# Day 1: Using Force as a Vectors and doing some problems

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## 0.1 Frame Break

0.1 Right Angled Triangle - The Triangle with a $90^\circ$.

## 0.2 Frame Break

0.2 Trig Because of some ratio of the right angled triangle we can say that,

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

## 0.3 Frame Break

0.3 Measuring Angles

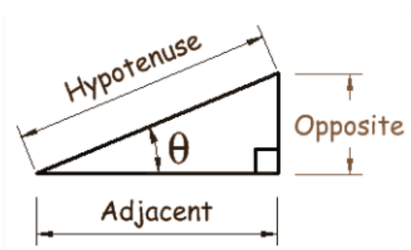



Figure 0.1: Right Angled Triangle

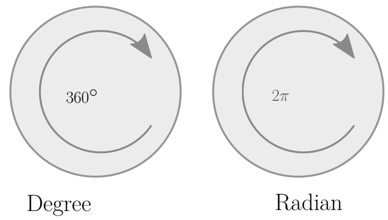



Figure 0.2: Comaparison

This means that,

$$180^\circ = \pi \text{ radians}$$

The reason we use radian can be shown when needed.

## 0.4 Frame Break

0.4 The fact $\alpha + \beta + \gamma = 180^\circ = \pi$ If you add all angles inside a triangle, the sum will always be $180^\circ$.

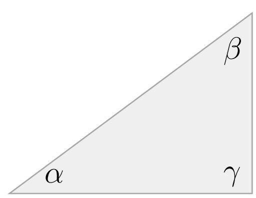



Figure 0.3: Summing up angles

$$\alpha + \beta + \gamma = 180^\circ = \pi$$

For a right angle triangle, $\gamma = 90^\circ = \frac{\pi}{2}$, hence,

$$\alpha + \beta + 90^\circ = 180^\circ$$

$$\alpha + \beta = 90^\circ$$

## 0.5 Frame Break

Newton‘s Second Law: $\vec{F} = m\vec{a}$

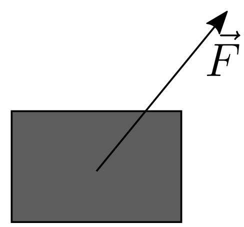



Figure 0.4: If you push a block

## 0.6 Frame Break

N2L

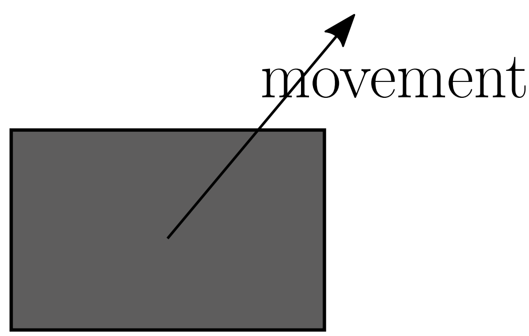



Figure 0.5: The movement will be in the same direction

The movement is Acceleration. Acceleration is increase/decrease in speed overtime.

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

## 0.7 Frame Break

A required relation of angles $\alpha$ and $\beta$.

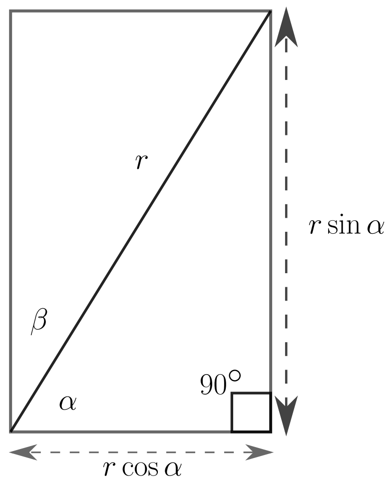



Figure 0.6:

## 0.8 Frame Break

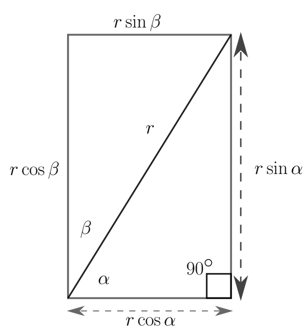



Figure 0.7:

This actually shows us that,

$$r\sin\alpha = r\cos\beta$$

$$r\cos\alpha = r\sin\beta$$

Because $\alpha + \beta = 90^\circ$, $\beta = 90^\circ - \alpha$. Hence,

$$r\sin\alpha = r\cos(90^\circ - \alpha)$$

$$r\cos\alpha = r\sin(90^\circ - \alpha)$$

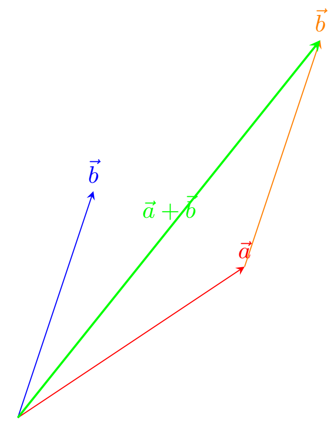
So, we now know that,

$$\sin(90^\circ - \alpha) = \cos\alpha$$
$$\cos(90^\circ - \alpha) = \sin\alpha$$

## 0.9 Frame Break

Warm Up: Cosine Law — For Vector We will use vectors very often in physics, so it is better we try it out before starting. At first, vectors can be added in this way,
Vectors can be added, like,

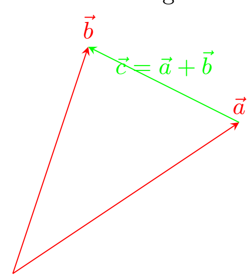
$$\vec{a} + \vec{b}$$




Which looks like one vector is put infront of another.

