# An Infant's Guide to Calculus

July 30, 2020

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# 1 What are we supposed to do?

One thing, if you are young, that might hit your chest is that you will not be given a formula to do the math. You will "Find" the formula with the help of the example and do the math in your own method.
In this small booklet, I intend to make an Introduction to Calculus for young physick minds, that's how the title gets the name "infant". But in a short and involving way. You are more likely to solve and do maths as you learn the Calculus of Physics rather than reading the text only. So start with a few things ready,

- A Pencil and an Eraser
- A plenty of Paper
- A calculator
- Your Physics reference book!

Throughout you are supposed to solve some simple maths and required to think some Physical matters. "Quick Check" has been added for insight, so as you read through the text and come across a "Quick Check" - do solve it!

# 2 Short Pre-Calculus

## 2.1 Drawing in Graph - Putting the Points

What is a graph and how to use it?

The lying down bold line is $x$ axis. Standing line is $y$ axis. Keep the word "Axis" in mind. Formally, $x$ axis is horizontal, $y$ axis is vertical. Numbers are printed in the manner shown. The 0 is called the origin, it is where both $x$ and $y$ axis take the value 0. Let us put a point. This point shall be 4 unit left from the origin. This point will be 5 units up the origin. That means that the point will be at $x = 4$ and $y = 5$. So, we move 4 unit along positive x and then move 5 unit above, along positive y. This gives the point. Formally we call, this point is at $(x, y) = (4, 5)$.
Like this, a point $(3, -2)$ can also be shown. **As a rule, remember**, in the form $(x, y)$, a point is placed (horizontal distance from the origin, vertical distance from the origin).

## 2.2 Graphs and Equations

What are the applications of a graph in mathematics?

Equation is just some algebraic thing that exists. As an example, some equation are

$$y = 2x + 5$$

$$5x + 2y - 10 = 0$$

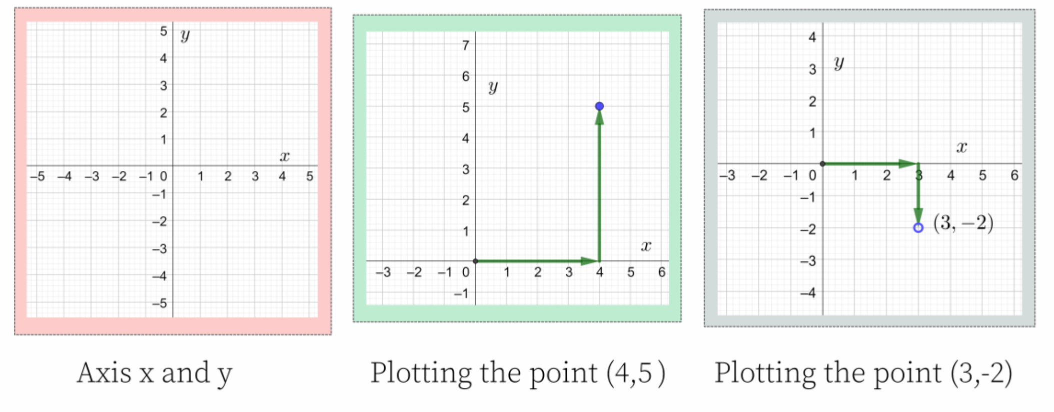


Figure 1: Three figures put in one place.

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Physicist follow a general rule. They say that the $x$ above are **variables.** And the $y$ above are **dependent variable**, because $y$ **almost always** depends on $x$.

You can put various numbers in $x$ and calculate what comes to be the value of $y$.

We can in general say that, *in these equations with $y$'s and $x$'s, $y$ depends on $x$.*

**Example 1.** We have an equation $3x + 5y - 8 = 9$. Let us find various characters of an equation using this.
**Solution:** At first let us see what happens to $y$ if we put $x = 1$.

$$3x + 5y - 8 = 9 \quad \rightarrow \quad 3(1) + 5y - 8 = 9$$

You can reduce it using your algebraic skills.

$$5y - 5 = 9$$

$$5y = 14$$

$$y = \frac{14}{5} = 2.8$$

So, putting $x = 1$ gave us that $y = 2.8$. We can place many other $x$ to do the same thing. Let us make a more use-able format to do the thing.
We write the equation so that $y$ is at one side alone. So we have,

$$5y = 17 - 3x$$

Dividing both side by 5, we have made $y$ totally dependent on $x$.

$$y = \frac{17 - 3x}{5}$$

You can see that if I place $x = 1$ here, the result is $y = 2.8$ as we did before. You can say, that we have found an 'algorithm' for the problem. Now, I will take a calculator (you should take one too) and put some $x$ in the above equation, like $x = 0, 1, 2, 3..$ and write the result in a graph.

| x | y |
|---|---|
| 0 | 3.4 |
| 1 | 2.8 |
| 2 | 2.2 |
| 3 | 1.6 |
| 4 | 1 |

Now what can you see? As the $x$ is increasing, the value of $y$ is decreasing. This is an important realization, an essense of what Calculus is. Using the Differential Calculus, we want to find, *how fast is $y$ changing with change of $x$.*

**Example 2.** The equation is $\frac{56x+2}{y} = 100$. Let us find various characters of an equation using this, again.
**Solution:** As the rule, we need to bring the $y$ at one side. To do this, we just multiply $\frac{y}{100}$ on both the sides that easily gives us,

$$y = \frac{56x+2}{100}$$

We input some $x$ values and make this chart.

| $x$ | $y$ |
|---|---|
| 0 | 0.02 |
| 1 | 0.58 |
| 2 | 1.14 |
| 3 | 1.7 |
| 4 | 2.26 |
| 5 | 2.82 |

This time we have the $y$ increasing with the increase of $x$.

Remember that we have made tables above for the equations.

**Quick Check 1.** *Do a numerical analysis for the following equation as above.*

$$x = \frac{2}{y+1}$$

*But remember that, if 0 is divided by anything, the answer is Undefined!*

### 2.2.1 Drawing the Graphs of the equations

How effective is a Graph?

Mathematicians thought more illustrative method than just making some tables with numbers. They thought that one way to visualize this nicely is to **Plot them on a Graph.**

The horizontal axis is called the $x$ axis, and the vertical one is $y$ axis. Simply, the lying one is $x$ and standing one is $y$. Various point can be given to the graph.

The next examples show these clearly.

**Example 3.** Make a graph of the equation $3x + 5y - 8 = 9$. We worked with this before.
**Solution:** We can use the values that we found in the previous example.

| x | y |
|---|---|
| 0 | 3.4 |
| 1 | 2.8 |
| 2 | 2.2 |
| 3 | 1.6 |
| 4 | 1 |

We have to locate the $x$ points and plot the points in the corresponding $y$ axis. We have to use this equation,

$$y = \frac{17 - 3x}{5}$$

Here, for each $x$, the value of $y$ found is put on. This gives us some points.
When the $x = 0$, then $y = 3.4$. So, a point will be placed in the point $(x, y) = (0, 3.4)$. This means that of the $x = 0$, the point will be made $y = 3.4$ units up. Consider the figure. Like so, for $x = 1$, we have put the point $y = 2.8$ units up. We shall call it the point $(x, y) = (1, 2.8)$. One by one, we do this for all the values. Remember how we put points in the graph is the first section?

Verify the points using example 3 chart.

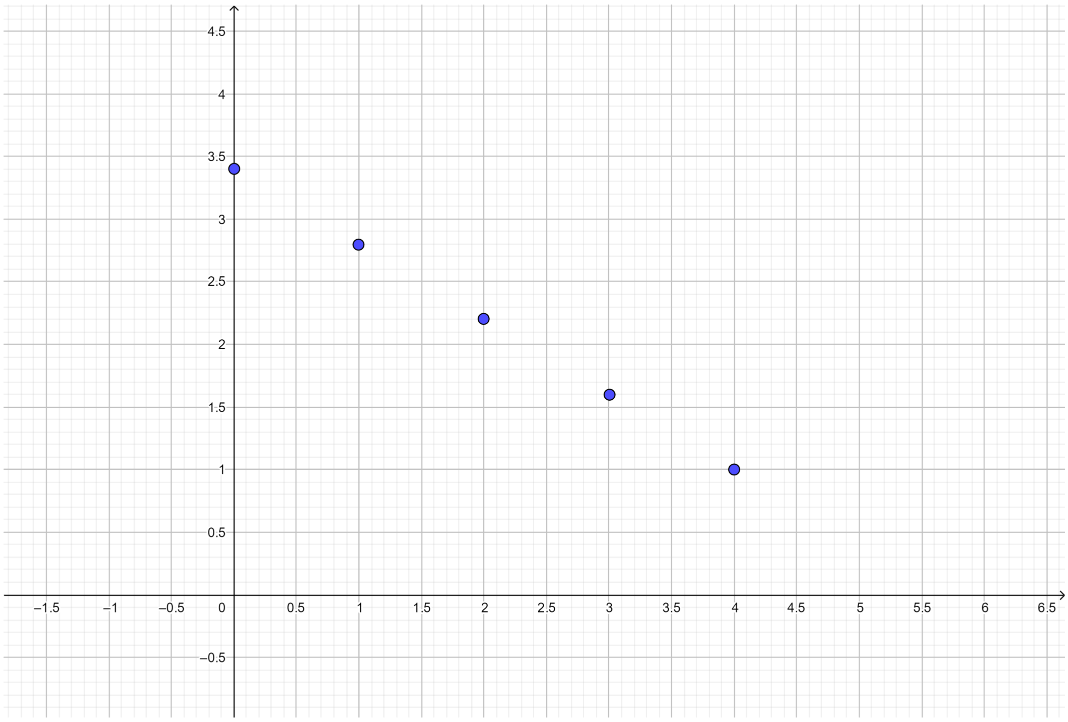



Figure 2: The plots for example 3, according to the chart.

So mathematicians were clever, what they did was, draw a straight line through all the points. This gave a line. This line can give much more insight to the equation.

The line that we can see in the figure is all what the equation wants to say. We can ask, "What is $y$ for the value $x = 6.5$?", looking at the line, we can say that it is $y = -0.5$. **If you see the line carefully, you can see that only having 2 points from the table of example 3 is enough to build the straight line in the graph.**

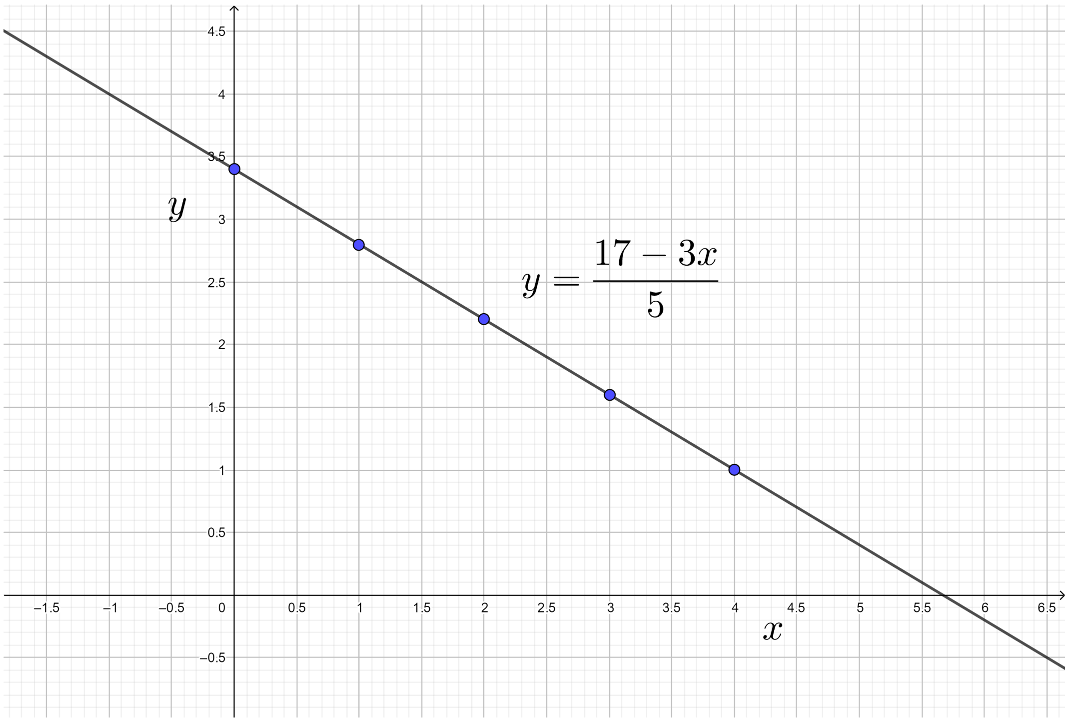



Figure 3: With the line of example 3.

**Example 3.1** Draw the graph of the function $y = f(x) = 0.6x + 3$
**Solution:** We will have to,

- Make the table for the function.
- Put the points on the function.
- Draw the straight line through the points, this will be the graph of the function.

Hence, let us make a simple table first.

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 1 | 3.6 |
| 2 | 4.2 |
| 3 | 4.8 |
| 4 | 5.4 |
| 5 | 6 |

Now, put these points in the graph and draw the line.

**Quick Check 2.** *Draw the graph for* $y = \frac{56x+2}{100}$.

## 2.3 Advancing the idea on Graphs

### 2.3.1 Function

Is there any easy way to relate Algebra and numbers? Can we make Mathematical Machines?

The $y$ in our above analysis is called technically, "The function of $x$." As we plot the graph, we don't say we are plotting $y$, we say that we are, "plotting the **function** of $x$." So, from

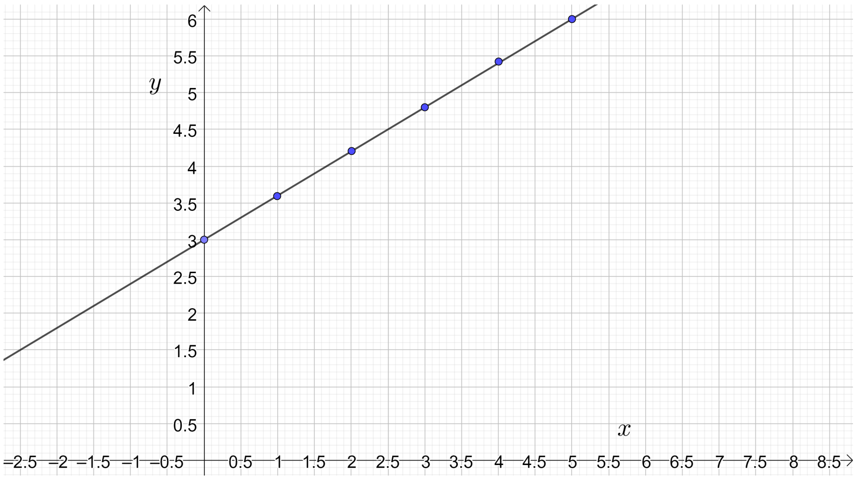



Figure 4: Graph for $y = f(x) = 0.6x + 3$

now on, we shall call the matter in this manner, relating to example 3.

$$y = f(x) \qquad \text{(y is the function of x)} \tag{1}$$

This is a fundamental statement.

$$y = f(x) = \frac{17 - 3x}{5}$$

Now learning maths you can define function very rigorously, but in Physics right now we don't need it. We just need to learn the practical use of Calculus as soon as possible. Function is the central idea.
**Common Confusion!** The $f(x)$ is **NOT** $f \times x$! It actually means "Function of $x$"! Function is like a machine, you can give ingredients in it and it shall give you some product. Let us make a "Cake" function.

$$\text{Flour, Eggs, Milk, Butter} \rightarrow \boxed{\text{Cake Function}} \rightarrow \text{Cake}$$
$$\text{Flour, Eggs, Milk, Butter, Chocolate} \rightarrow \boxed{\text{Cake Function}} \rightarrow \text{Chocolate Cake}$$

Mathematically, we can make a function, that will give the two times the square of a number. Like if we give 2, then square of 2 is 4, and twice of 4 is 8.

$$2 \rightarrow \boxed{\text{Twice of the Square Function}} \rightarrow 8$$
$$3 \rightarrow \boxed{\text{Twice of the Square Function}} \rightarrow 18$$
$$5 \rightarrow \boxed{\text{Twice of the Square Function}} \rightarrow 50$$
$$\text{-10} \rightarrow \boxed{\text{Twice of the Square Function}} \rightarrow 200$$

We can do this for every number. Now to make it able to deal with any number, let the number input be $x$. Then the two times the square of a number of $x$ is simply $2 \times x^2 = 2x^2$. This is a Function. So, this case we had,

$$f(x) = 2x^2$$

And it makes complete sense.

$$x \rightarrow \boxed{\text{Twice of the Square Function}} \rightarrow 2x^2.$$

Making a chart like example 1 and 2,

| $x$ | 2 | 3 | 5 | -10 |
|---|---|---|---|---|
| $f(x) = 2x^2$ | 8 | 18 | 50 | 200 |

As a matter of interest, the input numbers are called Domain and the output is called Range. Like, above, for Domain $-10$, the Range is 200. Interestingly, also for the Domain 10, Range is 200. You will also have to deal with these in Programming algorithms.

