

MATH 368/621 Fall 2020 Homework #4

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Problem 1

These exercises will give you practice with the gamma function.

- (a) [easy] Write the definition of $\Gamma(x)$. skip — duplicate
- (b) [easy] Prove $\Gamma(x+1) = x\Gamma(x)$. skip — duplicate
- (c) [easy] Write the definition of $\Gamma(x, a)$ without using the gamma function. skip — duplicate
- (d) [harder] Write the definition of $Q(x, a)$ without using the gamma function. skip — duplicate
- (e) [easy] For $a, c \in (0, \infty)$, prove the following:

$$\int_a^\infty t^{x-1} e^{-ct} dt = \frac{\Gamma(x, ac)}{c^x}$$

skip — duplicate

- (f) [easy] Let $X \sim \text{Gamma}(\alpha, \beta)$. Show that this r.v. is equivalent to $X \sim \text{Erlang}(k, \lambda)$ and find k and λ in terms of α and β . Are there any restrictions on the values of α and β for this relationship to hold?

skip — duplicate

Problem 2

These exercises will give you practice with transformations of discrete r.v.'s.

- (a) [easy] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = \ln(X+1)$.

$$g(x) = \ln(x+1) \rightarrow g^{-1}(y) = \exp(y) - 1$$

$$P_x(g^{-1}(y)) = P_x(\exp(y) - 1) = \binom{n}{\exp(y) - 1} p^{\exp(y)-1} (1-p)^{n-\exp(y)-1}$$

- (b) [harder] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = g(X) = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

$g(x)$ is not monotonic but it does not matter since the $\text{Supp}[X] = \{0, n\}$

$$g(x) = x^2 \rightarrow g^{-1}(y) = \sqrt{y}$$

$$P_x(g^{-1}(y)) = P_x(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}}$$

- (c) [difficult] Let $X \sim \text{Binomial}(n, p)$ where n is an even number. Find the PMF of $Y = g(X) = \text{mod}(X, 2)$ where “mod” denotes modulus division of the first argument by the second argument.
- (d) [difficult] [MA] Let $X \sim \text{NegBin}(k, p)$. Find the PMF of $Y = g(X) = \text{mod}(X, n)$ where $n \in \mathbb{N}$.

Problem 3

These exercises will give you practice with transformations of continuous r.v.'s and the quantile function.

- (a) [harder] Let $X \sim U(0, 1)$. Find the PDF of $Y = g(X) = aX + c$. Make sure you're careful with the indicator function that specifies the support. There are two cases.
- (b) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.
- (c) [E.C.] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$.
- (d) [harder] Let $X \sim U(0, 1)$. Find the PDF of $Y = g(X) = \ln\left(\frac{X}{1-X}\right)$. If this is a brand name r.v., mark it so and include its parameter values.
- (e) [easy] Find the Quantile function of X where $X \sim \text{Logistic}(0, 1)$.
- (f) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Logistic}(\mu, \sigma)$ where $X \sim \text{Logistic}(0, 1)$.
- (g) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

- (h) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^X$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.
- (i) [easy] Rederive the $X \sim \text{Laplace}(0, 1)$ r.v. model by taking the difference of two standard exponential r.v.'s.
- (j) [easy] Let $X \sim \text{Laplace}(0, 1)$. Prove that $\mathbb{E}[X] = 0$ without using the integral definition. There's a trick.
- (k) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Laplace}(\mu, \sigma)$ where $X \sim \text{Laplace}(0, 1)$.
- (l) [difficult] Show that $\mathcal{E} \sim \text{Laplace}(0, \sigma)$ is a reasonable error distribution.
- (m) [harder] [MA] Find the Quantile function of X where $X \sim \text{Laplace}(0, 1)$.
- (n) [difficult] [MA] Let $X \sim \text{ParetoI}(k, \lambda)$. Show that $Y = X \mid X > c$ where $c > k$ is also a ParetoI r.v. and find its parameter values.

Problem 4

We will now explore a couple of extreme distributions.

- (a) [harder] Let $X \sim \text{Exp}(1)$ and $Y = -\ln(X) \sim \text{Gumbel}(0, 1)$. Find the PDF of this standard Gumbel distribution. Make sure you include the indicator function throughout your proof.
- (b) [easy] Find the CDF of Y .
- (c) [easy] Let $G = \beta Y + \mu \sim \text{Gumbel}(\mu, \beta)$. Find the PDF of G , the general Gumbel distribution.
- (d) [easy] [MA] Show that for any r.v. X , if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.
- (e) [easy] Using the answer in the previous question, find the CDF of $G \sim \text{Gumbel}(\mu, \beta)$.

Problem 5

These exercises will give you practice with the Weibull distribution.

- (a) [easy] If $X \sim \text{Exp}(1)$ then show that $Y = \frac{1}{\lambda} X^{\frac{1}{k}} \sim \text{Weibull}(k, \lambda)$ where $k, \lambda > 0$.
- (b) [harder] Find $\text{Med}[Y]$.
- (c) [difficult] [MA] Prove that if $k > 1$ then $\mathbb{P}(Y \geq y + c \mid Y \geq c) < \mathbb{P}(Y \geq y)$ for $c > 0$.

- (d) [difficult] If $X \sim \text{Exp}(\lambda)$ then show that $Y = X^\beta \sim \text{Weibull}$ where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class (i.e. your answer in part a).
- (e) [easy] Using Y , the Weibull in terms of the parameterization we learned in class (i.e. your answer in part a), find the PDF of $W = Y + c \sim \text{Weibull}(k, \lambda, c)$ which is known as the “translated Weibull” or “3-parameter Weibull model”.

Problem 6

We will practice finding kernels and relating them to known distributions. The gamma function and the beta function will come up as well.

- (a) [easy] Find the kernel of the negative binomial PMF.

$$\begin{aligned}
 P(X = k) &= \binom{k+r-1}{k} (1-p)^r p^k \mathbb{1}_{k \in \{0,1,\dots\}} \\
 &= \frac{(k+r-1)!}{k!(k+r-1-k)!} (1-p)^r p^k \mathbb{1}_{k \in \{0,1,\dots\}} \\
 &= \frac{(k+r-1)!}{k!(r-1)!} (1-p)^r p^k \mathbb{1}_{k \in \{0,1,\dots\}} \\
 &\propto \frac{(k+r-1)!}{k!} p^k \mathbb{1}_{k \in \{0,1,\dots\}}
 \end{aligned}$$

- (b) [easy] Find the kernel of the beta PDF.

$$\begin{aligned}
 P(X = x) &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]} \\
 &\propto x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]}
 \end{aligned}$$

- (c) [easy] If $k(x) = e^{-\lambda x} x^{k-1} \mathbb{1}_{x>0}$ how would you know if the r.v. X was an Erlang (k, λ) or a Gamma (k, λ) ?

We know that X will be Gamma (k, λ) since the indicator functions tell us about their supports. We know that:

$$\text{Erlang}(k, \lambda) \rightarrow \text{Supp}[X] = [0, \infty)$$

$$\text{Gamma}(k, \lambda) \rightarrow \text{Supp}[X] = (0, \infty)$$

(d) [harder] If $k(x) = xe^{-x^2} \mathbf{1}_{x>0}$, how is X distributed?

Problem 7

We will now practice using order statistics concepts.

(a) [easy] If $X_1, \dots, X_n \stackrel{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the CDF of the maximum X_i and express the CDF of the minimum X_i .

$$\max \leftarrow 1 - (1 - F(x))^n$$

$$\min \leftarrow F(x)^n$$

(b) [easy] If $X_1, \dots, X_n \stackrel{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF of the maximum X_i and express the PDF of the minimum X_i .

$$\max \leftarrow nf(x)F(x)^{n-1}$$

$$\min \leftarrow nf(x)(1 - F(x))^{n-1}$$

(c) [easy] If $X_1, \dots, X_n \stackrel{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF and the CDF of $X_{(k)}$ i.e. the k th smallest X_i .

(d) [difficult] [MA] If discrete $X_1, \dots, X_n \stackrel{iid}{\sim} p(x)$, why would the formulas in (a-c) not be accurate?

$$\begin{aligned} f_{x_k}(x) &= \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \mathbf{1}_{x \in [0,1]} \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k+1-1} \mathbf{1}_{x \in [0,1]} \\ &= \text{Beta}(k, n-k+1) \end{aligned}$$

- (e) [harder] If $X_1, \dots, X_n \stackrel{iid}{\sim} U(0, 1)$, show that $X_{(k)} \sim \text{Beta}(k, n - k + 1)$.
- (f) [harder] Express $\binom{n}{k}$ in terms of the beta function.
- (g) [E.C.] If $X_1, \dots, X_n \stackrel{iid}{\sim} U(a, b)$, show that $X_{(k)}$ is a linear transformation of the beta distribution and find its parameters.
- (h) [harder] [MA] Show that $I_x(\alpha, \beta + 1) = \frac{\beta I_x(\alpha, \beta) + x^\alpha(1 - x)^\beta}{\alpha + \beta}$.

Problem 8

We will now practice multivariate change of variables where $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ where \mathbf{X} denotes a vector of k continuous r.v.'s and $\mathbf{g} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and is 1:1.

- (a) [easy] State the formula for the PDF of \mathbf{Y} .

$$f_{\vec{y}}(h(\vec{y})) = f_{\vec{x}}(h(\vec{y})) |J_n(\vec{y})|$$

- (b) [harder] Demonstrate that the formula for the PDF of \mathbf{Y} reduces to the univariate change of variables formula if the dimensions of \mathbf{Y} and \mathbf{X} are 1.

Recall that:

$$\begin{aligned}\mathbf{Y} &= [g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n)]^T \\ \mathbf{X} &= [h_1(y_1, \dots, y_n), \dots, h_n(y_1, \dots, y_n)]^T\end{aligned}$$

If the dimension of \mathbf{X} and \mathbf{Y} are 1 then $n = 1$ and

$$\begin{aligned}y &= g(x) \\ x &= h(y)\end{aligned}$$

This simply becomes the univariate formula

$$f_y(g^{-1}(y)) = f_x(g^{-1}(y)) \left| \frac{d}{dy} [g^{-1}(y)] \right|$$

- (c) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$.

$$f_{\vec{x}}(y_1, y_2) |y_2|$$

- (d) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent.

$$\int_{\mathbb{R}} f_{x_1}^{old}(ru) \mathbb{1}_{ru \in \text{Supp}[X_1]} f_{x_2}^{old}(u) |u| du$$

- (e) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent and have positive supports.

$$\int_0^\infty f_{x_1}^{old}(ru) f_{x_2}^{old}(u) |u| du$$

- (f) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1 + X_2}$.

$$\int f_{x_1}(ru) f_{x_2}(u - ru) |u| du$$

- (g) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1 + X_2}$ if X_1 and X_2 are independent.

$$\int_{\mathbb{R}} f_{x_1}^{old}(ru) \mathbb{1}_{ru \in \text{Supp}[X_1]} f_{x_2}^{old}(u - ru) \mathbb{1}_{u - ru \in \text{Supp}[X_2]} |u| du$$

- (h) [harder] State the formula for the PDF of $R = \frac{X_1}{X_1 + X_2}$ if X_1 and X_2 are independent and have positive supports. This should be a simpler expression than the previous.

$$\int_0^\infty f_{x_1}^{old}(ru) f_{x_2}^{old}(u - ru) |u| du$$

- (i) [difficult] Find a formula for the PDF of $E = X_1^{X_2}$ where $X_1, X_2 \stackrel{iid}{\sim} f(x)$.

$$\begin{cases} X_1 = Y_1^{\frac{1}{Y_2}} \\ X_2 = Y_2 \end{cases} \Leftrightarrow \begin{cases} Y_1 = X_1^{X_2} \\ Y_2 = X_2 \end{cases}$$

$$|J_n(\vec{y})| = \det \begin{vmatrix} \frac{Y_1^{\frac{1-Y_2}{Y_2}}}{Y_2} & -\exp\left(\frac{\ln(Y_1)}{Y_2}\right) \frac{\ln Y_2}{Y_2^2} \\ 0 & 1 \end{vmatrix} = \left| \frac{Y_1^{\frac{1-Y_2}{Y_2}}}{Y_2} \right|$$

$$\int f_x(y_1, y_2) \left| \frac{Y_1^{\frac{1-y_2}{y_2}}}{Y_2} \right| dY_2$$

Let $r = Y_1$ and $u = Y_2$, then we have

$$\int_{\mathbb{R}} f_x(r^{\frac{1}{u}}, u) \left| \frac{r^{\frac{1-u}{u}}}{u} \right| du$$

- (j) [difficult] Find the simplest formula you can for the PDF of $Q = \frac{X_1}{X_2} e^{X_3}$ where X_1, X_2, X_3 are dependent r.v.'s.
- (k) [difficult] Show that $R = \frac{X_1}{X_2} \sim \beta'(\alpha, \beta)$, the beta prime distribution, if $X_1 \sim \text{Gamma}(\alpha, 1)$ independent of $X_2 \sim \text{Gamma}(\beta, 1)$.

$$\begin{aligned} f_R(r) &= \int_0^\infty \frac{1^\alpha}{\Gamma(\alpha)} (ru)^{\alpha-1} \exp(-ru) \mathbf{1}_{ru \in [0, \infty]} \frac{(1)^\beta u^{\beta-1}}{\Gamma(\beta)} \exp(-u) |u| du \\ &= \frac{1^{\alpha+\beta}}{\Gamma(\alpha)} r^{\alpha-1} \mathbf{1}_{r>0} \int_0^\infty u^{\alpha+\beta-1} \exp(r+1+u) du \\ &= \frac{r^{\alpha-1}}{B(\alpha, \beta)(r+1)^{\alpha+\beta}} \\ &= \text{BetaPrime}(\alpha, \beta) \end{aligned}$$