## 0.10 Frame Break

Vector Triangle for Cosine Law If there is a side $a$ and $b$, then assuming the third side is $c$, the Law of cosine is to find the length of $c$ using the lengths of $a$ and $b$, and the angle $a$ and $b$ makes.
The easiest solution can be found using $\vec{a}$ and $\vec{b}$ vector methods. We know




angle $\theta$ between $a$ and $b$

## 0.11 Frame Break

Now let us have a triangle with side $\vec{a}$ and $\vec{b}$, then the third side is obviously $\vec{b}-\vec{a}$. This is same as we saw in last frame. Now, let the third side be $\vec{c}=\vec{b}-\vec{a}$, then can we find a scalar solution for the length of vector $\vec{c}$?

$$\begin{aligned}\vec{c} &= \vec{b}-\vec{a}\\ (\vec{c})^2 &= \left(\vec{b}-\vec{a}\right)^2\\ \vec{c}\cdot\vec{c} &= \vec{a}\cdot\vec{a}-2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}\end{aligned}$$

Now, $\vec{a}\cdot\vec{b}$, is the dot product. And for a dot product,

$$\vec{a}\cdot\vec{b} = |a|\ |b|\ \cos\theta$$

And if,

$$\vec{a}\cdot\vec{a} = |a||a|\cos(0)$$

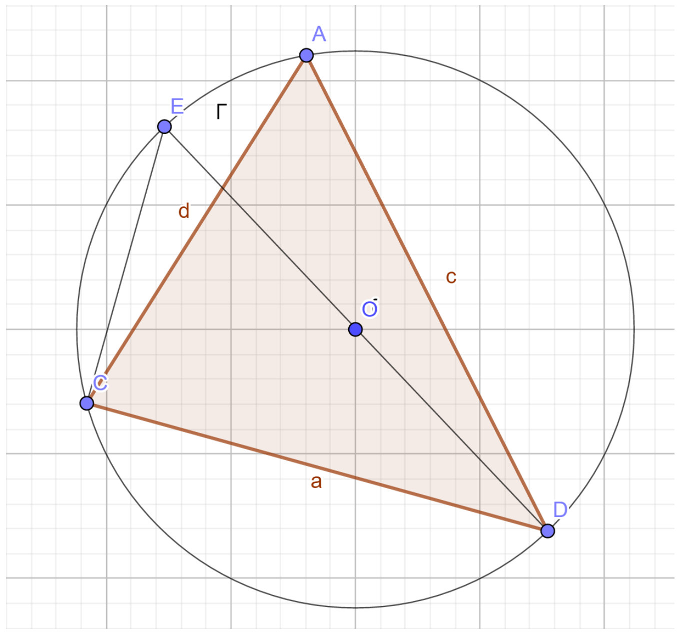



Figure 0.8: Sine Law

Because $\vec{a}$ makes $0^\circ$ with itself. Hence, we have,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

So, we have found length of third side,

$$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

## 0.12 Frame Break

Sine Law **The** $\sin\alpha$ **Law:** For a triangle, with lowercase denoting the angle and the uppercase the opposite side, the sides and angle follow the relation respect to the circumcircle is that,

$$\frac{a}{\sin A} = \frac{c}{\sin C} = \frac{d}{\sin D} = 2r$$

**Proof:** Let $\triangle ACD$ be the required triangle, we want to build a relation between the length of the sides and the opposite angles to them motivated from the fact that the opposite angle being large making the side project a larger length.

## 0.13 Frame Break

## 0.14 Frame Break

Let $\Gamma$ be the circumcircle of $\triangle ACD$ and we draw a diameter $DE$ that passes through the center of $\Gamma$ which is point $O$.

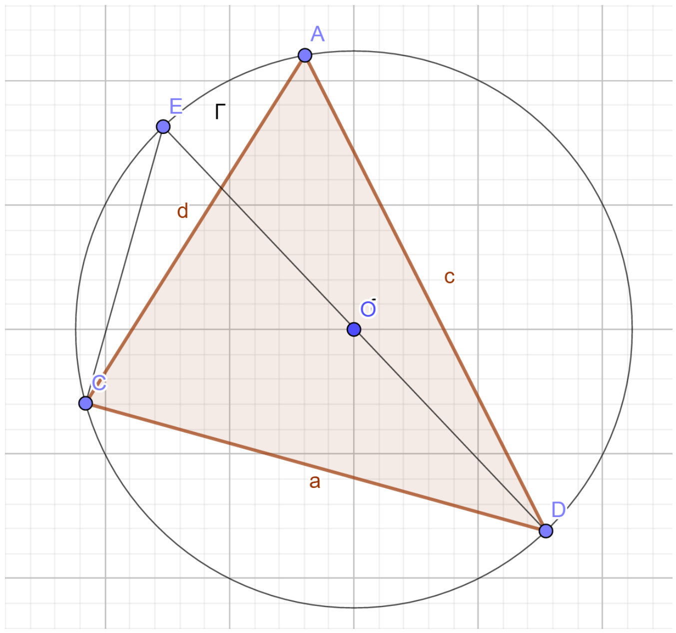



Figure 0.9: Sine Law

$\triangle ECD$ is a Right Angled Triangle because it meets the *Semi-circle* and thus,

$$\sin \angle CED = \frac{\bar{CD}}{\bar{ED}}$$

Using the *Arc and Angle Theorem* we can see that

$$\angle CED = \angle CAD = \angle A$$

As we know that $DE = 2r$ we are left to show that,

$$\sin \angle CED = \frac{\bar{CD}}{\bar{ED}}$$

$$\sin \angle A = \frac{a}{2r} \tag{1}$$

$$\frac{a}{\sin A} = 2r \tag{2}$$

And this can be again and again be proved for all the three remaining sides,

$$\frac{a}{\sin A} = \frac{c}{\sin C} = \frac{d}{\sin D} = 2r \tag{3}$$

And we are done with the proof.

## 0.15 Frame Break

Problem 1: The General Problem of Force

**Problem. 1.** There is a system made of an incline with two masses attached with string as shown. One is $m_1$ another is $m_2$, gravity works $\vec{g}$. *Find the acceleration of both bodies $m_1$ and $m_2$.*

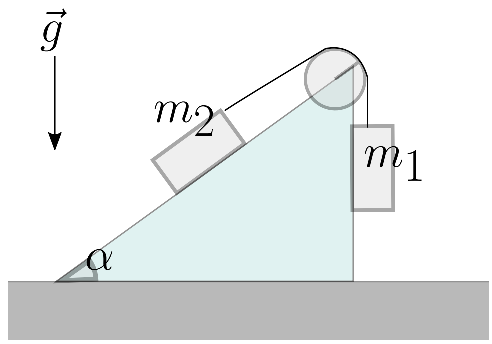



Figure 0.10: Problem 1 Diagram

## 0.16 Frame Break

Problem 1: The General Problem of Force

**Problem. 2.** There is a system made of an incline with two masses attached with string as shown. One is $m_1$ another is $m_2$, gravity works $\vec{g}$. *Find the acceleration of both bodies $m_1$ and $m_2$.*

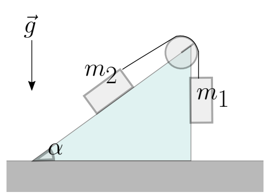



Figure 0.11: Problem 1 Diagram

he main force is caused by Gravity The rope can pull the $m_2$ and slow down $m_1$ when it tries to fall down. here is possibility that too much pull be $m_2$ can cause the $m_1$ to lift off the ground, instead of falling down. We have to be careful about all the possibilities.

