Chapter 1

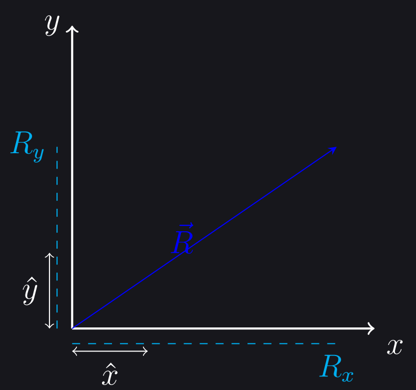
Finding the position

1.1 The differential Unit Vector

We can have any vector say $\vec{R}$ in space. Let us have a cartesian coordinate, then it has the form,

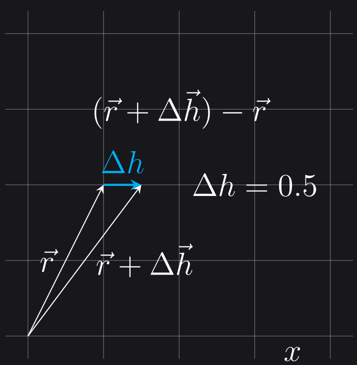
$$\vec{R}(x, y)$$

The basis vectors or the unit vectors are the directions where x and y is laid on. Those directions are, of course, necessary to build a position system.



We define the unit vectors the 1 unit length line on our coordinate lines. We call them as $\hat{x}$ or $\vec{e_x}$. Sometimes, we might also prefer to take the convention that, (x, y, z) is $(\vec{e_1}, \vec{e_2}, \vec{e_3})$.

There is a very slick way to actually define the unit vectors.



Let us have a vector $\vec{r}(x, y)$ and try to find what change occurs to it if we do a change like, $\vec{r}(x + \Delta x, y)$. This is very analogous to our analysis to partial differentials like $\frac{\partial f}{\partial x}$.

$$\frac{\partial \vec{r}}{\partial x} = \frac{\vec{r}(x + \Delta h) - \vec{r}(x)}{\Delta h}$$

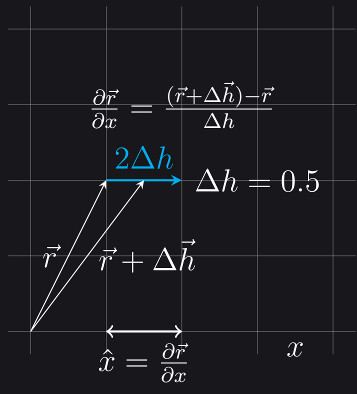
Let us do this for $\Delta h = 0.5$ for now. At first, we see that there is this vector called $\Delta\vec{h}$ in the figure that is actually $\vec{r}(x+\Delta h) - \vec{r}(x)$, the cyan color vector leftside. So,

$$\frac{\vec{r}(x + \Delta h) - \vec{r}(x)}{\Delta h} = \frac{\Delta\vec{h}}{0.5}$$

But, if we put the value $\Delta h = 0.5$, then we will be remained with,

$$\frac{\partial \vec{r}}{\partial x} = 2\Delta\vec{h}$$

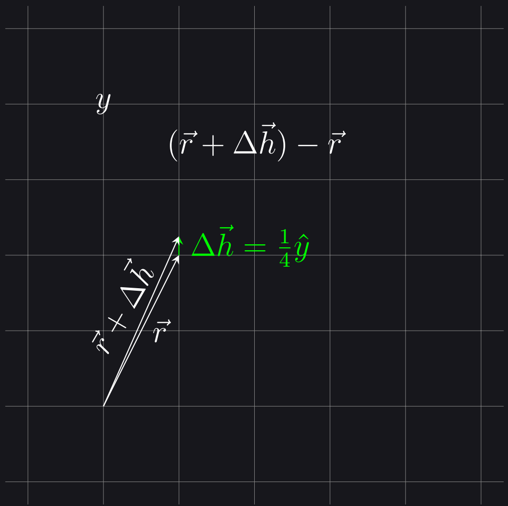
This actually is the unit vector, $\hat{x}$.

