Variational Calculus, Rediscovering π, Oscillating Force

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$$I = \int_A^B \mathrm{d}S = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,\mathrm{d}x$$

Chapter 1

Variational Calculus and Euler-Lagrange Equation

Good vibrations with freeball.

Powerful and Elegant. Similar to Differential calculus. Not finding the point that minimizes a function, we now need a function that minimizes another function.

In 1670 Calculus was made. 1686 Newton publish principia. Starts Classical Mechniacs 1696 Solves Brachistochone probelm 1733 Euler elaborated the problem. 1743 De Alembert made, reformulates the N2L. More involved. 1755 Lagrange made Tautochone problem. (age 19).

1756 Euler papers on Caclucus of Variations. (Central Point) 1788 Mechanic Analytic (Different method on analysis other than newton). Certain problems could be solved in minimization problem. 1834 Hamilton's Principle.

1.1 Motivation

Consider two points A and B. Coordinates (x_1, y_1) and (x_2, y_2). Minimize distance is going to be the straight line.

But what would be the shape of the chain that hung between the point? the shape will miinimizes the potential energy? Like, what is the shape of the trajectory in a g field moving betwee A and B that will minimize the time taken from A to B?

Idea is finding a path that will miinimizes a function. Path is kind of a function it self. We nened point in Differential, here we need path.

Minimize the distance

$$I = \int_A^B \mathrm{d}S = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,\mathrm{d}x$$

Brachistocrhome

$$I = \int_A^B \left(\frac{1 + y'^2}{2g(y_0 - y)} \right)^{\frac{1}{2}} \mathrm{d}x$$

Caternary Problem

Rope shape problem.

$$I = \int_{x_1}^{x_2} \rho Agy \mathrm{d}S = \rho A \int_{x_1}^{x_2} y \sqrt{1 + y'^2} \mathrm{d}x$$

$$l = \int_{xc_1}^{x_2} \sqrt{1 + y'^2} \mathrm{d}x$$

this is the constraint

Lagrangain

MInimize the lagrangian. This is the difference between the kinetic and energy potential. A system will follow the path of stationary points, that minimizes the lagrangain.

$$I = \int_{x_1}^{x_2} L[x, y, y'] \mathrm{d}x$$

I is a functional.

Derivation

Let us try to find the optimal path between x_1, y_1 and x_2, y_2, let us call $y(x)$ is an optimal path. Let there be another arbritrary path $n(x)$ is a random function that goes form one to another point. We can scale it using $\epsilon \nu(x)$. Here

$$\nu(x)$$

is a shape function and ϵ is a scale. So we can say a path,

$$\overline{y(x)} = y(x) + \epsilon \nu(x)$$

All is continuours. ν is arbitrary variation that satisfies the boundary. And,

$$\nu(x_1) = \nu(x_2) = 0$$

And,

$$\overline{y(x)'} = y'(x) + \epsilon \nu(x)'$$

This is similar to the differential calculus one where we consider $x + \mathrm{d}x$. Variation is like a like a differential, it's $\overline{y(x)}$.

$$I = \int_{x_1}^{x_2} F[x, y, y']\mathrm{d}x$$

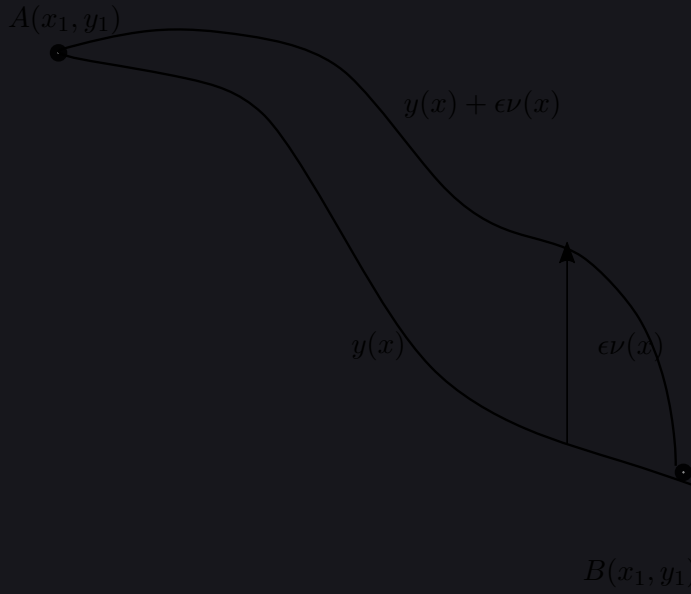Here is the foundational principle, which is that, if $y(x)$ is the stationary path, and some other path