## 0.17 Frame Break

This is the approach

- Find all the forces, it's the tricky part.
- Find the acceleration from the force using $\vec{F} = m\vec{a}$. i

## 0.18 Frame Break

Problem 1: Find All the forces

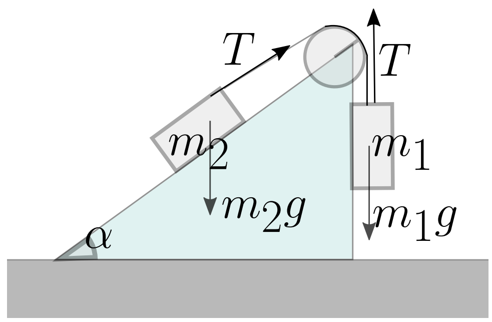



Figure 0.12: Sorry for this exploded word size.

We will start isolating the force on the bodies $m_1$ and $m_2$.

## 0.19 Frame Break

Problem 1: Let us find the forces on $m_2$ first

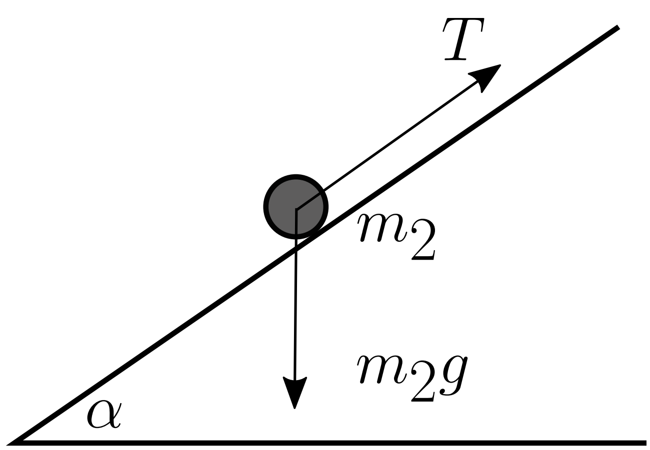



Figure 0.13:

## 0.20 Frame Break

Problem 1: Let us find the forces on $m_1$ then

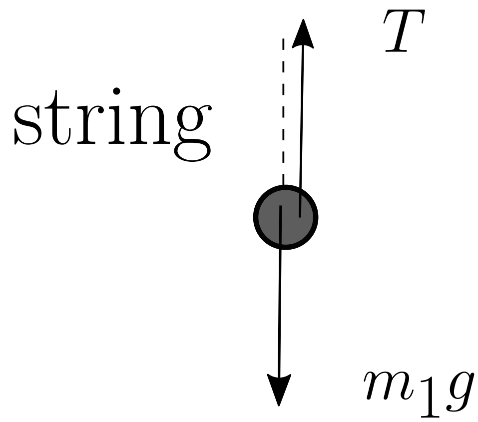



Figure 0.14:

## 0.21 Frame Break

What is $T$?

- $T$ is the Tension force, the "Tan" (in Bengali) because of the rope.
- $T$ is same for the whole rope, otherwise a non zero force on the rope would occur.

## 0.22 Frame Break

Let us look the total force on $m_2$ Note that motion of the body $m_2$ is constrained (limited) in the incline.

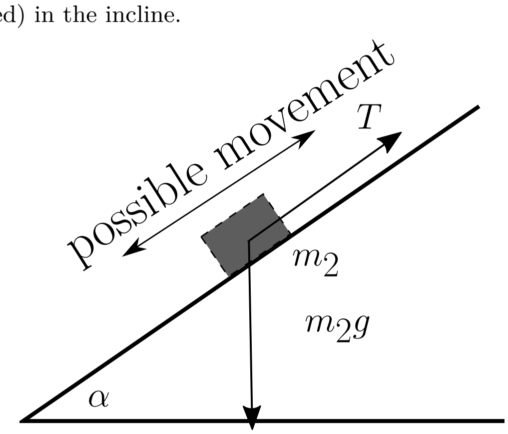



Figure 0.15:

Thus, there force on $m_2$ can only work along the incline. This is force is made from 2 components.

## 0.23 Frame Break

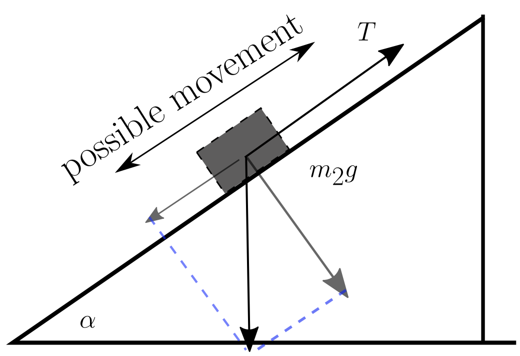



Figure 0.16:

## 0.24 Frame Break

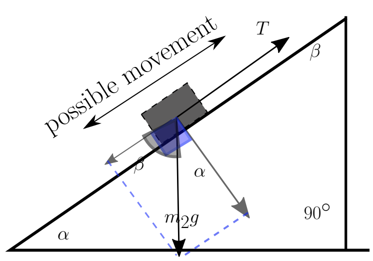



Figure 0.17:

$$\alpha + \beta = 90^\circ$$

And you can note that the projection of the $m_2g$ vector has similar angles.

## 0.25 Frame Break

On $m_2$

- We don't need the perpendicular force components.
- Because the block $m_2$ is constrained to move along the incline.
- So the force perpendicular to the motion are useless.

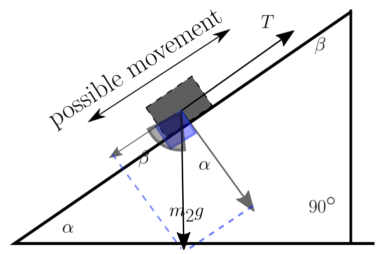



Figure 0.18:

## 0.26 Frame Break

Sign Conventions for Problem We will consider the sign convention so that,

- If $m_1$ falls down, then the motion is in positive direction.
- If $m_1$ lifts up, the the motion is in negative direction.
- $m_1$ falling down means that $m_2$ will lift up along the ramp. This motion, where $m_2$ will lift up along the inline/ramp, means the motion is "Positive".