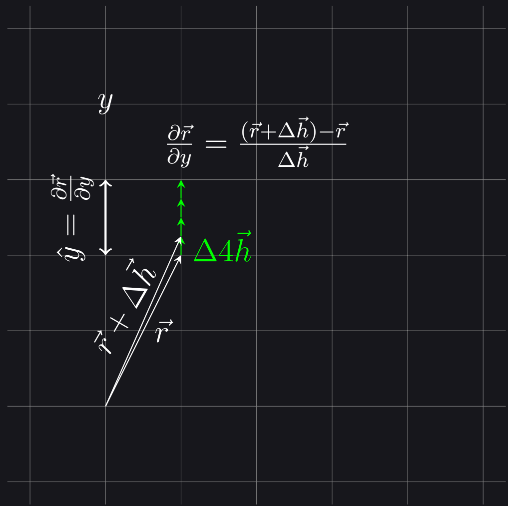


Just looking at such a diagram we can come to the conclusion,

$$\frac{\partial \vec{r}}{\partial x} = \hat{x}$$

This works for any coordinate. Next, we do the same with the y axis and find the solution for $\Delta h = 0.25 = 1/4$. We find that the change is a vector that has the length one-quarter of a unit along y axis. And when we solve the math with dividing the $\Delta h = 0.25$, we have $\frac{\partial r}{\partial y} = 1\hat{y}$.





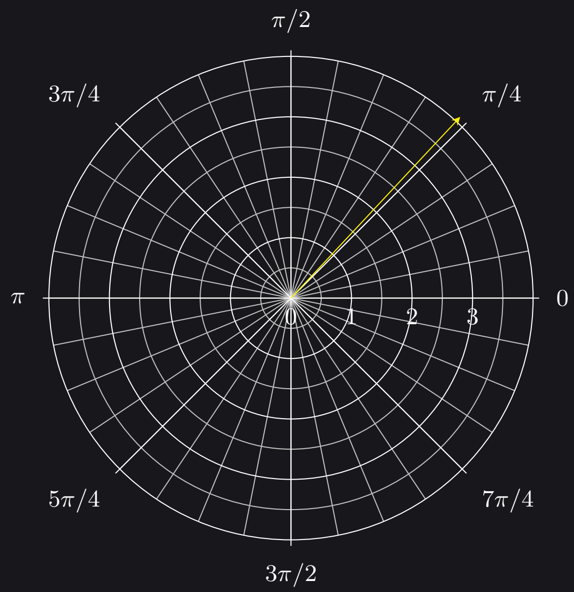
Note. The perfect equation I should I use is,

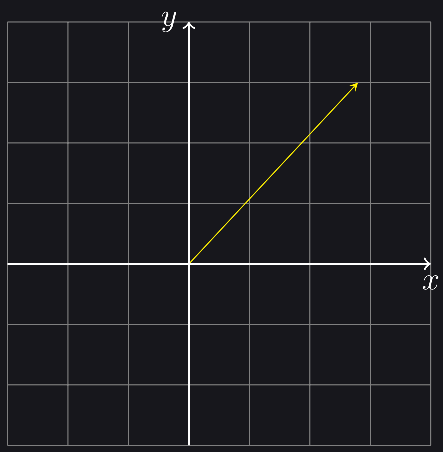
$$\frac{\partial \vec{R}}{\partial x} = \lim_{h\to 0}\frac{\vec{R}(x+h,y)-\vec{R}(x,y)}{h}$$

But I don't care about the limit much, it won't play any big role afterwards.

