Consider the ellipticity vector given by

$$\vec{\varepsilon} = \frac{\vec{L} \times \vec{v}}{GMm} + \vec{e}_r = \text{const.}$$

The Runge-Lenz vector is a scaled up version of the ellipticity vector by a factor of $GMm^2$. In terms of momentum $\vec{p}$, one can write the Runge-Lenz vector as

$$\vec{A} = \vec{p} \times \vec{L} - GMm^2\vec{e}_r.$$

Let us attempt taking the cross product of $\vec{A}$ with the angular momentum $\vec{L}$. Throughout the orbit of the comet $\vec{A}$ always points towards the negative $x$-axis while $\vec{L}$ always points to the positive $z$-axis. Hence, $\vec{L} \times \vec{A} = -lA\hat{y}$ where $l$ is the magnitude of $\vec{L}$.

$$\vec{L} \times \vec{A} = \vec{L} \times (\vec{p} \times \vec{L} - GMm^2\vec{e}_r) = \vec{L} \times (\vec{p} \times \vec{L}) - \vec{L} \times GMm^2\vec{e}_r.$$

We can use the double cross product vector identity to let us now write that

$$\vec{L} \times \vec{A} = \vec{p}(\vec{L} \cdot \vec{L}) - \vec{L}(\vec{L} \cdot \vec{p}) - \vec{L} \times GMm^2\vec{e}_r.$$

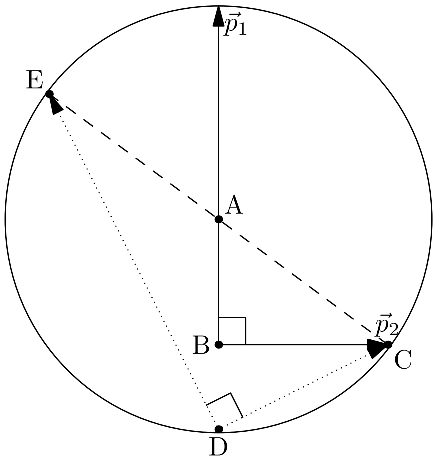
Since $\vec{L} \perp \vec{p}$, then $\vec{L} \cdot \vec{p} = 0$. Therefore, it can be written that

$$-lA\vec{e}_y = l^2\vec{p} - GMm^2\vec{e}_\theta.$$

Taking the magnitude of each side and then rearranging allows us to write

$$\left|\vec{p} + \frac{A}{l}\vec{e}_y\right| = \frac{GMm^2}{l} \implies p_x^2 + \left(p_y - \frac{A}{l}\right)^2 = \left(\frac{GMm^2}{l}\right)^2.$$

This shows us that the momentum vector $\vec{p}$ is confined to a circle with radius $GMm^2/l$ with the origin on the $y$-axis displaced a distance $A/l$ from the center of force. This is well known to be a hodograph for elliptical Keplers motion.




The above diagram shows a hodograph where the two momentum vectors are at the origin of force and the center of the circle is at a point $A/l$ above this. Since $\varepsilon = A/GMm^2$, one can write that $\cos\angle BAC = \varepsilon$. We see from the diagram that $|EC| = \sqrt{5}m|\vec{v}_2|$ from the Pythagorean theorem. Since the length of $|AC| = \sqrt{5}m|\vec{v}_2|/2$ and $|BC| = m|\vec{v}_2|$, we find through Pythagorean theorem that $|AB| = m|\vec{v}_2|\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - 1^2} = \frac{1}{2}m|\vec{v}_2|$. Hence,

$$\varepsilon = \cos\angle BAC = \frac{|AB|}{|AC|} = \frac{1/2}{\sqrt{5}/2} = \frac{\sqrt{5}}{5} \approx 0.447.$$