MATH 368/621 Fall 2020 Homework #1

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Problem 1

These exercises give you practice with sums and indicator functions.

(a) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \mathbb{1}_{x=17}$.

$$\sum_{x\in\mathbb{R}} \mathbb{1}_{x=17} = 1$$

(b) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} c\mathbb{1}_{x=17}$ where $c \in \mathbb{R}$ is a constant.

$$\sum_{x\in\mathbb{R}} c\mathbb{1}_{x=17} = c$$

(c) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}}$.

$$\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}} = 3$$

(d) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,3\}}$.

$$\begin{aligned}\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,3\}} &= 1(1) + 2(1) + 3(1) + 4(0) + 5(0) + \dots \\ &= 6\end{aligned}$$

(e) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{N}_0} x^{\mathbb{1}_{x\in\{1,2,3\}}}$.

$$\begin{aligned}\sum_{x\in\mathbb{N}_0} x^{\mathbb{1}_{x\in\{1,2,3\}}} &= 1^1 + 2^1 + 3^1 + 4^0 + \dots \\ &= 6 + \infty \\ &= \infty\end{aligned}$$

(f) [easy] Expand and simplify as much as you can: $\prod_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}}$.

$$\begin{aligned}\prod_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}} &= 1 * 1 * 1 * 0 * \dots \\ &= 0\end{aligned}$$

(g) [easy] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}}\mathbb{1}_{x\in\{4,5,6\}}$.

$$\begin{aligned}\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}}\mathbb{1}_{x\in\{4,5,6\}} &= (1)(0) + (1)(0) + (1)(0) + (0)(1) + (0)(1) + (0)(1) + \dots \\ &= 0\end{aligned}$$

(h) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} c\mathbb{1}_{x\in\{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x\in\mathbb{R}} c\mathbb{1}_{x\in\{1,2,\dots,t\}} &= c\sum_{i=0}^{t} i \\ &= tc\end{aligned}$$

(i) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} t\mathbb{1}_{x\in\{1,2,\dots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x\in\mathbb{R}} t\mathbb{1}_{x\in\{1,2,\dots,t\}} &= t\sum_{i=1}^{t} i \\ &= t^2\end{aligned}$$

(j) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,\ldots,t\}}$ where $c \in \mathbb{R}$ is a constant and $t \in \mathbb{N}$ is a constant.

$$\begin{aligned}\sum_{x\in\mathbb{R}} x\mathbb{1}_{x\in\{1,2,\ldots,t\}} &= \sum_{i=1}^{t} i \\ &= \frac{(t-1)(t)}{2}\end{aligned}$$

(k) [harder] Expand and simplify as much as you can: $\sum_{x\in\mathbb{R}} \frac{1}{x!}\mathbb{1}_{x\in\mathbb{N}}$.

$$\begin{aligned}\sum_{x\in\mathbb{R}} \frac{1}{x!}\mathbb{1}_{x\in\mathbb{N}} &= \sum_{x\in\mathbb{N}} \frac{1}{x!} \\ &= \exp(1)\end{aligned}$$

(l) [harder] Prove $\mathbb{E}[\mathbb{1}_{X\in A}] = \mathbb{P}(X \in A)$.

Problem 2

These exercises review convolutions.

(a) [easy] Is a JMF a type of PMF or PMF a type of JMF? Explain.

A JMF is a type of PMF because a JMF is derived from 2 or more PMF's

(b) [easy] Let $X_1, X_2 \overset{iid}{\sim} \text{Bernoulli}(p)$. Find the PMF of the sum of $T = X_1 + X_2$ using the appropriate discrete convolution formula that would make the problem easiest.

$$\begin{aligned}\mathbb{P}(t) &= \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t-x} \\ &= p^t (1-p)^{2-t} \sum_{i\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x} \\ &= p^t (1-p)^{2-t} \left(\binom{1}{t}\binom{1}{t-1}\right) \\ &= \binom{2}{t} p^t (1-p)^{2-t}\end{aligned}$$

(c) [easy] Let $X_1 \sim \text{Bernoulli}\,(p_1)$ independent of $X_2 \sim \text{Bernoulli}\,(p_2)$. Find the JMF of for X_1, X_2. Denote it using a 2×2 grid or the piecewise function notation.

$$\begin{cases} 1 & \text{w.p. } (p_1)(p_2) \\ 0 & \text{w.p. } (1-p_1)(1-p_2) \end{cases}$$

(d) [difficult] Let

$$X_1 \sim \begin{cases} 3 & \text{w.p. } 0.3 \\ 6 & \text{w.p. } 0.7 \end{cases} \qquad \text{independent of} \quad X_2 \sim \begin{cases} 4 & \text{w.p. } 0.4 \\ 8 & \text{w.p. } 0.6 \end{cases}$$

Find the PMF of $T = X_1 + X_2$ using a convolution. Denote it using the piecewise function notation.