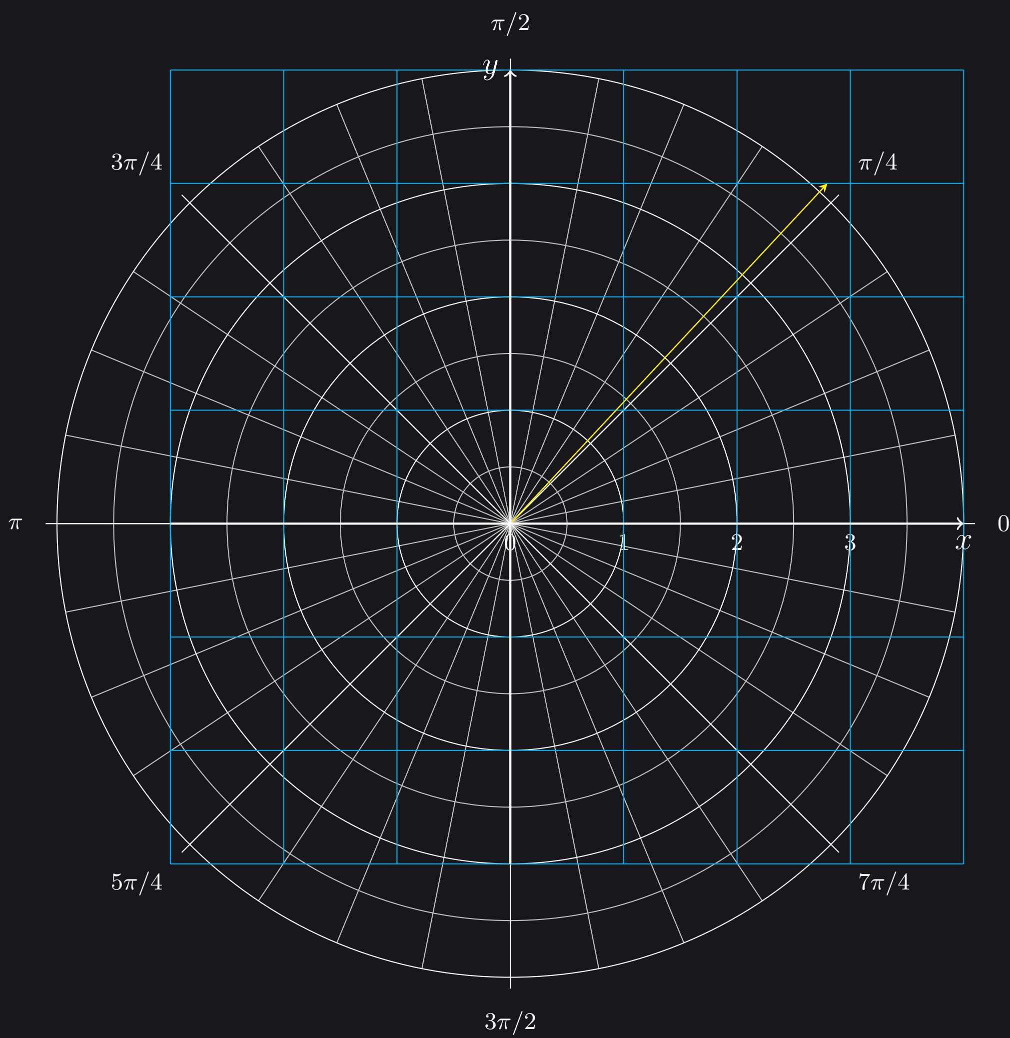
We can happily invoke the same condition for any coordinate systems, even for polar coordinates this holds quite well,

$$\frac{\partial \vec{R}}{\partial r} = \hat{r} \qquad \text{and} \qquad \frac{\partial \vec{R}}{\partial \theta} = \hat{\theta}$$





Of course, the vectors will be the same whether you have polar coordinate or rectangular.



The point is, measure in inches or centimeters, the length of the pencil won't change!

The diagram above showcases the two types of basis, though there is no change in the shape of the vector. While coding, I set the coordinate of the yellow vector in $47°$ angle and 4.1 unit in length. So, in cartesian, the component can be found using the $\sin\theta$ and $\cos\theta$. Usual things.

