MATH 368/621 Fall 2020 Homework #1

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Problem 1

These exercises give you practice with sums and indicator functions.

(a) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x=17}$.

$$\sum_{x \in \mathbb{R}} \mathbb{1}_{x=17} = 1$$

(b) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} c\mathbb{1}_{x=17}$ where $c \in \mathbb{R}$ is a constant.

$$\sum_{x \in \mathbb{R}} c\mathbb{1}_{x=17} = c$$

(c) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}}$.

$$\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} = 3$$

(d) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} x\mathbb{1}_{x \in \{1,2,3\}}$.

$$\sum_{x \in \mathbb{R}} x\mathbb{1}_{x \in \{1,2,3\}} = 1(1) + 2(1) + 3(1) + 4(0) + 5(0) + \dots$$
$$= 6$$

1

(e) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}}$.

$$\sum_{x \in \mathbb{N}_0} x^{\mathbb{1}_{x \in \{1,2,3\}}} = 1^1 + 2^1 + 3^1 + 4^0 + \dots$$
$$= 6 + \infty$$
$$= \infty$$

(f) [easy] Expand and simplify as much as you can: $\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}}$.

$$\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} = 1 * 1 * 1 * 0 * \dots$$
$$= 0$$

(g) [easy] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}}$.

$$\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{1,2,3\}} \mathbb{1}_{x \in \{4,5,6\}} = (1)(0) + (1)(0) + (1)(0) + (0)(1) + (0)(1) + (0)(1) + \dots$$
$$= 0$$

(h) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} c \mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\sum_{x \in \mathbb{R}} c \mathbb{1}_{x \in \{1,2,\dots,t\}} = c \sum_{i=0}^{t} 1$$
$$= tc$$

(i) [harder] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} t \mathbb{1}_{x \in \{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\sum_{x \in \mathbb{R}} t \mathbb{1}_{x \in \{1,2,\dots,t\}} = t \sum_{i=1}^{t} i$$
$$= t^2$$

(j) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,...,t\}}$ where $c\in\mathbb{R}$ is a constant and $t\in\mathbb{N}$ is a constant.

$$\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,...,t\}} = \sum_{i=1}^{t} i$$
$$= \frac{(t-1)(t)}{2}$$

(k) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \frac{1}{x!}\mathbb{1}_{x\in\mathbb{N}}$.

$$\sum_{x\in\mathbb{R}} \frac{1}{x!}\mathbb{1}_{x\in\mathbb{N}} = \sum_{x\in\mathbb{N}} \frac{1}{x!}$$
$$= \exp(1) - 1$$

(l) [harder] Prove $\mathbb{E}\left[\mathbb{1}_{X\in A}\right] = \mathbb{P}(X\in A)$.

Problem 2

These exercises review convolutions.

(a) [easy] Is a JMF a type of PMF or PMF a type of JMF? Explain.

A JMF is a type of PMF because a JMF is derived from 2 or more PMF's

(b) [easy] Let $X_1, X_2 \overset{iid}{\sim}$ Bernoulli (p). Find the PMF of the sum of $T = X_1 + X_2$ using the appropriate discrete convolution formula that would make the problem easiest.

$$\mathbb{P}(t) = \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x}(1-p)^{1-t-x}$$
$$= p^t(1-p)^{2-t} \sum_{x\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$
$$= p^t(1-p)^{2-t} \left(\binom{1}{t}\binom{1}{t-1}\right)$$
$$= \binom{2}{t} p^t(1-p)^{2-t}$$

(c) [easy] Let $X_1 \sim$ Bernoulli (p_1) independent of $X_2 \sim$ Bernoulli (p_2). Find the JMF of for X_1, X_2. Denote it using a 2×2 grid or the piecewise function notation.

$$\begin{cases} 1 & \text{w.p.} \quad (p_1)(p_2) \\ 0 & \text{w.p.} \quad (1-p_1)(1-p_2) \end{cases}$$

(d) [difficult] Let

$$X_1 \sim \begin{cases} 3 & \text{w.p.} \quad 0.3 \\ 6 & \text{w.p.} \quad 0.7 \end{cases} \qquad \text{independent of} \quad X_2 \sim \begin{cases} 4 & \text{w.p.} \quad 0.4 \\ 8 & \text{w.p.} \quad 0.6 \end{cases}$$

Find the PMF of $T = X_1 + X_2$ using a convolution. Denote it using the piecewise function notation.

(e) [difficult] Prove the PMF of a binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim}$ Bernoulli (p). You will need to use Pascal's Triangle combinatorial identity we employed in class.

