Poor Man's Introduction to Lagrangian Method

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Abstract

I initially thought that "Beggars Introduction..." would be a good title because of the light quality physics of the article, but being concerned of the unprofessional-ness I avoided it. A simple and example based introduction has been made on Analytical Mechanics for interested young non university students.

Before we start, we need to refresh our knowledge on some simple and math and physics things. I enlist it below.

- Idea on Derivatives and knowing what taking a derivative really is.
- Simple Mechanics and Kinematics.
- Having usual Halliday Resnick level Physics knowledge, or, the University Physics one.

Euler and Lagrange just learned Calculus and Mechanics from Newton and Leibnitz, so they want to mess with it. They know a formula,

$$\frac{dv}{dt} = a \tag{1}$$

Where v is velocity and a is acceleration, simply saying, rate of change of velocity v is acceleration a.
They also know that Kinetic Energy and Potential Energy (of a spring) are,

$$T = \frac{1}{2}mv^2 \qquad V = \frac{1}{2}kx^2 \tag{2}$$

We know all of it. Now, Lagrange and Euler thought, "What if we could find the acceleration from Energy?". Indeed, using derivatives this is possible. You know this example, when a ball falls from some height, then as its velocity increase, so increases its Kinetic Energy T by equation 2 and the Potential Energy decreases. To say easily, the Kinetic Energy is borrowed from the Potential Energy. So both have a relation during the free fall if we are concerned of Velocities and Accelerations!
Euler and Lagrange want to **Find acceleration from Energy.**
So Euler took a pencil and did this,

$$T = \frac{1}{2}mv^2 \quad \rightarrow \quad \frac{dT}{dv} = mv$$

And he again took a derivative, now with respect to time, because he knew equation 1.

$$\frac{d}{dt}\left(\frac{dT}{dv}\right) = m\,\frac{dv}{dt} = ma \tag{3}$$

They derived the Newton's Second law. But this is not quite complete though, we don't very specifically know acceleration, because the equation 3 needs to rely on potential energy somehow.
Lagrange took the pencil, and what he did was,

$$V = \frac{1}{2}kx^2 \quad \rightarrow \quad -\frac{dV}{dx} = -kx \tag{4}$$

But we know that $F = ma$ and $F = -kx$ for a spring, hence,

$$ma = -kx \tag{5}$$

And putting equation 3 and 4, we get,

$$\frac{d}{dt}\left(\frac{dT}{dv}\right) = -\frac{dV}{dx} \tag{6}$$

This awkward equation above is somewhat a form of the Euler Lagrange Equation, and is not quite useful in this form, yet. Euler and Lagrange are a bit clever, they write a stupid thing,

$$\mathfrak{L} = T - V \tag{7}$$

The stylish "L"" is called the Lagrangian. We can substitute this in place of the equation 6, notice that for this case,

$$\mathfrak{L} = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \tag{8}$$

Euler takes the derivatives,

$$\frac{d\mathfrak{L}}{dv} = \frac{d}{dv}\left(\frac{1}{2}mv^2 - \frac{1}{2}kx^2\right) = mv$$
$$\frac{d}{dt}\left(\frac{d\mathfrak{L}}{dv}\right) = \frac{d}{dt}mv = ma$$
$$\frac{d\mathfrak{L}}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2 - \frac{1}{2}kx^2\right) = -kx$$

So, from the equation 5, we just need to write that,

$$\frac{d}{dt}\left(\frac{d\mathfrak{L}}{dv}\right) = \frac{d\mathfrak{L}}{dx} \tag{9}$$

This is the Perfect Euler Lagrange Equation that is fundamentally important in "Analytical Mechanics", superior to Newtonian Mechanics. Now there are some thing that we need to talk, in other places you might find the Euler Lagrange Equation in a little different form, which is,

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{x}}\right) = \frac{\partial\mathfrak{L}}{\partial x} \tag{10}$$

The sign ∂ is same as d, but we use it when we are taking the derivatives of function that have multiple variables, in this case $\mathfrak{L}$ has variables v, x, t. The velocity even takes a new form, by the Newtons method of calculus, the Derivative of any variable, p is written as $\dot{p}$, a dot over (called "Fluxional Calculus"). As a rule of Thumb, we should note that,

$$\frac{dx}{dt} = \dot{x} = v \qquad \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} = a. \tag{11}$$

So, the equation 8 is written actually as,

$$\mathfrak{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{12}$$

You should be comfortable with this convention, or no problem, just continue with v instead of a $\dot{x}$. Later I will introduce a thing called **Equation of Motion** which is just the $\ddot{x}$. Also, you need to do maths for each coordinate separately, that I will show you through a example. Now we can show why the Euler Lagrange is better than the Newtonian Mechanics. It only required taking derivatives and magically give the answer. Remember that, The best way to master the Euler-Lagrangian Method is effectively understanding and reading a few example.
Problem: Find the acceleration a particle of mass m and h above ground using the Lagrangian method.