But the main point is this, trigonometry helps us to transform between basis. And to me this is quite amazing and fun.

Wraping up the discussion, the unit vector can be showed as a *partial differential* of any vector $\vec{r}$. Whatever vector we choose, the rate of change along a coordinate is going to be the unit vector. So,

$$\frac{\partial r}{\partial q} = \hat{q}$$

Where q is any basis, for example,

$$\frac{\partial \vec{r}}{\partial x} = \hat{x} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{y} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\theta} \qquad \frac{\partial \vec{r}}{\partial \phi} = \hat{\phi}$$

The cool fact that amazed me was that, whatever $\vec{r}$ you use, the answer is going to be the same - The Unit Vector respect to whom you take the derivative.

1.2 Using the $\frac{\partial \vec{r}}{\partial x} = \hat{x}$

We used to write that a vector is,

$$\vec{a} = a_x\hat{x} + a_y\hat{y}$$

But as we know this differential form, we replace that,

$$\vec{a} = a_x\frac{\partial \vec{r}}{\partial x} + a_y\frac{\partial \vec{r}}{\partial y}$$

But one thing starts to become weird now, what should be the $\vec{r}$ above? A point that scratches anybody armed with multivariable calculus is that this seems to be quite similar with the rule of chain differenciation of multivariable function,

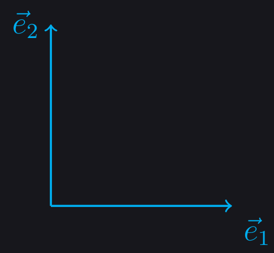
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

Chapter 2

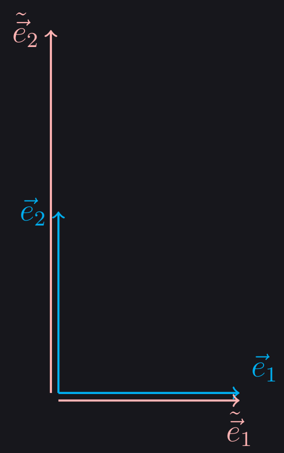
Building new Basis Vectors

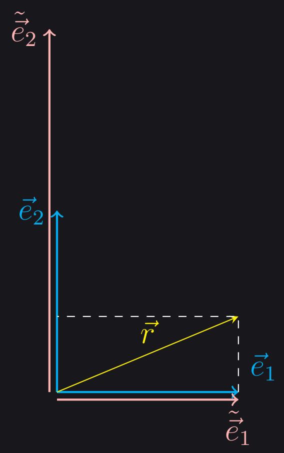
2.1 Making a new Basis

Let us start with a nice and tidy basis vector that has nice basis vectors. We will calculate the position of a random vector $\vec{r}$ using the $\vec{e}_1, \vec{e}_2$ system at first.



The next "tilde" basis has it's vertical twice times of the previous one. And the horizontal basis is the same.





So, in the diagram, the vector $\vec{r}$ in terms of basis $\vec{e}_1, \vec{e}_2$ is,

$$\vec{r} = 1\ \vec{e}_1 + 0.5\ \vec{e}_2$$

The vertical in the $\tilde{\vec{e}}_1, \tilde{\vec{e}}_2$ basis is 2 times the previous one. So, relatively the component of $\vec{r}$ in vertical becomes smaller, as basis is bigger.

$$\vec{r} = 1\ \tilde{\vec{e}}_1 + 0.25\ \tilde{\vec{e}}_2$$

Now let us ask the question, *What would be the relation between the new basis and old basis, and can we treat the new basis as a set of vectors and can we write it in terms of previous basis?*

Of course we can do that, we can write the new basis in terms of the old basis.

$$\tilde{\vec{e}}_1 = 1\ \vec{e}_1 + 0\ \vec{e}_2$$
$$\tilde{\vec{e}}_2 = 0\ \vec{e}_1 + 2\ \vec{e}_2$$

This seems very familar with the idea of using matrices, using the way of two matrices are multiplied, we can write that,

$$\begin{pmatrix} \tilde{\vec{e}}_1 \\ \tilde{\vec{e}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

This matrice with the number is actually transforming the thing into a new coordinate system, and so it is called the "Forward Transformation Matrix".