Figure 1.1: variation

$\overline{y(x)} = y(x) + \epsilon\nu(x)$ is another variation (other than the stationary) then, if $\epsilon \to 0$, then $\overline{y(X)}$ will start to become the stationary path, so, given that,

$$\lim_{\epsilon \to 0} \frac{\mathrm{d}I}{\mathrm{d}\epsilon} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_{x_1}^{x_2} F[x, \overline{y}, \overline{y'}]\mathrm{d}x = 0$$

Leibniz rule,

$$\int_{x_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(F[x, \overline{y}, \overline{y'}] \right) \mathrm{d}x = 0$$

This solves,

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}F}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}\epsilon} + \frac{\mathrm{d}F}{\mathrm{d}\overline{y}} \cdot \frac{\mathrm{d}\overline{y}}{\mathrm{d}\epsilon} + \frac{\mathrm{d}F}{\mathrm{d}\overline{y'}} \cdot \frac{\mathrm{d}F}{\mathrm{d}\epsilon} \right) = 0$$

$$\frac{\mathrm{d}\overline{y}}{\mathrm{d}\epsilon} = \nu(x)$$

$$\frac{\mathrm{d}\overline{y'}}{\mathrm{d}\epsilon} = \nu'(x)$$

$$\int_{x_1}^{x_1} \left(\frac{\mathrm{d}F}{\mathrm{d}} \cdot \nu + \frac{\mathrm{d}F}{\mathrm{d}\overline{y'}} \cdot \nu' \right) \mathrm{d}x = 0 \quad (\epsilon \to 0)$$

$\epsilon\nu(x)$ is a variation of the optimal $y(x)$ path.

$$\epsilon = 0, \overline{y} = y, \overline{y'} = y'$$

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}F}{\mathrm{d}y} \cdot \nu + \frac{\mathrm{d}F}{\mathrm{d}y'}\nu' \right) = 0$$

This is the weak form because there is this ν left. We need to integrate in parts,

$$\int_a^b u\mathrm{d}v = vu|_a^b - \int_a^b v\mathrm{d}u$$

$$\frac{\mathrm{d}F}{\mathrm{d}y'}\nu|_a^b + \int_{x_1}^{x_2} \left(\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right) \right) \nu\mathrm{d}x = 0$$

From the equation we talked about before, the first term is zero,

$$\frac{\mathrm{d}F}{\mathrm{d}y'}\nu|_a^b = 0$$

But it is given that ν is arbitrary, and doesnt change with time, is fixed, but choosed randomly, means that $\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right) = 0$.

This means, that,

$$\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right) = 0$$

Problem, I don't understand the last part, let us expand that and see what is it, So because we have $\frac{\mathrm{d}F}{\mathrm{d}y'}\nu|_a^b = 0$, we are left with the integral,

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right) \right) \nu\mathrm{d}x = 0$$

Now, Let us integrate them separately,

$$\int_{x_1}^{x_2} \mathrm{d}x\frac{\mathrm{d}F}{\mathrm{d}y}\nu - \int_{x_1}^{x_2} \mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right)\nu = 0$$

So this just means, okay problem solved,

$$\int_{x_1}^{x_2} \frac{\mathrm{d}F}{\mathrm{d}y}\mathrm{d}x = \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right)$$

$$\frac{\mathrm{d}F}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}y'}\right)$$

1.2   Final Thoughts

In many cases we want to minimize a path (funtion) that goes from $A(x_1, y_1)$ and $B(x_2, y_2)$. So, let the $y(x)$ path be the optimal path, that minimizes a quantity. So, a path, that is not optimal, is going to vary by, say another path,

$$\overline{y(x)} = y(x) + \epsilon\nu(x)$$

Here $\nu(x)$ is an arbitrary funciton (or path) that varies from $y(x)$ making another path $\overline{y(x)}$. And ϵ is required scale factor. Now, we need the boundary conditions,

$$\nu(x_1) = \nu(x_2) = 0$$

This integral I has to be minimized,

$$I = \int_{x_1}^{x_2} F[x, y, y'] \mathrm{d}x$$

This integral can be time t for the Brachistochrone, or the potential energy of a rope, that has to be minimized for a given $y(x)$. In first case of Brachistochrone,

$$y(x) \rightarrow \text{Ramp Shape}$$

And in hannging rope problem,

$$y(x) \rightarrow \text{Potential Energy}$$

Solving the math, we find that the optimal function F by minimizing I satisfies,

$$\frac{\partial}{\partial x} \frac{\mathrm{d}F}{\mathrm{d}y'} = \frac{\mathrm{d}F}{\mathrm{d}y}$$

Where we know that, $y(x)$ is a variable of the function F.

