Teaching Ariyan what is Center of Mass

Let there be an object, its shape be random and can be called $\Gamma = \Gamma(x, y, z)$. We are not interested in the shape for now.

Now, in a gravitational field that is uniform at all place by $\vec{g}$ field, this solid object is going to be forced along the gravity.

This gravity force will enable torque on the object alongside the overall force that we call weight. The total torque on the object because of gravity is the sum of all torques on a small points of mass Δm which feels a force $\Delta m\vec{g}$.

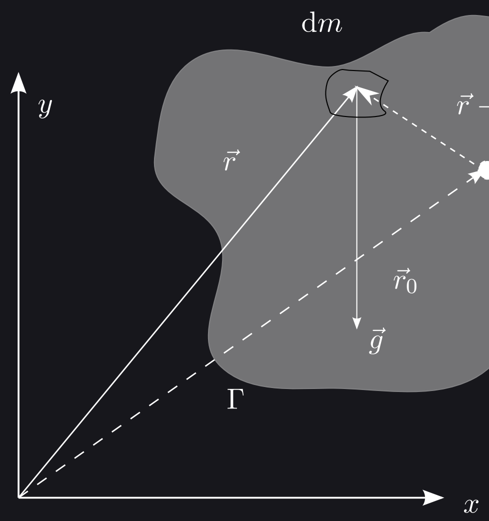

Figure 0.0.1: ariyanboka

Relative to the center of the coordinate, let there be an Δm_i mass, called the i-th mass. It has a position $\vec{r}_i$. Hence, using the equation $\vec{\tau} = \vec{r} \times \vec{F}$,

$$\vec{\tau} = \sum_i \vec{\tau}_i = \sum_i \vec{r} \times \vec{g}\Delta m_i$$

Now, let there be a point $\vec{r}_0$ that is inside the solid shape Γ. We will measure the torque respect to this point at $\vec{r}_0$. Thus,

$$\vec{\tau} = \sum_i \vec{r}_{\text{rel}} \times \vec{g}\Delta m_i$$

Now, using $\vec{r}_{\text{rel}} = \vec{r} - \vec{r}_0$,

$$\vec{\tau} = \sum_i (\vec{r} - \vec{r}_0) \times \vec{g}\Delta m_i$$

We shall define that, there is a point called Center of Mass, which is inside the shape Γ at $\vec{r}_{cm}$ where the net torque is zero. Thus, setting $\vec{r}_0 = \vec{r}_{cm}$,

$$\vec{\tau} = \sum_i (\vec{r} - \vec{r}_{cm}) \times \vec{g}\Delta m_i = 0$$

This sum can be written and solved as,

$$\sum_i \vec{r} \times \vec{g}\Delta m_i - \sum_i \vec{r}_{cm} \times \vec{g}\Delta m_i = 0$$

$$\sum_i \vec{r} \times \vec{g} \Delta m_i = \sum_i \vec{r}_{cm} \times \vec{g} \Delta m_i$$

Now this $\vec{r}_{cm}$ is a single vector that is not dependent on anything other than the shape of Γ, here $\vec{r}_{cm}$ is constant for a fixed shape, so it can be taken out of the sum. Both side can be then shown to be cross product with $\vec{g}$,

$$\left(\sum_i \vec{r} \Delta m_i \right) \times \vec{g} = \left(\vec{r}_{cm} \sum_i \Delta m_i \right) \times \vec{g}$$

Taking away $\times \vec{g}$ from both sides,

$$\sum_i \vec{r} \Delta m_i = \vec{r}_{cm} \sum_i \Delta m_i$$

But $\sum_i \Delta m_i$ is just the total mass M,

$$\sum_i \vec{r} \Delta m_i = \vec{r}_{cm} M$$

Now let us take the limits that $i \to \infty$, then, the sum becomes an integral, added Δm_i is so small that $\Delta m_i \approx \mathrm{d}m$,

$$\int_\Gamma \vec{r} \ \mathrm{d}m = M \vec{r}_{cm}$$

Thus, the equation of center of mass,

$$\boxed{\vec{r}_{cm} = \frac{\int_\Gamma \vec{r} \ \mathrm{d}m}{M}}$$

You can write this in a more usable form,

$$\boxed{\vec{r}_{cm} = \begin{pmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \int x \ \mathrm{d}m \\ \int y \ \mathrm{d}m \\ \int z \ \mathrm{d}z \end{pmatrix}}$$