### 2.3.2 Functions- Algebraically

HOW DOES ALGEBRA REPRESENT A FUNCTION? See some example of Functions,

$$f(x) = 6x - 4$$

$$f(x) = x^2$$

$$f(x) = \frac{x+1}{x-1}$$

These are also some lame equations. Here you can put $x$ as anything you want. But remember, you cannot divide anything by 0, like, putting $x = 2$ in the third function of next example 4 will give 3/0, that is undefined. Try dividing 0 with something in the calculator. It doesn't work.

**Example 4.** Input $x = 10$ in the following functions.

$$f(x) = 50x$$

$$g(x) = x^2 - 2x$$

$$f_t(x) = \frac{x+1}{x-2}$$

**Solution:** This is easy to do.

$$f(10) = 50(10) = 500$$

$$g(10) = 10^2 - 2(10) = 80$$

$$f_t(10) = \frac{10+1}{10-2} = 1.375$$

So, using the functions you can put various $x$ and have value of $y$. We know that we can plot equations in a graph, or draw an equation in the graph using the points $(x, y)$ as we did in previous examples. But I have said that $y = f(x)$, so the points of the equation that you want to plot in the graph is $(x, f(x))$. Remember, in $(x, f(x))$, the point is positioned $x$ distance away from the origin in horizontal direction (or along $x$ axis); and $f(x)$ is the distance along the vertical direction.

We don't just need to give numbers, we can also push in another equation.

**Example 5.** Define the functions given for $x = t^2, p + 5$. Functions are,

$$f(x) = 50x$$

$$g(x) = x^2 - 2x$$

$$f_t(x) = \frac{x+1}{x-2}$$

**Solution:** For $x = t^2$,

$$f(t^2) = 50(t^2) = 50t^2$$

$$g(t^2) = (t^2)^2 - 2(t^2) = t^4 - 2t^2$$

$$f_t(t^2) = \frac{t^2+1}{t^2-2}$$

For $x = p + 5$,

$$f(p+5) = 50(p+5)^2 = 50p^2 + 500p + 1250$$

$$g(p+5) = p^2 + 10p + 25 - 2p - 10 = p^2 + 8p + 15$$

$$f_t(p+5) = \frac{p+6}{p+3}$$

This seems little abstract and un-easy, but this is perfectly normal and okay.

**Quick Check 3.** *Given function is $p(t) = t^2 + 4t + 8$. Evaluate $p(x + h)$.*

**Quick Check 4.** *Given function is $f(w) = \frac{w+1}{w-1}$. Evaluate $f(1/k)$.*

**Quick Check 5.** ***IMPORTANT!*** *Make a table as we did in example 1,2 for the three functions below using a calculator.*

$$f(x) = 50x \quad g(x) = x^2 - 2x \quad f_t(x) = \frac{x+1}{x-2}$$

*Please think how much sense they make with their visualization.*

One realization to be made from this Quick Check is that we cannot put $x = 0$ in the $f_t(x)$. But yet, we can put x's like $x = 0.1, 0.01, 0.001,...$ and try to get **as close as possible to zero.** Notice that the function gives higher and higher value, so the height (y value in graph) goes up, we can assume that as near to zero we can make $x$, the result goes high, **approaches Infinity.**

### 2.3.3 Linear Function

ARE ALL THE FUNCTIONS PART OF A RULE? HOW IS A STRAIGHT LINE FUNCTION MADE? WHAT ARE IT'S RECIPE?

We have already seen what a Linear Function is, it is the function which makes a straight line in a graph. Every linear function has a similar format. The format is,

$$y = mx + c \tag{2}$$

For example, we can take equations from our previous examples,

$$y = \frac{17-3x}{5} \quad \to y = \frac{17}{5} - \frac{3x}{5} \to \quad y = -\frac{3x}{5} + \frac{17}{5}$$

Writing nicely, we have the format,

$$y = mx + c \quad \to \quad y = -\frac{3}{5}x + \frac{17}{5}$$

This tell that,

$$m = -\frac{3}{5} \quad c = \frac{17}{5}$$

For this form of equation, the equation makes a straight line in Graph.
We call the $m$ as **slope** and the $c$ as **constant.** Slope and Constants have physical meaning. But I will tell that after a little bit of maths.

**Example 6.** Find the Slope and Line Constant for the following equation,

$$5 - \frac{9x}{2y} = 0$$

**Solution:** Multiply both side by $2y$,

$$10y - 9x = 0$$

$$y = \frac{9}{10}x$$

As we know that the format of a straight line (Linear Function; same thing) is $y = mx + c$,

$$m = \frac{9}{10} \qquad c = 0$$

This is clear that, taking $y$ to one side and isolating $x$ is enough.

**Quick Check 6.** *We have an equation,*

$$18x + 9y + 4 = 0$$

*Find its Slope and Constant.*

### 2.3.4 Slope

Is SLOPE s IMPORTANT?

Slope is an important property of a Straight line equation or a Linear Function. This determines, **how much steep is a line when graphed. How much angle the straight line makes with the surface (x axis).**

If slope value is high, then the line in graph will make much angle with the horizontal line or the $x$ axis. If the slope is less, then the line will make low angle with the horizontal line. We can analyze it by the following.

Let us draw the graph of some linear functions. The colors represent each of the functions.

Notice, the negative slope equations have one characteristic that they point down with the increase of $x$. The positive slopes point somewhat upwards. And $1/2$ is $0.5$, $1/3$ is $0.33$. When $y = x$, then $m = 1$.

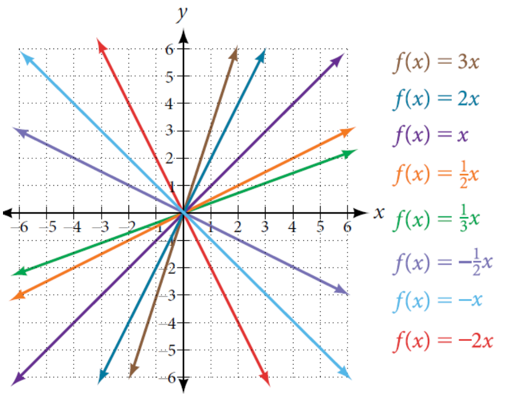
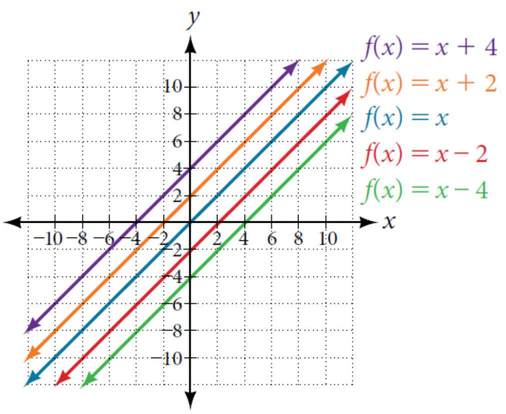



Figure 5: Various Slopes



### 2.3.5 The Line Constant

This is just for curiosity, we will not be dealing with this later.

The line constant determines **where to place the line**.You can see that the addition of constants drives the lines leftwards in negative $x$ direction.

This is difficult to explain directly, but you can do this small exercise to understand it clearly. Let one function be $f(x) = x$ and another function be $g(x) = x - 3$. Plot them on a graph paper to see the difference.

### 2.3.6 The Physical Meaning of Slope

If slope is important, does it have any real life meaning?

When we use the linear functions in daily life, then the appropriate name of Slope is **Rate of Change.** For example, you have a phone with no games in it. Suppose you started to download same amount of games everyday, and after 10 days, you had 20 games in it. This means that everyday you downloaded $20\,games/10\,days = 2\,games/day$. In words, you downloaded "2 Games per Day".

Going a little more numeric,
Day 0 $\rightarrow$ 0 Games.
Day 1 $\rightarrow$ 2 Games.
Day 2 $\rightarrow$ 4 Games.
Day 10$\rightarrow$ 20 Games.
You can also say that everyday, your number of games in phone changed by $+2$. That means that yesterday if you had $n$ games, today you will have $n + 2$ games. The next day you will have $(n + 2) + 2 = n + 4$ games, then next day $(n + 4) + 2 = n + 6$ games.

After $t$ days, you will have $n + 2t$ games. Slope comes to play an important role here.

Let the $y$ represent "Number of Games". Let the $x$ represent "Days". We know that the number of games in the phone depends on the days passed. So, mathematically, as number of games depend on the days, Number of games is a function of days.

Let the day with no games be day 0, then, if you download 2 games everyday, your Number of Games vs. Days graph can be made! This function of Games-Days will be linear. Let the function be,

$$y = mx + c$$

In day 0 or $x = 0$, number of games is 0 or $y = 0$. So,

$$0 = m \times 0 + c \qquad \rightarrow \qquad c = 0$$

In day 10 or $x = 10$, number of game is 20 or $y = 20$. So,

$$20 = m \times 10 + 0 \qquad \rightarrow \qquad m = 20/10 = 2$$

So the slope $m$ is 2, the number of games you download everyday!
Your graph will have the function,

$$y = 2x$$

Remember, y is Number of Games, x is days. We are able to show the number of games as the function of days. All I want to make you understand is that the number of games downloaded everyday is actually an important quantity. This is same as saying that the slope of your game downloading number is an important quantity. Because this makes possible for you to find how many games you will have in your phone in a random day. Using $y = 2x$, you can easily say that there were 8 games in the Day 4.

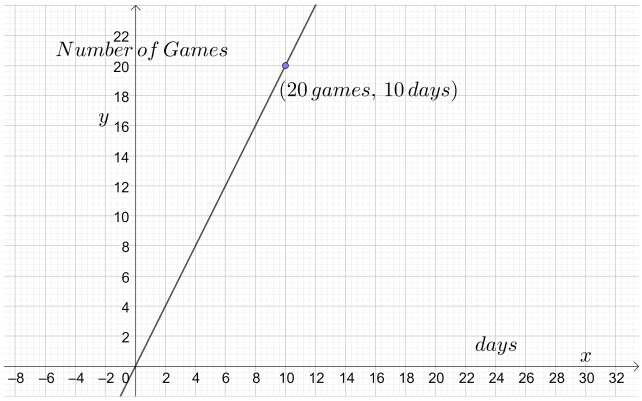



Figure 7: Number of Games as a function of Days

**Example 7.** Haydar had 1 music in his Ipod. But from one day he started to download music. Everyday he downloaded same number of music. 8 days later, he had 25 musics.
a) How many music he downloaded everyday?
b) How many music will he have after 14 days?
c) Draw a graph depicting Number of Music Downloaded as Function of Days.

**Solution:** We can solve the all problems together if we do it by writing the linear function of his Musics downloaded as function of Days.

$$y = mx + c$$

We need to one by one find the elements of the Linear Equation.

- $y$ axis will represent **Number of Music downloaded**.
- $x$ axis will represent **Days**, this is because **y is dependent variable on x**. As Number of Music downloaded is dependent on days.
- In day 0, when he haven't started downloading music, there is 1 music in his IPod. So,
$$y = mx + c \quad 1 = m \times 0 + c \quad c = 1$$
- In day 8, he has 25 musics. So,
$$y = mx + c \quad 25 = m \times 8 + 1 \quad m = 3$$
- We have $m = 3$ and $c = 1$. So, the function is,
$$y = 3x + 1$$

- Answer to a) Because, rate of change music everyday = music downloaded everyday = slope m, hence $\boxed{3}$.
- Answer to b) Just finding out the $y$ for $x = 14$ does the job. $y = 3(14) + 1 = 43$. So, $\boxed{\text{43 musics}}$.
- Answer to c) Come on, we already know how to do this!

We can now see another interesting example. Please try to understand this one clearly. If you don't feel confident, then skip the next part and come back to it later. No worry.

**Example 8.** Our friends from Biology Department have found a Bacteria. This Bacteria reproduces very quickly, everyday, it becomes twice than what it was. Firstly, there were 2 Bacterias. How many Bacteria will there be after 15 days?

I would like to discuss this one little more. If you have understood all the previous facts clearly, then you'll probably say, "Everyday the rate of increase is 2, then starting from 2 Bacterias, after 15 days, there will be $2(15) + 2 = 32$ Bacterias! Easy!". But are you sure? That quick answer is not bad right now, but it actually is incorrect. Read the example statement clearly. We have some Bacteria, tomorrow it will become TWICE what it is today. The Day after tomorrow, it will be TWICE than tomorrow, it will be 4 times than today!

Let me be more clear. Suppose we have $n$ bacteria right now. Let today be Day 0. Then clearly writing,
Day 0 $\to n$ bacteria
Day 1 $\to 2n$ bacteria
Day 2 $\to 2 \times 2n = 2^2 n$ bacteria
Day 3 $\to 2 \times 2 \times 2n = 2^3 n$ bacteria
Day 4 $\to 2 \times 2 \times 2 \times 2n = 2^4 n$ bacteria.
Day t $\to 2^t n$ bacteria
We have made a general formula, totally not same as the Linear Function. To be honest, this isn't a Linear Function! This function is close to what we call "Transcendental Function" and "Exponential Function".

The function for this situation will be,

$$\text{Number of Bacteria} = \text{Initial Number of Bacteria} \times 2^{day}$$

In $x - y$ coordinate graph,

$$y = n2^x$$

This looks silly, but in reality this is dangerous, why?
To answer this, let's solve the example first. It has been said we started with $n = 2$ Bacteria. So number of Bacteria in 15 days will be

$$y = 2 \times 2^{15} = 65,536$$

Under 15 days the Bacteria becomes 65 Thousand 5 Hundred 36! Astonishing results are for 1 month, that is 30 days. In 30 days, the number becomes,

$$y = 2 \times 2^{30} = 2,147,483,648$$

That is 2 Billion! There more like 8 billion people in the world. So this is much scary than it looks. Fact is, most of the Countries during Corona Virus pandemic showed Exponential number of Patient growths under 15-20 days.
To be on topic, the method of finding the rate of change will be a later discussed. We have to learn a bit more to come to that stage. I will make a fundamental announcement.

The slope $m$ of a **Linear Function** gives the **Rate of change** of the Function. If $m$ is positive, then the slope $m$ is same as **Rate of Increase**, if $m$ is negative, then slope $m$ is same as **Rate of Decrease**.

For example, your rate of change of games was 2 games per 1 day. Haydar's IPods rate of change of music was 3 musics per 1 day. Whenever we are dealing with Slopes, remember this Magic Spell,

| |
|---|
| The **Slope** is the rate of increase/decrease of ______________ per 1 ______________ |

Very soon you will have to change the "**Slope**" with something else. But the things will stay the same, no worry.

**Quick Check 7.** *Ankon perfectly solves 3 Astronomical Problems everyday. He perfectly solves 5 Mathematical Problems in the mean time. Suppose, you want to draw a graph showing the number of total problems solved by Ankon as a function of time. What shall be the slope of the line that you will draw?*

### 2.3.7 Slopes of lines that are not that Straight

Is a slope always the same for a function that doesn't look simple?

Sometimes (actually almost everytime in Physics), the graph of some situation is not a straight line. The equation $y = mx + c$ not quite works. This is because this equation can only make lines that look straight in the graph. Put any values you like for $m$, give anything you want to be $c$, then plot it. I bet you can never make it look curved. It will always be a straight line.

Let us try to understand it by the following example.

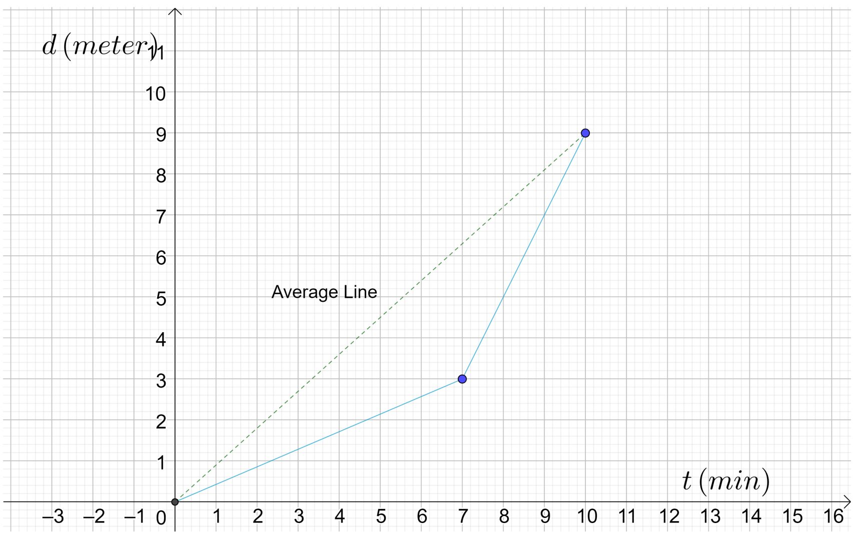



Figure 8: Your driving distance and time taken, for example 9.

