How to throw so that it doesn't return?

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I heard that there is a specific velocity in which if I throw an object directly into the sky, then the object won't return. It will keep going forever.
But isn't the Gravitational Force active at all distances upto infinity, so why doesn't the object slow down to zero and fall back?

Earth is a sphere that has mass in it. By Newton's Gravitational Theory, every mass should attract an other mass. The force is given by a simple equation. An object has a mass, so as the Earth. So Object and Earth attract each other. But is it possible to throw the object so fast that it never falls back and keeps going and going?

The thing that haunts me trying to think that case is Gravitational Force is literally present at all distance. Though it's magnitude or strength falls down with distance, but still, the force exists. Very generally we can just say that, because of deceleration a, for a u initial velocity,

$$F = ma \qquad v = u - at \tag{1}$$

So after a large time later, $at = u$ and then $v = 0$, the object comes to a rest.

So, what wrong here?

What we ignored in this problem is that the deaccelaration changes with increase of distance. Because the Force starts to decrease with the increase of Distance. The raw analysis should be made.

Let us try to answer this question at first. What is the energy needed to give to take away an object at a far distance r from Earth?

As we know that, $W = Fx$, there is a Gravitational Force and that is a function of distance. We can use the idea of an integration to find the amount of Work done to take the object from R to r, where R is Radius of Earth and r is a far distance from the center of the Earth.

By Newton's Law,

$$F = G\frac{Mm}{x^2} \tag{2}$$

Where M is Earth mass and m is the object mass, x is the distance.

From calculus point of view, for a small displacement dx, the Force stays almost constant. So, we do a small amount of Work dW.

$$dW = F\,dx \tag{3}$$

$$\int dW = \int_R^r G\,\frac{Mm}{x^2}dx \tag{4}$$

$$\Delta W = -GMm\left(\frac{1}{r} - \frac{1}{R}\right) \tag{5}$$

So, to take this thing into far outer space at a distance r, we have to give an Energy of $-GMm\left(\frac{1}{r} - \frac{1}{R}\right)$. What if r is so so so large that it is nearly infinite respect to R? Well, then,

$$\frac{1}{r} \to 0 \qquad \text{if } r >> R$$

Then $\frac{1}{r}$ is almost zero. Then we have the equation above,

$$W = G\frac{Mm}{R} \qquad \text{(if distance is large)} \tag{6}$$

Now let us try to decode what the math says.

If we take an object slowly slowly, so that it doesn't gather too much kinetic energy, then when we go really far, the amount of work that we have to input to do the job is about GMm/R. It seems that it doesn't depend on the large distance r. Why not?

Because at too much high distance, force becomes so weak that you don't even feel it. There are so many stars that are 100000000 times heavier than Earth itself, but do you ever feel bothered by their force? Nope. As they are so so far away. Hold a ball above ground and stay still, do you feel that the currently known heaviest star in the Universe is attracting it and moving it away? It's just not possible.

Same happens when you are far from Earth, there's essentially no Force present, so there's literally no a for $v = u - at$ to deaccelerate and you can keep going.

Okay, so, how do I know that velocity to make this happen?

Total amount of Work to put an object far away from Earth is GMm/R. So, we need to give this amount of Energy. And from Work Energy theorem, which tells that Amount of Work done is Change in Energy, putting total energy of GMm/R shall put the object in the way we want.

If that specific velocity is given, a kinetic energy will be gained. This kinetic energy if equal or greater than GMm/R, then the object shall totally leave Earth and never deacclerate and Fall down.

But how to find kinetic energy?

Newton's Second Law says that,

$$F = ma$$

So, describing it mathematically in terms of velocity v,

$$F = m\frac{dv}{dt} \tag{7}$$

$$\tag{8}$$

In the Work equation,

$$W = \int m\frac{dv}{dt}\,dx$$

You can see that,

$$\begin{aligned} W &= \int m\frac{dv}{dt}\,dx \\ W &= \int m\frac{dx}{dt}\,dv \\ W &= \int mv\,dv \\ W &= \frac{1}{2}mv^2 \end{aligned}$$

So, at objects v velocity, it has kinetic energy $E_k = \frac{1}{2}mv^2$. What if this is the same amount as GMm/R?

This will input enough energy to throw object away from Earth once and for all. It is possible now to show that,

$$\begin{aligned} \frac{1}{2}mv^2 &= G\frac{Mm}{R} \\ v^2 &= \frac{2GM}{r} \end{aligned}$$

$$v = \sqrt{\frac{2GM}{r}}$$

If you have ability to give a velocity $v = \sqrt{\frac{2GM}{R}}$, then we have that object away.

Most weird is that the velocity doesn't have the term m in the equation. This is absurd, whatever mass it is, $1\,kg$ or $100000\,kg$, if we can give that specific velocity, we have that object permanently away. G, M, R are all constants and we know their values. This gives,

$$
\begin{aligned}
G =& 6.67 \times 10^{-11}\, Nm^2\, kg^{-2} \qquad \text{Gravitational Constant} \\
M =& 5.97 \times 10^{24}\, kg \qquad \text{Earth Mass} \\
R =& 6.37 \times 10^{6}\, m \qquad \text{Earth Radius}
\end{aligned}
$$

So we finally get the special velocity,

$$v = 1.2 \times 10^4\, m/s \tag{9}$$

This velocity is called the **Escape Velocity** and is extremely high.

There is also one thing to notice. If it happens that some other object from a large distance away starts to move towards Earth and eventually hits the surface of Earth, then it shall also have the same velocity while falling as Escape velocity.

You can just imagine reverse case and find out why that can happen.