Chapter 2

The Newtonian π

The circumference of a circle is,
$$S = 2\pi r$$

The area of circle is,
$$A = \pi r^2$$

2.1   Derivation of Area

Consider a thin ring of radius r and it has a width $\mathrm{d}r$, hence the small area integrated from $r = 0$ to $r = R$,
$$A = \int_0^R \mathrm{d}r \ 2\pi r = \pi R^2$$

2.2   Calculate Pi

Take a hexagon with side 1, cut into 6 equilateral traingle. The circumference of the bouadry circle of the Hexagon is greater than the total length of 6 sides of Hexagons. The radius of the circle is $r = 1$, so circumference of the circle is $S = 2\pi$. The perimeter of the Hexagon (inside the circle) is 6, so,
$$6 < \pi \times 2 \to \pi > 3$$

Now consider a square that surrounds the circle, it's perimeter is 8, and similarly,
$$8 > \pi 2 \to \pi < 4$$

So,
$$3 > \pi > 4$$

This was known for a thousand of years.

Now break the Hexagon in Decagon, and make the outside square into a greater n-gon. Using this the pi can found. Using this π could be found till 4 decimal places. Its now a matter of muscle power. This was how it was for 2000 years.

Archimdes worked for around 96-gons, and Francois Viete later for a 393,216 sides polygon. Then Ludolph van Ceulen worked on polygon with 2^{62} sides,

$$2^{62} = 4,611,686,018.427,387,904$$

It was an astonishinly large calculation spanning over 25 years. That gave 35 correct decimal places.

Then came Issac Newton.

2.3   Newtons Rule

During Bubonic plage quarantine Newton Played with expressions,

$$(1+x)^2 = 1 + 2x + x^2$$

$$\left(1+x^3\right) = (1+x)\left(1+2x+x^2\right) = 1 + 3x + 3x^2 + x^3$$

There is pattern which is the *Pascal's Triangle*.

Pascal's Triangle can be used for finding the Coefficient's for expressions with any power n which is for positive integers.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

That is the Binomial theorem.

$$(1+x)^n = \sum \binom{n}{m} x^m$$

Newton broke the rule of n being integer. We can apply the theorem for n NOT being an integer.

$$(1+x)^{-1} = 1 - 1x + 1x^2 - 1x^3 + 1x^4 + \dots$$

This becomes infinite series. The math works, and acts like what we have seen in Taylor's series. It is justified if $(1+x)$ is multiplied with the series. This can be used to extend the Pascals triangle before the top node. And can be done in the $-\infty$, the negetive part looks like the pascal's triangle rothated.

We can exted this to,

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x}$$

This gives a series too. But between 0 and 1 there is infinite continuum of numbers making the Pascal's Triangle Continuous. Using this, square roots can be found.

We are definitely interested in $n = \frac{1}{2}$. Because of the parametric Equation of the circle.

2.4   Circle introduced

The equation that makes a circle is,
$$x^2 + y^2 = 1$$

And,
$$y = \left(1 - x^2\right)^{\frac{1}{2}}$$

This is the same expression where x has to be replaced with x^2, but this gives,

$$(1 - x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \ldots$$

Now the area of the one fourth of a circle is just,

$$A = \int_0^1 y(x)\mathrm{d}x = \int_0^1 (1 - x^2)^{\frac{1}{2}}\mathrm{d}x$$

As used, area of the full circle is $\pi r^2 = \pi$ given $r = 1$, hence,

$$\frac{\pi}{4} = \int_0^1 \left(1 - x^2\right)^{\frac{1}{2}}\mathrm{d}x$$

But using the series,

$$\frac{\pi}{4} = \int_0^1 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \ldots \ \mathrm{d}x$$

Now this can be evaluated directly, but this doesn't shrink as quick as the next small addition.

2.5   Limits not $0 \to 1$ but $0 \to \frac{1}{2}$

If the limit is set to integrate from 0 to $\frac{1}{2}$, then we can get the series to shrink quickly. *This can be understood fiddling around with the series a little more*. Hence, $x = 0$ to $x = \frac{1}{2}$, the region of the cirle can be thought to be a sector and a triangle. The sector has an area $\frac{1}{12}\pi$ that can calculated by using geometry and the other part is a triangle with area $\sqrt{3}/8$, using this pi can be calculated using just a few first terms,

$$\pi = 12\left(\frac{1}{2} - \frac{1}{2}\frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{1}{8}\frac{1}{5}\left(\frac{1}{2}\right)^5 - \ldots - \frac{\sqrt{3}}{8}\right)$$

