes the univariate formula

$$
f_y(g^{-1}(y)) = f_x(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} \left[g^{-1}(y)\right] \right|
$$

(c) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$.

$$
f_{\vec{x}}(y_1, y_2)\,|y_2|
$$

(d) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent.

$$
\int_{\mathbb{R}} f_{x_1}^{old}(ru)\mathbb{1}_{ru\in\mathrm{Supp}[X_1]} f_{x_2}^{old}(u)\,|u|\,du
$$

(e) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent and have positive supports.

$$
\int_0^\infty f_{x_1}^{old}(ru) f_{x_2}^{old}(u)\,|u|\,du
$$

(f) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$.

$$
\int f_{x_1}(ru) f_{x_2}(u - ru)\,|u|\,du
$$

(g) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent.

$$\int_{\mathbb{R}} f_{x_1}^{old}(ru)\mathbb{1}_{ru\in\text{Supp}[X_1]} f_{x_2}^{old}(u-ru)\mathbb{1}_{u-ru\in\text{Supp}[X_2]}\,|u|\,du$$

(h) [harder] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent and have positive supports. This should be a simpler expression than the previous.

$$\int_0^\infty f_{x_1}^{old}(ru) f_{x_2}^{old}(u-ru)\,|u|\,du$$

(i) [difficult] Find a formula for the PDF of $E = X_1^{X_2}$ where $X_1, X_2 \overset{iid}{\sim} f(x)$.

$$\begin{cases} X_1 = Y_1^{\frac{1}{Y_2}} \\ X_2 = Y_2 \end{cases} \Leftrightarrow \begin{cases} Y_1 = X_1^{X_2} \\ Y_2 = X_2 \end{cases}$$

$$|J_n(\vec{y})| = det\begin{vmatrix} \frac{Y_1^{\frac{1-Y_2}{Y_2}}}{Y_2} & -\exp\left(\frac{\ln(Y_1)}{Y_2}\right)\frac{\ln Y_2}{Y_2^2} \\ 0 & 1 \end{vmatrix} = \left|\frac{Y_1^{\frac{1-y_2}{y_2}}}{Y_2}\right|$$

$$\int f_x(y_1, y_2)\left|\frac{Y_1^{\frac{1-y_2}{y_2}}}{Y_2}\right| dY_2$$

Let $r = Y_1$ and $u = Y_2$, then we have

$$\int_{\mathbb{R}} f_x(r^{\frac{1}{u}}, u)\left|\frac{r^{\frac{1-u}{u}}}{u}\right| du$$

(j) [difficult] Find the simplest formula you can for the PDF of $Q = \frac{X_1}{X_2}e^{X_3}$ where X_1, X_2, X_3 are dependent r.v.'s.

(k) [difficult] Show that $R = \frac{X_1}{X_2} \sim \beta'(\alpha, \beta)$, the beta prime distribution, if $X_1 \sim \text{Gamma}\,(\alpha, 1)$ independent of $X_2 \sim \text{Gamma}\,(\beta, 1)$.

$$f_R(r) = \int_0^\infty \frac{1^\alpha}{\Gamma(\alpha)} (ru)^{\alpha-1} \exp(-ru)\, \mathbb{1}_{ru \in [0,\infty]} \frac{(1)^\beta u^{\beta-1}}{\Gamma(\beta)} \exp(-u)\, |u|\, du$$

$$= \frac{1^{\alpha+\beta}}{\Gamma(\alpha)} r^{\alpha-1} \mathbb{1}_{r>0} \int_0^\infty u^{\alpha+\beta-1} \exp(r+1+u)\, du$$

$$= \frac{r^{\alpha-1}}{B(\alpha,\beta)(r+1)^{\alpha+\beta}}$$

$$= \text{BetaPrime}(\alpha,\beta)$$