Similarly, the backward transformation too makes sense, we find that,

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{\vec{e}}_1 \\ \tilde{\vec{e}}_2 \end{pmatrix}$$

So, we can now use these transformation matrices to go from one coordinate to the other.

And using the method, we can find the components of a vector, like $\vec{r}$ in the new red basis. To do that, we can just use the backward transformation matrice to see that,

$$\vec{e}_1 = \tilde{\vec{e}}_1$$

$$\vec{e}_2 = \frac{1}{2}\tilde{\vec{e}}_2$$

Now our vector is,

$$\vec{r} = 1\ \vec{e}_1 + 0.5\ \vec{e}_2$$

And putting the new basis part in place of the unit vector, we see that,

$$\vec{r} = 1\ \left(\tilde{\vec{e}}_1\right) + 0.5\left(\frac{1}{2}\tilde{\vec{e}}_2\right) = 1\tilde{\vec{e}}_1 + 0.25\tilde{\vec{e}}_2$$

This perfectly satisfies our diagram.

But wait, we are missing the fun part, the $\vec{r}$ vector could be easily found just using the backward transformation matrix,

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 1\tilde{\vec{e}}_1 + 0.25\tilde{\vec{e}}_2$$

You can notice one thing, we have to use the backward transformation matrix to make the components of $\vec{r}$ from the previous basis turn to the new red basis. The method is simply multiplying the old basis component with the backward transformer.

2.2 The Differential method of Transformation

So let me ask this known question, how to transform from cartesian to polar coordinates?

To continue, we invoke some random vector $\vec{R}$ and we also know that polar basis vectors are,

$$\hat{r} = \frac{\partial \vec{R}}{\partial r} \qquad \text{and} \qquad \hat{\theta} = \frac{\partial \vec{R}}{\partial \theta}$$

If we want to come from the cartesian (rectangular), then first answer, what are those basis?

$$\hat{x} = \frac{\partial \vec{R}}{\partial x} \qquad \text{and} \qquad \hat{y} = \frac{\partial \vec{R}}{\partial y}$$

Now from the analogy of that chain rule that we were talking about, we can write the polar basis in terms of the old cartesian basis, so,

$$\frac{\partial \vec{R}}{\partial r} = \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\partial \vec{R}}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}r}\frac{\partial \vec{R}}{\partial y}$$

$$\frac{\partial \vec{R}}{\partial \theta} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\partial \vec{R}}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\partial \vec{R}}{\partial y}$$

This is just a fancy way to write,

$$\hat{r} = \frac{\mathrm{d}x}{\mathrm{d}r}\hat{x} + \frac{\mathrm{d}y}{\mathrm{d}r}\hat{y}$$

$$\hat{\theta} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\hat{x} + \frac{\mathrm{d}y}{\mathrm{d}\theta}\hat{y}$$

So, we can use the matrix form,

$$\begin{pmatrix} \frac{\partial \vec{R}}{\partial r} \\ \frac{\partial \vec{R}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}r} & \frac{\mathrm{d}y}{\mathrm{d}r} \\ \frac{\mathrm{d}x}{\mathrm{d}\theta} & \frac{\mathrm{d}y}{\mathrm{d}\theta} \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{R}}{\partial x} \\ \frac{\partial \vec{R}}{\partial y} \end{pmatrix}$$

And this forward transformation matrix is actually called "Jacobian".