**Example 9.** You are driving a car. The plot of the distance traveled versus time that you drive has been shown. Find the average speed of your drive.
**Solution:** There are two ways how you have made the journey, firstly, you traveled 3 meter in 7 minute. Then you traveled more 6 meter in next 3 minute. Your speed has not been the same in the two part.
At first, your speed was lesser. Because it took you more time to travel 1 meter. But then, you took lesser time to travel 1 meter as you drove fast. I will make a better analysis after this example. You can show it by this formula,

$$\text{speed} = \frac{\text{distance traveled}}{\text{time taken}} \tag{3}$$

But in this problem we need to find the "Average Speed", your overall speed. That can be dealt with formula of equation 3.
Notice closely, you drove for 10 minutes. You traveled 9 meter. So, your average speed is,

$$speed_{average} = \frac{9\,\text{meter}}{10\,\text{minute}}$$
$$s = 0.9\text{m/min}$$

As promised, I will make a different analysis to the problem. Note one thing, there are 2 straight line. We can separately measure the speed.
1st Speed, traveled 3 meter in 7 minute,

$$s_1 = \frac{3\,\text{meter}}{7\,\text{minute}} = 0.43\,\text{m/min}$$

2nd Speed, traveled $9 - 3 = 6$ meter in $10 - 7 = 3$ minute,

$$s_2 = \frac{6\,\text{meter}}{3\,\text{minute}} = 2\,\text{m/min}$$

So you have used 2 speeds. One is 0.43 meters per minute another is 2 meters per minute. This actually means at first, in 1 minute the distance traveled was 0.43 meter and later 2 meter.

But wait a second. Does our learning about slopes help in anyway. You'll be happy to know, sure it does!

Let us name the horizontal x axis as $t$, measured in minutes. Let the standing axis y be distance traveled $d$, measured in meters. There are two lines (ignore the average line for now). Each are straight so they should have a $y = mx + c$ format. Can we look for the line equations ?

For now, $y$ is $d$, distance. $x$ is $t$, time. So,

$$y = mx + c \quad \rightarrow \quad d = mt + c$$

**Line 01.**For the figure, at $t = 0$ min, $d = 0$ meter. Hence, as we did before, putting the values,

$$d = mt + c \quad \rightarrow \quad 0 = m \times 0 + c$$
$$c = 0$$

Then, at $t = 7$ min, $d = 3$ meter,

$$d = mt + 0 \quad \rightarrow \quad 3 = m \times 7$$
$$m = 3/7 = 0.43$$

Our equation is,

$$d = 0.43t$$

**Line 02.**For the second line, at $t = 7$ min, $d = 3$ meter. At $t = 10$ min, $d = 9$ meter. This time, we will have a little different way, but understandable.

$$d = mt + c$$

$$3 = 7m + c$$

$$9 = 10m + c$$

Subtract the two equations,

$$9 - 3 = 10m - 7m + c - c$$

This gives us,

$$6 = 3m \quad \rightarrow \quad m = 2$$

Now put $m = 2$ in any of the above equation, to find $c$,

$$3 = 7(2) + c \quad \rightarrow \quad c = -11$$

Or, put it in the other one, you will have same result,

$$9 = 10(2) + c \quad \rightarrow \quad c = -11$$

So, the equation for the second line is,

$$d = 2t - 11$$

What is happening, slopes of the lines are $m = 0.43$ and $m = 2$. But we know that the speeds are $s = 0.43$ and $s = 2$ meters per minute. Weird!

Actually this is okay. Remember, $y = mx + c$. So $y$ is dependent on $x$ using a set of mathematical stuff $m, c$. We know $y$ can represent the $function\ f(x)$, $y = f(x) = mx + c$. Here, $m$ is the **Rate of Change of y with x**. Similarly, we defined **distance** as a function of **time**. Hence, the **Rate of Change of distance with time is the slope**, but this slope takes the definition of speed either!

Note the line equations we made, $d = 0.43t$ and $d = 2t - 11$. This takes the form,

$$\text{distance} = \text{slope} \times \text{time}$$

But as we have seen, slope is same as speed,

$$\text{distance} = \text{speed} \times \text{time}$$

This is not new, we know that $s = d/t$, so just taking $t$ at another side, $d = st$. But this makes a very suitable position with the idea of line equation. Summarizing,

| |
|---|
| Rate of Change of distance with time is speed. Thus, the slope of distance vs. time graph is speed. |

**Quick Check 8.** *Nabab Ali Khan has been in Quarantine. He ate a lot in his home staying. The first day of quarantine, he weighed $80kg$. After $30$ days, his weight became $114kg$. You are his Personal Scientific officer. He wants to know how much weight per day he increased. What will you say him?*

**Quick Check 9.** *The water tank above my house has a leak. It has $200liters$ of water, but because of leak, every second, $0.4liter$ water leaks away. I need $20minutes$ to repair the leak. Will I be able to fix the leak before all the water leaks out?*

Now on that average line, we have joined the endpoints of the connected straight line. **The important understanding is that, when you join the endpoints of straight line, that straight line can show the average.** Look at the figure above. The slope of the line that joins the ends of the line is 0.9. That is actually the average speed. Another important learning here,

| |
|---|
| Join the straight lines end, and it will give an average measure. |

We can determine these average slopes either. I will later show a formula that can be used to find average slopes. Till then, join endpoints to get it. The next figure makes the sense.

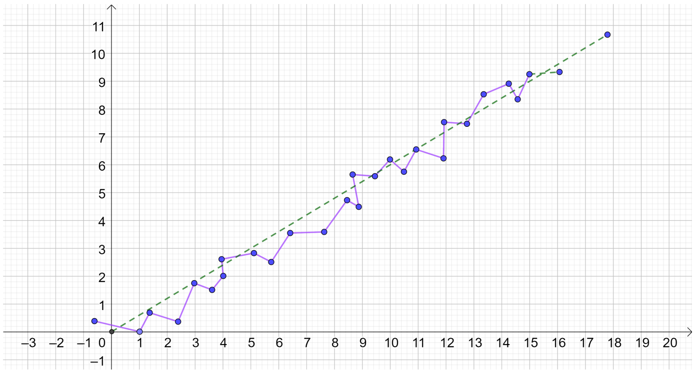


Figure 9: A Whole lot of small straight lines averaged into the Green dotted line. We can find the average slope from this method.

### 2.3.8 Slope Calculation - Quicker

Is there any simpler and faster way to calculate slope?

Look at the figure 2.3.8. You want to find the slopes of each of the lines. In the figure, there

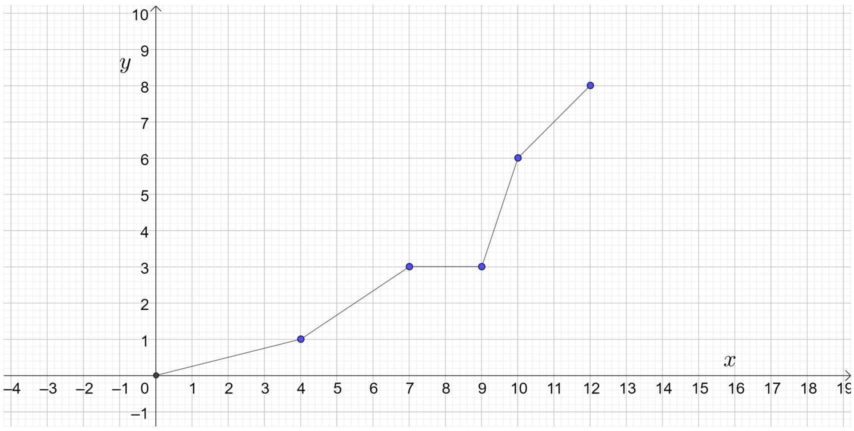


Figure 10: Irregular graph of lines.

are 5 such lines. We saw that we have to make at least two equations to find one slope. For slopes we have to solve ten equations. Daunting.

But in physics, you will later see that if things go too much problematic, then joining the end points and making calculation of the average is more than enough. But, we already know that finding slope is an important thing. We should find a method to calculate it faster. Here we can show the method. Our straight line is well known.

$$y = mx + c$$

Let, for a value $x_1$, $y$ take the value $y_1$. That is,

$$y_1 = mx_1 + c$$

For the same line, let another value of $x$ be $x_2$. Then,

$$y_2 = mx_2 + c$$

Subtract the two equation,

$$y_2 - y_1 = mx_2 - mx_1 + c - c$$

This gives us,

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{4}$$

What is this? See the next figure. We have taken a line. Here, randomly chosen, let

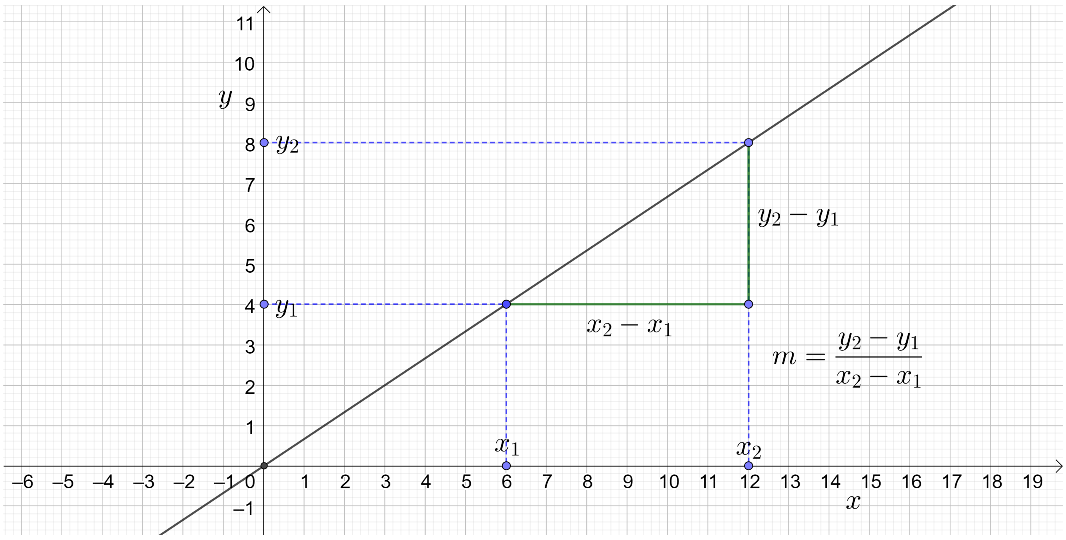



Figure 11: This figure is one of the most important one given in the whole Booklet

$x_2 = 12$, $x_1 = 6$. This shows that correspondingly $y_2 = 8$, $y_1 = 4$. According to the equation 4, $m = \frac{y_2 - y_1}{x_2 - x_1}$. Thus, this case,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-4}{12-6} = \frac{2}{3} = 0.666$$

The slope has easily been found. You can still use previous methods, but eventually you will find this actually is the same one.

$$y = mx + c \quad 0 = m(0) + c$$

So, $c = 0$.

$$8 = 12m + 0 \quad m = \frac{8}{12} = \frac{2}{3} = 0.666$$

So, $m = 0.666$. Now look at the figure. We have actually built a triangle whose two sides are $y_2 - y_2$ and $x_2 - x_1$. Carefully noticing $y_2 - y_1$ is actually the measure **how much the line has RISEN/STOOD UP moving from $x_1$ to $x_2$.** The **Rate of increment of height per unit increase along horizontal (x axis) is slope.**
Saying in another language, slope is just the division of the height length and base length. To be more specific, **ratio** of height and base. Mathematically,

$$m = height : base = \frac{height}{base} = \frac{\text{Rise along y}}{\text{Running along x}}$$

If you feel confused, don't worry, ignore this paragraph and re-read the last example and try to do the next Quick Check. Return when you feel confident, because a bit more discussion on this fact is done afterwards.

**Quick Check 10.** *Given figure is a function's straight line. Use the equation 4 to find the slope directly.*

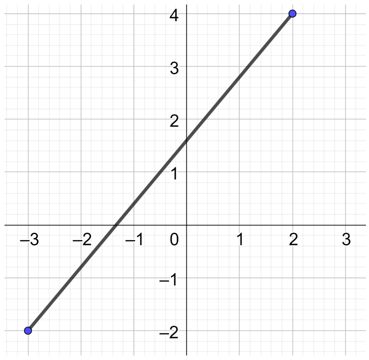



Figure 12: Find the slope for Quick Check 10

Now Physicist come across writing such $t_2 - t_1$ so much that they have found a different system to write this. They have made this thing,

$$\Delta s = s_2 - s_1$$
$$\Delta t = t_2 - t_1$$
$$\Delta y = y_2 - y_1$$
$$\Delta x = x_2 - x_1$$

What I am trying to say is,

$$\Delta\text{Value} = \text{Final Value} - \text{Initial Value}$$

Does it make the sense? Whenever we are concerned of the “Difference” or “Change” of a quantity, we use the sign uppercase delta $\Delta$ before the quantity. I will not waste time saying how important this sign is, you’ll later feel this yourself.

**Example 10. Using $\Delta$ in Physics.**
Suppose you are 10 meters away from me. You move straightly away and now your distance is 24 meter. I will say that your **Change** or **Difference** in distance is $24\,m - 10\,m = 14\,m$. Physicists hate the mathematics numerically, so, he would say your initial (first) distance was $d_1$. Then after moving away your distance is $d_2$. Overall change in position ,or say distance, is final distance - initial distance, $d_2 - d_1$. Physicist, with the special $\Delta$ will say, change in distance is $\Delta d$. This is because $\Delta d = d_2 - d_1$. So, for your case, $\Delta d = 14\,m$.

When you start your motion, time in my wrist watch was $t_1 = 10:24\,am$. When you reached distance $d_2 = 24\,m$, it was $t_2 = 10:28\,am$ in my watch. The time you took to move was $\Delta t = t_2 - t_1 = 10:28 - 10:24 = 4\,min$, so $\Delta t = 4\,min$.

I can say your speed, it is easily total distance traveled/total time taken, so,

$$s = \frac{\Delta d}{\Delta t} \tag{5}$$

Numerically,

$$s = 14\,m/4\,min = 3.5\,m/min$$

As a brief conclusion, you actually walked $\Delta d$ meters in $\Delta t$ minutes. That is 14 meters in 4 minutes.

To be honest, equation 5 formula of speed is more correct.
Now, our slope formula can be written using a $\Delta$.

$$m = \frac{\Delta y}{\Delta x} \tag{6}$$

**Important understanding** is that $\Delta y$ line shall always stand making $90°$ with $\Delta x$.

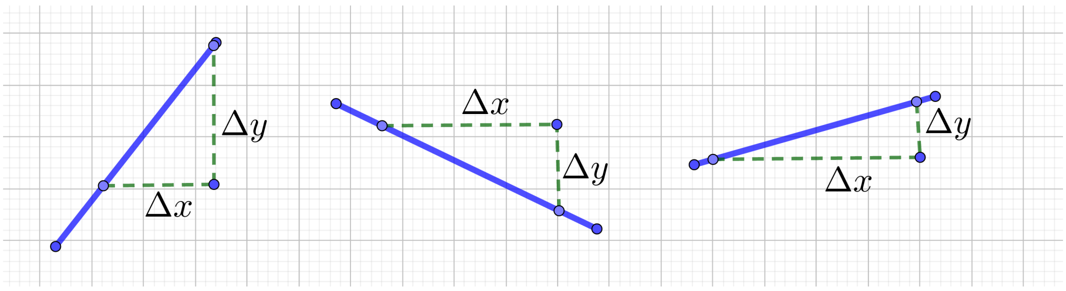



Figure 13: This is how various $m$ can be found, the side length are expressed. Note that when the $\Delta y$ is negative, pointing down, slope is negative too.

Meaning these lines shall always be perpendicular. It is tempting to let you know that many physics laws only require the difference, not actual value. A glorious example is Potential Differences in circuit. Later you will find something called Gauge Theory.

**Quick Check 11.** *In that leaking tank I was fixing, at one point water level came down to* $V_1 = 150$ *liter volume. After a while, it became* $V_2 = 100$ *liter. Express the change using the* $\Delta$.

Physicist also like to use the similar fashion of notation, where they use $i$ and $f$ for telling *initial* and *final* instead of $1, 2$. For example,

$$\Delta p = p_f - p_i$$

That is,

$$\Delta p = p_{initial} - p_{final}$$

### 2.3.9 Graph of functions that are not Straight

Is there functions that look curve and non straight in Graph? How to deal with em'?

Suppose you want to draw the graph of the function $f(x) = x^2$. This cannot exactly be shown in $y = mx + c$ form. Because here, $x$ has a power of 2, that is, $x^2$. Remember the learning of section 2.1, "Drawing in Graph - Putting the Points".

