

MATH 368/621 Fall 2020 Homework #1

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Problem 1

These exercises give you practice with sums and indicator functions.

(a) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x=17}$.

$$\sum_{x \in \mathbb{R}} \mathbb{1}_{x=17} = 1$$

(b) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} c \mathbb{1}_{x=17}$ where $c \in \mathbb{R}$ is a constant.

$$\sum_{x \in \mathbb{R}} c \mathbb{1}_{x=17} = c$$

(c) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}}$.

$$\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} = 3$$

(d) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1,2,3\}}$.

$$\begin{aligned} \sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1,2,3\}} &= 1(1) + 2(1) + 3(1) + 4(0) + 5(0) + \dots \\ &= 6 \end{aligned}$$

(e) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}}$.

$$\begin{aligned}\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}} &= 1^1 + 2^1 + 3^1 + 4^0 + \dots \\ &= 6 + \infty \\ &= \infty\end{aligned}$$

(f) [easy] Expand and simplify as much as you can: $\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}}$.

$$\begin{aligned}\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} &= 1 * 1 * 1 * 0 * \dots \\ &= 0\end{aligned}$$

(g) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}}$.

$$\begin{aligned}\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}} &= (1)(0) + (1)(0) + (1)(0) + (0)(1) + (0)(1) + (0)(1) + \dots \\ &= 0\end{aligned}$$

(h) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} c \mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x \in \mathbb{R}} c \mathbb{1}_{x \in \{1,2,\dots,t\}} &= c \sum_{i=0}^t i \\ &= tc\end{aligned}$$

(i) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} t \mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x \in \mathbb{R}} t \mathbb{1}_{x \in \{1,2,\dots,t\}} &= t \sum_{i=1}^t i \\ &= t^2\end{aligned}$$

- (j) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1, 2, \dots, t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned} \sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{1, 2, \dots, t\}} &= \sum_{i=1}^t i \\ &= \frac{(t-1)(t)}{2} \end{aligned}$$

- (k) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \frac{1}{x!} \mathbb{1}_{x \in \mathbb{N}}$.
- (l) [harder] Prove $\mathbb{E}[\mathbb{1}_{X \in A}] = \mathbb{P}(X \in A)$.

Problem 2

These exercises review convolutions.

- (a) [easy] Is a JMF a type of PMF or PMF a type of JMF? Explain.
- (b) [easy] Let $X_1, X_2 \stackrel{iid}{\sim} \text{Bernoulli}(p)$. Find the PMF of the sum of $T = X_1 + X_2$ using the appropriate discrete convolution formula that would make the problem easiest.
- (c) [easy] Let $X_1 \sim \text{Bernoulli}(p_1)$ independent of $X_2 \sim \text{Bernoulli}(p_2)$. Find the JMF of for X_1, X_2 . Denote it using a 2×2 grid or the piecewise function notation.
- (d) [difficult] Let

$$X_1 \sim \begin{cases} 3 & \text{w.p. } 0.3 \\ 6 & \text{w.p. } 0.7 \end{cases} \quad \text{independent of} \quad X_2 \sim \begin{cases} 4 & \text{w.p. } 0.4 \\ 8 & \text{w.p. } 0.6 \end{cases}$$

Find the PMF of $T = X_1 + X_2$ using a convolution. Denote it using the piecewise function notation.

- (e) [difficult] Prove the PMF of a binomial inductively using convolutions on the sequence of r.v.'s $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Bernoulli}(p)$. You will need to use Pascal's Triangle combinatorial identity we employed in class.
- (f) [difficult] [MA] Prove the PMF of a negative binomial inductively using convolutions on the sequence of r.v.'s $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Geometric}(p)$. You will need to use the "hockey stick identity" [click here].

- (g) [difficult] Let $X_1 \sim \text{Binomial}(n_1, p)$ independent of $X_2 \sim \text{Binomial}(n_2, p)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.
- (h) [easy] Prove the PMF of $X \sim \text{Poisson}(\lambda)$ using the limit as $n \rightarrow \infty$ and let $p = \frac{\lambda}{n}$.
- (i) [difficult] Let $X_1 \sim \text{Poisson}(\lambda_1)$ independent of $X_2 \sim \text{Poisson}(\lambda_2)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

Problem 3

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

- (a) [difficult] Let $X \sim \text{Geometric}(p_x)$ independent of $Y \sim \text{Geometric}(p_y)$. Find $\mathbb{P}(X > Y)$ using the method we did in class.
- (b) [easy] [MA] Prove this a different way by finding $\mathbb{P}(X = Y)$ and then using the law of total probability.
- (c) [easy] [MA] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)?
- (d) [difficult] Let $X \sim \text{Poisson}(\lambda)$ independent of $Y \sim \text{Poisson}(\lambda)$. Find an expression for $\mathbb{P}(X > Y)$ *as best as you are able to answer*. Part of this exercise is identifying where you cannot go any further.

Problem 4

These exercises will introduce the Multinomial distribution.

- (a) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}] = k$, what is the parameter space for both n and \mathbf{p} ?
- (b) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}] = k$, what is the $\text{Supp}[\mathbf{X}]$?
- (c) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}] = k$, what is $\dim[\mathbf{p}]$?
- (d) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}] = 2$, express p_2 as a function of p_1 .
- (e) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}] = 2$, how are both X_1 and X_2 distributed?
- (f) [easy] If $\mathbf{X} \sim \text{Multinomial}(n, \mathbf{p})$ and $n = 10$ and $\dim[\mathbf{X}] = 7$ as a column vector, give an example value of \mathbf{x} , a realization of the r.v. \mathbf{X} .

- (g) [easy] If $\mathbf{X} \sim \text{Multinomial}\left(9, [0.1 \ 0.2 \ 0.7]^\top\right)$, find $\mathbb{P}\left(\mathbf{X} = [3 \ 2 \ 4]^\top\right)$ to the nearest two decimal places.
- (h) [difficult] [MA] If $\mathbf{X}_1 \sim \text{Multinomial}(n, \mathbf{p})$ and independently $\mathbf{X}_2 \sim \text{Multinomial}(n, \mathbf{p})$ where $\dim[\mathbf{X}_1] = \dim[\mathbf{X}_2] = k$. Find the JMF of $\mathbf{T}_2 = \mathbf{X}_1 + \mathbf{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of x_1, \dots, x_K . Finally, use Theorem 1 in this paper: [\[click here\]](#) for the summation.