We can do the inverted method of coming to cartesian from polar,

$$\begin{pmatrix} \frac{\partial \vec{R}}{\partial x} \\ \frac{\partial \vec{R}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}r}{\mathrm{d}x} & \frac{\mathrm{d}\theta}{\mathrm{d}x} \\ \frac{\mathrm{d}r}{\mathrm{d}y} & \frac{\mathrm{d}\theta}{\mathrm{d}y} \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{R}}{\partial r} \\ \frac{\partial \vec{R}}{\partial \theta} \end{pmatrix}$$

I want to introduce a general form of these transformations, look they are all some fancy summation. If we tell the transformation matrix components to be F_i^j, then,

$$\hat{r}_1 = F_1^1 \vec{e}_1 + F_2^1 \vec{e}_2 + F_3^1 \vec{e}_3$$

Where I tell that, e_1, e_2, e_3 are just x, y, z, and samely r_1, r_2, r_3 is just r, θ, ϕ. Although till now I have been working with 2 Dimensional cases, the same will work for literally any size of dimension.

The above summation in compact form,

$$\hat{r}_j = \sum_{i=1}^{3} F_i^j \vec{e}_i$$

The notation should make some sense, the way they go, it doesn't need explanation. Using the "Einstein" summation convention, when there is two indices repeating, like $F_i^j \vec{e}_i$, then we can of course assume there is surely a summation (I am loosely stating it, later will make it more clear). So, using Einstein's Summation Convention,

$$\hat{r}_j = F_i^j \vec{e}_i$$

And in the differential form,

$$\frac{\partial \vec{R}}{\partial r_j} = F_i^j \frac{\mathrm{d}\vec{R}}{\mathrm{d}x_i}$$

$$\frac{\partial \vec{R}}{\partial r_j} = \frac{\mathrm{d}x_i}{\mathrm{d}r_j} \frac{\partial \vec{R}}{\partial x_i}$$

In the previous method without the convention, the thing should look like,

$$\frac{\partial \vec{R}}{\partial r_j} = \sum_{i=1}^{3} \frac{\mathrm{d}x_i}{\mathrm{d}r_j} \frac{\partial \vec{R}}{\partial x_i}$$

But let us forget this for a moment and look at the geometry. Can we build another kind of basis vector just by rotating our axis?

We can tilt the old white basis to new basis. If we measure the coordinte of a vector in old basis, say $a_1\vec{e}_1 + a_2\vec{e}_2$, how would it look like in the new basis?

At first we need to express the new basis in terms of the old basis, so, we can build some old basis vectors that will make the new basis.

The new basis,

$$\vec{e}_1 = \cos\theta\vec{e}_1 + \sin\theta\vec{e}_2$$

$$\vec{e}_2 = -\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2$$

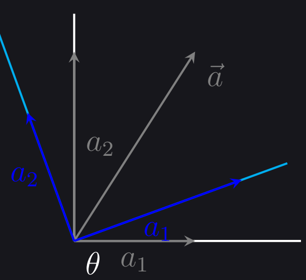
As we smell matrix,

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

Now as we started off,

$$a_1\vec{e}_1 + a_2\vec{e}_2$$

If we want to take this vector in terms of the new basis,



This has some idea how do we go from one to another. As before, we have to at first write the equation of vector $\vec{a}$, then change the old basis to new basis.

Or we can find the inverse of the transformation matrix that we see.

2.3 Spherical Astronomy

Let's have two cartesian frames, x, y, z and x', y', z', here, we get the other frame by rotating x, y, z along x axis by an angle ζ. We can imply our tranformation rules.

So,

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

This actually gives us,

$$\begin{aligned} x' &= x \\ y' &= y\cos\zeta + z\sin\zeta \\ z' &= -y\sin\zeta + z\cos\zeta \end{aligned}$$

Which is relevant to what we have learned so far. Now, in Astronomy, the way we view the World is spherical, we view around all angles in all axes. We measure stuffs in angles.

Let's now introduce **Polar Coordinates**, where only angles matter, so we can assume $r = 1$, you can still keep r in the equations as you solve, but at the end, both sides of the equations will have r so it will get factored out.

There's a point P which is located in $P = P(1, \psi, \theta)$, according to the figure.

Here, the plane relating to which ψ and θ is being measured is the xy plane.

We can measure ψ' and θ' with respect to the new $x'y'$ plane we get from the ζ rotation.