But still we can plot the points given from the numerical values. It turns out that the function doesn't produce a single straight line as we saw before, these functions cannot even be described in $y = mx + c$, so we call these functions as **Non linear functions.** You can think "Non Line-ar Functions", functions that don't produce a "line".

| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 | -4 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y = f(x) = x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 | 16 |

Now in a graph paper, plotting each of the points, This doesn't seem to be part of 1 straight

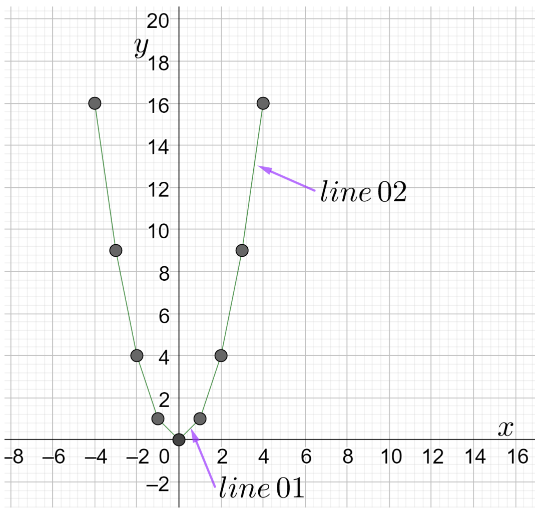



Figure 14: Some points of $f(x) = x^2$

line, it seems that many straight lines make up the functions graph. Note the first line "line 01" that is joining the point $(x, y) = (1, 1)$. This 1 is not seen on the axis, but it is midway between 0 and 2. Then see the "line 02", second line is steeper than first line.

That is, the slope of the second line is more than line 01. You will notice as the $x$ increase, the line becomes steeper and steeper. That means, the slope increases with the line. Slope $m$ is not thus constant.

Let's calculate the slope of line 01. Line 01 joins (0,0) with (1,1). Let $y_2 = 1$, $y_1 = 0$ and $x_2 = 1$, $x_1 = 0$. Then the slope is,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = 1$$

Similarly, let us measure the slope of line 02. Line 02 joins (3,9) with (4,16). Let $y_2 = 16$, $y_1 = 9$ and $x_2 = 4$, $x_1 = 3$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = 7$$

Thus, the slope increases. But what if, I take much more number of points. See, I can also take $x = 3.5$ and in the y axis it will be $y = 12.25$; that shall be plotted as (3.5,12.25). I can also take a point $x = 2.3$ and it shall give $y = 5.29$. Like, so, I can make whatever number of points. Randomly choosing some $x$ values,

| $x$ | 1.1 | 1.8 | 2.3 | 2.7 | 3.3 | 3.7 | 3.5 | 1.4 | 2.5 | 3.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = f(x) = x^2$ | 1.21 | 3.24 | 5.29 | 7.29 | 10.89 | 13.69 | 12.25 | 1.96 | 6.25 | 11.56 | 2.25 |

| $x$ | -1.1 | -1.8 | -2.3 | -2.7 | -3.3 | -3.7 | -3.5 | -1.4 | -2.5 | -3.4 | -1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = f(x) = x^2$ | 1.21 | 3.24 | 5.29 | 7.29 | 10.89 | 13.69 | 12.25 | 1.96 | 6.25 | 11.56 | 2.25 |

This gives us, with also the previous points, There are 31 points all along. Any number of

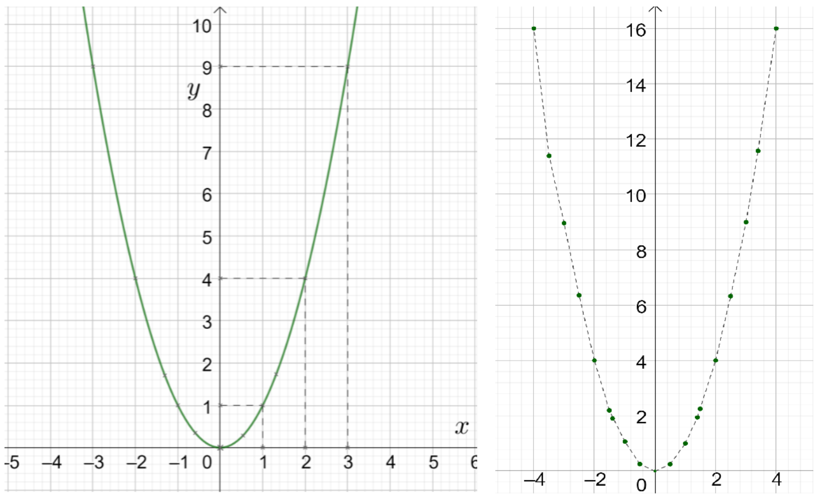



Figure 15: With much more points the graph seems to become more "smooth"

points can be given, and the more we will give, the more smooth it shall become. Adjacent to the above figure, there is somewhat 10 billion points plotted, by a computer. That gives us a curve. Verify that for $x = 2$, $y = 2^2 = 4$ in the curve.

**Quick Check 12.** *Draw the curve of $f(x) = x^2 - 2x$. Draw it roughly.*

Most of the Physics functions make curves in graph. We studied some functions before, in graph paper, they look like the figure 2.3.9. **Major challenge is finding the slopes of the functions.** We already know that slopes are an important quantity. The idea of Tangents can help in such case.

## 2.4 Tangent line - Idea

What is the angle made at a random point of the curve? Every curve can be seen as pieces of thousands of lines joined together. For example, in last section we at first found 10 points to plot in the function of $f(x) = x^2$. A few lines joined the points, among them, one line was line 01 and another line 02 (figure 2.3.9). There we in total only 8 lines in that figure. We increased the number of points and so as increased the number of lines in next part, taking 31 points, giving us almost 18 lines. That started to look smooth.

Taking infinite (extremely large) number of points makes the function become a perfect curve. It doesn't anymore look "pixelated"

The lines that initially made up the function had a slope. Like line 01 had slope 1 and line 02 had slope 7.

**We can zoom into a curve and try to see what line makes that certain part of the curve.**

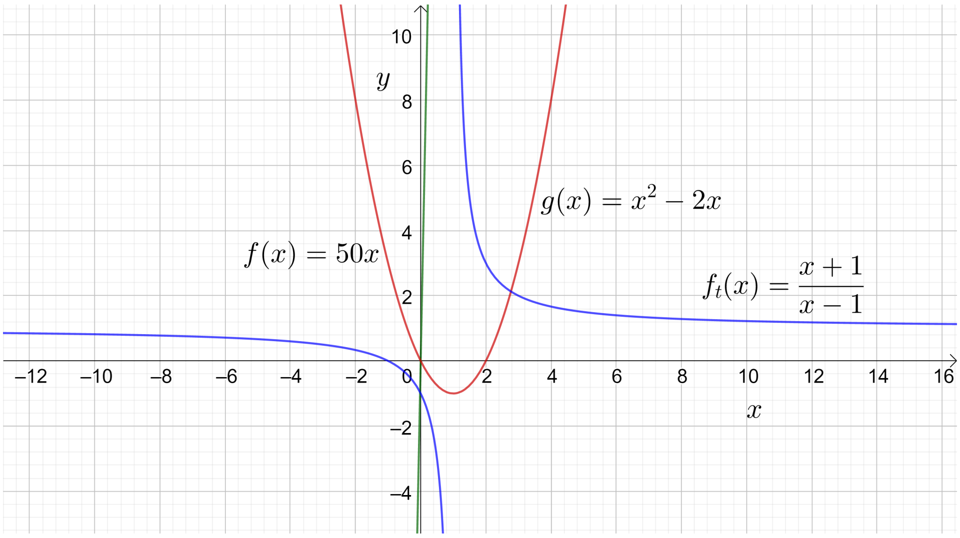



Figure 16: Functions and their Graphs

For now, you should just concentrate on these diagrams below and make a clear understanding what is going on, note all the **bold** lines in Purple color. These are the tangent lines. **The figure gives a mental image. Tangent is an ordinary line that touches**

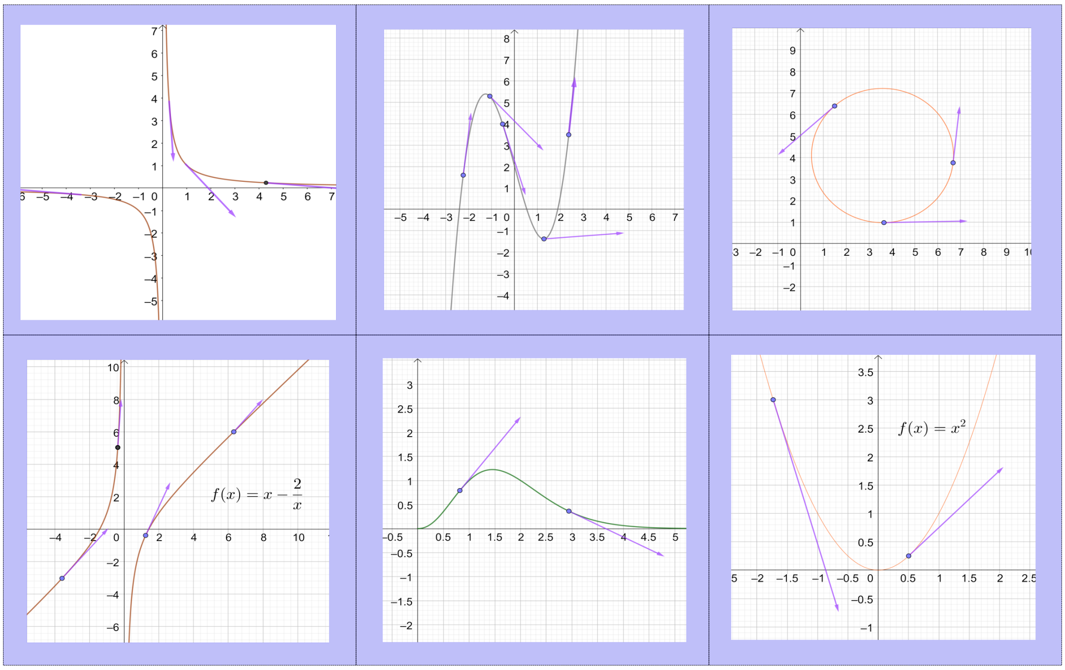



Figure 17: Various Tangents of Functions, in purple color

**a point of the non linear function, or the curve. This line will not cross the curve, will only touch the curve.**

# 3 Simple Calculus - Differential Calculus

How to look for the slope of curves at random points? Is there any method to do that?

To simplify everything, using Derivative, you can find the Slope. And if you know slope, using Integration you can find the function, you can reverse the thing !

## 3.1 The idea of Derivatives

### 3.1.1 The Slope problem with Non Linear Functions

Compare the three functions given. We have dealt with such functions many times, we return back it again.

The first function is easy to understand, the line $y$ increases with the increase of $x$, the thing is linear.

The second function is not a straight line. The graph of second function starts from the left by decreasing rapidly, then begins to decrease more slowly and level off. Then finally begins to increase. Slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

Figure 18: The three functions, one is a simple $y = mx + c$ format line, second one is $y = x^2$ and last one is $y = x^3$ type.

So as with the third function. This seems to rise up from the left part, then at one point it becomes horizontal and after it, the function again starts to rise up rapidly.

**Now we want to move our concentration in finding the Slope.** Remember, slope is rate of change.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For the first function, like the figure 2.3.8, we can find the slope. Here the function is a straight line. But the later two function as in the figure are not a single straight line. So we cannot actually use the formula of $m$ to find the slope. This leads us to the Tangent Problem, and it was calculus that helped us solve this paradox.

One thing is important to note. The non linear functions above show that they don't have a single slope. Because, the second function becomes level in the middle, there, the slope must be zero ($m = 0$ means that the line is horizontal or parallel to $x$ axis). And as we proceed, the slope increases, recall the fact of figure 2.3.9. There, the line 1 and line 2, though being part of the same function, has different slope. One thing is clear,

> The slope in non linear functions are somewhat different for different points

### 3.1.2 Idea of Approximating - Limits

What is the use of Approximation to find answers?

Let us do it using an illustrative example.

> **Example 11.** Raisa wants to calculate the area of a circle that has been drawn on a land. She has a stick to draw lines on the land and only knows that the Radius of the circle is 7 meter. She doesn't believe on $\pi$, so she would not use the formula of circle area $A = \pi r^2$.
> Though Raisa seem to have no want to use the formula, she draws some boxes and counts the number of boxes and surprisingly gets extremely well numerical value of area. How ?

Here, the well known idea of Limits has been applied by Raisa. But we idea of limits is almost of no use in Physics, Physics has it's own meaning on it. We shall illustrate her method in the diagram below. The idea is simple, for the first grid of box, we shall draw a grid of lines

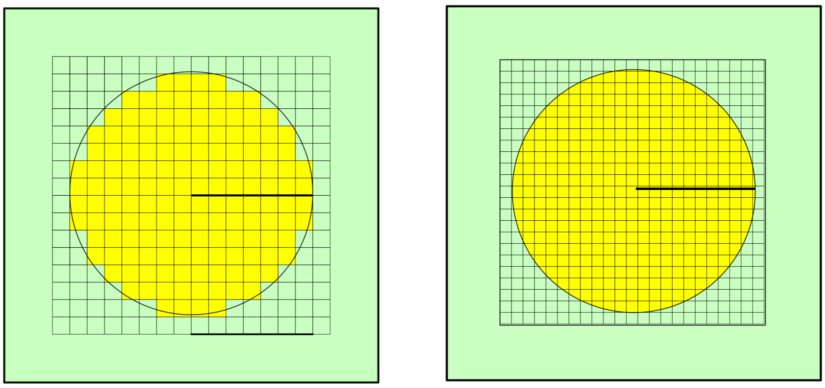


Figure 19: The circle drawn into a grid of boxes. The second one has a few hundreds of boxes more than the first one.

that make squares, each with side 1 meter. The area of each of the box is $1 \times 1\,m^2 = 1\,m^2$. If we count all the boxes inside the circle, then we easily get the approximate area!

There are in total 148 boxes inside the circle. Some part of the box sticks out, but we ignore it. Area of 1 box is $1\,m^2$, hence, approximate area of the circle is Area of circle = Number of boxes inside $\times$ Area of a single box $= 148\,m^2$. Using $A = \pi r^2$, we get $A = 3.14 \times 7^2 = 153.94\,m^2$. Quite the same.

> **Example 12.** How can she make her numerical value of Area more near to the exact Area?

Looking at the figure with smaller boxes make the sense. The smaller boxes we keep, the lesser margin of error we have. What shall happen if we make the number of boxes inside nearly infinite? We will then have the exact answer of Area. But this requires Raisa to draw billions of boxes. Though we can easily do it using a computer program.

I have used Vector graphics, so it might not be problematic for you to zoom in the figure above and count the small boxes. Before that keep in mind a fundamental learning,

Many things that are hard to calculate can be approximated into something small. Like we did in example 11 and 10. The smaller we are going to go, the better answer we will have.

### 3.1.3 Idea of Approximating - Numbers

Suppose, you were doing a calculation and in the calculator it came,

$$1.23654762$$

You probably don't need to use so many numbers after the decimal, what you do is,

$$1.23654762 \approx 1.24$$

Here the sign $\approx$ is called approximately equal. While doing math, we do such things.

But there are some cases when we need to use the next level of approximation. Before that remember,

In Physics, during calculation, some "too much small than required" quantities are ignored. For now, also remember that Power of small numbers are smaller. For example a small numbers square is very small.

This will later be elaborated.

Let us have a small number, say 0.001. What is the square of this small number? $0.001^2 = 0.000001$. You can also do this in this professional manner,

$$0.001^2 = \left(\frac{1}{1000}\right)^2 = \frac{1}{1000000} = 0.000001$$

The square of small number is smaller. Because we consider 1.23654762 as 1.24, ignoring whatever number is there after two digits after the decimal, we can just assume,

$$0.000001 \approx 0$$

How can we use it? See the next example.

**Example 13.** Evaluate $(23.003)^2$.
**Solution:** We can simply use a calculator, and get,

$$23.003^2 = 529.138009$$

In Physics, we don't need to write the whole thing. This motivates the next example.

**Example 14.** Evaluate $(23.003)^2$, but now like a Physicist.
**Solution:** Like a Physicist, we will say that,

$$23.003^2 \approx 529.14$$

Now, Physicist's are clever. So they use another method to do the same thing. This is demonstrated below.