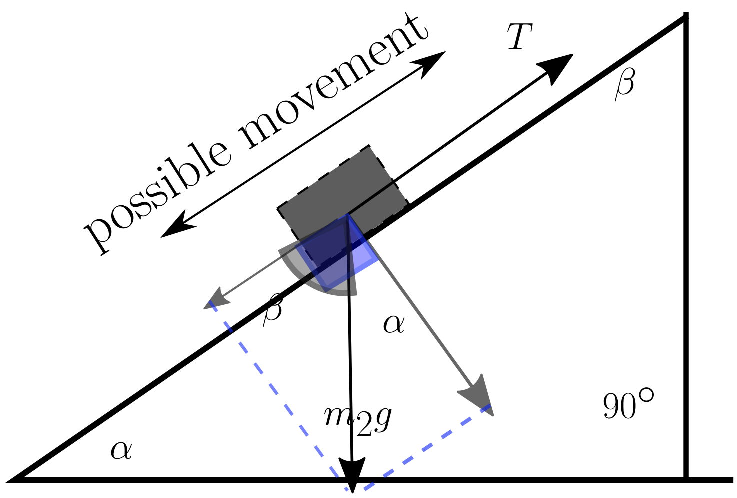



Figure 0.19:

## 0.27 Frame Break

The force by gravity, parallel to the direction of possible motion,

$$m_2 g \cos\beta = m_2 g \sin\alpha = F_g$$

This is what we need.

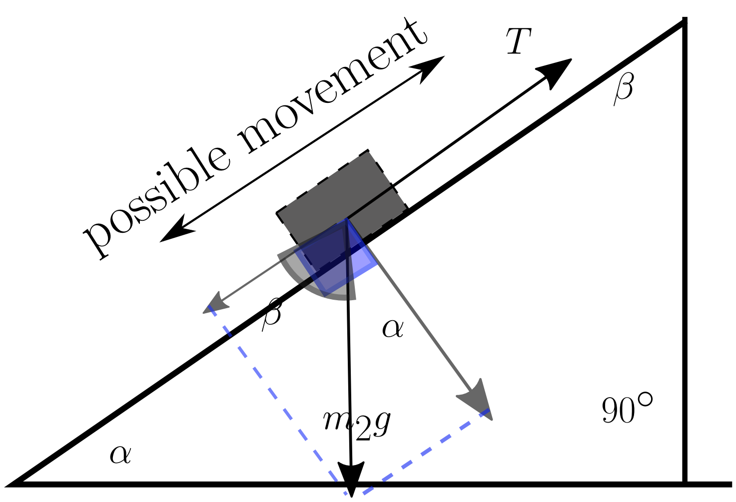



Figure 0.20:

## 0.28 Frame Break

Now, the total force on $m_2$,

$$T - F_g$$

This total force can let us know about the accelearation of the $m_2$ using $F = ma$,

$$m_2 a_2 = T - F_g$$

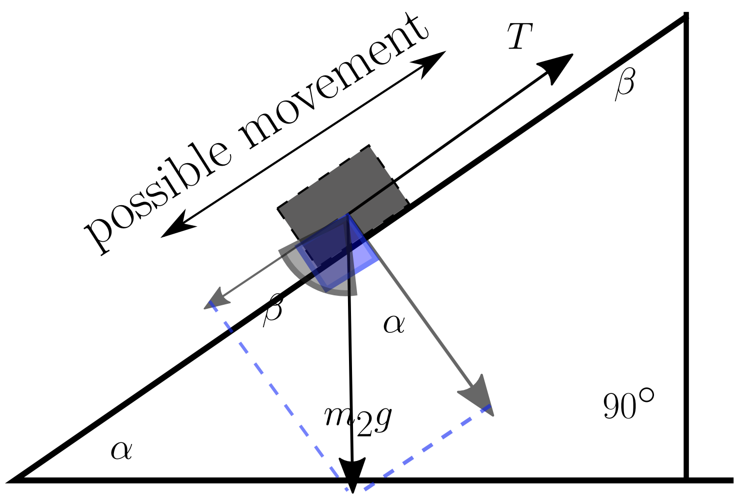



Figure 0.21:

## 0.29 Frame Break

Now, we need the total force on $m_1$.

## 0.30 Frame Break

The gravity pulls by $m_1 g$ and the tension pulls against $T$, so total force,

$$m_1 a_1 = m_1 g - T$$

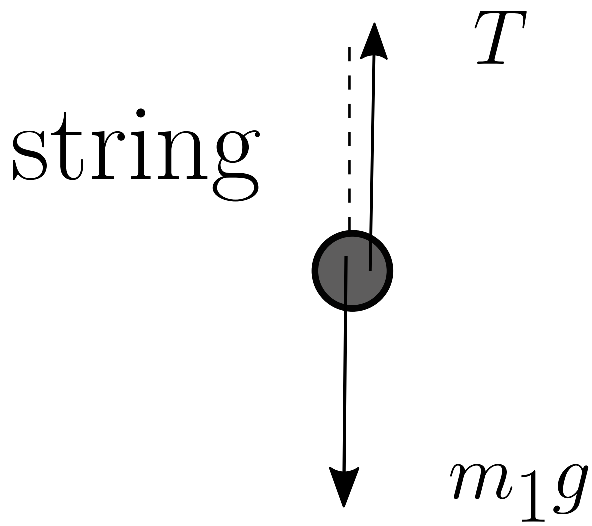



Figure 0.22:

## 0.31 Frame Break

We have built up two equations,

$$
\begin{aligned}
m_1a_1 &= m_1g - T \\
m_2a_2 &= T - m_2g\sin\alpha
\end{aligned}
$$

- There are 2 equations
- But there are 3 unknowns. They are, $a_1, a_2, T$.