2.6   The series for π

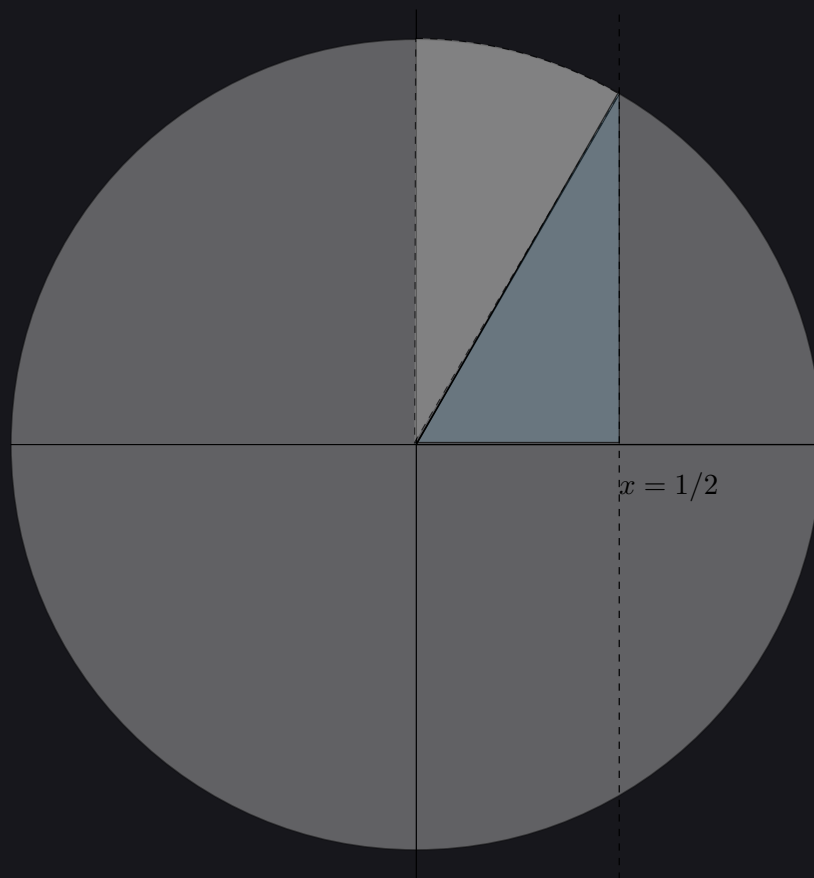Using the integral over there the function that give $\frac{\pi}{4}$ can be found, which is the series.