We can easily see that there is a kind of relation in 23.003.

$$23.003 = 23 + 0.003$$

We have to find the square,

$$(23.003)^2 = (23 + 0.003)^2$$

We remember that,

$$(a + b)^2 = a^2 + 2ab + b^2$$

Hence,

$$(23 + 0.003)^2 = 23^2 + 2 \times (23 \times 0.003) + 0.003^2$$

We are okay with this till here. But see that $0.003^2$ term, this equals 0.000009, too too much small. So we shall assume 0.000009 is approximately 0. Thus,

$$0.000009 \approx 0$$

But you should be careful, don't use equal sign $=$ in place of approximate sign $\approx$!

$$(23 + 0.003)^2 = 23^2 + 2 \times (23 \times 0.003) + 0.003^2$$

$$(23 + 0.003)^2 \approx 23^2 + 2 \times (23 \times 0.003) + 0$$

$$(23 + 0.003)^2 \approx 529 + 0.138$$

So finally,

$$23.003^2 = (23 + 0.003)^2 \approx 529.138 \approx 529.14$$

This also leads us to a new kind of formula,

$$(a + b)^2 = a^2 + 2ab \qquad \text{(If b is too much small)} \tag{7}$$

In Physics texts, sometimes you will find $\approx$ written as . And very small numbers are denoted by $b << 1$, which means $b$ is too much small than 1.

**Quick Check 13.** *Use the above equation 7 to evaluate these.*

$$3324.0001^2 \quad 230.002^2 \quad 9.0008$$

Usually in Physics you are required to keep things okay until 2 decimal places. Like, 9.872 can be written as 9.87. But you should not write 9.872 as 9.9. Keep things alright until 2 decimal places. Next example is unimportant, so you can avoid it, but it shall aware you about unwanted confusion.

**Example 15.** Evaluate $9.8755^2$
**Solution:**

$$9.8755 = 9.87 + 0.0055$$

Because, $0.0055^2 = 0.00003025 \approx 0$,

$$9.8755^2 = (9.87 + 0.0055)^2 \approx 9.87^2 + 2(9.87 \times 0.0055)$$

$$\approx 97.4169 + 0.10857 \approx 97.56$$

Another extremely important approximation can be derived from the binomial theorem (class 9 folks), avoid the next equation,

$$(x + y)^n = \sum_{i=0}^{n} nC_i \, x^{n-i} \, y^i \tag{8}$$

We can write it's simple form, note the next part carefully. Let us try to approximate $(a+b)^n$. Before, we tried to understand what happened when $n=2$, that is, $(a+b)^2$. Now we reduce it for every kind of power $n$.

$$\begin{aligned}(a+b)^n &= \left(a\left(1+\frac{b}{a}\right)\right)^n \\ &= a^n\left(1+\frac{b}{a}\right)^n\end{aligned}$$

Now with the help of Binomial theorem, if it is that $a$ is extremely big than $b$, which is, $a >> b$, then

$$(a+b)^n \approx a^n \times \left(1+n\frac{b}{a}\right) \qquad \text{(Binomial Approximation)} \tag{9}$$

You might wonder what is this. Look at the reduction if $n=2$, and $a >> b$, $a$ is extremely big than $b$.

$$(a+b)^2 \approx a^2\left(1+2\frac{b}{a}\right) = a^2 + 2\frac{a^2}{a}b$$

$$(a+b)^2 \approx a^2 + 2ab$$

So, this works. We will use Binomial Approximation to learn differential calculus.

**Example 16.** Evaluate $345.002^4$ like a Physicist.
**Solution:** As we can see, $345.002 = 345 + 0.002$, from Binomial Approximation formula, $a = 345.002$, $b = 0.002$, $n = 4$.

$$(a+b)^n \approx a^n \times \left(1+n\frac{b}{a}\right)$$

$$(345+0.002)^4 \approx 345^4 \times \left(1+4\frac{0.002}{345}\right) \approx 14200000000 = 1.42 \times 10^{10}$$

**Example 17.** Evaluate $345.002^{1/2}$ like a Physicist.
**Solution:** We know all about it.

$$(345+0.002)^{1/2} \approx 345^4 \times \left(1+1/2\frac{0.002}{345}\right) \approx 18.57$$

### 3.1.4 Secant Line - An Approximate method to calculate slopes of Non Linear Function

Why is a Calculus introductory notebook babbling approximations?

We are at the beginning of the idea of a Derivative. We know that the slope of a non linear function is quite difficult. So, we will use our Approximation techniques to make it possible.

**Now, we will try to approximate one point with a another point on the same function and try to build a line and calculate it's slope. This shall approximately depict a slope for the point of the curve. This line will be called the *Secant line.***

Before, we have to make slight modifications in our formula of slope. We have learnt, $y = f(x)$. So, if we say that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

**Attempt 01:**
Let us draw a nice function, $y = f(x) = 0.15x^3$. We can take any kind of function as we

like, I've chosen this one because this is easy to visualize in computer. Please keep looking to the diagram.

Let us try to approximately find the slope at $x_1 = 1$ in the graph. Randomly choosing $x_2 = 2.5$. See the figure. Rather than using $x_2$, let us concentrate on $\Delta x$. Because,

$$\Delta x = x_2 - x_1 \qquad \rightarrow \qquad x_2 = x_1 + \Delta x$$

Here, $\Delta x$ is the separation between the two points. With $x_1 = 1$ and $x_2 = 2.5$, $y_1 = f(x_1) = f(1) = 0.15(1)^3 = 0.15$. Similarly, $y_2 = f(x_2) = f(2.5) = 0.15(2.5)^3 = 2.34$.

$$x_1 = 1 \qquad x_2 = 2.5 \qquad \Delta x = 1.5$$

$$f(x_1) = 0.15 \qquad f(x_2) = 2.34$$

The red line here is a secant line, we calculate it's slope, approximately.

$$m = \frac{2.34 - 0.15}{2.5 - 1} = 1.46$$

**Attempt 02:**
Let us take $x_2$ closer to $x_1$. I mean reduce the separation, that is $\Delta x$. Now let $\Delta x = 1$. This gives,

$$x_1 = 1 \qquad x_2 = 2 \qquad \Delta x = 1$$

$$f(x_1) = 0.15 \qquad f(x_2) = 1.2$$

Same as before finding the slope.

$$m = \frac{1.05}{1} = 1.05$$

**Attempt 03:**
We again reduce the $\Delta x$. This time, $x_1 = 1$ and $x_2 = 1.1$. This means that the point in the curves will be more close to each other.

$$m = \frac{0.15(1.1)^3 - 0.15(1)^3}{0.1}$$
$$m = 0.49$$

**Attempt 04:**
We again reduce the $\Delta x$. This time, $x_1 = 1$ and $x_2 = 1.01$. This means that the point in the curves will be more close to each other.

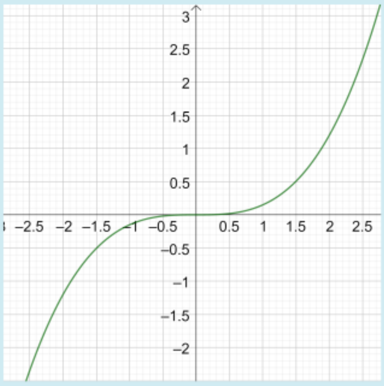
$$m = \frac{0.15(1.01)^3 - 0.15(1)^3}{0.01}$$
$$m = 0.45$$

The calculation is not important. Important thing is that zooming the function at point $x = 1$ makes the function as if it is a straight line. And straight line means that we can calculate the slope. The requirement is that $\Delta x$ of the secant line should be small. We should tabulate our answers.

| | |
|---|---|
| $\Delta x = 1.5$ | $m = 1.46$ |
| $\Delta x = 1$ | $m = 1.05$ |
| $\Delta x = 0.1$ | $m = 0.49$ |
| $\Delta x = 0.01$ | $m = 0.45$ |

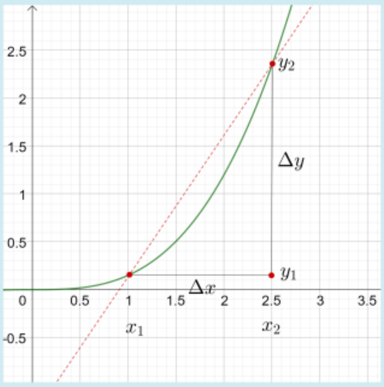
So, we have finally a solution! If we can bring the two points as shown in the next figure as close as possible, then the curve somewhat becomes a straight line between the points. Again see the figure, the last one specially, the curve seems to become a straight line. We can go infinitely small and calculate the slope of that particular small line, this should be the slope of the curve at that point!

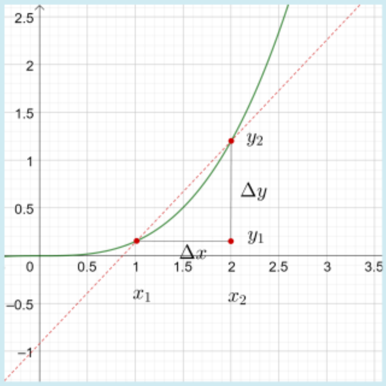
If you extend the small line, make it longer, then this line should Touch the curve, but will not cross it. This special line that touches a curve, but never crosses it (geometrically speaking, never intersects), then that line is the Tangent line. Remember the figures from that section named Tangent?

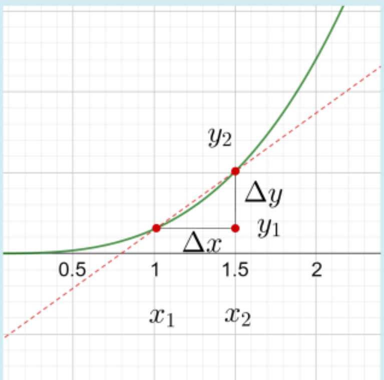


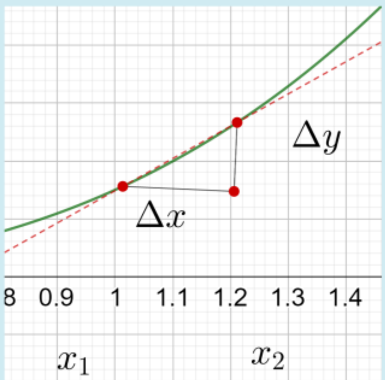

The function is same. We are changing the separation between the points. Note that when the separation is low, the secant line approximately becomes the part of function at x = 1.

Going infinitely small, we can understand the slope at x = 1.



## 3.2 Derivative

ARE WE JUST STARTING CALCULUS? Again writing the formula for slope,

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Because,

$$\Delta x = x_2 - x_1$$

Now, we have seen that, if we can reduce the separation between the points $x_2$ and $x_1$, then the secant line can give better results of the slope for the curved function. Which is same as saying "reduce $\Delta x$ as much as possible".

You might wonder $\Delta x$ might be written as 0.001. Or you can even go crazy and do 0.000000001. In fact, Newton went to impossible, so many zeros that it might be literally impossible to write here.

$$\Delta x = 0.0000000000000000000000000......0000000000000000000000000000001$$

Physicist's don't have so much time to write zeros, they just say $\Delta x \to 0$. This means, $\Delta x$ goes near to zero. Simple.

What if we use our knowledge on approximations and use it to make calculation of slope for a curve?

$$m = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{10}$$

We require $\Delta x \to 0$ in the last equation. Mathematicians use this method to write it,

$$m = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{11}$$

The "lim" means that the $\Delta x$ should become as small as 0 in the equation. Commonsense.

We have seen how to deal with small numbers. The last equation above can give the slope at the particular point $x_1$. We can choose any point as $x_1$, so just remove the 1 from from below the $x_1$. We shall call $x_1$ as $x$.

$$m = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{12}$$

How can we use this formula? Let us again try to find the slope of the last section of Secant lines. We analyzed $f(x) = 0.15x^3$. What will the slope be at $x = 1$? Let us put this function in the equation above. Remind, $f(x + \Delta x) = 0.15(x + \Delta x)^3$ for the function $f(x) = 0.15x^3$.

$$m = \lim_{\Delta x \to 0} \frac{0.15(x + \Delta x)^3 - 0.15x^3}{\Delta}$$

How will $0.15(x + \Delta x)^3$ react for $\Delta x \to 0$; which means smaller $\Delta x$. Remember we saw $(a+b)^n \approx a^n \left(1 + n\frac{b}{a}\right)$ for $b << 1$. Here $a$ shall be $x$ and $b$ shall be $\Delta x$, as it is small. Then,

$$0.15(x + \Delta x)^3 \approx 0.15 \times x^3 \left(1 + 3\frac{\Delta x}{x}\right) = 0.15 \left(x^3 + 3\frac{x^3 \Delta x}{x}\right)$$

Doing the math a little more gives that,

$$0.15(x + \Delta x)^3 \approx 0.15x^3 + 0.45x^2 \Delta x$$

Now we can place this thing into our slope formula. We will not write the $\lim_{\Delta x \to 0}$, anymore, as we already have approximated the function for small $\Delta x$

$$\begin{aligned} m &= \frac{0.15(x+\Delta x)^3 - 0.15x^3}{\Delta x} \\ &= \frac{0.15x^3 + 0.45x^2\Delta x - 0.15x^3}{\Delta x} \\ &= \frac{0.45x^2\Delta x}{\Delta x} \\ &= 0.45x^2 \end{aligned} \tag{13}$$

It seems that we have a formula. We wanted to find the slope at $x = 1$.

$$m = 0.45(1)^2 = 0.45$$

You should be happy to know we have exactly found the slope at point $x = 1$ for the function $f(x) = 0.15x^3$. But how can we know that this is actually correct. We are ready to make a formula for any kind of function. Like, above we worked for $f(x) = 0.15x^3$, this method can also be applied for $f(x) = 232x^5$,$f(x) = x^2$,$f(x) = 34x^3$, anything.

We can assume a type of function in the form $f(x) = \alpha x^n$. For $f(x) = 0.15x^3$, $\alpha = 0.15$ and $n = 3$.

If $f(x) = \alpha x^n$, then $f(x + \Delta x) = \alpha(x + \Delta x)^n$. Using approximation formula,

$$\alpha(x+\Delta x)^n \approx \alpha\left(x^n\left(1 + n\frac{\Delta x}{x}\right)\right)$$

Reduced into the form,

$$\alpha(x+\Delta x)^n \approx \alpha x^n + n\alpha\frac{x^n\Delta x}{x}$$

$$\begin{aligned} m &= \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{\alpha x^n + n\alpha\frac{x^n\Delta x}{x} - \alpha x^n}{\Delta x} \\ &= \frac{n\alpha\frac{x^n\Delta x}{x}}{\Delta x} \\ &= \frac{n\alpha x^n}{x} \end{aligned} \tag{14}$$

We know from Elementary algebra that $x^3/x^2 = x = x^1$ which is same as $x^3/x^2 = x^{3-2} = x^1 = x$, and $x^4/x^2 = x^2 = x^{4-2}$, that means that for a power $n$, $x^n/x = x^n/x^1 = x^{n-1}$. So,

$$m = n\alpha x^{n-1} \tag{15}$$

This is much more easy to work with. Verifying, for $f(x) = 0.15x^3$,

$$m = 3 \times 0.15x^{3-1} = 0.45x^2$$

For $x = 1$, $m = 0.45$.

Congrats to take the First Derivative (unless you're just revising).

### 3.2.1 Accepting Slope as Derivative

How is a slope and how is a derivative different? From now on we shall not utter the word "Slope", replace the word slope as "Derivative". We will not find the "Slope" of any function, we will find the "Derivative". Now there is a reason why we do that. This requires us to recall our past experiances,

$$m = \frac{\Delta y}{\Delta x}$$

Where the $y = f(x)$, thus,

$$m = \frac{\Delta f(x)}{\Delta x}$$

But we know that when dealing with Non Linear Functions, the slopes can only be found for a single point. And also if $\Delta x$ is extremely small, $\Delta x \to 0$, $\Delta x$ almost becomes zero, but not exactly zero. We write it as,

$$m = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} \tag{16}$$

But Physicist's don't have so much time to write $lim_{\Delta x \to 0}$, because in Physics, almost everytime you have to look for Rate of Change (slope, unless you've forgotten). It was Leibniz who thought that we can replace $\Delta$ symbol with a $d$, when separations are small ($\Delta x \to 0$).

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx} \tag{17}$$

This is not fair that you cut $d$ in both part of fraction like ordinary algebra, it is a symbol that tells that we have small $\Delta x$ in the slope formula. Physicist are more lazy to be honest, you will also find them writing,

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx} = \frac{df}{dx}$$

This reduction makes sense, because anybody knows that in $\frac{df(x)}{dx}$ the upper part must be the function and the lower part must be the variable.

As a recap, we also knew slope formula in graph, in new symbols.

$$m = \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \qquad \text{(If the separation small)} \tag{18}$$

Showing how to use this, let us try to make sense of these with the $f(x) = \alpha x^n$.

$$m = \frac{df(x)}{dx} = \frac{d}{dx}\alpha x^n = n\alpha x^{n-1} \tag{19}$$

Using this, we can find the derivative of the function $f(x) = 0.15x^3$. Speaking same, find slope.

$$m = \frac{df(x)}{dx} = \frac{d}{dx}0.15x^3 = 0.45x^2 \tag{20}$$

We will not write $m$ from now on, slope $m$ is replaced by derivative $\frac{dy}{dx}$.