(e) [difficult] Prove the PMF of a binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}\,(p)$. You will need to use Pascal's Triangle combinatorial identity we employed in class.

$$\text{Let } T_n = X_1 + X_2 + \ldots + X_n \text{ and } T_n = X_n + T_{n-1}$$

$$\begin{aligned}
\mathbb{P}\,(t) &= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{n-1}{t-x} p^{t-x} (1-p)^{n-1-t+x} \\
&= \sum_{x \in \{0,1\}} p^t (1-p)^{n-t} \binom{n-1}{t-x} \\
&= p^t (1-p)^{n-t} \sum_{x \in \{0,1\}} \binom{n-1}{t-x} \\
&= p^t (1-p)^{n-t} \left(\binom{n-1}{t} + \binom{n-1}{t-1} \right) \\
&= p^t (1-p)^{n-t} \binom{n}{t}
\end{aligned}$$

(f) [difficult] [MA] Prove the PMF of a negative binomial inductively using convolutions on the sequence of r.v.'s $X_1, \ldots, X_n \overset{iid}{\sim} \text{Geometric}\,(p)$. You will need to use the "hockey stick identity" [click here].

(g) [difficult] Let $X_1 \sim \text{Binomial}(n_1, p)$ independent of $X_2 \sim \text{Binomial}(n_2, p)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

$$\begin{aligned}
\mathbb{P}(t) &= \sum_{x \in \mathbb{R}} \binom{n_1}{x} p^x (1-p)^{n_1 - x} \binom{n_2}{t-x} p^{t-x} (1-p)^{n_2 - t + x} \mathbb{1}_{t - x \in \text{Supp}[X_1]} \\
&= p^t (1-p)^{n_1 + n_2 - t} \sum_{x \in \{0, \ldots, t\}} \binom{n_1}{x} \binom{n_2}{t-x} \\
&= p^t (1-p)^{n_1 + n_2 - t} \binom{n_1 + n_2}{t}
\end{aligned}$$

(h) [easy] Prove the PMF of $X \sim \text{Poisson}(\lambda)$ using the limit as $n \to \infty$ and let $p = \frac{\lambda}{n}$.

(i) [difficult] Let $X_1 \sim \text{Poisson}(\lambda_1)$ independent of $X_2 \sim \text{Poisson}(\lambda_2)$. Find the PMF of the sum of $T = X_1 + X_2$ using a convolution.

Problem 3

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

(a) [difficult] Let $X \sim \text{Geometric}(p_x)$ independent of $Y \sim \text{Geometric}(p_y)$. Find $\mathbb{P}(X > Y)$ using the method we did in class.

(b) [easy] [MA] Prove this a different way by finding $\mathbb{P}(X = Y)$ and then using the law of total probability.

(c) [easy] [MA] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)?

(d) [difficult] Let $X \sim \text{Poisson}(\lambda)$ independent of $Y \sim \text{Poisson}(\lambda)$. Find an expression for $\mathbb{P}(X > Y)$ *as best as you are able to answer*. Part of this exercise is identifying where you cannot go any further.

Problem 4

These exercises will introduce the Multinomial distribution.

(a) [easy] If $\boldsymbol{X} \sim \text{Multinomial}(n, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is the parameter space for both n and $\boldsymbol{p}$?

(b) [easy] If $\boldsymbol{X} \sim \text{Multinomial}(n, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is the $\text{Supp}[\boldsymbol{X}]$?

(c) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is $\dim[\boldsymbol{p}]$?

(d) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = 2$, express p_2 as a function of p_1.

(e) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = 2$, how are both X_1 and X_2 distributed?

(f) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and $n = 10$ and $\dim[\boldsymbol{X}] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$.

(g) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(9,\, [0.1\ 0.2\ 0.7]^\top\right)$, find $\mathbb{P}\left(\boldsymbol{X} = [3\ 2\ 4]^\top\right)$ to the nearest two decimal places.

(h) [difficult] [MA] If $\boldsymbol{X}_1 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and independently $\boldsymbol{X}_2 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}_1] = \dim[\boldsymbol{X}_2] = k$. Find the JMF of $\boldsymbol{T}_2 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of $x_1, \ldots, x_K$. Finally, use Theorem 1 in this paper: [click here] for the summation.