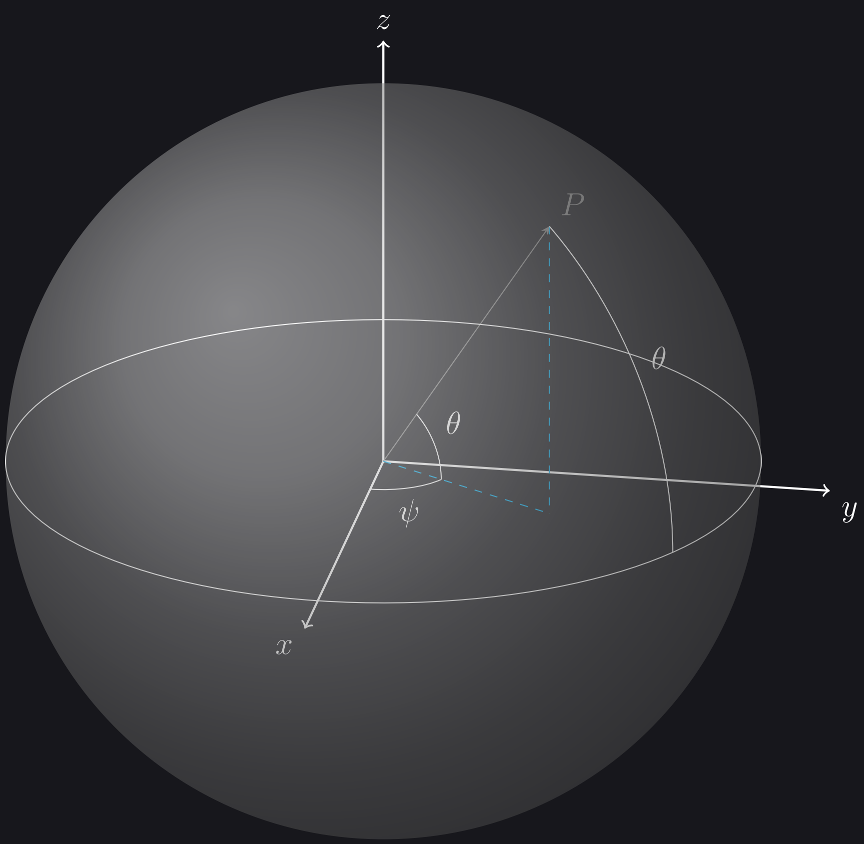
So we can have,

$$P = P(x', y', z') = P(1, \psi', \theta')$$

There must be some way how ψ, θ are related to x, y, z, so as the new coordinate after ζ rotation.

Well, they are related. Just taking projections of the vector that joins P with the center of the Sphere, we can find the vectors components along x, y and z.

$$\begin{aligned} P_x &= (1 \ \ \cos\theta)\cos\psi \\ P_y &= (1 \ \ \cos\theta)\sin\psi \\ P_z &= 1 \ \ \sin\theta \end{aligned}$$