**Example 18.** Find the derivative of the following functions at the points $x = 3$.

$$y = x^2 \qquad y = \frac{3}{4}x^3 \qquad y = 3x^4 \times \frac{1}{x^2} \qquad y = 9x$$

**Solution:** Now we are said to find the derivative at the point $x = 3$ for these given functions. This is same as saying find the slope for these non linear functions at the point $x = 3$. The required formula has been found by us.

$$m = \frac{df(x)}{dx} = \frac{dy}{dx} \qquad y = \alpha x^n \qquad \frac{dy}{dx} = n\alpha x^{n-1} \tag{21}$$

So,

$$y = x^2 \qquad \frac{dy}{dx} = 2x$$
$$y = \frac{3}{4}x^3 \qquad \frac{dy}{dx} = \frac{9}{4}x^2$$
$$y = 3x^2 \qquad \frac{dy}{dx} = 6x$$
$$y = 9x \qquad \frac{dy}{dx} = 9$$

There is one thing that need reflection. The function $y = f(x) = 9x$ tells that function is in the form of $y = mx+c$. On this analysis, the $c = 0$ and $m = 9$, but $m$ is slope, again we have already learnt that the $m$ is similar to $\frac{dy}{dx}$. This makes sense right here, slope $m = 9$ and also the derivative $\frac{dy}{dx} = 9$.
Now we can just put the position of the point in our found derivatives. For $x = 3$

$$y = 2x = 6$$
$$y = \frac{9}{4}x^2 = \frac{81}{4}$$
$$y = 6x = 18$$
$$y = 9 = 9$$

A linear function has same derivative everywhere. Because, it has same slope at any point, that is, it has the same steepness everywhere.

There is one interesting point to notice. We know that the slope for linear function is same everywhere. So it has the same value of $dy/dx$ everywhere. Thus, we can make the new form of linear functions,

$$y = mx + c$$

Turned down to,

$$y = \frac{dy}{dx}x + c \tag{22}$$

Interesting enough.

**Quick Check 14.** *Find the derivatives of the following functions. With practice you can even do this in your mind.*

$$y = \frac{x^2}{3x} \qquad y = \frac{9x^2}{8} \qquad y = 3x \times 4x$$

If you find that you have easily done these problems, then you are at the stage of doing more. Some reference books on calculus will be given, I request you to have them from the net or buy them and try to do some more same problems. Because this topic here, you are going to do all through your life (yes, if you choose Physics to be the objective, otherwise Good bye).

And if it happens you still aren't confident, no problem. Keep going on with this booklet.

### 3.2.2 How can I SEE a Derivative?

Is there any physical or mathematical importance of derivative?

Let us again return to a function. I will again choose the $f(x) = 0.15x^3$ as you are somewhat familiar with it. We will now find the derivative of this function at $x = 1$.

$$\frac{d}{dx}0.15x^3 = 0.45x^2$$

For $x = 1$,

$$0.45x^2 = 0.45$$

So the slope (or derivative) at $x = 1$ is 0.45.

Let also imagine a line with the same slope (I again mean Derivative) $m = 0.45$. This line should be at the form $y = mx + c$. The constant $c$ is not yet determined, but we should remember that the line constant determines the position of the line. Our linear function is thus,

$$y = 0.45x + c$$

In my computer system, I can plot the graph and set various values for $c$ to see which one works.

Let me put $c = 0$ and there is nothing special in it. It is just a line. Let us put the $c = -0.3$.

$$y = 0.45x - 0.3$$

Now a there is a relation, you can perfectly see that the line is touching the point $x = 1$ of the curve without crossing it. A clear and perfect definition of Tangent. You can also imagine that the point of the curve has been enlarged and made longer.

Why did we choose $c = -0.3$ ? We tried to look for the derivative of the function $f(x) = 0.15x^3$ at the place, $x = 1$. The point of the curve is located $y = f(1) = 0.15$ height for $x = 1$.

We found the derivative, which is the slope at the point $(x, f(x)) = (1, 0.15)$. If we are able to draw another line, that has same slope, and one of the points of the line is located at $(1, 0.15)$. So, the new straight line function that we are trying to find, should have a point at $(1, 0.15) = (x, y)$. Hence,

$$y = 0.45x + c$$

As we said,

$$0.15 = 0.45 \times 1 + c$$

That easily gives,

$$c = 0.15 - 0.45 \qquad c = -0.3$$

This special line $y = 0.15x - 0.3$ is tangent to the function $f(x) = 0.15x^3$ at $x = 1$.

### 3.2.3 Physically thinking derivative

Is there Derivative in real life?

Let us use logic, recalling previous learning.

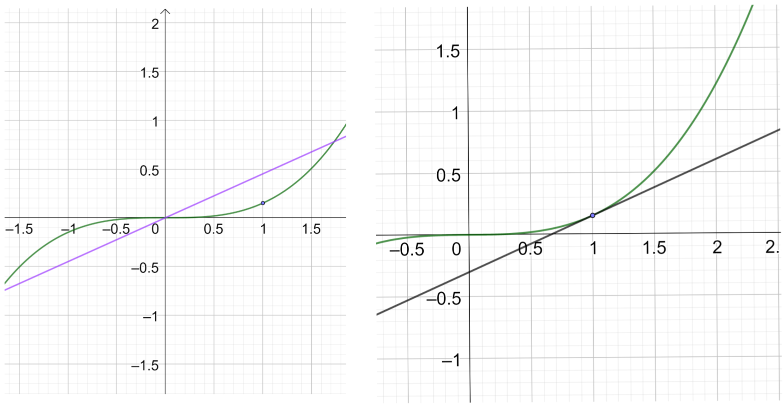
$$\text{Rate of Change of a function} = \text{ Slope}$$

$$\text{Slope} = \text{Derivative}$$

$$\text{Rate of Change of a function} = \text{Derivative}$$

**Rate of Change of almost anything is Derivative.**

Match the above sentences with me.




$y = 0.15x$ $\qquad$ $y = 0.15x - 0.3$

Figure 20: The two straight lines that make up the derivative.

- The Rate of Change of Distance of a moving body with time is Speed.
- The Rate of Change of Speed of a moving body with time is Acceleration.
- The Distance traveled per unit time (1 second) is Speed. *Also said, the distance traveled in 1 second is Speed.*
- The Change in Speed in unit time is Acceleration. *Also said, the speed changed every 1 second is Acceleration.*
- The amount of Energy used every 1 second is Power (measured in Watts).
- Electric Current is Charge flown per unit time through a circuit.
- Amount of Money extra got from bank per 1 year is profit.

Now because you know calculus, you can sound cool in a party saying,

- Derivative of Distance with respect to Time is Speed.

$$s = \frac{dd}{dt}$$

- Derivative of Speed with respect to Time is Acceleration.

$$a = \frac{ds}{dt}$$

- Derivative of Energy with respect to Time is Power.

$$P = \frac{dE}{dt}$$

- Derivative of Electric Charge through a circuit with respect to time is Electric Charge.

$$I = \frac{dQ}{dt}$$

- Derivative of Money with respect to Time is Profit.

Sorry, we are not doing Economics

## 3.3 The Derivative of Distance - Physics

Now you can forget the ideas of slopes and graphs for a minute. Let us exactly think what a Derivative can offer you. Let us think an example.

Suppose a particle is moving in such a way that its speed is not same. It sometimes changes and it occurs in a straight line along the $x$ axis. It travels different $\Delta x$ distance is same time interval $\Delta t$.

Thus we have to say that this particle goes through a Variable Speed. We seek out to find its speed at just a moment instantly. Like, answering the question, " What is the speed of the particle at time $t$ moment?"

To solve this, we come to the definition of speed.

$$s = \frac{\Delta x}{\Delta t}$$

But what does a moment mean? A moment means that the time interval is really short. So short that it almost becomes instantaneous. We have seen how to define it mathematically.

$$s = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

This definition is okay, but how can we use it? Described in the following example.

**Example 19.** A particle moves so that its distance from a point is given by $x = 2t^2\,m$. This means that, for example, at $t = 2\,sec$, the particle is located $x = 2(2)^2\,m = 8\,m$ away. At $t = 0$, the particle is at the point, so distance is 0. Find the velocity of the particle at $t = 4\,sec$.
**Solution:** We know, derivative of distance with time is velocity. So,

$$s = \frac{dx}{dt} = \frac{d}{dt}2t^2$$

As we know, that

$$\frac{d}{dt}\alpha t^n = \alpha n t^{n-1}$$

So,

$$\frac{d}{dt}2t^2 = 2 \times 2t^{2-1} = 4t$$

Thus we know that the speed is,

$$s = 4t$$

You can notice that the speed is a function of time. At $t = 4$,

$$s = 4(4) = 16$$

The units are $m/sec$, so speed at 4 second is $s = 16\,m/sec$.

Now, you can yourself verify the fact recalling graphs. If you put $t$ at the x axis and $x$ (distance) at y axis, you can draw the function $x = 2t^2$. Now, you can move to the place in the graph where $t = 4$, it is the moment when the particle is located $32\,m$ away.

Draw the tangent line, the line that touches the function at $t = 4$ but never crosses it. As it came that speed is 16, the slope of the function at $t = 4$ will also be 16.

**Example 20.** Again, the water tank of my home is leaking. This time the water leaks in a different fashion. The volume of water decreases as a function of time. At time $t$, the volume of water contained by the tank is $100 - \sigma t^{3.2}$.
Find the function of water coming out as a function of time.
**Solution:** The amount of water leaking out per unit time is equal to the rate of change of water level of the Tank. Here, the water level decreases in the tank, as water exits.
We just need to find a derivative of water level to know the rate of change. Let the rate of change be $e$. Then,

$$e = \frac{dV}{dt} = \frac{d}{dt}100 - \sigma t^{3.2} = \frac{d}{dt}100 + \frac{d}{dt}(-\sigma)t^{3.2}$$

What is the derivative of 100 ? 100 is a number, it is a constant, so it doesn't change. So, its derivative is 0.

$$e = 0 - 3.2\sigma t^{3.2-1}$$
$$e = -3.2\sigma t^{2.2}$$

$e = -3.2\sigma t^{2.2}$ is the answer. Note that the answer is negative, which is okay, because water is reducing in the tank.

### 3.4 Differenciation Techniques - 01

Now I can write some formula that makes complete sense. YOU SHOULD NEVER MEMORIZE THESE. Return to this page later while doing maths and see the formula when needed. This is not a NCTB Textbook or School stuff.

Let $u$ and $v$ be separately be a function of $x$. Let $c$ be a number that is constant and NOT a function of anything. Then, some rules apply.

- $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
- $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$
- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}(c+u) = \frac{du}{dx}$

Some more formula will be there later. For example, $\frac{d}{dx}u^v$, $\frac{d}{dg}f(g(x)) = \frac{df}{dx}\frac{dx}{dg}$ and more complicated ones.

## 4 Simple Calculus - Integral Calculus

### 4.1 Summation

How can we write mathematical addition series in a short form?

Suppose we have a series of numbers that are added.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + ... + 20$$

Now if I ask "What is the 5th number?", then by the series we can easily say that it will be 5. "What is the 19th number?", because this series is increasing in a simple manner, we can say that the 19th number will be 19. Simply, any n-th number above is $n$.

We can do the same for a series of even numbers, like,

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + ... + 40$$

The 1st number is 2, the second number is 4, the third number is 6. So, here we find a pattern. The n-th number in the above series is $2n$. Like, the 9th number above is 18.

Now we can also do it with odd numbers.

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + ... + 31$$

The 1st number is 1, the 2nd number is 3, the third number is 5. We again find a pattern, the n-th number is $2n - 1$. Which is one less than the twice of $n$. Like, the 2nd numbers value is $2(2) - 1 = 3$, and samely, the 6th number will be $2(6) - 1 = 11$.

Using a calculator you can find the sum of the series. Like,

$$1 + 2 + 3 + 4 + 5 + 6 + .... + 97 + 98 + 99 + 100 = 5050$$

But this way writing the series is tough, and in calculus, we have to add such series later. Thus, to make this become simple, mathematicians use a uppercase Sigma $\Sigma$,

$$\sum$$

There is a rule how to use this.

In the series,

$$1 + 2 + 3 + 4 + 5 + 6 + .... + 97 + 98 + 99 + 100 = 5050$$

any n-th number is n. Like, the 5th number is 5. In the Sigma $\Sigma$ notation, we have to write this series as,

$$\sum_{n=1}^{n=100} n = 5050$$

Below sigma $\Sigma$, we write the lower limit: as we start the series from 1st number. Above we write the upper limit: we write 100 because our series ends at 100th number there. Then, the n after the sigma $\Sigma$ denotes the general formula for the n-th value.

For the series,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + ... + 20$$

We have to write,

$$\sum_{n=1}^{n=20} n$$

For the series,

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + ... + 40$$

We have the general formula for any n-th term as $2n$, so,

$$\sum_{n=1}^{n=20} 2n$$

We give the lower limit $n = 1$ because our 1st number in the series is $2(1) = 2$. Like so, we give the upper limit from the very fact that the last number is 40, which is the 20-th number. For the series,

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + ... + 31$$

We write

$$\sum_{n=1}^{n=16} 2n - 1$$

As a summary, we have another example,

$$3 + 6 + 9 + 12 + 15 + 18 + ... + 27 + 30 = \sum_{n=1}^{n=10} 3n$$

You can verify it yourself.

An algebraic sum is also possible.

$$\sum_{i=0}^{i=5} ar^i = a + ar + ar^2 + ar^3 + ar^4 + ar^5$$

$$\sum_{i=3}^{i=7} i^i = 3^3 + 4^4 + 5^5 + 6^6 + 7^7$$

$$\sum_{k=1}^{k=4} \frac{c}{k} = c + \frac{c}{2} + \frac{c}{3} + \frac{c}{4}$$

$$\sum_{c=1}^{c=4} \frac{c}{k} = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k}$$

We can do many more examples, but you don't need to seriously do summation right now. This just helps to make things understand quickly.

**Quick Check 15.** *What the series?*

$$\sum_{n=0}^{n=3} (x + y)^n = ?$$

## 4.2 Integration

How to do the reverse of Differentiation? Or, how can we find the function from the derivative?

We now want to find the reverse way. Suppose we know the derivative, so is it possible in some way to know the function?

Another example is like, a particle's speed along a straight line is given by $s = 2t$. What is its position with respect to time?

As you might have guessed, we know that speed is the derivative of distance traveled, and we also know that the derivative of $t^2$ is $2t$.

Hence the position with respect to time is $d = t^2$

Integral calculus tries to deal with it. In differential calculus, we break things into infinitesimal small pieces. In Integral calculus, we do the opposite. Add the all small things together so that we can find the function from it's derivative.

Integration is also called Anti Derivative, where Anti means reverse. It is well understood.

**Derivative**

Function $\rightarrow$ Rate of Change

**Integration**

Rate of Change $\rightarrow$ Function

### 4.3 Geometrically defining Integration

Suppose, $y = F(x)$ and the derivative is,

$$\frac{dy}{dx} = \frac{d}{dx}F(x) = f(x)$$

What can we do to reverse the Differentiation? We can bring $x$ at other side like below.

$$dy = f(x)dx$$

This has an important geometric meaning.
If you plot the function $f(x)$, which is the derivative of $F(x)$, you will find that, $f(x)dx$ is the small area under the function at $x$.
So, $dy$ is a small area. If we can add up all the area, we surely can find $y$ which is needed to be found.

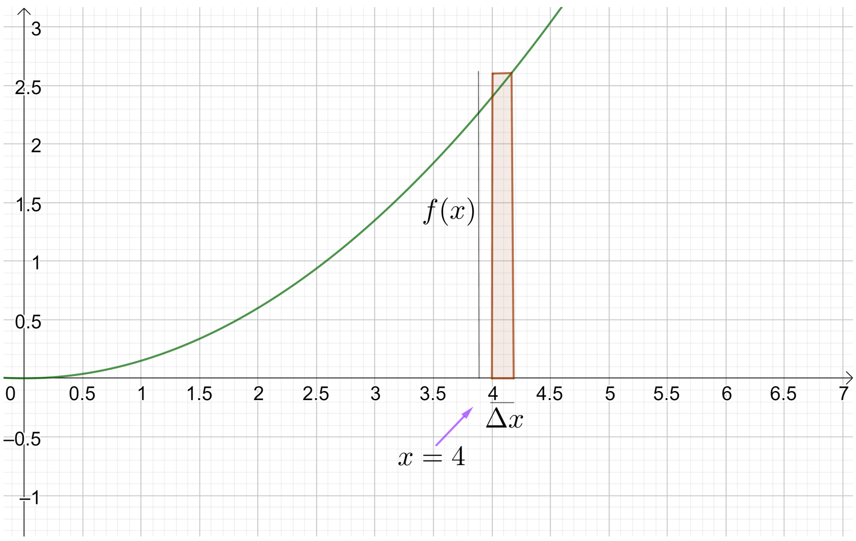



Figure 21: A small area $dy = f(x)\,dx$ segment at $x = 4$ of a random function.