This system of equation can only be solved if, The number of Unknown variable match with the number of equations.
This means,
We need *another* equation

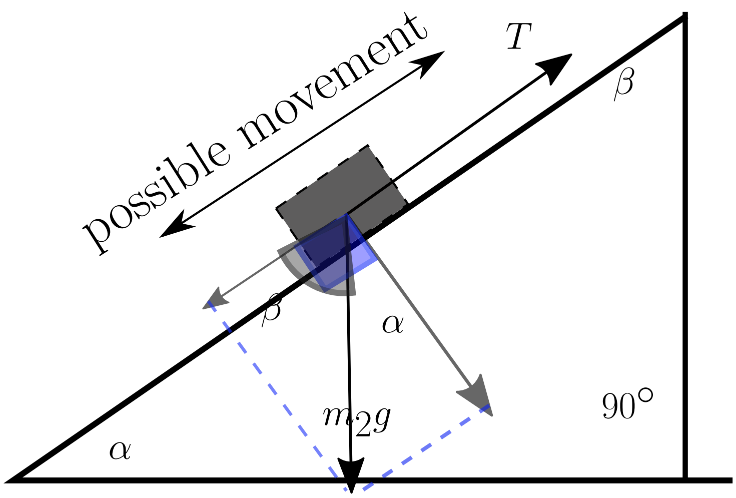



Figure 0.23:

## 0.32 Frame Break

Please look at $a_1$ and $a_2$, note that, because of the string,
both of the blocks $m_1$ and $m_2$ should together, in the same motion.
Because, if one is faster than the other, the rope will resist that.
So we shall have $a_1 = a_2$.

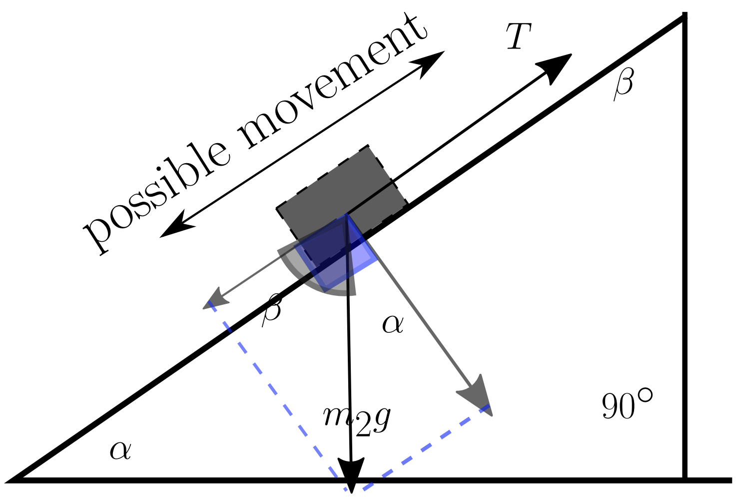



Figure 0.24:

## 0.33 Frame Break

Answer final Now using the fact $a_1 = a_2 = a$,

$$m_1 a = m_1 g - T$$
$$m_2 a = T - m_2 g \sin\alpha$$

To solve this equations,

- $T = m_2 a + m_2 g \sin\alpha$
- Put this in second equation, $m_1 a = m_1 g - (m_2 a + m_2 g \sin\alpha)$.
- Solve this for, $a$, you will have, $m_1 g - m_2 g \sin\alpha = (m_1 + m_2)a$
- We have, $a = \frac{g(m_1 - m_2 \sin\alpha)}{m_1 + m_2}$

Note that, if $m_1 > m_2 \sin\alpha$, then $a > 0$, which means $a$ is positive and the acceleration is positive, which follows the convention $m_1$ falling down is positive. Hence,

$$\boxed{a = \frac{g\,(m_1 - m_2 \sin\alpha)}{m_1 + m_2}}$$

## 0.34 Frame Break

Add something to Problem 1 Text **What is the Accelearation if there is friction?**

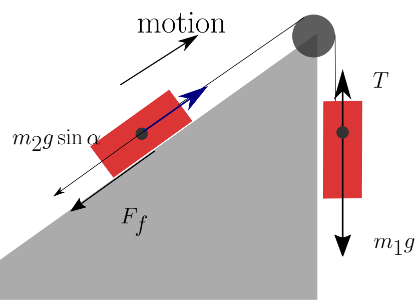



Figure 0.25:

The friction is to oppose motion.

## 0.35 Frame Break

- Now, we find the required equations.
- $m_2 a = T - m_2 g \sin\alpha - F_f$.
- $m_1 a = m_1 g - T$.
- Friction is always $F_f = \mu N$, where $N$ is the Normal component of force. Normal force is the force that the incline gives perpendicularly to the block.
- Normal force is discussed in next frame.

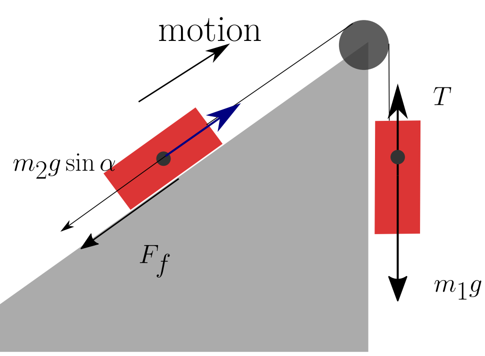



Figure 0.26:

## 0.36 Frame Break

We need friction force $F_f$. The normal force is the perpendicular projection of the gravitational force.

- There is no ability to move perpendicular to the inclination.
- So the force is balanced along the perp $\perp$.
- Hence, the $N = m_2 g \cos\alpha$.
- Friction is, $F_f = \mu N$, here, $\mu$ is the friction coefficient. .

$$F_f = m_2 g \mu \cos\alpha$$

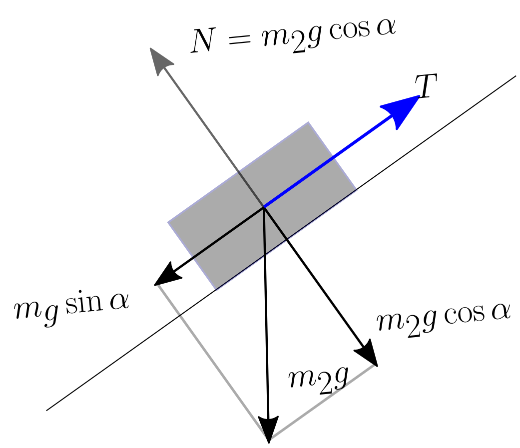



Figure 0.27:

## 0.37 Frame Break

The equations that we had,

- $m_2 a = T - m_2 g \sin\alpha - F_f = T - m_2 g \sin\alpha - (m_2 g \mu \cos\alpha)$.
- $m_1 a = m_1 g - T$.
- Please note that this equation only hold if there is motion where $m_2$ lifts up. Otherwise the sign of friction will change.