Figure 2.1: Newton's Circle

$$m\ddot{x}(t) + kx(t) = F_0 \cos \omega t$$

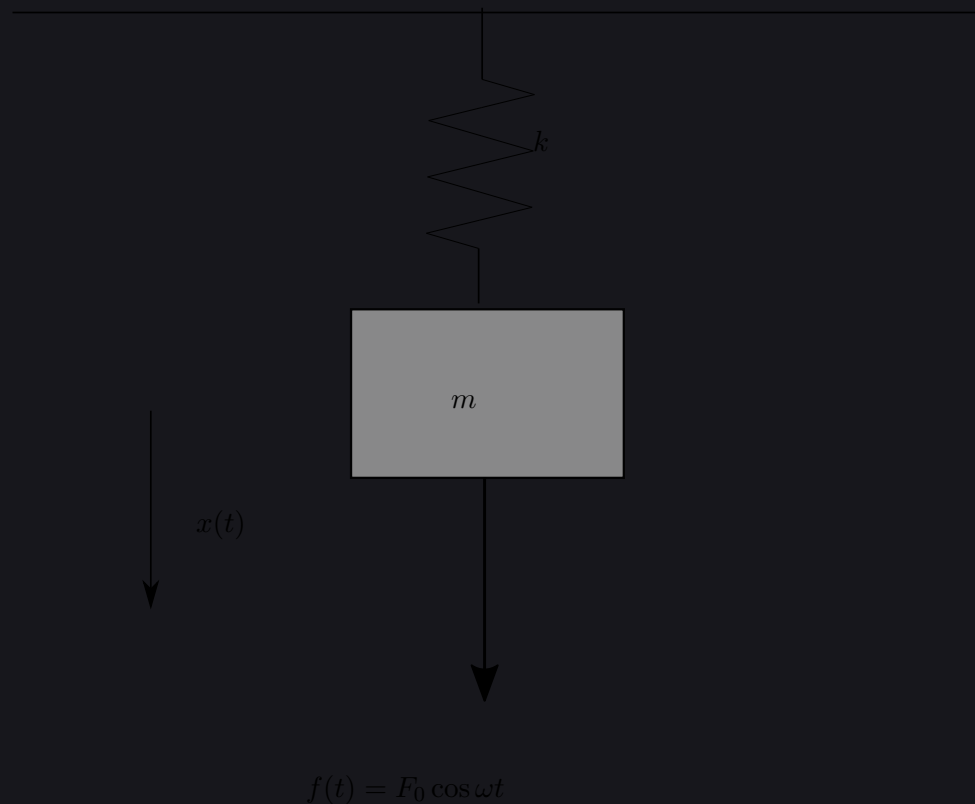$$x(t) = x_h(t) + x_p(t)$$

Chapter 3

Forced Vibrations

Figure 3.1: Forcedspringsystemdiagram

A non constant force is being applied on the mass. The equations of motion,

$$m\ddot{x}(t) + kx(t) = F_0 \cos \omega t$$

So, combination of the homogenous and particular solution,

$$x(t) = x_h(t) + x_p(t)$$

3.1   Homogenous solution

Homogenous is transient or steady state solution. For homogenius, we assume a free vibration problem. So,the **Homogenous Solutioin**

$$m\ddot{x}(t) + kx = 0$$

Solution,

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

3.2   Particular Solution

this is a particule soution . **Particular Solution**, method of undetermined coefficients, for oscillating force,

$$x(t)_p = d_1 \cos \omega_p t + d_2 \sin \omega_p t$$

$$\dot{x}(t)_p = -\omega_p d_1 \sin \omega_p t + \omega_p d_2 \cos \omega_p t$$

$$\ddot{x}_p(t) = -\omega_p^2 d_1 \cos \omega_p t - \omega_p^2 d_2 \sin \omega_p t$$

Substitute, the equations, we get, putting $\ddot{x}$, $\dot{x}$ in the general $x(t)$,

$$d_1(k_1 - \omega_p^2 m) \cos \omega_p t + d_2(k - \omega_p^2 m) \sin \omega_p t = F_0 \cos \omega t$$

Note that, for some mathematical reason $\omega_p = \omega$ that is the oscillation of the force.

Now, looking at the last equations,

$$d_2 \left(k - \omega^2 m\right) = 0 \rightarrow d_2 = 0$$

$$d_1 \left(k - \omega^2 m\right) = F_0$$

This means, that,

$$d_1 = \frac{F_0}{k - \omega^2 m}$$

Hence,

$$x_p(t) = \frac{F_0}{k - \omega^2 m} \cos \omega t$$

So the solution ot $x(t)$,

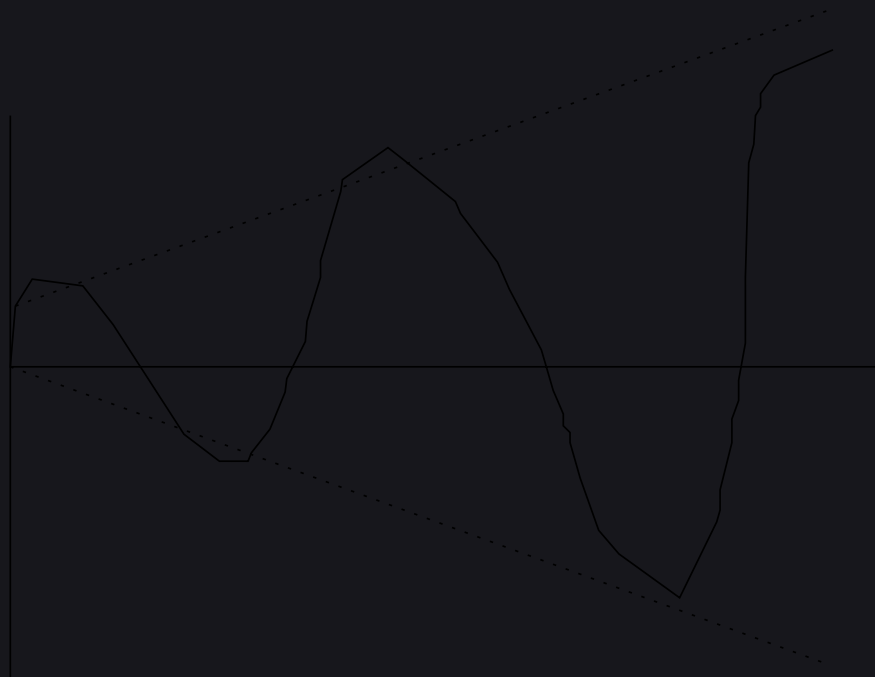$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - \omega^2 m} \cos \omega t$$

Now, what if $\omega_n = \omega$? If this is true,

Figure 3.2: dynamicsinstab