But let us do an approximate method.

#### 4.3.1 Approximate integration

For a function, approximately the derivative is $\Delta y/\Delta x$. Let $y = F(x)$ So,

$$\frac{\Delta y}{\Delta x} = f(x)$$

Bring the $\Delta x$ at one side.

$$\Delta y = f(x)\Delta x$$

Now this is a small area under the function $f(x)$. The total area under the function $f(x)$ can be written as another function. And from above equation, we know that this function is $F(x)$ for whom we are looking for.

One method to find the total area is summing up all the small areas along $x$. So,

$$\text{Area } = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + .... + f(x_n)\Delta x$$

Simply,

$$\text{Area } = \sum_{i=0}^{i=n} f(x_i)\Delta x$$

And we know that this area is the $F(x)$. So, approximately,

$$F(x) \approx \sum_{i=0}^{i=n} f(x_i)\Delta x$$

How can we make the approximation better? If we make the $\Delta x$ as small as possible. Recall the idea of limits. So, we have to keep in mind one thing.

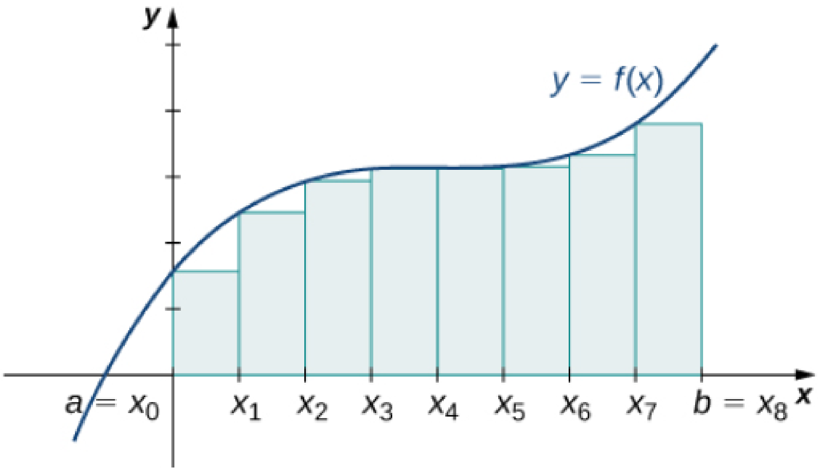



Figure 22: Finding approximate area, we are trying to add up all the small rectangles. Here $n = 8$, so there are 8 rectangles formed.

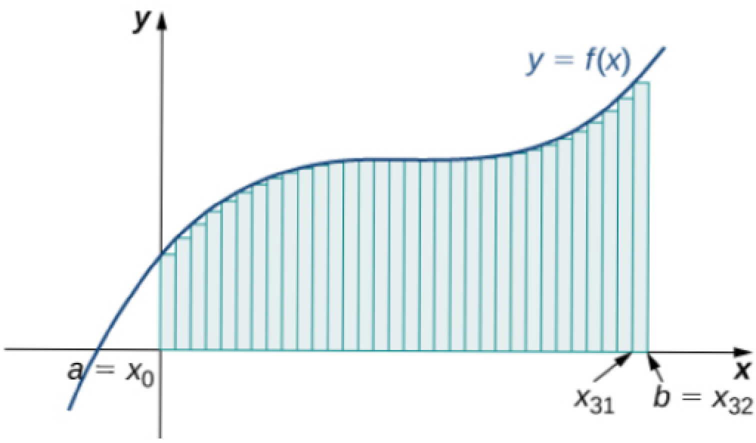



Figure 23: In this case, we have increased the number of partition, so our area is going to be more accurate. Here are 32 rectangles. We would try to take a billions of rectangles.

> The area under the curve of the derivative will be a function of $x$. As in the above two figure, we took the area from $x = a$ upto $x = b$. We are free to choose any other things that comes.

### 4.3.2 Finding definitely

In figure, let $x_n$ be $b$ and $x_0$ be $a$, then,

$$\Delta x = \frac{b-a}{n}$$

If we increase the partition infinitely, or extremely high, or, increasing the number of divisions, then we start to get better values in area. If we increase $n$, then $\Delta x$ will go smaller and smaller. So, we can see, if $n$ is extremely high, almost infinite, then $\Delta x$ is very small.

$$dx = \lim_{n\to\infty} \frac{b-a}{n}$$

Thus, using it in our approximate integration formula,

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{i=n} f(x_i)\,dx$$

This $\lim_{n\to\infty} \sum_{i=0}^{i=n}$ is hard to write frequently, we have introduced a stylish $S$ to show it, $S$ stands for Sum.

$$F(x) = \int_a^b f(x)\,dx$$

The $a$ and $b$ means we are taking area under the boundary $x = b$ and $x = a$. These are called the limits of the integration. You can also do maths without them, it will shown later.

$$F(x) = \int_a^b f(x)\,dx \qquad \text{Fundamental Theorem of Calculus} \tag{23}$$

You don't have to worry if you feel lost and confused. Perhaps, there are lot more examples to come to make the thing more concrete and understand-able.

#### 4.3.3 Quick look what Integral really is

Let us do this. Let $F(x) = x^2$. Then, it's derivative $f(x) = dy/dx = 2x$. Using the Fundamental Theorem of Calculus, it's noticeable,

$$x^2 = \int 2x\,dx$$

Let $F(x) = \frac{3x^3}{7}$. Then, the derivative is $\frac{9x^2}{7}$. We can thus see,

$$\frac{3x^3}{7} = \int \frac{9x^2}{7}\,dx$$

### 4.4 Algebraically defining Integration

We can see that for some function $f(x) = \alpha x^n$,

$$\frac{d}{dx} f(x) = n\alpha x^{n-1}$$

From definitions above, we can see,

$$f(x) = \int n\alpha x^{n-1}\,dx$$

Now, let us try to integrate another thing, that is in the form same as $f(x)$.

$$\int px^t\,dx$$

Last two equations can be similarly seen,

$$n\alpha x^{n-1} = px^t \quad \rightarrow \quad p = n\alpha \quad \text{and} \quad t = n-1$$

Which tells that,

$$n = t + 1$$

$$p = n\alpha = (t+1)\alpha \quad \rightarrow \quad \alpha = \frac{p}{t+1}$$

So, put these in the $f(x)$.

$$f(x) = \alpha x^n$$

$$f(x) = \frac{p}{t+1} x^{t+1}$$

Why did we do all these? To show that,

$$\alpha x^n = \int n\alpha x^{n-1}\, dx$$

$$\frac{p}{t+1} x^{t+1} = \int p x^t\, dx$$

Does this work? Let us do this small check.

Derivative of $x^2$ is $2x$. Thus Anti Derivative or Integration of $2x$ is $x^2$. To verify,

$$f(x) = px^t = 2x$$

So,$p = 2$ and $t = 1$. Now,

$$\int 2x\, dx = \int px^t = \frac{p}{t+1} x^{t+1} = \frac{2}{1+1} x^{1+1} = \frac{2}{2} x^2$$

So,

$$\int 2x\, dx = x^2$$

That is verified. We shall remember the overall idea of integration.

$$\boxed{\int \alpha x^m = \frac{\alpha}{m+1} x^{m+1}} \tag{24}$$

## 4.5 Area and Algebraic Definition together

Algebraically we know that for some function that is in the form $\alpha x^n$, the integration of it should be $\frac{\alpha\, x^{n+1}}{n+1}$.
Geometrically, the area under the $\alpha x^n$ is going to be the integration. Two are equivalent. But how?

Let us draw the function $f(x) = 2x$. Let us find the area upto $x = 10$ from $x = 0$. We can use the formula, $A = \frac{1}{2} \times base \times height$. That is $A = (1/2)bh$. Here, the height is $h = 20$ and base is $b = 10$ (see figure). So area is,

$$A = \frac{1}{2}(10)(20)$$

$$A = 100$$

Now how to use Integration formula?

$$\int 2x\, dx = x^2$$

We need the area under $f(x) = 2x$ from $x = 0$ to $x = 10$. So,

$$x^2 = (0)^2 = 0 \quad (x = 0)$$

$$x^2 = (10)^2 = 100 \quad (x = 10)$$

Area is $A = 100 - 0 = 100$. Both gave same answer.

We can also do this to find algebraic solution. At any $x$, the height is $h = f(x) = 2x$. The base is $b = x$. Thus, area is $A = (1/2)bh = (1/2)x \times 2x$, so $A = x^2$.

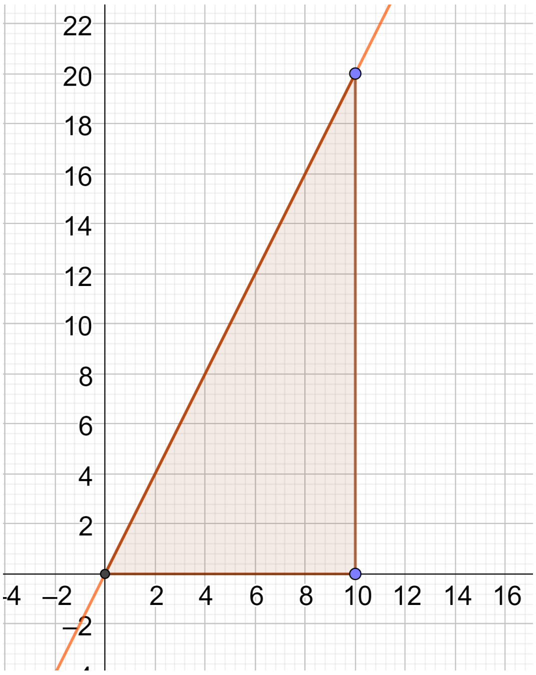



Figure 24: Finding area of the Triangle.

## 4.6 Integral Maths

We keep doing a chain of integrals. **Remember, we call these Indefinite Integrals.** You will easily understand this makes sense later. Remember,

$$\int \alpha x^n = \alpha \frac{x^{n+1}}{n+1} \tag{25}$$

**Example 21.** Solve the following integrals,

- $$\int \alpha x\, dx$$
- $$\int x^3\, dx$$
- $$\int \frac{c}{x^{1/2}}\, dx$$
- $$\int \frac{g}{a} x^a\, dx$$
- $$\int x t^2\, dt$$
- $$\int x^2 y^2 z^2\, dz$$

**Solution:** You should read the problem and try to find whether there are any traps in there.

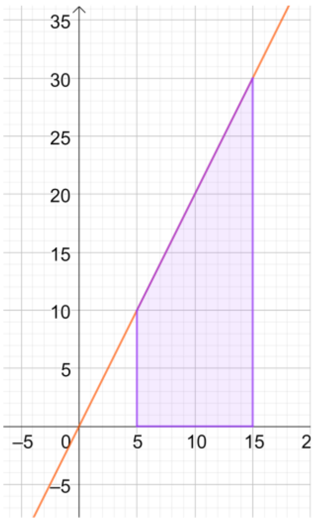
- $$\int \alpha x\, dx = \alpha \frac{x^{1+1}}{1+1} = \alpha \frac{x^2}{2}$$
- $$\int x^3\, dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}$$
- $$\int \frac{c}{x^{1/2}}\, dx = c \frac{x^{1+1/2}}{1+1/2} = \frac{2}{3} c x^{3/2}$$
- $$\int \frac{g}{a} x^a\, dx = \frac{g}{a} \frac{x^{a+1}}{a+1} = \frac{g}{a(a+1)} x^{a+1}$$
- Trap. Notice that the integral has $dt$ instead of $dx$. So, we have to take integral in this manner,
$$\int x t^2\, dt = x \frac{t^3}{3}$$
- Also a trap.
$$\int x^2 y^2 z^2\, dz = x^2 y^2 \frac{z^3}{3} = \frac{x^2 y^2 z^3}{3}$$

**Quick Check 16.** *Solve this set of Integrals (Anti Derivative) mentally or on paper.*

- $$\int 3x^2\, dx$$
- $$\int xyz\, dx$$

- $$\int xyz\,dz$$
- $$100\sqrt{x}\,dx$$

### 4.7 Definite Integral



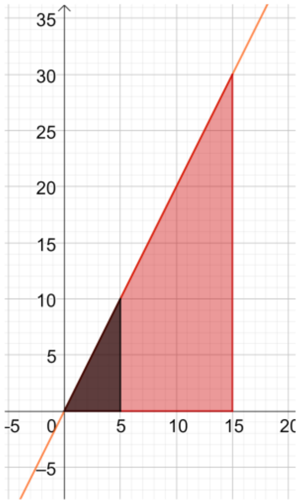




Figure 25: The problem that we want to find the area of Purple region of the 1st figure, and 2nd one up is the visual diagram for the problem.

No need to give an explanation. Let us try to work out this problem first.

> **Example 22.** Suppose we want to find the area of this purple portion of the figure as shown, the area from $x = 5$ to $x = 15$ region. What can we do?
> As we have found, we can just subtract as specific black portion from the whole red region as shown to find the purple region area.
> Thus, the area of the red region, $A = (1/2)bh = (1/2) \times 15 \times 30 = 225$
> The figure in the black region, $A = (1/2)bh = (1/2) \times 5 \times 10 = 25$.
> The area of the purple region from problem is going to be the difference, which is Red Region - Black Region. So, $A = 225 - 25 = 200$.
> The area of the purple region is 200.

This above example is now evaluated in the language of Calculus.

**Example 23.** The equation of the straight line is $f(x) = 2x$. Thus, the integral gives the derivative beneath it.

$$\int 2x\,dx = x^2$$

But, this area is not positioned where we want the area (purple region), to do that, we notice that we have to take the area from $x = 5$ to $x = 15$. So, in **Definite Integrals**, the area of the Purple region is,

$$\int_5^{15} 2x\,dx$$

This above integral is solved by taking the difference as below.

$$\int 2x\,dx = x^2$$

The definite integral will thus be,

$$\int_5^{15} 2x\,dx = x^2]_5^{15}$$

$$\int_5^{15} 2x\,dx = 15^2 - 5^2 = 225 - 25 = 200$$

The area of the purple region is 200.

Finally, I can give a raw formula by the two below, if,

$$F(x) = \int f(x)\,dx \tag{26}$$

then,

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{27}$$

**Quick Check 17.** *Try to solve these definite integrals.*

- $$\int_0^3 \frac{2}{3}x^3\,dx$$
- $$\int_{-5}^5 \frac{100}{x^2}\,dx$$
- $$\int_{20}^{40} 100\sqrt{x}\,dx$$

## 4.8 Practically starting to use Integrals

I want a clarification about Integrals very first.

☐ Integrals give the Area beneath a function.
☐ Integrals give the function from the derivative.

Both are same statement and dependent on each other, this should not be too confusing after you start to read some of the next examples.

**Example 24. Problem 01:**
There is a crow whose mass is a function of time. It's mass is given by $m = 2.5\sqrt{t}$. Mass measured in kilograms and time in years. It lives for 2 years.
Find the rate of change of it's age in $0.5\,years$ age and in $2\,years$.
**Solution:** The derivative is alone required,

$$\frac{d}{dt}2.5t^{1/2} = 1.25t^{-1/2} = \frac{1.25}{\sqrt{t}}$$

At $t = 0.5\,years$, rate of change is $putvalue$!!!
At $t = 2\,years$, rate of change is $putvalue$!!!

**Problem 02:** What is the mass of a crow as a function of time ? Find using the rate of change.
**Solution:** Integrate.

$$\int \frac{1.25}{\sqrt{t}}\,dx = 1.25\frac{t^{1-1/2}}{1-1/2} = 2.5t^{1/2}$$

Thus the function is $2.5\sqrt{t}$.

**Example 25.** Velocity of a particle is given as a function of time, $v = 2t$. What is the position of the particle at any $t$ time?
**Solution:** We know that,

$$\frac{ds}{dt}v \qquad ds = v\,dt$$

Integrating both sides,

$$\int ds = \int v\,dt$$

Integrated,

$$s = \int 2t\,dt \quad \rightarrow \quad s = t^2$$

So, distance from origin as a function of time,

$$s = t^2$$

**Example 26.** You have a particle, you put Force on it as a function of distance traveled $x$, that is $F = \alpha/x^2$. We define the quantity work by,

$$W = \int_{x_1}^{x_2} F\,dx \tag{28}$$

So, what is the work moving the particle from a distance $a$ to distance $b$?
**Solution:** Using the above formula for work,

$$W = \int_a^b \frac{\alpha}{x^2}\,dx$$

Here, the variable is the distance $x$, and we take the limits from $a$ to $b$. Thus, evaluating the integral,

$$W = \int \frac{\alpha}{x^2}\,dx = \frac{\alpha x^{-2+1}}{-2+1} = -\frac{\alpha}{x}$$

So,

$$W = -\frac{\alpha}{x}$$

Now, for the definite integral,

$$W = \int_a^b \frac{\alpha}{x^2}\,dx = -\frac{\alpha}{x}]_a^b = -\alpha\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{\alpha}{a} - \frac{\alpha}{b}$$

So, we have,

$$W = \frac{\alpha}{a} - \frac{\alpha}{b}$$

This is the work done to move the particle from $a$ to $b$ with a force of $F = \frac{\alpha}{x^2}$.