Similarly solving we get, only for positive motion case,

$$\boxed{a = \frac{g\,(m_1 - m_2 \sin\alpha - m_2 \mu \cos\alpha)}{m_1 + m_2}}$$

## 0.38 Frame Break

If the motion is reversed,

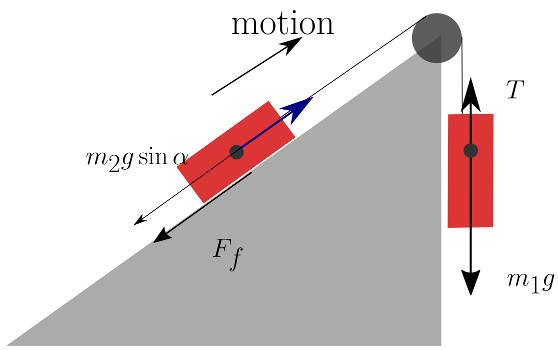



Figure 0.28:

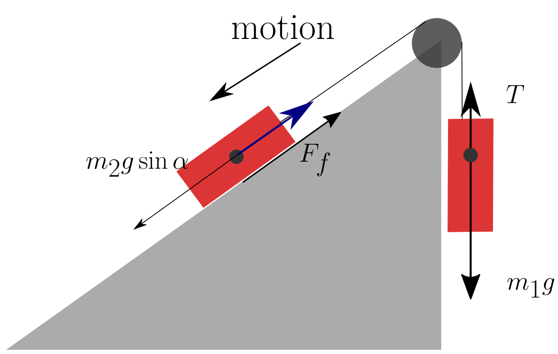



Figure 0.29:

We need to write the equation again. Because the direction of friction will change for different motion, where $m_2$ falls down along the rame. We have to write,

- $m_2 a = -T + m_2 g \sin\alpha - F_f$.
- $m_1 a = T - m_1 g$.

This is solved to get,

$$\boxed{a = \frac{g(m_2 \sin\alpha - m_2\mu\cos\alpha - m_1)}{m_1 + m_2}}$$

## 0.39 Frame Break

For 2 cases of friction, we have

$$\boxed{a = \frac{g(m_2 \sin\alpha - m_2\mu\cos\alpha - m_1)}{m_1 + m_2}}$$

$$\boxed{a = \frac{g\left(m_1 - m_2 \sin\alpha - m_2\mu\cos\alpha\right)}{m_1 + m_2}}$$

These are two cases, first for $m_2$ getting down, second for $m_2$ getting up. We can define the condition of getting up, down, or resting in position because of friction.
To $m_2$ get down, we need,

$$m_2 \sin\alpha - m_2\mu\cos\alpha - m_1 > 0 \quad \rightarrow \quad m_1 - m_2 \sin\alpha + m_2\mu\cos\alpha < 0$$

For $m_2$ get up, we need,

$$m_1 - m_2 \sin\alpha - m_2\mu\cos\alpha > 0$$

## 0.40 Frame Break

Problem 2

**Problem. 3.** There is a ring of radius $R$ and there is a bead that can move along the circle of the ring. The ring rotates in $\omega$ angular speed, find a position of the bead in which case it stays in *rest*.

This position we are going to define with the angle $\theta$ respect to rotation axis.

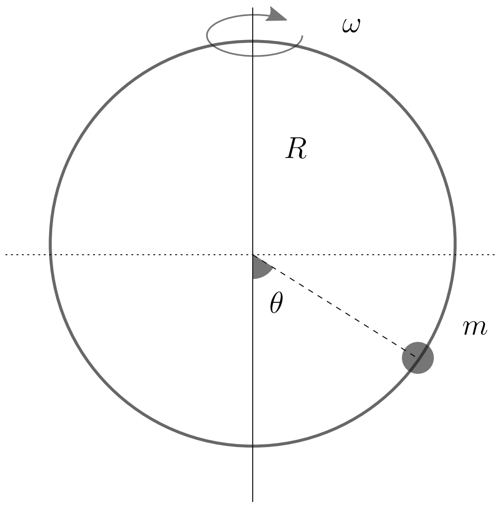



Figure 0.30:

## 0.41 Frame Break

> **Idea 1.** If anything is in a constant velocity or rest, then the total force on it is zero

Because $F = ma$, if $a$ is $a = 0$, then there is a constant velocity or it is zero. Because,

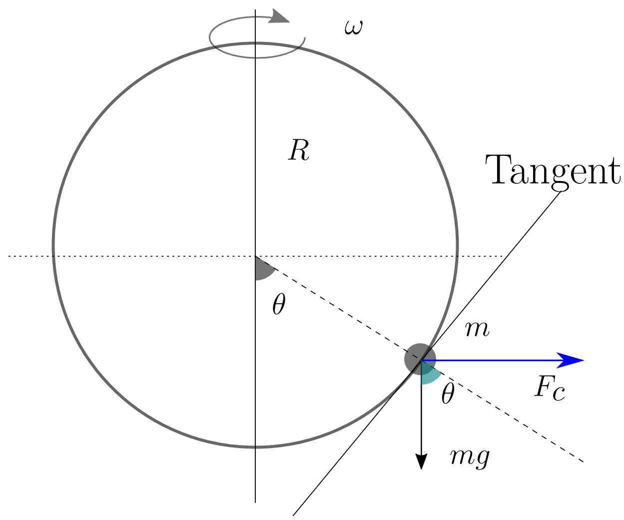



Figure 0.31: We will isolate the forces.

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \rightarrow \quad a = 0 \quad \text{means} \quad v = \text{ const}$$

## 0.42 Frame Break

Problem 2: Force

- $\theta^c$ is $90^\circ - \theta$.
- $F_C$ is the Centrifugal force $F_C = m\omega^2 r$, where $r$ is distance from $axis$.
- The motion of the bead is constrained along the tangent; if there is a component of force along the tanget, the bead will move.
- Projection of gravity on the tangent is $mg \sin\theta$. .
- Projection of Centrifugal force $F_C$ on the tangent is $F_C \cos\theta$.
- From the diagram evidently two projection are opposite to each other.

$\omega$



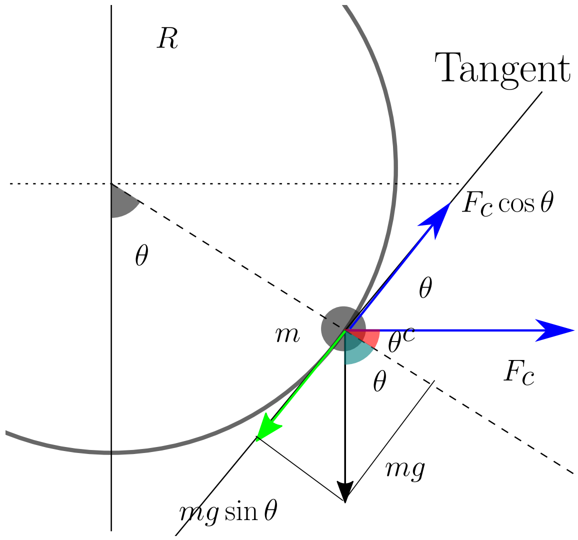



Figure 0.32:

$\omega$

## 0.43 Frame Break

Problem 2: Close look at the forces.

