Rigouorous Derivation of Force equation

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Force between two aligned bar magnets was found out to be,

$$F \propto \frac{1}{d^4}$$

Hi, few days ago, we found the equation of force between two magnets along the same line,

$$F = \gamma \frac{\mu_1 \mu_2}{r^4}$$

I told you,

$$|\gamma| = 6 \times 10^{-7}$$

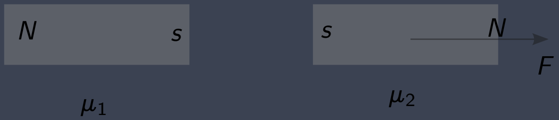

Figure: forcefigure

Magnetic field from a bar magnet,

$$\vec{B}_\mu = \frac{\mu_0 \mu}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

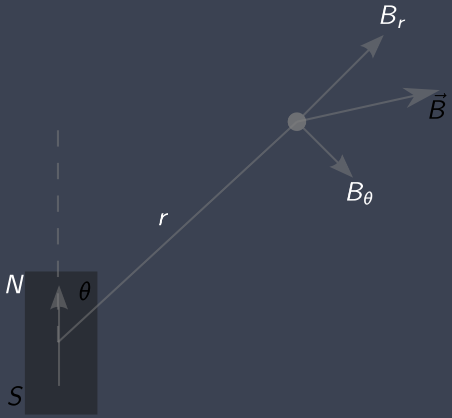

Figure: Magnetic Field from a Bar Magnet Visualized

Force on a Magnetic Dipole $\vec{m}$ because of exterior field $\vec{B}$ is,

$$\boxed{\nabla\left(\vec{m}\cdot\vec{B}\right)}$$

Here, we have the dipole moment μ_2, on which force $\vec{F}$ is applied, the external field B_{μ_1} is caused by the 1st Magnet with Dipole Moment μ_1

$$\vec{F} = \nabla\left(\vec{\mu_2}\cdot\left(\mu_0\frac{\mu_1}{4\pi r^3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\right)$$

But,

$$\vec{\mu_2} = \mu_{2r}\hat{r} + \mu_{2\theta}\hat{\theta}$$

Becuase our μ_2 is alinged horizontally,

$$\mu_{2\theta} = 0$$

For this case.

$$\vec{F} = \nabla\left(\vec{\mu_2}\cdot\left(\mu_0\frac{\mu_1}{4\pi r^3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\right)$$

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For this case. So, the dot product is,

$$\vec{F} = \nabla\left(\mu_0\frac{\mu_1}{4\pi r^3}\left(2\mu_{2r}\cos\theta + \mu_{2\theta}\sin\theta\right)\right)$$

Gradient in Polar,

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

So, using that, we calculate the gradient of the above expression,

$$\vec{F} = \frac{\partial}{\partial r}\left(\mu_0 \frac{\mu_1}{4\pi r^3}\left(2\mu_{2r}\cos\theta + \mu_{2\theta}\sin\theta\right)\right)\hat{r} +$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_0 \frac{\mu_1}{4\pi r^3}\left(2\mu_{2r}\cos\theta + \mu_{2\theta}\sin\theta\right)\right)\hat{\theta}$$

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We get,

$$\vec{F} = -\frac{3\mu_0 m_1}{4\pi r^4}\left(2\mu_{2r}\cos\theta + \mu_{2\theta}\sin\theta\right)\hat{r}$$

$$+\frac{\mu_0\mu_1}{4\pi r^4}\left(-2\mu_{2r}\sin\theta + \mu_{2\theta}\cos\theta\right)\hat{\theta}$$

Now, in our above problem, the magnets are aligned along an axis, thus,

$$\theta = 0$$

This means the equation becomes easier to manage.

We get,

$$\vec{F} = 6\mu_0 \frac{\mu_1 \mu_2}{4\pi r^4} = \frac{3}{2\pi} \mu_0 \frac{\mu_1 \mu_2}{r^4} \hat{x}$$

So, for two bar magnets aligned, the force is,

$$F = \frac{3\mu_0}{2\pi} \frac{\mu_1 \mu_2}{r^4}$$

Here the equivalent γ is,

$$3\mu_0/2\pi = 3\left(4\pi \times 10^{-7}\right)/2\pi = 6 \times 10^{-7}$$

$$\boxed{F = \frac{3\mu_0}{2\pi} \frac{\mu_1 \mu_2}{r^4}}$$