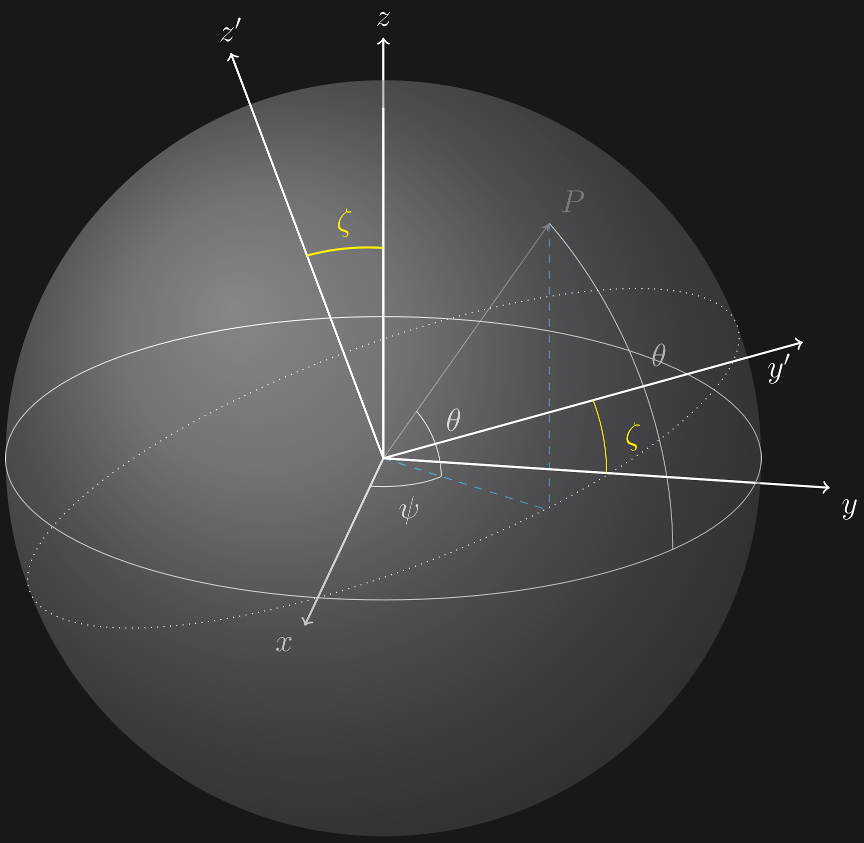
You can carefully look at the diagram and check that it's true. So, clearly writing,

$$P_x = \cos\theta\cos\psi$$
$$P_y = \cos\theta\sin\psi$$
$$P_z = \sin\theta$$

Now, let the position of point P be $P = P(x, y, z)$, then it's obvious that $P_x = x$, which is the position of P along x axis. So we finally have this 3 equations that relate the coordinate with ψ and θ.

$$x = \cos\theta\cos\psi$$
$$y = \cos\theta\sin\psi$$
$$z = \sin\theta$$

Now what will be ψ' and θ' in the new coordinate that's rotated by ζ? Well, we know how to transform from xyz to $x'y'z'$, so we can plug our equations according to that and solve the problem.



$$
\begin{aligned}
x' &= x \\
y' &= y\cos\zeta + z\sin\zeta \\
z' &= -y\sin\zeta + z\cos\zeta
\end{aligned}
$$

$$
\begin{aligned}
\cos\psi'\cos\theta' &= \cos\psi\cos\theta \\
\cos\theta'\sin\psi' &= \cos\theta\sin\psi\cos\zeta + \sin\theta\sin\zeta \\
\sin\theta' &= -\cos\theta\sin\psi\sin\zeta + \sin\theta\cos\zeta
\end{aligned}
$$

Let's input a bit Astronomy here.

The xyz coordinate is the Observer coordinate. The $x'y'z'$ is the Equatorial Coordinate, which is the coordinate with which stars position are measured. Because earth is rotating, xyz isn't a good choice to measure stars position.

Thus,

$$\theta = a$$

$$90^\circ - \psi = A$$

The angle θ is the **Altitude** of P, let's assume it's a star in the sky. The ψ is $90^\circ -$ Azimuth, because it's measured with respect to y axis.

This is what we basically need, in astronomy, the convention are just a little different. Instead of measuring ψ with respect to x, we will measure it with respect to y axis. Because, x can correspond to W (west) and thus y is S (south). With respect to the South, the angle h (Hour Angle) is measured. Hence,

$$90^\circ - \psi' = h$$

And,

$$\theta' = \delta$$

Where δ is the declination. Using these,

$$\begin{aligned} \sin h \cos\delta &= \sin A \cos\delta \\ \cos\delta \cos h &= \cos a \cos A \cos\zeta + \sin\theta \sin\zeta \\ \sin\theta' &= -\cos\theta \sin\psi \sin\zeta + \sin\theta \cos\zeta \end{aligned}$$