Solution: Start knowing that we are interested in only the y axis, the height. The motion occurs along y only, and we regard the y distance (height) as h.
Now we can write the Lagrangian $T - V$, taking the potential energy with respect to the ground,

$$\mathfrak{L} = \frac{1}{2}mv^2 - mgh \tag{13}$$

We should write this in the usual Fluxional Convention,

$$\mathfrak{L} = \frac{1}{2}m\dot{h}^2 - mgh \tag{14}$$

This case the Euler Lagrange Equation is,

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{h}}\right) = \frac{\partial \mathfrak{L}}{\partial h} \tag{15}$$

Plug it in our equation 10. We get,

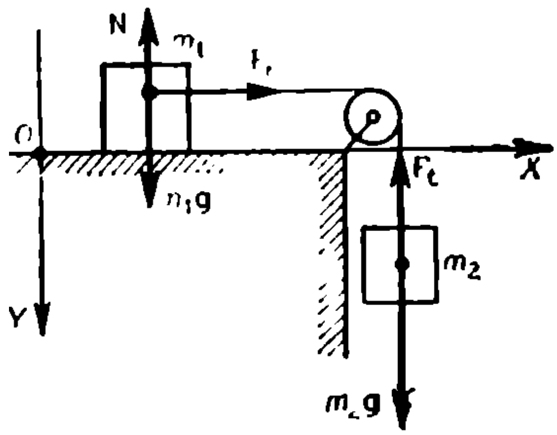
$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{h}}\right) = m\ddot{h} \tag{16}$$

And also for the second part,

$$\frac{\partial \mathfrak{L}}{\partial h} = mg \tag{17}$$

If we write the Equality by the Euler Lagrange Equation, we have,

$$\ddot{h} = g \tag{18}$$



Answer: $\ddot{h} = g$

This is commonsense, the particle falls down in g acceleration. But the Euler Lagrangian is far more superior.

Problem: A mass m_1 sits on a table and another mass m_2 hangs with an inextensible string attached with m_1 and supported by a light pulley. What would be the acceleration of the masses if there is no friction?

Solution: You might have solved such problems from Irodov or such places before. But this time we will complete this using the Euler Lagrange Equation.

If the m_2 decents by Δy_2, then if the string is strong and inextensible, then the mass m_1 should also move forward by Δx_1. So, $\Delta y_2 = \Delta x_1$. This is a part of simple commonsense. String length is conserved. Now if you take the derivative of them, the string causes the relation,

$$\dot{x} = \dot{y} \quad and \quad \ddot{x} = \ddot{y} \tag{19}$$

This implies that both the masses shall move with equal speed and acceleration. If this wasn't the case, the string would get cut or extended. Commonsense. But what is not commonsense is that there should only the y axis (vertical) to be used, because Potential Energy is not working in the x axis (horizontal), and any motion in horizontal axis is just related to the motion of the y axis. The "Degree of Freedom" is about y axis only, not x because x depends on y axis. You are expected to learn further in future. All I want to say for now is that we want to write the Euler Lagrange for y axis only.
Writing the Lagrangian, taking the $V = 0$ at the surface of the Table, $V = -mgy$. This is for, if the distance of the m_2 and surface of table y.

$$\mathfrak{L} = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + m_2 gy \tag{20}$$

Using the Euler Lagrange,

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}}\right) = (m_1 + m_2)\ddot{y} \tag{21}$$

$$\frac{\partial \mathfrak{L}}{\partial h} = m_2 g \tag{22}$$

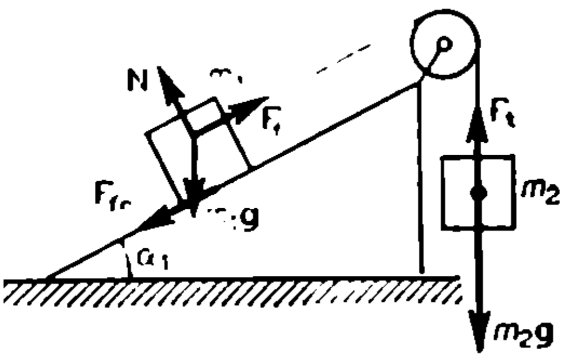
And we find the acceleration of both the bodies.
Answer:

$$\ddot{y} = \frac{m_2}{m_1 + m_2} g$$

Problem: Now as the previous problem, the wedge makes an angle θ. And the friction coefficient of the surface of wedge is μ. Find for the accelerations.

Solution: Everything is quite same as the previous problem, but keep in mind, by string length fact we discussed, if the m_1 moves y distance along the wedge, then it lifts itself by $\Delta h = y \sin\theta$. You can draw the small y and show the height increment. So potential energy of m_1 will increase. Another fact, for friction, if the m_1 moves by y along the wedge, then Frictional Work is $m_1 g \cos\theta \mu y$. You have to subtract it from the kinetic energy. The signs will change if m_1 is so heavy that the system slides by the wedge lifting up m_2, don't worry about this now.