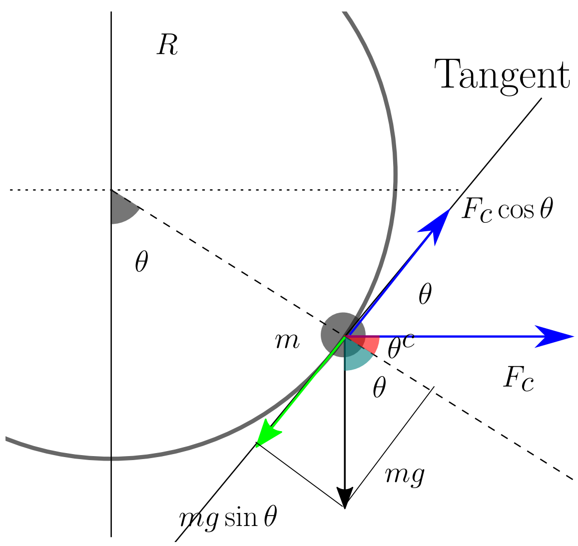



Figure 0.33: A closer look

## 0.44 Frame Break

Problem 2: Centrifugal Force We need the Radius of Rotation, which is found in the figures, we calculate it,

$$r = R\sin\theta$$

Using these,

$$F_c = m\omega^2 r = m\omega^2 R\sin\theta$$

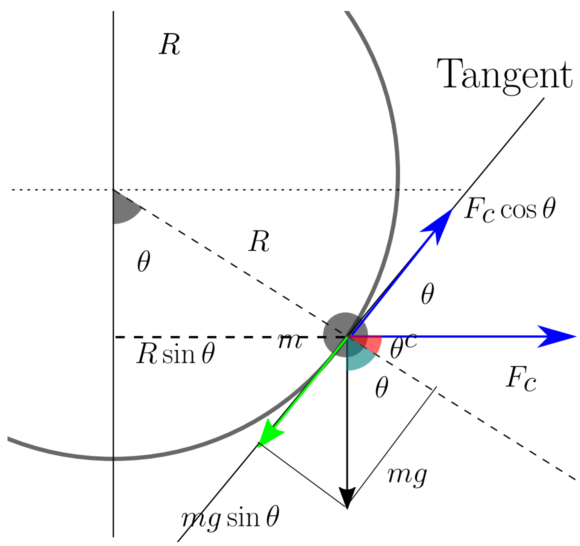



Figure 0.34:

## 0.45 Frame Break

Problem 2: Balance force for Stationary condition We need total force 0,

$$mg\sin\theta - \left(m\omega^2 R\sin\theta\right)\cos\theta = 0$$

$$g\sin\theta = \omega^2 R\cos\theta\sin\theta$$

And we have,

$$\cos\theta = \frac{g}{\omega^2 R}$$

The angle $\theta$ is,

$$\boxed{\theta = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)}$$

## 0.46 Frame Break

Rotational Mechanics (Quick) **Torque**
Torque is rotational force.

Like force, Torque causes acceleration of rotation,

$$\tau = I\alpha$$

Where, $I$ is Moment of Inertia, which is like Mass for rotating bodies.
$I$ depends on the Geometry and Mass of the system.

$$\tau = r \times F$$

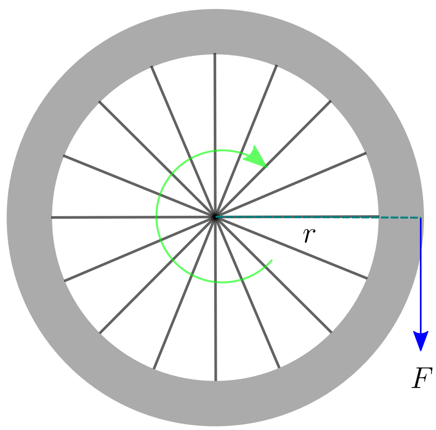



Figure 0.35:

## 0.47 Frame Break

Extending Torque Idea In case the $F$ makes an angle with the radius vector $r$, we have to add a $\sin\theta$ factor for the perpendicular projection,

$$\tau = r \times F \sin\theta$$

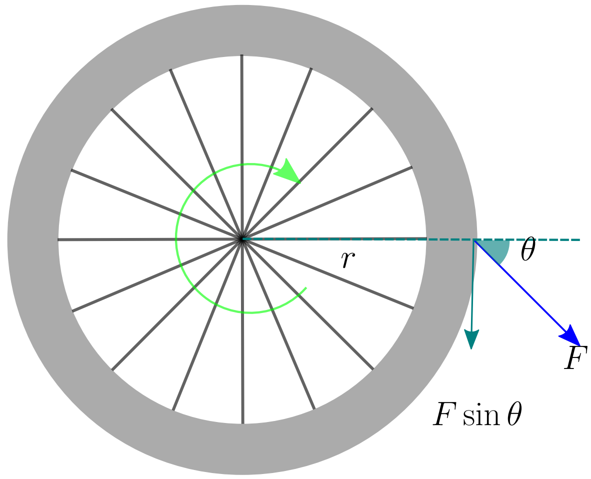



Figure 0.36:

> **Idea 2.** Most of the rules of Force can be used for Torque too, for example, torque is balanced too.

## 0.48 Frame Break

Angular Momentum Like the momentum we learned few minutes ago, angular momentum is also like that,

$$p = mv$$

And similarly

$$L = I\omega$$

Where $I$ is known, and $\omega$ is Angular Speed.
Please note that these are vector, but we don't need the vector law for Angular Dynamcics right now.

## 0.49 Frame Break

Problem 3: Some Idea on Torque using last Idea

> **Problem. 4.** Three rods are connected by hinges to each other, the outermost ones are hinged to a ceiling at point $A$, and $B$, the distance between these are equal of a rod. A weight is hanged at $C$, find the minimum force applied at $D$ to keep the system in Balance.