Let $T_n = X_1 + X_2 + \ldots + X_n$ and $T_n = X_n + T_{n-1}$

$$\begin{aligned} \mathbb{P}(t) &= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{n-1}{t-x} p^{t-x} (1-p)^{n-1-t+x} \\ &= \sum_{x \in \{0,1\}} p^t (1-p)^{n-t} \binom{n-1}{t-x} \\ &= p^t (1-p)^{n-t} \sum_{x \in \{0,1\}} \binom{n-1}{t-x} \\ &= p^t (1-p)^{n-t} \left(\binom{n-1}{t} + \binom{n-1}{t-1} \right) \\ &= p^t (1-p)^{n-t} \binom{n}{t} \end{aligned}$$

(f) [difficult] [MA] Prove the PMF of a negative binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim}$ Geometric (p). You will need to use the "hockey stick identity" [click here].

4

$$\text{Let} T_n = X_n + T_{n-1}$$

$$\mathbb{P}(t) = \sum \mathbb{P}(X_n) \mathbb{P}(T_n - X_n)$$

$$= \sum_{x \in \{0,1,\dots\}} (1-p)^x \binom{(t-x)+n-2}{n-2} (1-p)^{t-x} p^{n-1} \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$= (1-p)^t p^n \sum_{x \in \{0,1,\dots\}} \binom{(t-x)+n-2}{n-2}$$

$$= (1-p)^t p^n \sum x = 0^t \binom{t+n-x-2}{n-2}$$

$$= (1-p)^t p^n \binom{t+n-1}{n-1}$$

(g) [difficult] Let $X_1 \sim \text{Binomial}(n_1, p)$ independent of $X_2 \sim \text{Binomial}(n_2, p)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

$$\mathbb{P}(t) = \sum_{x \in \mathbb{R}} \binom{n_1}{x} p^x (1-p)^{n_1-x} \binom{n_2}{t-x} p^{t-x} (1-p)^{n_2-t+x} \mathbb{1}_{t-x \in \text{Supp}[X_1]}$$

$$= p^t (1-p)^{n_1+n_2-t} \sum_{x \in \{0,\dots,t\}} \binom{n_1}{x} \binom{n_2}{t-x}$$

$$= p^t (1-p)^{n_1+n_2-t} \binom{n_1+n_2}{t}$$

(h) [easy] Prove the PMF of $X \sim \text{Poisson}(\lambda)$ using the limit as $n \to \infty$ and let $p = \frac{\lambda}{n}$.

$$\text{Let } X \sim \text{Binomial}(n, p) \text{ where } n \to \infty \text{ and } p = \frac{\lambda}{n}$$

$$\mathbb{P}(x) = \lim_{n \to \infty} \frac{n!}{x!(n-x)!} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x} \mathbb{1}_{x \in 0,\dots,n}$$

$$= \lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{1-x} \mathbb{1}_{x \in 0,\dots,n}$$

$$= \frac{\lambda^x}{x!} \lim_{n \to \infty} \frac{n!}{(n-x)n^x} \lim_{n \to \infty} (1 - \frac{\lambda}{n})^n \lim_{n \to \infty} (1 - \frac{\lambda}{n})^{-x} \mathbb{1}_{x \in 0,\dots,n}$$

$$= \frac{\lambda^x}{x!} \lim_{n \to \infty} \frac{n(n-1)\ldots(n-x+1)}{n \cdot n \ldots \cdot n} \exp\left(-\lambda\right) \mathbb{1}_{x \in \{0,1,\ldots\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,\ldots\}}$$

$$= \text{Poisson}\left(\lambda\right)$$

(i) [difficult] Let $X_1 \sim \text{Poisson}\left(\lambda_1\right)$ independent of $X_2 \sim \text{Poisson}\left(\lambda_2\right)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

$$\mathbb{P}\left(t\right) = \sum_{x_1 \in \{0,\ldots\}} = \sum \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1}}{x_1!} \frac{\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{(t-x_1)!} \mathbb{1}_{t-x \in \text{Supp}[X_1]}$$

$$= \sum_{x_1 \in \{0,\ldots,t\}} \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1}}{x_1!} \frac{\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{(t-x_1)!}$$

$$= \sum_{x_1 \in \{0,\ldots,t\}} \left(\frac{1}{x_1!(t-x_1)!}\exp\left(-\lambda_1\right)\lambda_1^{x_1}\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}\right)\frac{t!}{t!}$$

$$= \sum_{x_1 \in \{0,\ldots,t\}} \binom{t}{x_1} \frac{\exp\left(-\lambda_1\right)\lambda_1^{x_1}\exp\left(-\lambda_2\right)\lambda_2^{t-x_1}}{t!}$$

$$= \frac{\exp\left(-\lambda_1-\lambda_2\right)}{t!} \sum_{x_1 \in \{0,\ldots,t\}} \binom{t}{x_1}\lambda_1^{x_1}\lambda_2^{t-x_1}$$

$$= \frac{\exp\left(-\lambda_1-\lambda_2\right)\left(\lambda_2+\lambda_1\right)^t}{t!}$$

Problem 3

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

(a) [difficult] Let $X \sim \text{Geometric}\left(p_x\right)$ independent of $Y \sim \text{Geometric}\left(p_y\right)$. Find $\mathbb{P}\left(X > Y\right)$ using the method we did in class.

(b) [easy] [MA] Prove this a different way by finding $\mathbb{P}\left(X = Y\right)$ and then using the law of total probability.

(c) [easy] [MA] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)?

(d) [difficult] Let $X \sim \text{Poisson}\left(\lambda\right)$ independent of $Y \sim \text{Poisson}\left(\lambda\right)$. Find an expression for $\mathbb{P}\left(X > Y\right)$ *as best as you are able to answer*. Part of this exercise is identifying where you cannot go any further.

$$\mathbb{P}\left(X > Y\right) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} \mathbb{P}\left(x\right) \mathbb{P}\left(y\right) \mathbb{1}_{x > y}$$

$$= \sum_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} \frac{e^{-\lambda} \lambda^x}{x!} \frac{e^{-\lambda} \lambda^y}{y!} \mathbb{1}_{x > y}$$

$$= e^{-2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^y}{y!} \sum_{x \in \{y+1, \dots\}} \frac{\lambda^x}{x!}$$

$$= e^{-2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^y}{y!} \sum_{x\prime \in \mathbb{N}_0} \frac{\lambda^{x\prime + y + 1}}{x\prime!}$$

$$= \lambda e^{2\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^{2y}}{y!} \sum_{x\prime \in \mathbb{N}_0} \frac{\lambda^{x\prime}}{x\prime!}$$

$$= \lambda e^{-\lambda} \sum_{y \in \mathbb{N}_0} \frac{\lambda^{2y}}{y!}$$

Problem 4

These exercises will introduce the Multinomial distribution.

(a) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the parameter space for both n and $\boldsymbol{p}$?

$$n \in \mathbb{N}, \quad \boldsymbol{p} \in \{\vec{v} : \vec{v} \cdot 1, v_1 \in (0, 1), \dots, v_k \in (0, 1)\}$$

(b) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the $\text{Supp}\left[\boldsymbol{X}\right]$?

$$\text{Supp}\left[\boldsymbol{X}\right] = \{\vec{x} : \vec{x} \cdot 1 = n, x_1 \in \{0, \dots, n\}, \dots, x_k \in \{0, \dots, n\}\}$$

(c) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is $\dim\left[\boldsymbol{p}\right]$?

$$\dim\left[\boldsymbol{p}\right] = k$$

(d) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = 2$, express p_2 as a function of p_1.

$$p_1 = 1 - p_2$$

7

(e) [easy] If $\boldsymbol{X} \sim$ Multinomial $(n,\, \boldsymbol{p})$ where dim $[\boldsymbol{X}] = 2$, how are both X_1 and X_2 distributed?

$$X_1 \sim \text{Binomial}\,(n,\, p_1)\,, \quad X_2 \sim \text{Binomial}\,(n,\, p_2)$$

(f) [easy] If $\boldsymbol{X} \sim$ Multinomial $(n,\, \boldsymbol{p})$ and $n = 10$ and dim $[\boldsymbol{X}] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$.

$$\boldsymbol{x} = [1, 1, 1, 1, 1, 2, 3]^T$$

(g) [easy] If $\boldsymbol{X} \sim$ Multinomial $\left(9,\, [0.1\ 0.2\ 0.7]^\top\right)$, find $\mathbb{P}\left(\boldsymbol{X} = [3\ 2\ 4]^\top\right)$ to the nearest two decimal places.

$$
\begin{aligned}
\mathbb{P}\left(X = [3, 2, 4]^T\right) &= \binom{9}{3,\, 2,\, 4} \cdot (0.1) \cdot (0.2) \cdot (0.7) \\
&= \frac{9!}{3!2!4!} \cdot (0.1) \cdot (0.2) \cdot (0.7) \\
&= 17.64
\end{aligned}
$$

(h) [difficult] [MA] If $\boldsymbol{X}_1 \sim$ Multinomial $(n,\, \boldsymbol{p})$ and independently $\boldsymbol{X}_2 \sim$ Multinomial $(n,\, \boldsymbol{p})$ where dim $[\boldsymbol{X}_1] = \dim [\boldsymbol{X}_2] = k$. Find the JMF of $\boldsymbol{T}_2 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of $x_1, \ldots, x_K$. Finally, use Theorem 1 in this paper: [click here] for the summation.

$$
\begin{aligned}
\mathbb{P}\,(t_1, \ldots, t_k) &= \sum_{x_1, \ldots, x_k} \binom{n}{x_1, \ldots, x_k} \prod_{k=1}^{k} p_k^{x_k} \binom{n}{(t_1 - x_1), \ldots, (t_k - x_k)} \prod_{k=1}^{k} p_k^{t_k - x_k} \\
&= \prod_{i=1}^{k} p_i^{t_i} \sum_{x_1, \ldots, x_k} \binom{n}{x_1, \ldots, x_k} \binom{n}{(t_1 - x_1), \ldots, (t_k - x_k)} \\
&= \prod_{i=1}^{k} p_i^{t_i} \binom{2n}{t_1, t_2, \ldots, t_k}
\end{aligned}
$$