**Quick Check 18.** *Professor David Morin's bicycle has got leak and air is coming out of the tire. Thus, the mass of the air in the tire is reducing with time. Mr. Morin, instead of fixing the leak, calculated that the rate of change of mass of air inside tire is,*

$$\frac{dm}{dt} = -4.534t$$

*Where $m$ is mass of air in the tire and $t$ is time.*
*What is the mass of air inside the tire as a function of time? Find the $m(t)$.*

# 5 Finding the Calculus from Physics

This topic introduces Calculus in the fastest manner. The previous topics are broader and none the less contain in depth Maths, that you will probably need soon. For the general idea of Calculus in Physics, this is the pedagogically viewed topic. You should read a few topic from previous part before hand,

- "Summation" part.
- Knowing what $\Delta$ is from the "Slope Calculation - Quicker" Part

## 5.1 Energy

Energy can be found in different forms. When a particle has a mass and it is moving, then it has *Kinetic Energy*, when we push or extend a spring, then what stores in the spring is *Spring Energy*.

You should read the Energy and Work topics from your Physics textbook. A through understanding is necessary, though some theory are still given.

We can write various kinds of energy,

$$T = \frac{1}{2}mv^2 \tag{29}$$

The above is the kinetic energy.

So let us know what usually happens when we try to exploit the system. We can have a particle of mass $m$ and let it move in a straight line as a velocity $v$. Then the kinetic energy is given by the formula above. What will happen if we input $Work$ in it?

The Work is just force times distance it has been applied. So, if you apply Work on the particle, its kinetic energy will increase and become $T'$.

$$T' = T + W$$

This is well understood. Doing the Work is same as pouring energy in the system. Now, work is defined by, simple sense,

$$W = Fx \tag{30}$$

The force cannot is constant. So, our equation is,

$$\frac{1}{2}mv'^2 = \frac{1}{2}mv^2 + Fx$$

This equation tells that if we have a particle moving at $v$ velocity, then applying $Fx$ amount of work shall make the velocity change to $v'$.

So to work with this formula, you have to have a **Constant force.** Why?

The formula is defined as

$$W = Fx$$

Now, if there is constant force, if we try to draw this in a Force-Displacement graph, then the graph looks like a rectangle. And the area of the rectangle is length times base. Here length will be the $F$ and $x$ will be the breadth. Then from $W = Fx$, we see that the area under the graph is equal to the work done on an object. So, we can now know that the work

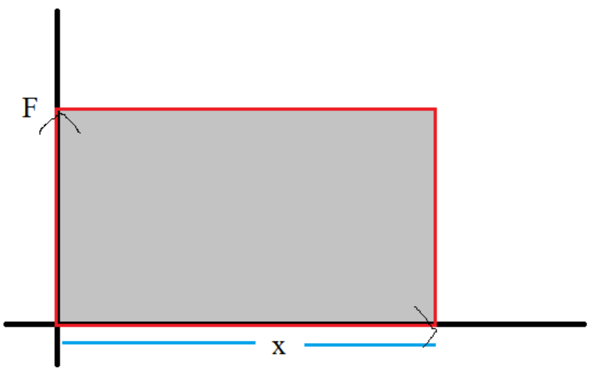



Figure 26: Graph of Force and Displacement

done is just a kind of an area. Indeed, many things in Physics can be shown as an area. How?

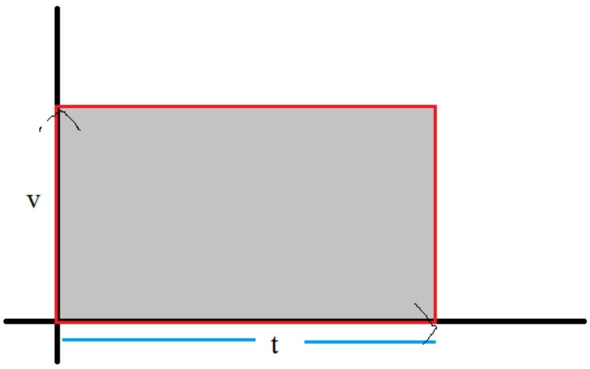



Figure 27: Speed time graph and it's area

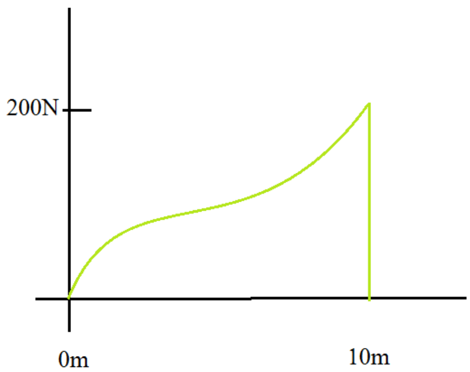



Figure 28: The block is being pushed on a floor

Consider the distance traveled when particle is moving, then if it's speed is $v$, then after $t$ time, it will travel $vt$ distance.[1] So, distance traveled $s$ is shown as,

$$s = vt$$

Now in the same manner look at the graph of the Work above. In case the $v$ is constant, the area under the graph is $vt$, and this is equal to the distance traveled. So, distance is also the area under the graph *speed* and *time*. The Area interpretation is helpful many times. But what can happen in the work[2] if the force non-constant? If the force is non constant then the force will change with the increase of $x$ . And some times it will be more, sometimes it will be less. Then how can we put forward the idea of Work from that? We have to re-imagine a general case. Suppose we are pushing a block and making it move along the

[1]You can think this in this manner too, so suppose, a particle covers 5 meters in 5 seconds. Then what distance will it travel in 10 seconds? Well, because 5 meter is covered in 5 seconds, in every 1 second it travels 1 meter. This is speed, the distance you cover in 1 second (unit time). So, in 10 seconds, as you travel 1 meter every second, you will cross 10 meters.

[2]which is defined now to be the area under the graph of Force versus Displacement

ground. Thus, it displaces and as it displaces, we increase the level of force. We start from the $0\,m$ position and keep increasing the force in a non linear manner[3] and stop at $10\,m$. At the end, we had the force $200\,N$ when stopping. The graph is in figure 28.

Now we want to find the total amount of work done. But at first let us answer this question, why should work be done? Or say, what has happened for which we are wanting to find the work?

The reason is, we were pushing the block- that is same as saying applying force on the block. And that force could displace the block from it's position, so we did put work. And this work has came out from our body's metabolic energy.

So keep in mind we want to find the work done. The formula was,

$$W = Fx$$

But the problem now is that the force is not constant. So what can we do?

We can cheat a little. Let us stay at some position between $0\,m$ and $10\,m$. At that random point, let us displace a very small distance $dx$. What is this $d$ ? It stands for a small change, and thus $dx$ means a small change over displacement $x$. Then the change in force is negligible, and we also get a very small piece along the region. Consider the small

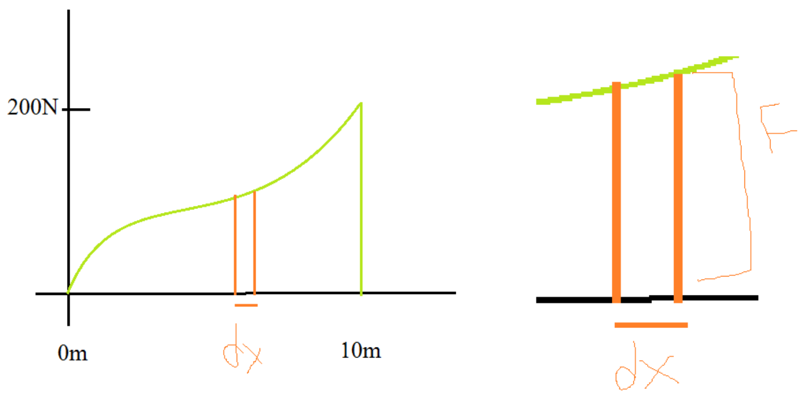



Figure 29: Cutting a small portion and zooming it

portion for now, look at the force side, in the figure 29. The two force after the change in displacement $dx$ are the two orange lines, parallel. But is there any noticeable difference in their height? Nope. And recall the height along this graph is just the force. So, the change in height between the two neighboring orange lines, that prescribe the force at two different position, is negligible.

The change in force after a small displacement is essentially zero, so the force is kind of constant, as it doesn't change.

Now this portion of region can be given a Work formula. The Force being constant at this narrow region, the work will be,

$$dW = F(x)dx$$

You can also see that the region is a tiny strip, that is why it has a small area. The area of this small region is also the work (as we found). And thus, the work is also small as displacement is small, that's why we write $dW$. Now, how can we find the total work from 0 to 10 meters?

We can divide all the graph portions and add them up, all of them. We know for each area of small region, the work is $dW$. So, you can see we can ADD up all the strips and

[3]For this reason the Force-Displacement Graph looks kind of curved

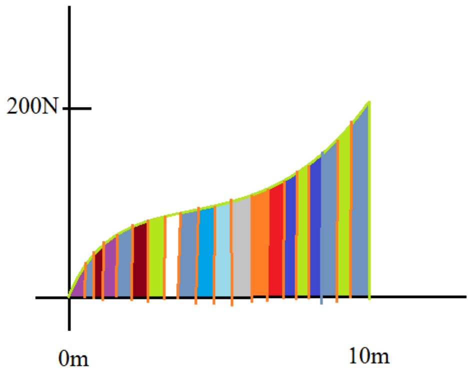



Figure 30: Note down every small part and then add them up.

that will be denoted for the figure as,

$$\sum F(x)\,dx$$

## 5.2 Speed

Speed is defined to be the distance traveled per unit time. So, we can try to measure the speed at an instant, the speed of a car just at the moment we want, instantaneous speed. How can we do it? So speed is simply written in the form,

$$v = \frac{\Delta x}{\Delta t}$$

The change in position divided by the change in time. With instantaneous we mean the $\Delta t$ will be as small as possible, quite small. So, from the same motivation of Energy and Work from last section, make this to infinitesimal,

$$v = \frac{dx}{dt}$$

In small time $dt$ the car shall travel as small distance $dx$. You can revise the part on slopes to get this straight, the speed is just the slope in the *distance* and *time* graph.

Let us turn down the question to something else, how can we find the *distance* from the formula above if the velocity is non-constant?

# 6 How to read this booklet

I have tried to make this booklet so that every part and everything here can be used effectively in Physics. You are introduced to some different sorts of maths in the beginning, these will help you in Experimental Data Analysis later (if you are just starting to learn Physics). But the main objective is to make a more understand-able (comprehensive) guide to Calculus.

There is no danger saying that a well Physics book will always use derivatives and integrals.

I have no given too much explanation like normal physics texts. Rather, I have directly given simple example so that you can directly see, why we use a concept in what way. So, you SHOULD write the examples in your note book and try to copy the maths there and also read the text to understand WHY have we did that.

After reading most of the sub-section, you will notice that a Quick Check exercise has been given. Try to do it, as it uses the past topics and doesn't take too much time.

Whenever you are feeling stuck, less confident, de-motivated, or just frustrated of being unable to do any problem, what you should do is just skip the part and read the next sections throughly and return to the problem later. Why? There is more information and ideas that you can learn from the next examples, they will surely give you a deeper understanding, and hopefully the confidence to do the problem.

And never hesitate to E-Mail me! `ahmedsabit02@gmail.com`

And some parts can be ignored totally for the beginning if you are convinced of the formulas directly and are ready to use it. Like, Approximations parts could be ignored, though they are the fundamental recipes of Derivatives and Integrals, you don't need to understand that. Until you know what calculus do, we are okay.

# 7 Answers to Quick Checks

- QC 1: For $(0, 1, 2, 3, 4, 5)$ $x$ values, the $y$ are $(undefined, 1, 0, -0.33, -0.5, -0.6)$.
- QC 2: Note the figure.

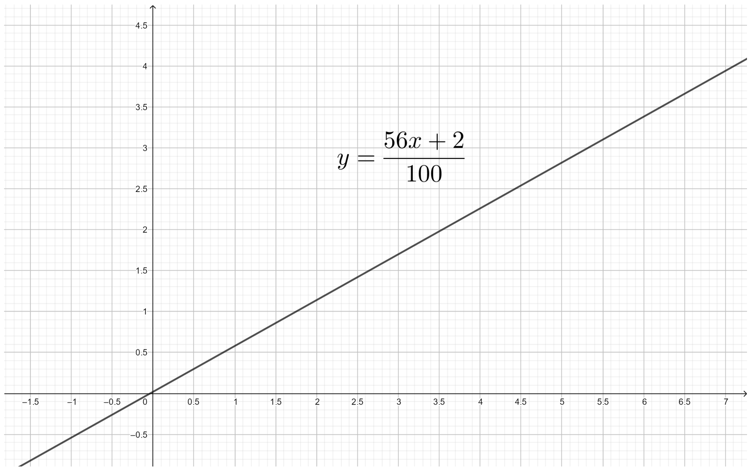



Figure 31: QC 2

- QC 3: $x^2 + h^2 + 2xh - 4x + h + 8$
- QC 4: $\dfrac{1+k}{1-k}$
- QC 6: $18x+9y+4 = 0 \to 9y = -18x-4 \to y = -2x-4/9$. So, $m = -2$ and $c = -4/9$
- QC 7: Here, total 3+5=8 problem is solved **per day**. That is, *rate of increase of solved problem per day* is 8. By definition, the rate of change is the slope. So, the slope is 8.
- QC 8: Total Increase of weight is $(114 - 80)kg = 34kg$. Total time 30 days. So rate of increase of weight $34kg/30day = 1.13kg/day$. That means, everyday his weight increased 1.13 kg. A Professional Eater, indeed.
- QC 9: You are required to use the learning of example 9. Let it take $t$ time for all the water to leak away. There will be no water when amount of leaked water is 200 liter. So, $200 = 0.4t$. This gives 500seconds. 60second is 1minute, so it takes $500/60 = 8.33$minute for all the water to leak. I take 20minute to repair leak, so the water will all go away before I fix it. Bariwala will surely be very angry on this.
- QC 10:Height of the triangle is 6 and base is 5. So, $m = height/base = 6/5 = 1.2$. Speaking in other method, $y_2 - y_1 = 4-(-2) = 6$ and $x_2 - x_1 = 2-(-3) = 5$. Hence, $m = (y_2 - y_1)/(x_2 - x_1) = 6/5$. Both method are just the same thing. You might even take other values of $x$ and $y$'s, it is all right if answer $m = 1.2$ is correct.
- QC 11: $\Delta V = 50$ liter. That V is for Volume of water.
- QC 12: The graph should just roughly look as the computer generated figure.
- QC 14: $1/3$, $\frac{9}{4}x$, $24x$
- QC 15: $1 + x + y + (x+y)^2 + (x+y)^3$

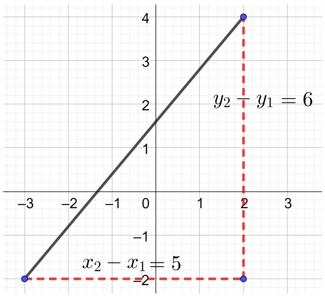



Figure 32: Quick Check 10 answer. Note that your values can vary.

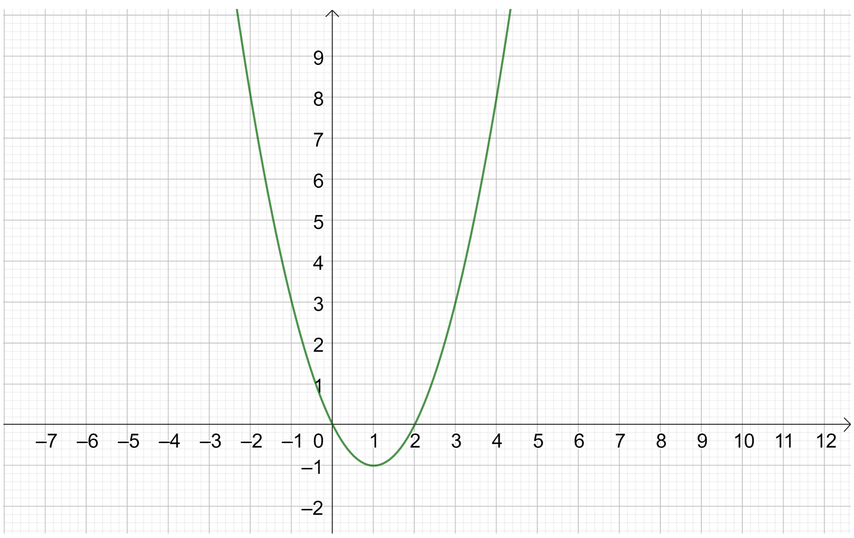


Figure 33: Quick Check 12