So, we will have,



$$\mathfrak{L} = \frac{1}{2}(m_1 + m_2)\dot{y}^2 - m_1 g \cos\theta \mu y - (-m_2 g y + m_1 g \sin\theta) \tag{23}$$

Our derivatives yield,

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{y}}\right) = (m_1 + m_2)\ddot{y} \tag{24}$$

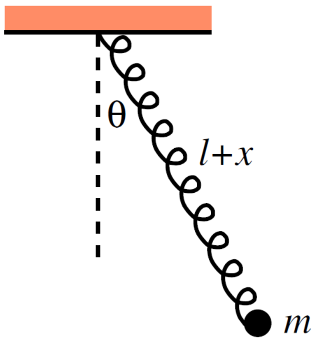
$$\frac{\partial \mathfrak{L}}{\partial h} = (m_2 - \mu m_1 \cos\theta + m_1 \sin\theta) g \tag{25}$$

We come to the shortly found answer,
Answer:

$$\ddot{y} = \frac{m_2 - \mu m_1 \cos\theta + m_1 \sin\theta}{m_1 + m_2} g$$

We come to the same answer writing some equations in typical Newtonian Force Balance methods.

Problem: A pendulum has been made with its string replaced by a spring with spring constant k. Mass m hangs on one of it's end. The length of the spring is given by $l + x(t)$ and the angle it makes with the vertical is given by $\theta(t)$. We have to look for the accelerations of $\theta(t)$ and $x(t)$.



Note, the same statement is "find the equation of motion of the system".
Solution: The kinetic energy should be broken into two independent parts, the radial part and the tangential part.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2 \tag{26}$$

Now we have to determine the potential energy, it comes from the gravity and the spring. Recall that in potential energy, we are always concerned of the difference in Potential Energy, thus, if we assume the top pivot point to have potential energy $V = 0$, then

$$V = -mg(l+x)\,\cos\theta + \frac{1}{2}kx^2 \tag{27}$$

So, the Lagrangian is, $\mathfrak{L} = T - V$,

$$\mathfrak{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2 \; + \; mg(l+x)\,\cos\theta - \frac{1}{2}kx^2 \tag{28}$$

Now, to make math by the two coordinate, radial θ and tangential x, we write Euler-Lagrange Equation for both coordinate, like,

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{x}}\right) = \frac{\partial\mathfrak{L}}{dx} \quad and \quad \frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathfrak{L}}{d\theta} \tag{29}$$

Now you just need to take some derivatives and if doing it we get,

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{x}}\right) = m\ddot{x} \tag{30}$$

$$\frac{\partial\mathfrak{L}}{\partial x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx \tag{31}$$

With the Euler Lagrange Equation, we get finally for Radial Coordinate,

$$m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx \tag{32}$$

Similarly for Tangential Coordinate,

$$\frac{d}{dt}\Big(m(l+x)^2\dot{\theta}\Big) = -mg\,\sin\theta \tag{33}$$

Reducing this to,

$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta \tag{34}$$

So, finally, we have the answers, the Radial and Tangential Accelerations are,

Answer:
Radial Accelerations,

$$m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx$$

Tangential Accelerations,

$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta$$

A Rule of Thumb: There is always something common in all Euler Lagrangian Solutions to Problems. We shall recall this checklist whenever we are attacking a problem using this method.

1. **Appropriate Coordinate System.**Choose the Appropriate coordinate system and ensure the coordinates used in the Euler Lagrange Equation are the Free degrees, it is, the Degree of Freedom axis. For instance, remind how we took θ and radial part x for the Spring Pendulum problem. So as neglecting x axis in the First problem because it depended in y.

2. **Potential Zero Point.**Choose the point where Potential would be zero. Sometimes you need to be careful in case the Potential Energy system is not clear. Write the potential energy as a function of your free coordinates (degrees of freedom), generously take constant, because they'd cancel out.

3. **Velocities for Kinetic Energy.** Find out the velocities (actually speed) for the kinetic energy. This sometimes is the toughest part, then you need to write the position as a function of time and take derivatives to get velocity. Methods in the reference books said below, you can for now look at Morin's Mechanics book.

4. **Lagrangian.** Write the Lagrangian. This most the times is the longest equations in the Problem, but it's not a concern though.

5. **Euler Lagrange input.** Carefully take the derivatives of the system according to the Euler Lagrange Equation, equation 10.

6. **Equation of Motion.** You should have the $\ddot{r}$ if r is your working coordinate, it is the acceleration of the system in the r coordinate, use this to come to required solution.

7. **Alienated things.** You are not supposed to look only for the $\ddot{r}$ in the problem. Look what actually the problem requires. Be ready for doing things that usually are not usual. Check IPhO 2003 Taiwan Problem 01 [Mechanics Problem] for example.