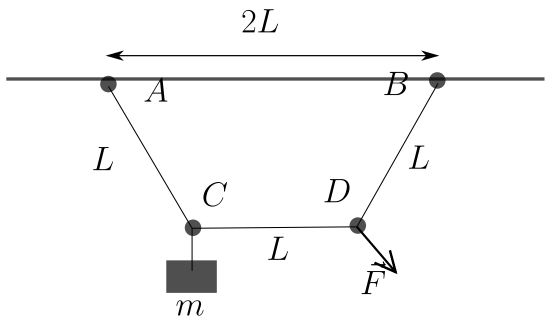



Figure 0.37:

## 0.50 Frame Break

If there is no force, then

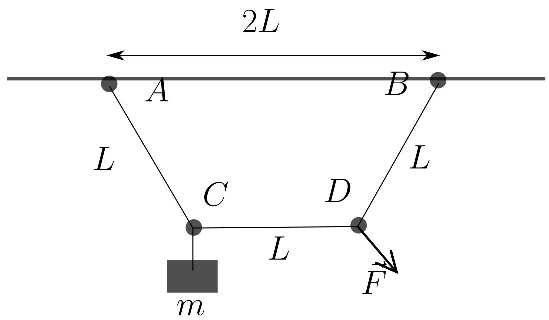



Figure 0.38:

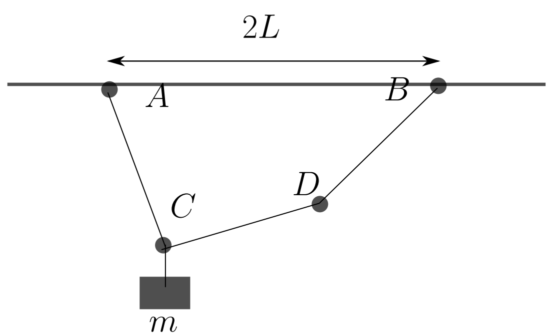



Figure 0.39:

Please note, for the pivot $B$, if we can stop rotation by axis of $B$, then the system can be balanced.

## 0.51 Frame Break

We need to know the shapes angles

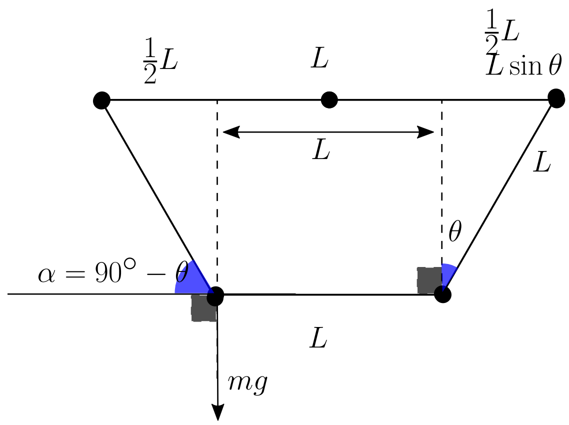



Figure 0.40:

## 0.52 Frame Break

Finding angles for Problem 3 This is sure that,

$$L\sin\theta = \frac{1}{2}L$$

Thus,

$$\theta = 30^{\circ}$$

From here,

$$\alpha = 90^{\circ} - \theta = 60^{\circ}$$

Now, the angle that the $mg$ vector makes with the left side of the rod is,

$$90^{\circ} + \alpha = 90^{\circ} + 60^{\circ} = 150^{\circ}$$

So, angle $m\vec{g}$ makes with the left rod is $150^{\circ}$.

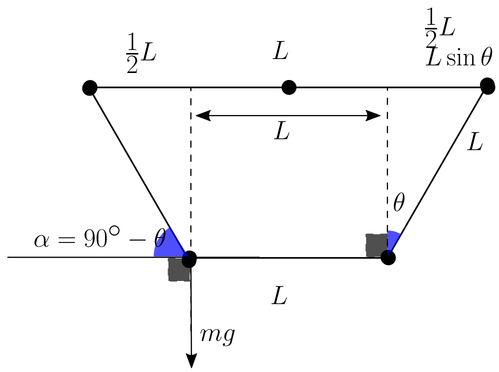



Figure 0.41:

## 0.53 Frame Break

Torque balance rules for Problem 3.

Like force balance, we have to invoke Torque balance, because,

- Force balance is similar to Torque Balance
- In this problem, we need to apply force at $D$, and if you notice, the motion of $D$ is like the rotation of right rod pivoting $B$.
- To balance the system, we need to balance the *Rotation*.

## 0.54 Frame Break

So, the torque on the left side of the rod,

$$\tau = mgL\sin(150^\circ)$$

And $\sin 150^\circ = \frac{1}{2}$, hence,

$$\tau = \frac{mg}{2}L$$

I have something to add here too,

- In the $mg$ force vector, some force is $\perp$ perpendicular to the rod (prjection)
- Some force is along the rod.

The force components that are along the rod are uimportant, because they pull the left rod and do nothing.

## 0.55 Frame Break

Final thoughts on Problem 3 Now we add some logic.

If left side has torque $\tau$ where $\tau = \frac{mg}{2}L$, then if we apply the same torque on the right side, then we have a torque balance, which will also balance the system, hence, on the right side, we also apply a torque $\tau'$ which is,

$$\tau' = \frac{mg}{2}L$$

And $\tau = FL$, thus,

$$F = \frac{mg}{2}$$

This force is perpendicular to the rod, and solely works to balance the system, without any component along the rod, hence, no force is wasted and $F = \frac{mg}{2}$ is the minimum force possible.

$$\boxed{F = \frac{mg}{2}}$$

## 0.56 Frame Break

Problem 3 Solution $F = \frac{mg}{2}$

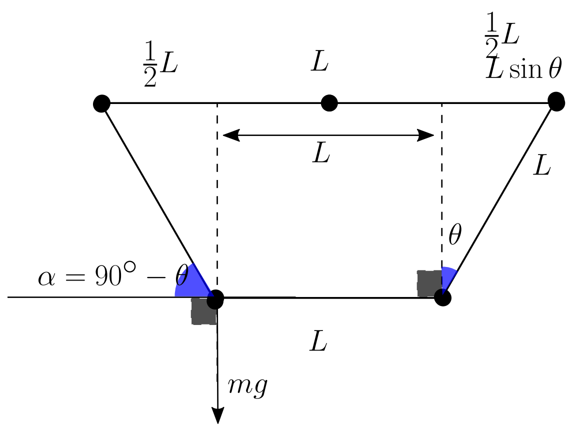



Figure 0.42: