Problem Journal - Physics

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The writing of this text started 1 or 2 days after the Eid Ul Fitr of 2020. Contains cheers and mental breakdown like gonkee making a physics engine from scratch.

I've tried all the problems, I'd continue to add after my computer becomes stable in some Debian based distribution.

Contents

Chapter 1

Experiments

1.1 Electromagnetism Experiments

I think, Magnetostatic experiments are more basic to do than doing the Electrostatic one, the difficulty is aided by the things I actually don't have any possible access to.

I should make the Experiment environment to look different.

Problem 1 (Magnetic Mass). The system consists of a rod and two identical magnetics. The magnets are ring in shape and they ha ve their poles on the plane containing the hole. Find the Magnetic Dipole Moments aka. "Magnetic Mass".

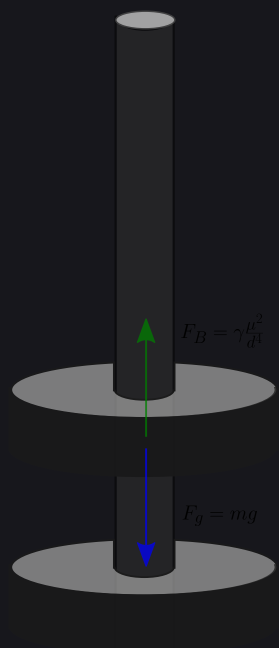

Figure 1.1.1

Solution. Keep the magnets like the figure so that one is upon another, given the magnetic force is large enough, the magnet put above should levitate (just a word). Now, because the top magnet is not supposed to fall into any other external force (given with the fact there is no net acceleration), there is a force balance. The system is assumed steady without any oscillation whatsoever.
In one of the tutorials we found the *Bar Magnet Forces* when they are aligned. So, we can tell that,

$$F_B = \gamma\frac{\mu^2}{d^4}$$

Hopefully a derivation is made in the Magnetism part. From force balance,

$$mg = \gamma\frac{\mu^2}{d^4}$$

We have to make measurements of mass and distance. We can find the mass from calculating the volume of the magnet and multiply density ρ, because I don't have a mass measurement machine. So, if we have mass,

$$\mu = \sqrt{\frac{mg\ d^4}{\gamma}}$$

Chapter 2

Kinematics

2.1 Linear Kinematics

Rate of change of distance is speed. And when we are interested on the direction, then it is velocity. It is a vector and the length (magnitude) of the vector equal to the speed. If speed is s, then velocity $\vec{v}$ relation with it is,

$$|\vec{v}| = s$$

We define velocity by,

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

Where $\vec{r}$ is the position vector of the object, and as the position of the object is changing with respect to time, surely the velociyt $\vec{v}$ is non zero. To think bit more easily, we can assume every thing is one dimensional, hence the particle moves along the x axis, and distance of the particle with respect to the origin be x. Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{x} = \vec{v}$$

And thus speed is,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

If we multiply $\mathrm{d}t$ both sides, then we get[1],

$$\mathrm{d}x = v\ \mathrm{d}t$$

Integrating both sides we get,

$$x = \int_{t_0}^{t_1} v\,\mathrm{d}t$$

As we know that integration is similar to finding the area under a curve found from plotting a function, it tells us that,

> **Theorem 1** — The area under the $v-t$ velocity - time graph is the overall distance covered between the time interval we measure. Because,
>
> $$x = \int_{t_0}^{t_1} v\,\mathrm{d}t$$

From the equaiton of speed and distnace, starting from x_0 with uniform speed and ending up at x,

$$v = \frac{x - x_0}{t}$$

Hence,

$$vt = x - x_0$$

[1] This is not very correct though, who cares when we are doing things informally?

$$x = x_0 + vt$$

Hence this is the equation that can help us find where a particle will end up if it moves with constant speed after a certain time.

As velocity is the rate of change of distance, rate of change of velocity is accleration. Hence,

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

Considering one dimention,

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v-u}{t}$$

Where v is the final speed and u is the initial speed. The difference of speed divided by the time taken for the change to take place. Solving that,

$$at = v - u$$

$$v = u + at$$

Which is the equation of the final velocity when there is acceleration.

We have learned that the area under the curve of v vs. t is the distnace covered. Let us start moving with speed u and after t time we are at speed v. Acceleration is a, which is supposed to be constant now, what is the total distance covered?

We have to plot the speed every moment of time, and the area under it will be the sum of the area of the rectangle zone and the triangular zone. The area of each adds to,

$$\text{Area} = (u \times t) + \left(\frac{1}{2}t \times (v-u)\right)$$

Recalling that triangle area is "half into base into height". We see from $\frac{\mathrm{d}v}{\mathrm{d}t} = a$ that,

$$v - u = at$$

So,

$$\text{Area} = (u \times t) + \left(\frac{1}{2}t \times at\right)$$

So, adding the information that area is the distance covered,

$$s = ut + \frac{1}{2}at^2$$

s is the distance covered.

We can build another equation using,

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

Square the final speed equation,

$$v^2 = (u + at)^2 = u^2 + a^2t^2 + 2uat$$

Take $2a$ common so that we get a familar $\frac{1}{2}at^2 + ut$,

$$v^2 = u^2 + 2a\left(\frac{1}{2}at^2 + ut\right)$$

Finally,

$$v^2 = u^2 + 2as$$

Yao, this is a nice one.

Problem 2. The angle function of a particle with respect to an origin is,

$$\phi = 4t - 3t^2 + t^3$$

Find the angular acceleration between the time period $t = 2$ and $t = 4$ seconds.

Solution. The idea is to find the derivative, that is $\dot{\phi} = 4 - t + 3t^2$ and then using the idea,

$$\langle \alpha \rangle = \frac{\Delta \dot{\phi}}{\Delta t}$$

Which gives us,

$$\langle \alpha \rangle = 12$$

Problem 3. There is a mass that has been thrown at an angle $\theta = 30°$ with a speed of $v = 20\ m/s$. Find the length of the curve of the projectile.

Solution. The idea is a subtle one (at least for me). We can tell that an elementary length $\mathrm{d}l$ over the curve is,

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

Now, taking a factor $\mathrm{d}x^2$,

$$\mathrm{d}l = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Now, we know the equation of a projectile,

$$y = x\tan\theta_0 - \frac{gx^2}{2(v_0\cos\theta)^2}$$

Taking the derivative, we shall have a long equation to integrate,

$$\int \mathrm{d}s = \int_0^{l_0} \sqrt{1 + \left(\tan\theta_0 - \frac{g}{v_0^2\cos^2\theta}x\right)^2}\,\mathrm{d}x$$

> I wanted to solve this, but actually we are given with the values and there is no hurt of solving this using a calculator for the numerical result, recalling that the *exact solution must be hard, but there is always some way that might be more handy and easier than the hard one.* What I get is,
>
> $$s = 36.479\ m$$

Problem 4. A cyclist rides along the circumference of a circular horizontal plane of radius R, the friction coefficient being dependent only on distance r from the center O of the plane as $k = k_0\left(1 - \frac{r}{R}\right)$, where k_0 is a constant. Find the radius of the circle with the maximum velocity. What is the velocity?

It's quite that I had to toil a lot once to do these, coming till here was always so fail-full. The equation I initially derived is,

$$\sqrt{gk_0 r - gk_0 \frac{r^2}{R}} = v$$

> **Solution.** The centrifugal acceleration should be balanced by the friction force, if centrifugal acceleration is greater than the friction force, the cyclist will slip. If it is smaller than friction force, the cyclist can increase his speed to get near the maximum speed.
>
> $$f_r = \frac{mv^2}{r}$$
>
> Solving with the equation in the problem,
>
> $$gk_0\left(1 - \frac{r}{R}\right) = \frac{v^2}{r}$$
>
> $$v^2 = gk_0\left(r - \frac{r^2}{R}\right)$$
>
> Now when the speed is maximum, the variable term $\left(r - \frac{r^2}{R}\right)$ is maximum, so,
>
> $$\frac{\mathrm{d}}{\mathrm{d}r}\left(r - \frac{r^2}{R}\right) = 0$$
>
> Solving this for enough time, we come to,
>
> $$r = \frac{R}{2}$$
>
> We can ensure that this is max by taking the second derivative,
>
> $$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(r - \frac{r^2}{R}\right) < 0$$
>
> And it is. The velocity is then,
>
> $$v = \frac{1}{2}\sqrt{Rk_0 g}$$

Problem 5. A car moves with a constant tangential accelaration w_τ along a horizontal surface cricumscribing a circle of radius R. The coefficinet of sliding friction between the wheels of the car and the surface is k. What distance will the car ride without sliding if at the initial moment of time velocity is zero?

Solution. There are two forces that are needed to be balanced by the friction force before slipping. The centrifugal force and the force for the ta ngential acceleration.

$$\vec{F} = m\left(w_\tau\ \hat{\theta} + \frac{v^2}{R}\ \hat{r}\right)$$

We require the magnitude,

$$F = m\sqrt{\left(w_\tau^2 + \left(\frac{v^2}{R}\right)^2\right)}$$

Friction force,

$$f = mgk$$

When the car is just about to slip, the friction force will become equal to the force needed to keep the car moving; so far the force keeping the car moving was lesser than the friction and it has been increasing with speed.

$$\sqrt{\left(w_\tau^2 + \left(\frac{v^2}{R}\right)^2\right)} = gk$$

We notice that $v^2 = 2w_\tau s$, where s is the distance travelled in a circular path. Hence,

$$w_\tau^2 + \left(\frac{2w_\tau s}{R}\right)^2 = g^2k^2$$

Solving for s,

$$\boxed{s = \frac{1}{2}R\sqrt{\left(\frac{kg}{w_\tau}\right)^2 - 1}}$$

This one is a super cool problem I don't know where to keep, just for the reason the kinematics section is being heavily relied on maths (I don't know why), I will put this here.

Problem 6. The standard kilogram is in the shape of a circular cylinder of fixed volume. What should be the ratio between the height of the cylinder and the radius to give the smallest surface area, thus minimizing the effects of surface contamination and wear?

The main fact is that I have thought about this problem for long, and tried many times without great amount of success, that day was the one when I could finally do some tricks to find the thing. The answer is also, awesome.

Solution. I will start from basics, just to make sure the elegance is not reduced. The volume is going to be constant, so,

$$V = \pi r^2 h$$

Now, this can be written in terms of a simplification. Look, we want the Area to be the least when the volume is constant. So, there is a specific ratio of the height and radius of a cylinder that makes it different in shape from other cylinders. So, we may define,

$$\eta = \frac{h}{r}$$

Hence the volume is,

$$V = \pi r^3 \eta$$

Now the area, that needs to be minimized, is,

$$A = 2\pi r h + 2\pi r^2$$

And in our η language,

$$A = 2\pi r^2 \eta + 2\pi r^2$$

We can make this happen that, this area is just the function of a si ngle quanitity, hence, we need to get rid of the r from the equation, so using the volume equation,

$$r^2 = \left(\frac{V}{\pi\eta}\right)^{\frac{2}{3}}$$

Putting this in place,

$$A = \left(\frac{2\pi V^{2/3}}{\pi^{2/3}}\right)\left(\frac{\eta+1}{\eta^{2/3}}\right)$$

For the maximal conditions, we invoke that,

$$\frac{\mathrm{d}A}{\mathrm{d}\eta} = 0$$

And when I was solving, I messed up not putting a negative sign where it was supporsed to and ruined some of my time. Later I used symbolab to solve for the roots because I was too tired doing mistakes. The solution, I could not derive, but at least, came that,

$$\eta = 2$$

Which is a root of a quadric equation appearing while trying to solve above condition. And thus,

$$\boxed{r = \frac{h}{2}}$$

So, when the radius is half the height, then the cylinder has the least surface area per max volume.

At least this problem is so long sought by me that I can be at least proud of myself that I invoked η in the equation with my own wisdom then.

Chapter 3

Mechanics

3.1 General Mechanics

Alert through out the doc I use formal mathematical methods and statements very casually. All it matters is the Physics and it's ideas.

Newton's Laws gives us a massive kickstart when trying to understnad mechanics, the three sentences generalizes most of the situations, and I include them in our very own "Theoretical Theorem". As we all are familiar with the statements, I will rather write my translation of the laws instead of the official ones.

> **Theorem 2** — If no net force is acting on a body, then either the body is at rest or moving at a constant velocity.

The statement of the theorem is mathematically simple to describe, here we have the speed either a constant or 0, hence,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0$$

All it means is that the acceleration is zero.

Gallileo once did the experiment. The theory of motion was that anything moving will come to rest, no matter if there is force or not. Most of the philosophers before the time of him thought that "force" was needed to keep a body moving. For a body to move in a straight line at a constant speed, they believed that some external agent had to continually propel it. Otherwise it would "naturally" stop moving.[1]

Galileo tried to do the experiment and he rolled a block on a table. The table was bit rough so it was natural to think that the roughness "impelled" the body to move. Thus to decreae the impelling, some lubricant on the surface were added, that had proportionately decresed the amount of impelling and the block slid further. Using better quality of lubricant, the velocity of the block would decrease much slower. We can see there is some property that is being diminished as we keep adding lubricant.

We can extrapolate and say that if all the "property" could be diminished, then the block would continually keep moving. And of course as we have guessed, the "property" thing we are talking about is friction.

Galileo asserted that some external force was necessary to change the velocity of a body but that no external force was necessary to maintain the velocity of a body.

Now we come to the question, then what is the thing that can change the "velocity" of the body? By doing experiments with a spring and fiddlilng around with small cars and objects, we can come to the point that,

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[1] I copied some parts of it from HR without anybody's consent, I could have done no better sentence than that.

> **Theorem 3** — When several forces act on a boyd, each produces its own acceleration independently. The resulting acceleration is the vector sum of the several independent acclerations.

This casually motivates Newton's second Law. But here is a problem.

A baseball will be accelerated more by a given force than will an automobile. In order to define the relation of acceleration with force, we need to understand the property that "resists" motion.

This property of resisting motion is *Inertia*. It's the property that objects want to stay in their place. If any object, for example a spring, tries to force a mass, the mass also applies some force on the spring.

We define a quantity called *mass*, it will be account of inertia of the body, and if two bodies are acted on by the same force, each of their accelerations will be inversely proportional to their mass,

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

Newton's second Law is just a form that defines the above relation,

> **Theorem 4** (N2L) — Force is a vector $\vec{F}_i$ and if there is several forces *acting on the same body*, then the total force is,
>
> $$\sum_i \vec{F}_i = \vec{F}$$
>
> And the acceleration will be equal to,
>
> $$\boxed{\vec{F} = m\vec{a}}$$
>
> where m is the mass that takes care of the inertia of the body. To be honest, the proper way to define N2L is,
>
> $$\vec{a} = \frac{\vec{F}}{m}$$
>
> That makes use of the relation, more the force, more the acceleration, and more the mass (inertia), less the acceleration.

Another way of stating the N2L nicely is using some vector language,

$$\vec{F} = ma_x\ \hat{x} + ma_y\ \hat{y} + ma_z\ \hat{z}$$

Newton's Third Law took me a while to understand, either good textbooks had large testaments, and other places had useless ideas, until our college teacher in one of the first classes spoke "the forces are acted on two objects", that day finally the idea clicked.

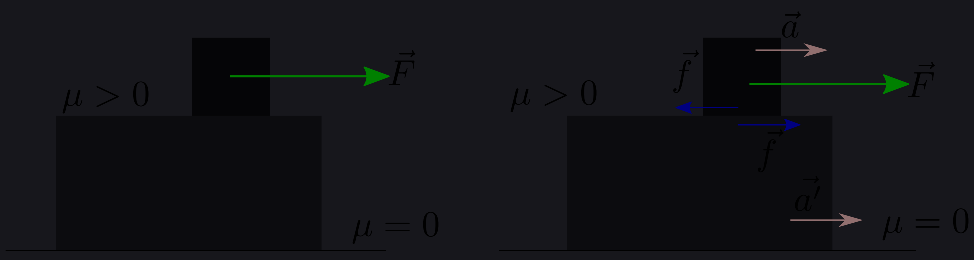


Figure 3.1.1: Needs a white background, I know.

Let us have this case motivated from a problem in Jaan Kalda Mechanics booklet. There is a large block and above it a small block. The large block and the ground have no mutual friction, but the block and the smaller block has some mutual friction.

Now if the upper block is acted on with some force, not very great at the moment, there will be some friction force acting on both of the masses. Both of them will have to move.

The friction force on upper block is f, and it is caused by the lower block. But as the lower block is causing the force, it will too be acted on by an equal but opposite force of magnitude f. That's what N3L says, if two masses are interacting, the same force will be applied on each of the bodies.

Upper block has $\vec{f}$ and the lower block faces $-\vec{f}$. The $-\vec{f}$ is all the force acting on the lower mass, hence its motion is dependent on force $-\vec{f}$. But the upper mass has net force $\vec{F} - \vec{f}$. This will cause the motion of upper block. And of course, given $\vec{f}$ is smaller than F, the upper mass is accelerating more than lower one. There is this kind of slipping effect.

Problem 7 (Independent Paths). There is a random hill that extends from $x = 0$ to $x = l$. Now, when a boy, pulls a block from the most bottom to a hight h of the hill, the hill and block has always a friction coefficient μ. Now, find the total work done by the boy, if he pulls the block very slowly.

Solution. So let he pull a very little amount of $\mathrm{d}l$ amount along the hill, so what will happen is that he shall have to do some amount of work against friction. That will be,

$$\mathrm{d}W_f = \mu N \mathrm{d}l = \mu mg \cos\theta \mathrm{d}l = \mu mg \mathrm{d}x$$

Taking the integral from $x = 0$ to $x = l$, we find that, this is simply,

$$W_f = \mu mgl$$

And for the potential energy he has to work,

$$W_p = mgh$$

For similar reasons. Total work thus needed,

$$\boxed{W = mg(\mu l + h)}$$

Problem 8 (Independent Paths). The previous problem, but the kid is moving with speed of v constant, starts off at an angle $\theta = o$ and ends at $\theta = 30°$.

Solution. We will have the curve of the hill to be responsible for some centripetal acceleration, so making some cetrifugal force reducing the force of gravity, hence overall fricion work. Reduction of the force is given by,

$$N' = N - m\frac{v^2}{R}$$

Now we have to know the radius of curvature. At any point, we can tell that from some origin, it is always feasible that,

$$\mathrm{d}l = R\mathrm{d}\theta$$

So, we can say that,

$$N' = N - mv^2\frac{\mathrm{d}\theta}{\mathrm{d}l}$$

Integrating, we find that,

$$W_f = mg\mu l + mv^2\int_{\theta_0}^{\theta_f} \mathrm{d}\theta$$

Adding to the last solution, this one is too solved.

Problem 9 (Speedbraker lite). The platform is made that is horizontal, and at one point, this suddenly bends down to an angle α to the horizontal. A cylinder of radius R and mass m is rolling along the path, and it rolls down the incline made. What is the limit of α so that the cylinder does not jump off from the track?

Solution. Let us call the centripetal acceleration (and force) as C. So,

$$C = \frac{mv^2}{R} = \frac{2\left(\frac{mv_0^2}{2} + mgR(1-\cos\theta)\right)}{R}$$

The normal force be called,

$$N = C - mg\cos\theta \geq 0$$

When the normal force is equal to zero, we shall have that the cylinder jump off the track. So, $N \geq 0$. Now, using this, and getting rid of m,

$$mv_0^2 + 2mgR(1-\cos\theta) - mgR\cos\theta \geq 0$$

A little more bit of solving and coming to the limiting case of the jumping off, we can tell that,

$$\frac{v_0^2}{gR} + 2 - 3\cos\theta = 0$$

We can think that when there is maximum angle made, then there is the least normal force, so, the max angle, is α.

$$\frac{v_0^2}{gR} + 2 = 3\cos\alpha$$

The solution is then straighforward,

$$\frac{1}{3}\left(\frac{v_0^2}{gR} + 2\right) = \cos\alpha$$

And thus the angle α is, at the limiting case,

$$\boxed{\alpha \geq \cos^{-1}\left(\frac{1}{3}\left(\frac{v_0^2}{gR} + 2\right)\right)}$$

I find the next problem quite fundamental.

Problem 10 (`Series springs equivalence`). There are two springs, one of k_1 and another of k_2 constant. If they are put in series, find the equivalent spring constant.

Solution. Let us take the system of two springs as mentioned in the problem and apply a force F on them. This will increase the length by Δx. We can tell that $\Delta x = \Delta x_1 + \Delta x_2$, the total increment of two springs.
The system is in equilibrium, hence, there is a force balance established. The force is balanced by the *tendency of the springs to contract, they pull both sides and strive to contract.* Hence, the point of spring where we apply F, there is this spring force F_2 that zeroes that and establishes a balance. So,

$$F_2 = F$$

At the junction of the two springs, there is also a force balance, the pull be the spring 1 is balanced by the opposing pull of spring 2. This means that,

$$F_1 = F_2$$

We have enough information to solve for the equivalent spring constant, the equivalent spring will have expansion,

$$\Delta x = \frac{F}{k}$$

Where k is the equivalent k constant. From the first derived equation,

$$k_2\Delta x_2 = k\Delta x \to \Delta x_2 = \frac{k}{k_2}\Delta x$$

And from $F_1 = F_2$,

$$\Delta x_1 = \frac{k}{k_1}\Delta x$$

We can solve for total expansion,

$$\Delta x = \Delta x_1 + \Delta x_2 = \frac{k}{k_1}\Delta x + \frac{k}{k_2}\Delta x$$

That resolved,

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}}$$

And my comment after seeing that was "why are springs capacitors?". Indeed there is something about, as $V = C\Delta q$, we have a similar case and analogy.

Problem 11. There is a disk of mass M and radius R that is precessing around a circle axis-ing a vertical line at a radius of r. The disk is hosted on a rod of length L and it is roatating around L at a speed of ω. Now, find the angular speed of the precession, for the case of ω is great enough than the precessing speed.

Solution. We don't derive the precession, let us assume that the object is already rotating around the vertical axis. So, we can assume the vertical angular speed to be $\mathrm{d}/\mathrm{d}t(\alpha)$. Suppose there is a $\Delta\alpha$ motion. So there will be a small change in the vector $\vec{L}$ that is the angular momentum.
Let r be the radius of the circle cut by the COM of the mass. Hence,

$$r = l\sin\theta$$

There is also the torque acting from the gravitational force. The torque is $\tau = mg\sin\theta l$. And we know,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau = mg\sin\theta l$$

Now, the small change in the angular momentum,

$$\mathrm{d}L = L\Delta\beta \quad \rightarrow \quad \tau = L\frac{\mathrm{d}\beta}{\mathrm{d}t}$$

Now, we need to find this β, we can just apply sin law, or just see the diagram carefully,

$$r\Delta\alpha = l\beta \quad \rightarrow \quad \beta = \frac{r\alpha}{l}$$

Now, let the object rotate at an angular speed ω, so,

$$L = I\omega$$

Altogether with the math, we wrap up,

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\tau}{L} = \frac{mg\sin\theta l}{I\omega}$$

Using the relation between α and β,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{mg\sin\theta l^2}{I\omega l\sin\theta}$$

So, we get that the angular vertical speed,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{mgl}{I\omega}$$

Now, the moment of inertia of the disk,

$$I = \frac{1}{2}mR^2$$

So, altogether, we can have,

$$\boxed{\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{2gl}{\omega R^2}}$$

Problem 12. There is a disk of mass M and radius R that is precessing around a circle axis-ing a vertical line at a radius of r. The disk is hosted on a rod of length L and it is roatating around L at a speed of ω. Now, find the angular speed of the precession, for the case of ω is NOT great enough than the precessing speed.

Problem 13. There is a cylindrical shape that has a block above it. The block starts to slip from the place and when it reaches x distance below it's initial height, it takes off. What will be the value of x if the radius of the cylinder is $R = 6$?

Solution. From the balance of accelerations, we can easily say that,

$$\frac{v^2}{R} = g - g\frac{x}{R}$$

Where I have given the x/R in place of the $\cos\theta$. Now, as Masruk vaiya has said, *centripetal acceleration is just the kinetic energy*, $v^2 = 2gx$,

$$\frac{2gx}{R} = g\left(1 - \frac{x}{R}\right)$$

Solving this, we get that,

$$x = \frac{R}{3} = \frac{6}{3} = 2$$

Indeed the result is quite interesting,

$$\boxed{x = \frac{R}{3}}$$

Problem 14. There is a sphere that hosts another sphere (small), or radius r, now the larger sphere is fixed, find the position where the smaller sphere would jump off the sphere, if it just starts from the top.

Solution. The smaller ball should take off only when the Normal force from gravity is overbalanced by the increasing centripetal acceleration. That is,

$$mg\cos\theta \leq m\frac{v^2}{R'}$$

Where, the $R' = R + r$, though this won't matter much later. Now, we need to know v^2 for the above equation and solve for the $\cos\theta$ for the desired case. If there is *negligible rotation*, then, there will also be negligible rotational energy, then we shall have,

$$\frac{1}{2}v^2 = gR'(1-\cos\theta)$$

But in case the rotation is not negligible, we have that, some energy be distributed to the linear speed and some for rotation, and the rotation contributes to the linear speed as well,

$$r^2\omega^2 = v^2$$

,

$$\frac{1}{2}v^2 = gR'(1-\cos\theta) - \frac{1}{2}\eta r^2\omega^2$$

Solving for the speed, we can have,

$$v^2 = \frac{2gR'(1-\cos\theta)}{1+\eta}$$

And putting this in the condition of jumping off,

$$\cos\theta = \frac{2}{3+\eta}$$

In the problem we have a uniform ball, so η is equal to $2/5$, so this gives the angle,

$$\theta = 54^\circ$$

Problem 15 (Swinging Arms)**.** When a man suddenly goes a little bit off the step while moving up by a stairway, he starts to fall. But, if he rotates his hand in that case, will it be fruitful?

Problem 16 (Rod detached)**.** A rod rests on a wall, almost vertically. The length is l, now after given a tiny kick over the top, it starts to fall down. Tell when the rod detaches its end from the wall.

Solution. The rod is more likely to move off the wall when its x axis speed is maximum.

Problem 17 (Slipping to Rotating, Edition 1.0). A ball with linear speed v is put on the ground. If it has moment of inertia coefficient η, then find the final speed when the ball has started to rotate and move without slipping.

Solution. The unknown friction force F causes the ball to slow down. It affects the linear speed v.

$$F = ma$$

There is also a torque. The torque is more responsible to rotate the ball eventually.

$$Fr = I\alpha$$

$$F = \eta m r \alpha$$

Now, there will be a change in speed $v' - v_0 = \Delta v$. From regular definitions,

$$\Delta v = at = \frac{F}{m}t = \eta r \alpha t$$

We recognize that $\alpha t = \omega$, so,

$$\Delta v = \eta r \Delta\omega = \eta v'$$

$$v' - v = \eta v$$

Solving this final equation tells us the speed, and interestingly (and intuitively) enough, it doesn't depend on the friction at all.

$$\boxed{v' = \frac{v}{1+\eta}}$$

Problem 18 (Slipping to Rotating, Edition 2.0). If the friction coefficient is μ, then how much time will it take to reach the non slipping stage of motion?

Solution. The friction force is then, $F = \mu m g$ This gives us that

$$a = \mu g$$

From the analysis of speed,

$$v - \mu g T = \frac{v}{1+\eta}$$

And if just solve for T, the result is,

$$\boxed{T = \frac{v}{\mu g}\left(\frac{\eta}{1+\eta}\right)}$$

Problem 19 (`Slipping to Rotating, Edition 3.0`)**.** How much distance should the ball cover before reaching the non slipping stage of motion?

Problem 20 (`Slipping to Rotating, Edition 4.0, (and last)`)**.** What is the loss of energy?

Solution. Just using the results from *Ed 1.0*, we can easily find that,

$$\boxed{\Delta KE = \frac{1}{2}mv^2\frac{\eta}{1+\eta}}$$

Problem 21 (`Wrapping around a pole`)**.** A hockey puck, attached to a piece of string, lying on the surface, moving on ice, is given a ta ngential kick and it starts to wrap the string around the pole. The puck is gradually spiraling in, so, from the angular momentum's conservation the speed should increase as the puck comes closer to the pole. Is it true?

Problem 22 (`Oscillation in Cylinder`)**.** A small ball wiht radius r is rolling without any slip inside a cylinder of radius R. What is the frequency of the small oscillations?

Problem 23 (`Oscillation in Oscillation`)**.** A hollow cylinder of mass M_1 and radius R_1 rolls without slipping on the inside surface of another cylinder of mass M_2 and radius R_2. For a moment assume that $R_1 << R_2$. Everything can move and everywhere friction is infinite. What is the frequency of the small osciallations?

Problem 24 (`Irodov 1.248, Ashmit Dutta`)**.** A uniform cylinder of radius R is spun about its axis to the angular velocity ω_0 and then placed into a corner. The coefficient of friction between the corner walls and the cylinder is equal to μ. How many turns will the cylinder accomplish until it stops?

Solution. Comparing forces in the x and y direction gives us the two relations

$$\begin{aligned} \mu N_2 + N_1 &= mg \\ \mu N_1 &= N_2 \end{aligned}$$

Substituting (2) into (1) gives us

$$\begin{aligned} \mu^2 N_1 + N_1 &= mg \\ N_1(\mu^2+1) &= mg \\ N_1 &= \frac{mg}{\mu^2+1}. \end{aligned}$$

From equation (2) we have the two relations of

$$\begin{aligned} N_1 &= \frac{mg}{\mu^2+1} \\ N_2 &= \frac{\mu mg}{\mu^2+1}. \end{aligned}$$

Examining torques yields

$$f_1R + f_2R = \mu N_1 R + \mu N_2 R = \mu R\left(\frac{mg}{\mu^2+1} + \frac{\mu mg}{\mu^2+1}\right).$$

We know two things as well. Namely

$$E_{\text{initial}} = \frac{1}{4}mR^2\omega^2, \qquad \text{work done: } \tau\theta = 2\pi\tau n$$

Therfore by combining these two equations we get

$$\frac{1}{4}mR^2\omega^2 = \mu R\left(\frac{mg}{\mu^2+1} + \frac{\mu mg}{\mu^2+1}\right)\cdot 2\pi n$$

$$\boxed{n = \frac{\omega^2 R(1+\mu^2)}{8\pi\mu g(1+\mu)}}$$

Problem 25 (Ashmit Dutta). Consider a 'rimless wheel', which consists of $N > 4$ thin radial homogeneous rods, similar to the spokes of a normal wheel. The rods are fixed to each other such that their ends point to the vertexes of a regular polygon. This object is placed on a slope with an angle α. Which is the minimum coefficient of friction, such that the body does not slip during rolling down?

Solution. The coefficient of friction can be found from considering the boundary case of static friction. From

$$ma = mg\sin\theta - F_s$$

and

$$I\alpha = F_s r$$

we get

$$a = g\sin\theta - \frac{F_s}{m}$$

and

$$\alpha = \frac{F_s r}{I}$$

With static friction there is no slipping thus we combine using $a = \alpha r$ to get

$$F_s = \frac{mI\sin\theta}{I + mr^2}$$

Since $F_s \leq F_{s,max} = \mu mg\cos\theta$, the angle where the "rimless wheel" stops rolling without slipping can be found as

$$\frac{mI\sin\theta}{I+mr^2} = \mu mg\cos\theta \implies \tan\theta = \mu\frac{I+mr^2}{I}$$

Let us now calculate the rotational inertia of the "rimless wheel" at it's center. Let us generalize the inertia for any number of rods n that is greater than 4. The rotational inertia a single spoke about it's end is $\frac{1}{3}mr^2$, applying this for n connected rods about it's center, we will then get the rotational inertia of the "rimless wheel" to be $\frac{n}{3}mr^2$. Substituting this value of I into the equation we found we get

$$\tan\theta = \mu\frac{\frac{n+3}{3}mr^2}{\frac{n}{3}mr^2} \implies \tan\theta = \mu\frac{n+3}{n}$$

This implies that μ is then

$$\boxed{\mu = \frac{n}{n+3}\tan\theta}$$

Problem 26 (Who pushed the Chimney?). A Chimney is standing upright, but some guy (probably from the nearby Physics degree college)gives a small kick to the top of the Chimney. And it topples over (!). At what point (relative to the bottom of the object)is it more likely to break?

This problem is little bit more dense, I would have included it in the advanced section, but general seems to be fine though.

Problem 27 (The Bridge, Ed 01). The **Euler-Bernoulli Beam Theory** describes the torque at a specific point on the rod as,

$$\tau = D\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

where the E is the Young's modulus, I is the second moment of area of the cross section, measured in mm, and w is the deflection at a certain location x along the rod.

When a beam is compressed, it tends to buckle. Show that the first mode of buckling occurs at a force of,

$$P_1 = \frac{\pi^2 EI}{L^2}$$

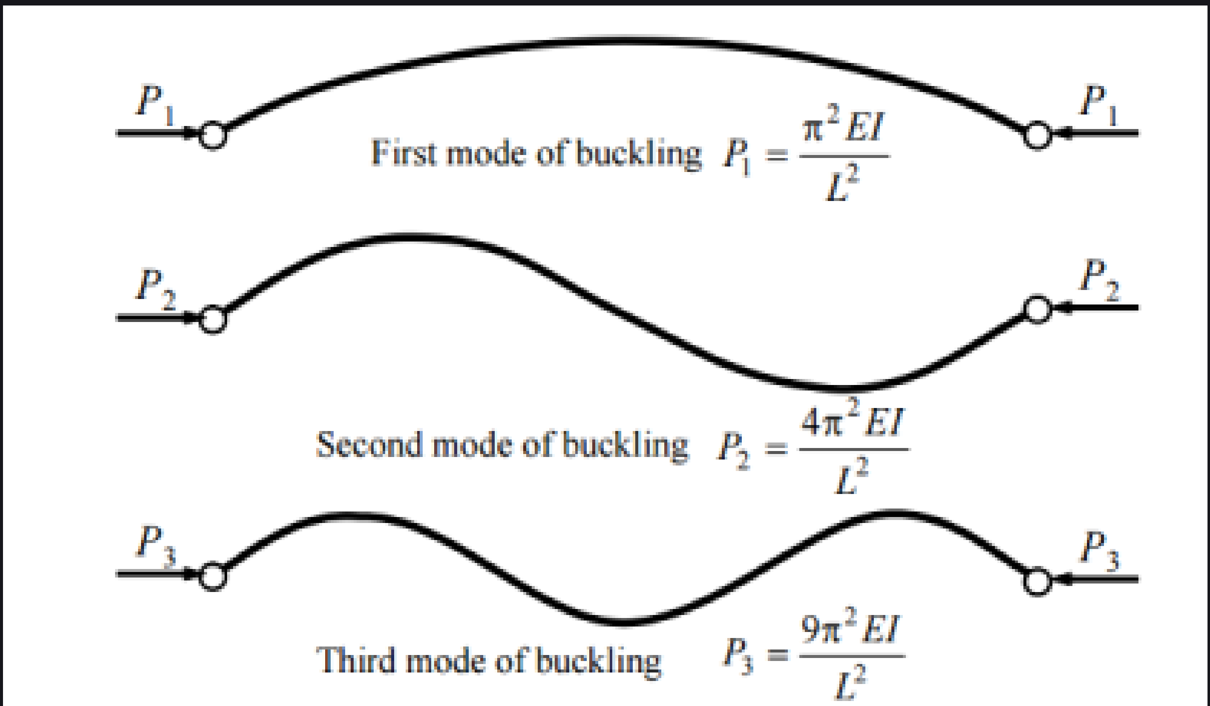


Figure 3.1.2: The standing modes of buckling, it reminds me about IPhO 2019 Thermoacoustic Engine.

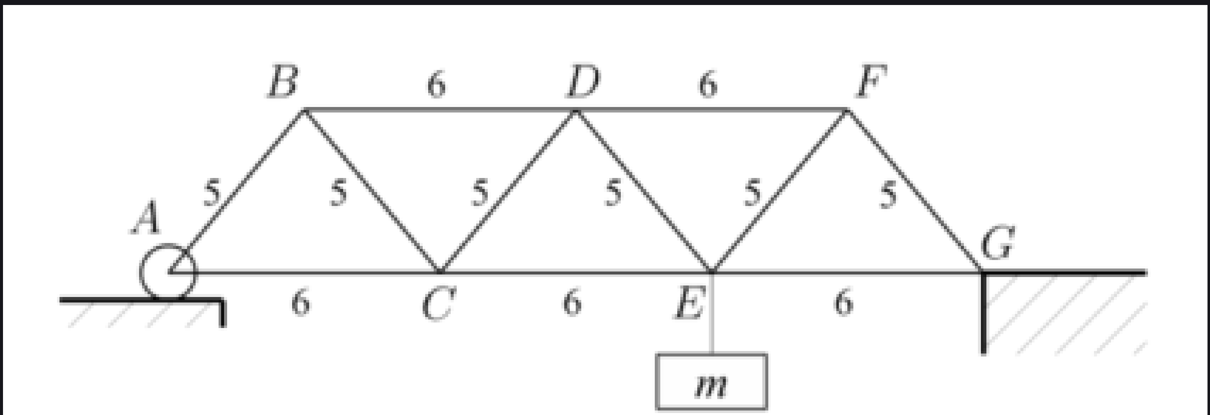


Figure 3.1.3

Problem 28 (The Bridge, Ed 02 by Qilin Xue).

Note. When designing safe bridges, it is vital to take into account buckling effects. They generally occur at smaller forces and when they do buckle, it happens unexpectedly. In 1907, it was the failure of the American engineer Theodore Cooper to properly take into account this effect that caused the Quebec Bridge to collapse.

This incident started the iron ring tradition among Canadian engineering students, and legend has it that the rings are made from the material of the failed Quebec bridge as a constant reminder of the lives at stake. Taking this in consideration, we go through the next problem.

Refer to the bridge in the diagram, it is made of massless structural steel with the following properties:

- The cross sectional area of 4250 mm^2.
- The moment of inertial $I = 20.6 \times 10^6\ mm^4$.
- The Young's Modulus $E = 200,000\ MPa$.
- The maximum tensile stress 1860 MPa.
- The maximum compressive strength of 1650 MPa.

At what mass m will the first member break? List the three most important memebers that should be reinforced.

What will be the deflection of member E?

This is a challenging problem (as Qilin goes), to start, how can you solve a staticss problem by viewing it from an energy standpoint?

Problem 29 (The Wedge which Slips). This is that epic legendary Jaan Kalda problem where there is this wedge and two mass that can rest supporting on the tension of the wire.

Solution. Solution by **yojan_sushi**, duh, I won't be able to solve these things.

1. Shift to the non-inertial reference frame of the wedge, and use the FBDs of m_1, m_2 to get 2 equations. These will involve a_0 (acceleration of wedge in ground frame), a_1 (acceleration of pulley system with respect to wedge), and T (tension in string). While doing the FBDs, you need to add an inertial force $-m_1 a_0$ to mass m_1 and an inertial force $-m_2 a_0$ to mass m_2. Also, as was pointed out above, T is constant because the pulley is massless. This provides us with 2 equations for 3 variables; we need 1 more equation!

2. Shift back to the ground frame, and use the fact that the center of mass of the system has no x-acceleration. (This is because $F_{ext,x} = 0$.) In order to find a_{cm}, we must find the acceleration of m_1, m_2, and M in the ground frame. By definition, the acceleration of M is a_0. However, the key idea here is that the accelerations of m_1, m_2 are not just $a_1 \cos\alpha_1$ and $a_1 \cos\alpha_2$: they are actually $a_1 \cos\alpha_1 + a_0$ and $a_1 \cos\alpha_2 + a_0$, respectively, because we need to add back the relative acceleration a_0 when we shift out of the wedge frame! If we call $a_1' = a_1 \cos\alpha_1 + a_0$ and $a_2' = a_1 \cos\alpha_2 + a_0$, we can find our desired 3rd equation by using $a_{cm} = \frac{m_1 a_1' + m_2 a_2' + M a_0}{m_1 + m_2 + M} = 0$.

To actually solve the equations, I believe the fastest way is to add the 2 equations from Step 1 (eliminating T), then write a_1 in terms of a_0 using the 3rd equation, and finally plugging this expression for a_1 back into the equation we found by adding the 2 equations from Step 1. This should give you an acceleration a_0 of

$$a_0 = \frac{(m_1 \cos\alpha_1 + m_2 \cos\alpha_2)(m_1 \sin\alpha_1 - m_2 \sin\alpha_2)}{(m_1 + m_2 + M)(m_1 + m_2) - (m_1 \cos\alpha_1 + m_2 \cos\alpha_2)^2},$$

as in the official solution.

Problem 30 (4 Blocks Stack). There are four block as in the diagram. The mutual friction between blocks has the coefficient μ but there is no friction of the blocks with the ground. If a force F is put on these masses M, m, then what will be each of their accelerations?

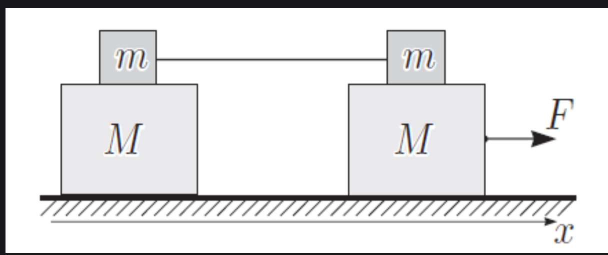


Figure 3.1.4

Solution. Let us first start the problem assuming there is no slipping, hence μ is high enough. Then we can tell,

$$F - T = (m + M)a$$

For first set (right side, force acting on them). For the left side, all the force possible is just the tension,

$$T = (m + M)a$$

The two equation solves that,

$$F = 2(m + M)a$$

Hence,

$$\boxed{a = \frac{F}{2(m + M)}}$$

But what condition is it for? We have to certify that too. Look, there is a friction of between the blocks m and M at the right side. The friction force on the top m acts leftwards and N3L states that a same force will be applied on M but on the rightways direction. The friction force is,

$$f = \mu N = \mu m g$$

Another, and final force that acts on the right side block m is the tension of string, naturally while pulling the string, the left part of the system will cause some resistance, here it will be T.
At the end of the day, the right side m actually accelerates, that means that,

$$f - T = ma = m\frac{F}{2(m+M)}$$

Tension is half of the force (from first two equation), hence,

$$\mu mg - \frac{F}{2} = m\frac{F}{2(m+M)}$$

Solving this, we come to the point that the friction force must be equal or lesser to,

$$\mu mg \geq \frac{mF + F(m+M)}{2(m+M)}$$

Solving that,

$$\boxed{2\mu mg\frac{m+M}{2m+M} \geq F}$$

So, this is the condition, the first one.
The second condition can be there is slipping, and if you look closely, the right m will slip. It's slipping will cause lesser tension T on the left part, so the left m will not have enough force to slip, as it starts to move with the larger leftside M on frictionless surface.
Let us look for the acceleration of the larger right block,

$$F - f = MA_1$$

Solves to,

$$\frac{F - \mu mg}{M} = A_1$$

Force on the right side m,

$$\mu mg - T = ma_1$$

The tension will cause the left side m and M to move together, slipping on surface,

$$\frac{T}{m+M} = a_1 = \frac{\mu mg - T}{m}$$

And the acceleration is equal because of the string,

$$T = m\mu g\frac{M+m}{2m+M}$$

And this case, acceleration of the right side m and leftside blocks ,

$$\boxed{a = \mu mg\frac{1}{2m+M}}$$

And this condition is; when the 1st. non-slip condition isn't satisfied, or you can say,

$$\boxed{2\mu mg\frac{m+M}{2m+M} \leq F}$$

Problem 31 (Raindrop through the Fog). In a very dense fog, there are many tiny water drops, that 'float' in the air with negligible speed. If one of the water drops, which is a little larger than the rest, begins to sink, it absorbs those smaller drops that lie in tis path (figure).

The ever growing drop, which can be regarded as spherical, is found to be accelerating uniformly, despite the air drag - proportional to the square of the speed of the drop and also to the cross sectional area of the drop.

What is the maximum value for the acceleration?

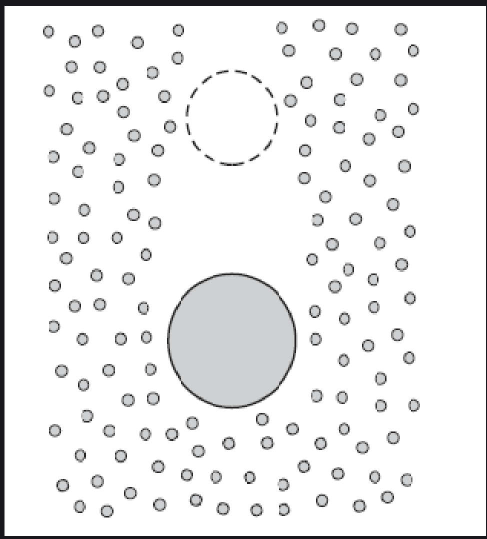


Figure 3.1.5

Solution. As the drop is falling through the fog, it increases it's mass by taking the fog-particles. The volume increases, and as the drop is always spherical, the radius is also suppossed to increase with time.
Now, there is a constant acceleration, that means the forces are kind of balanced, that gives the net force always the same.
But we cannot use $F = ma$, because the mass of the drop is not a constant, thus, we have to go a little bit more fundamental and use the very own,

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}$$

By using the rule,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}\,(mv)}{\mathrm{d}t} = v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Now let us take accounts of all the external force that works. If we consider the vertical, coming down towards the ground as mg, then the drag force is of course pointing the opposite direction, $F_d = -bv^2 \times A$.

$$F = mg - bv^2\left(\pi r^2\right)$$

Now we can write,

$$v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv^2\left(\pi r^2\right)$$

For sure we know that,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a$$

We need to know what $\frac{\mathrm{d}m}{\mathrm{d}t}$ is. So, this is the rate of change of the mass; or more directly the increase of the mass per unit time.
What is supposed to be the increase of mass?
So say the speed of the drop is v, thus in $\mathrm{d}t$ time, it crosses a distance $v\mathrm{d}t$. And the cross section is $A = \pi r^2$. This volume,

$$\mathrm{d}V = (v\mathrm{d}t)\left(\pi r^2\right)$$

is the volume of 'fog' that is intaken by the drop.
Let the density of the fog be n. Then, mass of the volumetric region $\mathrm{d}V$ crossed by the drop in $\mathrm{d}t$ time is,

$$\mathrm{d}m = n\mathrm{d}V = vn\left(\pi r^2\right)\mathrm{d}t$$

We get,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = vn\left(\pi r^2\right)$$

Let's put the pieces together, remembering $m = \rho V$,

$$v^2 n\left(\pi r^2\right) + \frac{4}{3}\pi r^3 \rho \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{4}{3}\pi r^3 \rho g - b\left(\pi r^2\right) v^2$$

Taking the acceleration $\frac{\mathrm{d}v}{\mathrm{d}t} = a$ at oneside,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\frac{4}{3}\pi r^3 \rho g - v^2 (b+n)\pi r^2}{\frac{4}{3}\pi r^3 \rho}$$

So, after solving a little more,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{3v^2 (b+n)}{4r\rho}$$

Now, we have an unknown r in the equation, that actually solely depends on the motion of the drop (as in the path of motion, the drop keeps including fog in it).
So, we know,

$$\mathrm{d}m = \rho\mathrm{d}V = \rho\left(4\pi r^2 \, \mathrm{d}r\right)$$

As we know, $\mathrm{d}m = vn\left(\pi r^2\right)\mathrm{d}t$,

$$vn\left(\pi r^2\right)\mathrm{d}t = \rho\left(4\pi r^2 \, \mathrm{d}r\right)$$

So, $\mathrm{d}r$ is,

$$\mathrm{d}r = \frac{n}{4\rho} \, v \, \mathrm{d}t$$

Integrating, we have,

$$\int \mathrm{d}r = \frac{n}{4\rho} \int v \, \mathrm{d}t$$

This solves,

$$r = \frac{n}{4\rho} \Delta x$$

Here, Δx is the displacement of the drop (I should have used a Δh or Δy though, anyways, who cares?). Now, we can put r in it's place,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{3v^2\,(b+n)}{4\rho\left(\frac{n}{4\rho}x\right)}$$

And as we already know, $v^2 = 2a\Delta x$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{3v^2\,(b+n)\times 2a}{v^2 n}$$

Solving it,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a = g - \frac{6a\,(b+n)}{n}$$

What we have is now,

$$a = g\left(\frac{n}{6b+7n}\right)$$

Now, there is something critical going on. The factor b is something of a replacable thing above. Because from advanced fluid dynamics, assuming that the flow of the air is laminar, the constant b is,

$$b = \frac{1}{2}c_d\rho_{air}$$

Here c_d is the drag coefficient, and I get the idea of this from the footnote 29 of the problem source book $200p^3$.

$$a = g\frac{n}{n\left(3c_d\frac{\rho_{air}}{n}+7\right)}$$

In optimal cases, the max is $g/7$, given the conditions meet. Maximum possible,

$$\boxed{a = \frac{g}{7}}$$

Problem 32 (`Kalda Mech Pr. 25`). There is a wedge that has two incline of α at two sides (respect to the horizontal). A ball of m is at the center (valley) and another ball M is above by the incline along the right side. If the ball M is let go and if it hits with m, then what is the condition for the m to lift off?

Solution. The m is at rest, and it's gravitational force is balanced by the two normal force N_1 and N_2, initially they are equal.
Now, for the m to roll above the ramp, it has to overcome the force that pulls it down, which is the gravitational forces component on the incline, which, obviously faces downwards.
When the M hits the m, then there will be blow of F force, that $\vec{F}$ is parallel to

right side of the incline, which makes an angle 2α with the left incline. *Component of this force along the incline is the force that will force the ball m to ramp up along the incline.* The component is,

$$F \cos 2\alpha$$

But what is this F force? It is the component of the weight of the M mass along the right incline.

$$F = Mg \cos \alpha^c = Mg \sin \alpha$$

Notice that $\alpha^c = \frac{\pi}{2} - \alpha$, that stands for complementary, useful notation while dealing with right angled triangle. So solving for that previous force component $F \cos 2\alpha$,

$$F_c = (Mg \sin \alpha) \cos 2\alpha$$

This should be equal, or greater than the resistng force to lift the mass, the force that pulls the ball back is just,

$$F_p = mg \cos \alpha^c = mg \sin \alpha$$

Hence, solving this,

$$F_c \geq F_p$$

$$Mg \sin \alpha \cos 2\alpha \geq mg \sin \alpha$$

That is just,

$$\boxed{M \cos 2\alpha \geq m}$$

Problem 33 (Rock Climber). NBPHO problem

Solution. The maximum sustainable acceleration is $5g = a_{max}$. The graph shows the stress-strain in kilo-newtons and strain in form of %
Climber falls, then expansion of the rope in percentage is going to be,

$$\varepsilon = \frac{l - L}{L + H}$$

Now, let the climber fall, then,

$$\sigma(\varepsilon) - mg = ma$$

This gives us,

$$\sigma(\varepsilon) = 6mg = 4.7\ Kn$$

Now, $\sigma = 4.7\ Kn$, which is at $\epsilon = 32\%$ so,

$$\varepsilon = 0.32 \to \frac{l - L}{L + H} = 0.32$$

Hence,

$$\boxed{l = 0.32\,(L + H) + L}$$

Now, we have found the maximum distance of l, if l is more, stress would be greater and $5g$ acceleration cannot be true.

$$\boxed{l \leq 0.32\,(H+L)+L}$$

Problem 34. There are 3 balls that are attached with two rods with hinges and contrained to move in 1 plane. ONe of the ball is given a blow of velocity v.

Find the minimum distance between the two ends balls possible after some time.

Solution. First of all, it is clear that because of the blow, the shape of the structure is going to change. The blow has speed v, hence the center of mass will move along the vector $\hat{v}$ in,

$$3m_{cm} = mv \rightarrow v_{cm} = \frac{v}{3}$$

The direction is same along the line. The whole system will move along with time in this constant speed, because no other thing is causing an external force.
Now, the shape of the structure changes, why would it?
I thought about it for some time, the first thing that came in my head was that it's probably rotating, that's causing the system to have some change in shape.
This way of thinking was resonable because the angular momentum input into the system when the blow is given,

$$L = mvl$$

And angular momentum is going to be constant. We can try solvig the other side of the quation (i.e solve for L) from an other system of equation and use it to solve for the required quantity.
I was still confused whether the system will rotate or not. There is only one way to check.
The total energy initially is going to be equal to the total rotational energy (if it is present) and the kinetic energy that accounts for the translational motion.
So, if the latter translational motion of after the blow is equal to the total energy, there is no rotational motion.

$$E = E_r + E_t$$

Now, the Translations energy E_t is,

$$E_t = \frac{1}{2}(3m)v_{cm}^2 = \frac{3}{2}m\frac{v^2}{9} = \frac{1}{6}mv^2$$

The total energy is,

$$E = \frac{1}{2}mv^2$$

The rotational is,

$$E_r = E - E_t = \frac{1}{2}mv^2 - \frac{1}{6}mv^2 = \frac{1}{3}mv^2$$

Which is non zero.
Now, it took me a while to find this rotational energy, to be extremely honest, I was about to give up and look at the solutions, but after I peeked a little something came in my minds that I thought would work.
As we now have this energy, we can find the rotation. So,

$$E_r = \frac{1}{2}I\omega^2$$

We can reduce ω using, $L = I\omega$,

$$\frac{2E_r}{I} = \omega^2 = \left(\frac{L}{I}\right)^2 \to I = \frac{L^2}{2E_r}$$

Yes, we are going to use this thing, it's the pivotal equation that we will use to solve the problem finally.
We have to find the moment of inertia. There are ways to do it, the most straight forward is just finding out the equation directly. Taking the shape of the structure, we can see it's an isoceles triangle. The median is COM and the height the COM is located at height is $y = \frac{H}{3}$ above the 3rd (uequal) side. Using this, the distance is calculated to the mass from center,

$$I = m\left(\frac{2}{3}l\cos\varphi\right)^2 + 2m(\sqrt{d^2 + (\frac{1}{3}l\cos\varphi)^2})^2$$

Here $d = l\sin\varphi$. The angle φ is half the angle between the two equal sides. It takes some time solving it, but the result is,

$$I = ml^2\left(2 - \frac{4}{3}\cos^2\varphi\right)$$

But we had found another equation ofr I,

$$I = \frac{L^2}{2E_r} = \frac{m^2v^2l^2}{\frac{2}{3}mv^2} = \frac{3}{2}ml^2$$

Now,

$$\frac{3}{2} = 2 - \frac{4}{3}\cos^2\varphi$$

Solving it, we need $\sin\varphi$, because the distance $2d = 2l\sin\varphi$,

$$\sin\varphi = \sqrt{\frac{5}{8}}$$

Now if we plug this in $2d$,

$$\boxed{\sqrt{\frac{5}{2}}l}$$

This is the solution.

Problem 35 (Irodov 1.73). In the arrangement shown in figure, find the accelerations of the body m_1 because there's no frictions. The masses are stated, m_2, m_0 the rest.

Solution. Let's work in the frame of the world (or just the table). Here, the acceleration of the two mass hanging be a_1 and a_2.
m_1 and m_2 are actually falling because of the pulley configuration.
Let the hanging pulley accelerate at a_0, and because of string, the m_0 should also accelerate at a_0. Now, in the accelerating frame (a_0), where the pulley is stationary, let the acceleration of the masses be a' and $-a'$, they must be equal. This means,

$$a_1 = a' + a_0$$

$$a_2 = -a' + a_0$$

From this, we get the result, $a_0 = \frac{a_1+a_2}{2}$, by adding the two equation.
Now, the equation of motion for m_0,

$$m_0 a_0 = T_0$$

The equation of motion for m_2,

$$m_2 g - T = m_2 a_2$$

The equation of motion for m_1,

$$m_1 g - T = m_1 a_1$$

The total force on the hanging pulley must be zero, that means,

$$F_p = T_0 - 2T = 0 \rightarrow T = \frac{T_0}{2}$$

So, finally, we need to solve,

$$m_2 g - \frac{m_0 a_0}{2} = m_2 a_2$$

$$m_1 g - \frac{m_0 a_0}{2} = m_1 a_1$$

$$\frac{a_1 + a_2}{2} = a_0$$

This will give a_1 as a result.

Method 2: Let's move to the frame of the hanging pulley, there the acceleration of m_1 be a' and for m_2 be $-a'$. Which is feasible.
This frame is accelerating downwards with respect to the world in a_0 acceleration. So, for the frame, there will be a fictitious force $m_1 a_0$ on the mass m_1. Same to m_2.
Thus, in the accelerated frame, the equation of motion,

$$m_1 g - m_1 a_0 - \frac{m_0 a_0}{2} = m_1 a'$$
$$m_2 g - m_2 a_0 - \frac{m_0 a_0}{2} = -m_2 a'$$

Here we have to solve for a_0 and a' then the result we need is just,

$$a_0 + a' = a_1$$

I put thing in wolfram, it solves the equation,

$$a' = \frac{-m_0 m_1 g + g m_0 m_2}{m_0 m_1 + m_0 m_2 + 4 m_1 m_2}$$
$$a_0 = \frac{4 m_1 m_2}{m_0 m_1 + m_0 m_2 + 4 m_1 m_2} g$$

From this,

$$\boxed{a_1 = \frac{4 g m_1 m_2 + m_0 g (m_1 - m_2)}{4 m_1 m_2 + m_0 (m_1 + m_2)}}$$

Problem 36 (Irodov 1.74)**.** In the arrangement shown in the figure, the mass of the rod is M and the mass of the bead is m. The bead can slide along the string with some friction. At initial moment $t = 0$, the bead is at the lower end height of the rod. Aftr time t, the ball reaches the top height of the rod, the rods length is l. Find the friction between the string and the bead. The bodies moved at constant acceleration.

Solution. The mass m is pulled up by friction force f. By N3L, the tension of the rope is also just f. Hence, the equations of motion,

$$f - mg = ma$$

$$Mg - f = MA$$

Now, let in t time, the m shift upwards by distance s_1. And the rod's top end shift down by s_2, for m to reach the top end of rod in t time, the s_1 and s_2 should follow a rule,

$$s_2 + s_1 = l$$

It can seen by drawing a nice diagram. This is,

$$\frac{1}{2}(a + A) t^2 = l$$

Solving these three equations, we can get,

$$\boxed{f = \frac{2l}{t^2} \frac{mM}{M - m}}$$

3.2 Esoteric Lagrangian and Energy

Problem 37. There is a cone with a height h, and a base circle of radius r formed with a sector shaped paper. The sheet is such the size and shape ahtat its two aight edges alsmost touch on the sloping surface of the comne. In this state the cone is stress-free.

The cone is placed on a horizontal, slippery table-top, and loaded at its apex with a vertical force of magnitude w, without collapsing. The splaying of the cone is opposed by a pair of focred of magneitude F acting tangentially at the join in the base circle. Find the value of F.

Solution. The work done to push δh of the cone from the tip,

$$\delta W = w\delta h$$

The exactly smae amount of work is needed to be done by the force F to resist splaying for δh, the caused mothion of ends,

$$l^2 = h^2 + r^2 e$$

δh cause δr increase in the splay.

$$l = const$$

$$2h\mathrm{d}h + 2r\mathrm{d}r$$

This solves to,

$$\mathrm{d}r = \frac{h}{r}\mathrm{d}h$$

Now, we have,

$$2\pi\mathrm{d}r = \mathrm{d}x$$

So, the solution,

$$\therefore \quad 2\pi\frac{h}{r}\mathrm{d}r = \mathrm{d}x$$

Staticity require,

$$\delta W = \delta W = w\mathrm{d}h = F\mathrm{d}x$$

So, we can see that,

$$w\mathrm{d}h = F\ 2\pi\frac{h}{r}\mathrm{d}h \qquad \rightarrow \qquad \boxed{F = \frac{wr}{2\pi h}}$$

Problem 38. Two masses, m_1 and m_2 are connected with a spring of constant k and find out the natural frequency of their oscillation.

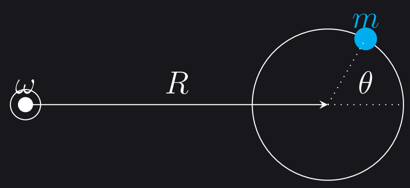
Solution. The work of reduced mass is two make the mathematical base of the system act as if it is just a si ngle mass system.
The mass of the si ngle mass analog is μ that is called the reduced mass.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

The frequency will be same as of the system, thus, that is,

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} = \frac{1}{2\pi}\sqrt{k\frac{m_1 + m_2}{m_1 m_2}}}$$

Problem 39 (Bead on ring and rotating). There is a ring of radius r where a small bead of mass m can freely move. The ring is rotating with ω with axis R, find the equation of motion of the bead, and the frequency of small oscillation about the equilibrium point.



Solution. The distance of the mass from the axis,

$$r^2 + R^2 - 2Rr\cos\theta'$$

As,

$$\theta' = 180 - \theta$$

we have,

$$r^2 + R^2 + 2Rr\cos\theta$$

The lagrangian is,

$$\mathcal{L} = T - V = T\frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(r^2 + R^2 + 2Rr\cos\theta)\omega^2$$

Here is no potential energy of gravity. The EL first part,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = mr^2\ddot{\theta}$$

Second part,

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mRr\sin\theta\omega^2$$

So we have,

$$\boxed{\ddot{\theta} = -\frac{R}{r}\sin\theta\omega^2}$$

The equilibrium is where, $\ddot{\theta}$ is 0. So,

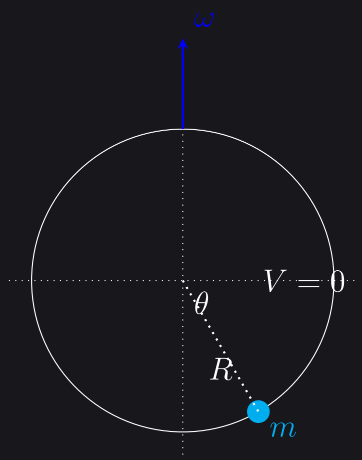
$$\theta_{equilibrium} = 0$$

The oscillation is,

$$f = \frac{\omega}{2\pi}\sqrt{\frac{R}{r}}$$

This problem was solved quite quickly.

Problem 40 (Bead in a rotating Ring). There is a ring of mass m that is put in a vertically kept ring of radius R. The ring is rotated in a vertical axis, find the stable equilibrium.



Solution. Assuming that the zero potential is the horizontal line through the center, we can tell that,

$$\mathcal{L} = \frac{1}{2}m\omega^2 R^2 \sin^2\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

Taking the derivatives of Euler Lagrange quickly tells us,

$$\ddot{\theta} = \sin\left(\omega^2\cos\theta - \frac{g}{R}\right)$$

For equilibrium we require the Force is zero, hence the accleration is zero,

$$\sin\theta = 0$$

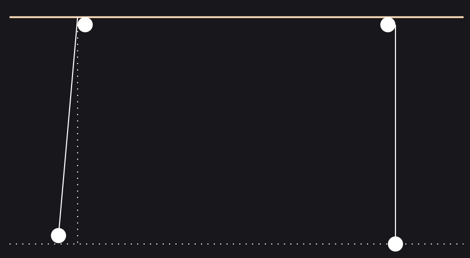
Or,

$$\omega^2\cos\theta - \frac{g}{R} = 0$$

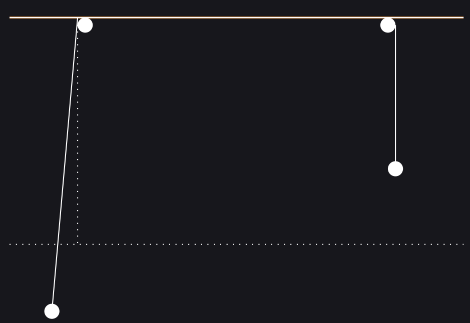
The equilibrium that can give oscillation,

$$\theta = \arccos\frac{g}{\omega^2 R}$$

Stable equilibriums...



Problem 41 (One swings). There are two pulleys and using a string two equal masses m are hung. They are initially both l away from the pulley. One of them is given a tiny kick imparting a speed v and amplitude of oscillation ϵ. Which of the mass will hit the pulley first?



We can do some pre thinking, look, if one of the mass is oscillating, then it must be in some circular trajectory and have some angular speed, now if this is the case, then there is also some centripetal acceleration by the mass.

This acceleration is likely to cause increased tension in the string. So we can think that the mass that is oscillating will try to pull the string a little bit more, hence, the mass that is initially kept stationary is likely to hit the pulley.

Solution. Let there be some movement in the system, then if there is a displacement of y of the masses vertically, and assuming the ceiling to be the $V = 0$, distance of the oscillator from pulley,

$$(l + y)\cos\theta$$

And the other one,

$$l - y$$

Here y can be either positive or negative we don't care for now, all we need to assume is there is some motion after $t = 0$ because I was confused about this simple fact when trying to solve this. Oh yes, I forgot to mention I could not solve this either!

We can write the potential to be,

$$V = -mg(l + y)\cos\theta - mg(l - y)$$

Putting this to the lagrangian, after assuming there exists some motion of the string that contributes to the kinetic energy of both the masses,

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{y}^2 + (l+y)^2\dot{\theta}^2) + mg(l+y)\cos\theta + mg(l-y)$$

Remind we need the negative of the potential energy, and there is some extra kinetic energy because one of the mass is subject to oscillation, resulting in a rotation θ as $\dot{\theta}$, here θ is the angle made with the vertical line by the rope.
There are two “generalized coordinates” or say independent coordinates our depend on,

$$y, \theta$$

So we are to take the EL twice,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = 2m\ddot{y}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \ddot{\theta}}\right) = m(l+y)^2\ddot{\theta}^2$$

Now the derivatives on y, θ,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} &= 2m((l+y)\dot{\theta}^2) + mg\cos\theta - mg \\ \frac{\partial \mathcal{L}}{\partial \theta} &= -mg(l+y)\sin\theta\end{aligned}$$

So at the end of the day what we have is,

$$\begin{aligned}2m\ddot{y} =& 2m((l+y)\dot{\theta}^2) + mg\cos\theta - mg \\ m(l+y)^2\ddot{\theta}^2 =& - mg(l+y)\sin\theta\end{aligned}$$

The first order approximation for $t = 0$ gives us,

$$2\ddot{y} = 0$$

$$\ddot{\theta} + \frac{g}{l+y}\sin\theta = 0$$

Now on we use d for $l+y$ because this merely just the distance. For unknown reason (of course what I am writing is true, I just don't understand that why), Morin takes the 2nd order approx to say that,

$$2\ddot{d} = 2d\dot{\theta} - g\frac{\theta^2}{2}$$

$$\ddot{\theta} + \frac{g}{l+y}\theta = 0$$

The solution of the $\ddot{\theta}$ is given by,

$$\theta = \epsilon\cos(\omega t)$$

$$\omega = \sqrt{\frac{g}{d}}$$

Hence,

$$\dot{\theta} = -\omega\epsilon\cos(\omega t)$$

This result when put in the $\ddot{d}$,

$$2\ddot{d} = 2d\left(\omega^2\epsilon^2\sin^2(\omega t)\right) + \frac{g}{2}\left(\epsilon^2\cos^2(\omega t)\right)$$

This results in,

$$2\ddot{d} = g\epsilon^2\left(\sin^2\omega t - \frac{\cos^2\omega t}{2}\right)$$

Now Morin and people who can solve problems better than me will take the average, I still don't know why, how can a small approximation be further used for a full cycle seeming tends to be invalid to me, If I had done that, that probably must have been wrong. Taking the average, both $\sin^2\omega t$ and $\cos^2\omega t$ are $1/2$, so, the average is ,

$$\ddot{d} = \frac{g\epsilon^2}{8}$$

Now this actually tells that y is increasing, that means the non oscillating mass is coming up, that means the oscillating cup will eventually go down.

Problem 42 (Mass on Edge of a Wheel). There is a mass at the edge of the cylindrical wheel with radius R and the wheel has mass centered at its center. There is an angle θ made between the vertical and the line joining the mass at the edge with cylinder center. Let the mass be m and cylinder mass M. Find the frequency of oscilaltions.

Solution. We must go the the referance point on the ground where rests the CM, distance from the CM to the mass m is,

$$(x, y)_m = (R\theta - R\sin\theta, R - R\cos\theta)$$

We get the $R\theta$ term if the cylinder rolls to the CM initially making the angle θ with vertical. If we differenciate the $(x, y)_m$, we find the speed, then adding the squares (vector addition), we get the velocity,

$$v^2 = 2R^2\dot{\theta}^2(1 - \cos\theta)$$

The speed M has is ,

$$V^2 = R^2\dot{\theta}^2$$

The lagrangian,

$$\mathcal{L} = \frac{1}{2}MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1 - \cos\theta) + mgR\cos\theta$$

Taking the EL gives us,

$$\left(MR^2 + 2mR^2(1-\cos\theta)\right)\ddot{\theta} = -mR^2\dot{\theta}^2\sin\theta - mgR\sin\theta$$

Knowing that for small angles, first order,

$$1-\cos\theta \cong 0 \qquad\qquad \sin\theta = \theta$$

We have,

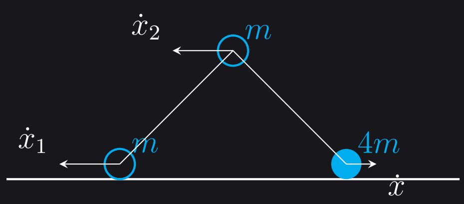
$$MR\ddot{\theta} = -mg\theta$$

So,

$$\omega = \sqrt{\frac{m}{M}\frac{g}{R}}$$

Let us have another nice problem after a tedious 2 hour work.

Problem 43 (M65, Kalda). There are two cylinders of mass m and another of $4m$. They are joined with hinged l length rods, with the $4m$ in the ground. The angle made with the rods is $90°$. If it is left from that postion, find the acceleration of the $4m$ just after release.



Solution. There is a thing we need to consider, the mass at one corner is $4m$, for that reason the center of mass will not be aligned in the centroid of the isosceles trinagle made by the cylinders. CM will only be in the centroid if the masses are equal. I made the mistake of not thinking about the unbalanced mass $4m$ that took me an hour to figure out.
Now, we have the center of mass a little raised up from ground, but as it is let go, it will fall back to ground, but there is no x axis (horizontal) component of force, so the x coordinate of the CM will not change.
Let us call the position of the $4m$ as x and the other mass on ground as x_1. The x axis position of the mass above the ground would be x_2. By the Conservation of the position of the center of mass along x axis, and moving our referance frame to the center of mass

$$0 = \frac{mx_1 + mx_2 + 4mx}{6m}$$

This actually tells us that,

$$x_1 + x_2 = -4x$$

And taking the derivatives we know,

$$\dot{x}_1 + \dot{x}_2 = -4\dot{x}$$

Here $\dot{x}$ is the velocity, so I will ease my sense of notation a bit,

$$v_1 + v_2 = -4v$$

We are putting that negative just because the $4m$ moves in the other direction the other two mass does.
If the system was symmetric, instead $4m$, we had it equal to other masses m, then the mass that is put above the ground would just come down but didn't move along the x axis.
Here this is not the case, the m that is above the ground will be moving to the right (according to the figure), keeping position of the CM intact (along x axis). And the $4m$ will move to the left, balancing the whole system.
Let us write the kinetic energy T for the lagrangian,

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{4}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

The last term $\dot{y}$ comes because the mass above the ground is also falling down, as it pushes the masses on the ground apart using the rod.
Now we shall focus to find the $\dot{y}$, that requires us to move to the frame of the mass above the ground.
In that frame, we notice something nice, in the frame of the mass above the ground (let us call it second mass from now as we have named it with position x_2), the shape of rod is always an isosceles triangle. And if the masses $4m$ and m on the ground are not moving in same and opposite speeds, in this frame, the triangle won't be isosceles, one of the mass has to lift off.
In the frame of the second mass, the speed of the masses on ground will be equal but pointing to the opposite directions.
To think this we need to think the second mass is resting, and the two mass below are moving, on the ground, if one of the mass moves faster, the one moving faster will have to either leave the ground, or pull the second mass to motion, which is contradictory. We also know none of the masses on the ground going to lift off.
This speed equality helps us to find the vertical speed of second mass and also an equation that will make the mass easier. In the new frame we just discussed, the speeds of bodies are same, so just looking at the magnitude of them,

$$v_1 - v_2 = v + v_2$$

Which is,

$$v_1 = v + 2v_2$$

We will use this equation a bit later, let us quickly find the vertical speed of the second mass, we shall stay in the frame of the second mass's x axis position, so we can view it's vertical speed,

$$h^2 = l^2 - x^2$$

Here, just for now, x is half of the distance between the two masses on ground, so differentiating, with respect to time,

$$2h\frac{\mathrm{d}h}{\mathrm{d}t} = -2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

Of course we use the Chain rule above, which says,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Now, we know that the speed $\frac{\mathrm{d}x}{\mathrm{d}t}$ is speed of the end moving away in the second mass frame, and that certifies,

$$\dot{h} = -\frac{x}{h}\left(\dot{x} + \dot{x}_2\right)$$

Hence, the lagrangian is,

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{4}{2}m\dot{x}^2 + \frac{1}{2}m\frac{x^2}{h^2}\left(\dot{x} + \dot{x}_2\right)^2$$

And we also have some equations that relate the x axis position of masses, from the CM position conservation and just consider the magnitude of velocity,

$$v_1 + v_2 = 4v$$

And we found few minutes ago,

$$v_1 = v + 2v_2$$

Solving them, we actually find a decent relationship!

$$\begin{aligned} v_2 &= v \\ v_1 &= 3v \end{aligned}$$

Very funny, our lagrangian will just shrink now, put things in place and taking the repeating common factor $\frac{1}{2}m$,

$$\frac{1}{2}m\dot{x}^2\left(9 + 1 + 4 + 4\frac{x^2}{h^2}\right) = \frac{1}{2}\left(18m\right)\dot{x}^2$$

Now we should think a thing for a moment, we have to solve the Euler Lagrange Equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}$$

Or the method 6 from Mechanics by Kalda is similarly useful,

$$\ddot{x} = -\frac{V'(x)}{\mathcal{M}}$$

Which is if the total energy can be written as,

$$\frac{1}{2}\mathcal{M}\dot{x}^2 + V(x) = \text{Const}$$

Hence, at the end of finding the T,

$$T = 9m\dot{x}^2$$

The potential energy is, considering that half of the distance between masses on ground is x,

$$V = mg\sqrt{l^2 - x^2}$$

The full lagrangian in its form,

$$\mathcal{L} = 9m\dot{x}^2 - mg\sqrt{l^2 - x^2}$$

We can use the take the derivative of the Euler Equation now,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\left(9m\dot{x}^2 - mg\sqrt{l^2 - x^2}\right) = \frac{\partial}{\partial x}\left(9m\dot{x}^2 - mg\sqrt{l^2 - x^2}\right)$$

When we take the second part, $\frac{\partial \mathcal{L}}{\partial x}$, then there is this term of Kinetic energy, $\frac{1}{2}m\left(4\frac{x^2}{h^2}\right)\dot{x}^2$, that will get differentiated, but notice that this will have the coefficient $\dot{x}^2$ with it self, and by the condition of the problem, initially there is no motion so $\dot{x}^2 = 0$. Hence, while considering the potential energy, we don't bother to write it. This solves,

$$18m\ddot{x} = -\frac{mg(-2x)}{\sqrt{l^2 - x^2}} = \frac{2mgx}{h}$$

As there is the 45° angle, we are free to say that $x = h$, thus, ultimately,

$$\boxed{\ddot{x} = \frac{g}{9}}$$

When solving, I started with some wrong assumptions but using various methods that kept giving answers $g/3$.

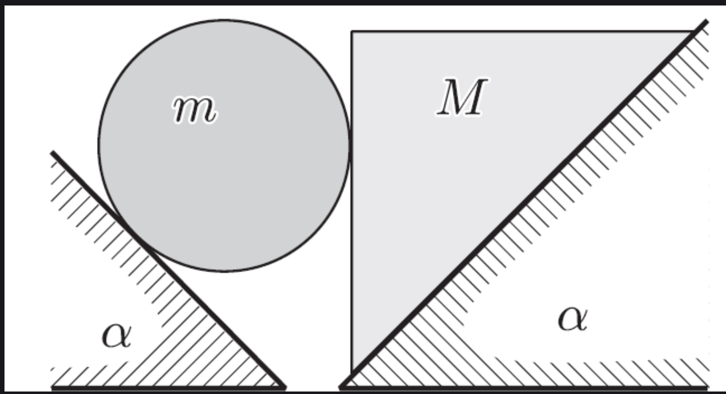


Figure 3.2.1

Problem 44 (M64, Kalda). There is this cylinder like the diagram and this wedge, mass m and M. Find the initial accleration of the masses. No friction anywhere. The inclines don't move.

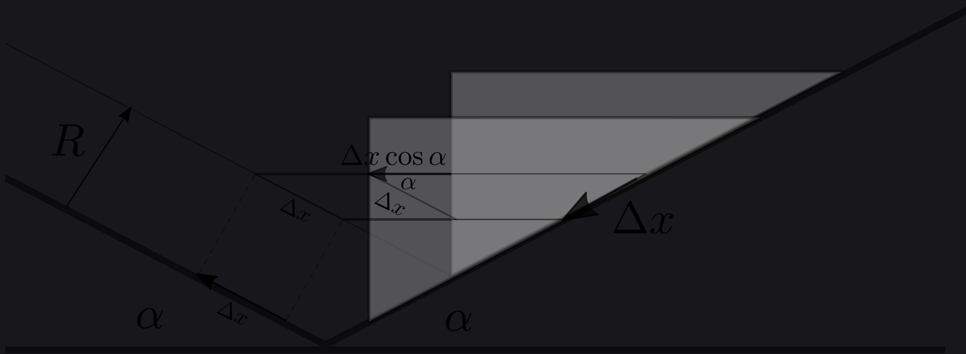


Figure 3.2.2

Solution. Let the wedge slid down a little bit by Δx, the cylinder could slide down either, we just assume one case. Now one thing we need to get rid of from the mind is that the cylinder is not to rotate. It will slid off in with the center tracing the line R above the inclined path.

Now if the wedge comes down, there will be vertical shift $\Delta x \sin\alpha$, and horizontal shift $\Delta x \cos\alpha$, of the wedge. The cylinder will move off due to the shifts, and if we assume that the contact between cylinder and wedge is always there, then from the shift of wedge, we find shift of the cylinder, as it is rigid and needs to make it's space somewhere up if wedge takes the room.

The cylinder is to move along the line we drew, and the shift seems to be evidently $\Delta x \cos\alpha \frac{1}{\cos\alpha}$. Please look at the diagram in case you don't get it I am in scarcity of words to explain that.

It turns out if the wedge comes down by Δx then the cylinder should get along the incline upwards by the same Δx.

The kinetic energy will be T,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{x}^2$$

In case of potential, if we consider any initial postion as $V = 0$, then after some time, the block might be down by $\Delta x \sin\alpha$, and the cylinder up the ramp by height $\Delta x \sin\alpha$. All what is important is that one goes up, one goes down,

$$V = mg\Delta x \sin\alpha - Mg\Delta x \sin\alpha$$

The lagrangian,

$$\mathcal{L} = T - V = \frac{1}{2}(m+M)\dot{x}^2 - mg\Delta x \sin\alpha \left(1 - \frac{M}{m}\right)$$

Take the Euler Lagrangian derivatives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m+M)\dot{x}^2 - mg\Delta x \sin\alpha\left(1 - \frac{M}{m}\right)\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{1}{2}(m+M)\dot{x}^2 - mg\Delta x \sin\alpha\left(1 - \frac{M}{m}\right)\right)$$

Solves to,

$$\ddot{x} = g\sin\alpha\left(\frac{\frac{M}{m} - 1}{\frac{M}{m} + 1}\right)$$

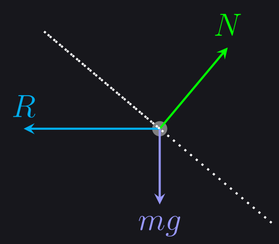
With the convension the cylinder going up is positive x.
To be honest the method 6 is a little bit quicker because the answer in that case is obviously clear. We just need to take derivative of potential and by rule,

$$\mathcal{M} = M + m$$

Hence,

$$\ddot{x} = -\frac{-mg\sin\alpha\left(1 - \frac{M}{m}\right)}{(m+M)}$$

Problem 45 (M64, Kalda). There is this cylinder like the diagram and this wedge, mass m and M. Find the Reaction force between the masses. No friction anywhere.



Solution. All the force on the cylinder can be shown with the diagram above. Of course, the angle $\vec{R}$ makes with the surface is α as in the problem, and carefully looking, these are the only forces acting on the cylinder. There imbalance causes the acceleration $\ddot{x}$ to occur.
One important habit to remember (and forget from HRK) is not to just blindly write $N = mg\cos\alpha$ as I might do and spent more hours to get to the point. Because the work of normal force is to keep the formality and logic so that nothing "gets into" the surface perpendicularly due to force. So, in balanced cases, the normal force is equal to all the forces put perpendicularly against the surface.
Who is the primary reason of moving off of the cylinder? The wedge on incline. And how do he do it? By pushing the cylinder. What push? $\vec{R}$.
The net force along the incline,

$$F = R\cos\alpha - mg\cos(\alpha') = R\cos\alpha - mg\sin\alpha = \sin\alpha\,(R\sec\alpha - mg)$$

From N2L,

$$ma = m\ddot{x} = mg\sin\alpha\left(\frac{\frac{M}{m}-1}{\frac{M}{m}+1}\right)$$

So,

$$\sin\alpha\,(R\sec\alpha - mg) = mg\sin\alpha\left(\frac{\frac{M}{m}-1}{\frac{M}{m}+1}\right)$$

Solving this, we get an agreement with the solution sheet,

$$\boxed{R = g\tan\alpha\frac{2Mm}{m+M}}$$

$$\ddot{\theta} = \frac{-10m^2(R+r)^2\sin\theta\cos\theta\frac{10mg\left(1-\cos\theta(M+m)^2\right)}{5mM(R+r)^2\cos^2\theta+7(M+m)^2R^2} - 5g(R+r)(M+m)^2\sin\theta}{5(R+r)^2(M+m)^2 + 2m(M+m)^2R^2 + 5m^2(R+r)^2\cos^2\theta}$$

Problem 46 (`Cone and Bead`). Let there be a cone, with the top end pointing to the $-\hat{z}$, and the half angle is α. Now, there is a bead that is constrained to move inside the inner surface of the cone. Find the oscillation frequency possible by the bead in teh system.

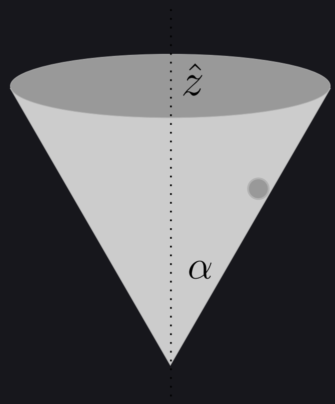


Figure 3.2.3

Solution. The main work is to write the Lagrangian,

$$T = \frac{1}{2}m\left(\dot{z}^2 + \dot{r}^2 + r^2\dot{\theta}^2\right)$$

Which we have written in cylindrical coordinates. Here, $r = z\tan\theta$. Using that,

$$T = \frac{1}{2}m\left(\dot{z}^2 + z\,\dot{\tan\theta}^2 + z^2\tan^2\theta\dot{\theta}^2\right)$$

And respect to the pointed end of the cone, we write the gravitational potential,

$$V = mgz$$

Using this, the Lagrangian,

$$L = \frac{1}{2}m\left(\dot{z}^2 + \dot{z}^2\tan^2\theta + z^2\tan^2\theta\dot{\theta}^2\right) - mgz$$

Here, the $L = L(z, \theta)$, two coordinats, surely z and θ are respectively independent. Now, taking the derivative we can find the required equations directly.

Problem 47. There is a rod of length l that is hanging by a string of length $2l$. If the rod is nudged, find the maximum possible frequency of the mode of oscillation.

Solution. Let's write the position of the center of mass,

$$x = 2l\sin\phi + \frac{l}{2}\sin\theta$$

$$\dot{x} = 2l\cos\phi\dot{\phi} + \frac{l}{2}\cos\theta\dot{\theta}$$

That's the x coordinate, we need y now,

$$y = 2l\cos\phi + \frac{l}{2}\cos\theta$$

$$\dot{y} = -2l\sin\phi\dot{\phi} - \frac{l}{2}\sin\theta\dot{\theta}$$

Now, we aim to solve for the lagrangian, so, what we have is,

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2$$

We have to compute $\dot{x}^2 + \dot{y}^2$. After we do compute then we will approximate.

3.3 Mechanics with Momentum and Collision

Problem 48. Coefficient of restitution is $C = 0.9$, for a system where a ball falls on the ground and hops off. The C means that if the ball falls with v speed, it will bounce up with $0.9v$ speed. So, at first the ball is vertically thrown upward with speed 50 m/s, at $t = 0$ time. After what time the ball comes to rest and does not bounce anymore?

Solution. Every time the ball hops off, then there is a time it spends on air. That time is,

$$T = \frac{2v}{g}$$

Given the ball hops off with the speed v. Initially speed is $v_0 = 50\ m/s$, but when it bounces next time, it will have speed $v = 0.9 \times 50\ m/s$, or we can say, $v = v_0 C$. After it returns to ground again, it will bounce of with $v = C(v_0 C)$ that is, $v = C^2 v_0$. And this keeps going. For every bounce n, there will be bounce speed $v_0 C^n$. We have to sum up all the time taken by the ball in air.

$$T = \sum T_i = \sum_{i=0}^{\infty} 2\frac{v_0}{g} C^i$$

There will be alot of small bounces made, so we sum up until infinity, but that does not mean that time will also be infinite, because with every bounce the speed decreases, so as the time the ball spends on air.

$$T = \frac{2v_0}{g} \sum_{i=0}^{\infty} C^i$$

We have $C < 1$ that helps us to take this sum, it will be,

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

Our sum would be,

$$\boxed{T = \frac{2v_0}{g}\frac{1}{1-C}}$$

That gives the numerical value,

$$T = 100\ s$$

Not infinite.

The next problem builds up the necessary foundations on Elastic and Inelastic Collsion.

Problem 49. There is a mass m that is moving with speed v_0 towards a resting M mass. We define f as the absolute value of the momentum of M after collision divided by the initial momentum of m.

What would be the condition for f to be maximum if the collision is *Inelastic*?

Solution. Initial momentum is mv_0. Final momentum should be $mV_1 + MV_2$. Now let us come to a point, what do we mean by the term "Inelastic Collision"? It is that the energy is not at all conserved, but this is quite loose.

The outcome of the statement that energy is not conserved actually tells that, if the masses collide, afterwards they will move off sticking to each other. To be specific, if we move to the frame of the center of mass before the collision, the masses will be progressing towards the center of mass.

They will eventually collide at the center of mass. But after collide, they will lose all the energy in the center of mass frame, so they will stick to each other and don't move at all in the center of mass frame.

I use the center of mass frame phrase several times above to make sure we don't mess things up.
As they will move together, their final speed will be the same, hence, $V_1 = V_2$, and by conservation of momentum,

$$mv_0 = (m + M)V$$

Thus,

$$V = \frac{mv_0}{m + M}$$

Now, we can calculate f,

$$f = \frac{MV}{mv_0} = \frac{Mmv_0}{(m + M)mv_0} = \frac{M}{m + M}$$

What can be the maximum value f can take? Look, if m is small enough, all we can have is $f = 1$, not more. Otherwise f is not maximized. Hence,

$$\boxed{m << M}$$

is the condition for f to be maximum when there is Inelastic collision.

Problem 50. There is a mass m that is moving with speed v_0 towards a resting M mass. We define f as the absolute value of the momentum of M after collision divided by the initial momentum of m.

Find f's maximum if the collision is *Elastic*?

Solution. We will make the same resoning as before but this time as the Energy is conserved (elastic collision), we might need to solve more equations.
Momentum conservation and energy conservation gives a nice answer to this. We can solve the two equations,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$$

With the 1/2 terms shaved off. This gives us the solution,

$$u_1 + u_2 = v_1 + v_2$$

Hence for this problem,

$$v_0 = v_1 + v_2$$

Where the v_1, v_2 are the speed of m, M after collision. But the problem is less demanding than thought.
In CM frame, the speed of the lab frame resting M is going to be v_{cm}. After collision, in CM frame, it will move off in the opposite direction with the exact same speed it came to the CM with, $-v_{cm}$. In lab frame, we will measure that the speed will be $2v_{cm}$, the superball problem has some idea of it when beginning, I hope to write a detailed explanation later. Okay, the speed after collision M has,

$$2v_{cm} = \frac{2mv_0}{M+m}$$

Thus, f will be,

$$f = M\frac{2mv_0}{(M+m)(mv_0)} = \frac{2M}{M+m}$$

If m is small, we can have $f = 2$. And that is probably the maximum.

Problem 51. There is a ramp that is upon a slippery surface, and a block of mass m is running towards it as speed v. The mass of the ramp is M. What shall be the max height attained by the block as it leaps off the ramp?

Solution. After the block lifts off, it will have an x component of speed v_{go}, and by the cons of mom,

$$mv = mv_g + Mv_g$$

In ramp frame this is suitable to think either.

$$v_g = \frac{m}{M+m}v$$

Now by the cons of en,

$$v^2 = v_i^2 + v_g^2 + \frac{M}{m}v_g^2$$

> We shall put the v_g in place and here, the v_i will be the vertical speed of the block so that the block can lift, now,
> $$v_i^2 = 2gh$$
> Solving the equation gives us that,
> $$h = \frac{M}{M+m}\frac{v^2}{2g}$$

Problem 52 (Superball!). One ball is atop of the other, h above the ground. Lowest ball has the mass m_0 and the one above it has mass $m = fm_0$. And $f < 1$. Atop the fm_0 ball is $f(fm_0) = f^2m_0$. There is in total $n+1$ balls. The system is let fall. The balls are very close apart, and all collisions are elastic. Find the speed of the topmost ball.

> **Solution.** At first moving the center of mass frame, we know the speed with which the ball comes towards the COM is exactly what the speed the ball leaves with. Bringing this analogy to the Laboratory frame, we find that, the speed after collision is,
> $$v_A' = -v_a + 2V_{com}$$

Problem 53 (Superball Numerical). If $f = 0.5$ and $n = 10$, how much higher the last ball will go off relative to the falling distance h?

Problem 54 (Hitting twice). A ball of mass M strikes a two ball system joined with a spring of k constant. The M has initial speed of v_0 and the two balls of the spring dumbell have mass m each. Find the minimum value of M such that the mass M collided with the spring twice. Crude and sensible approximations are to be made.

> **Solution.** We shall start of with assuming the positions of the masses. After collision, we require that the position of the dumbell end and M become equal again, for the second collition. The spring will be subject to an oscillation. This can make up for the situation to occur.
> (Yay, tried solving this again and as usual did mistake, hurray!)
> Alright, so I am a bit tired of doing this for last half an hour or so, anyways. There will be collision between the M and m at first.
> The collision will cause some exchange of momentum as it happens. This momentum delivery is trivial to think, this is why the dumbell would move off. But the thing I made a mistake is that the end mass m away from the collision will not move at all. So initially it will be the m leftside of the dumbell to experiance motion. I thought that the right m will also get the equal share of momentum as

leftside, but fortunately the right side does not get that (or unfortunately, if you have sympathy for me as I was trying this problem wrongly for half an hour).
We know for a collision, the post-speed,

$$v' = -v_0 + 2V_{cm}$$

This case,

$$V_{cm} = \frac{Mv_0}{M+m}$$

Meanwhile I was dealing with $V_{cm} = \frac{Mv_0}{M+2m}$, I am so frustrated with these mistakes I make everyday. If we say that,

$$\frac{m}{M} = \gamma$$

Speed of M,

$$v_M = \frac{1-\gamma}{1+\gamma}v_0$$

Speed of m,

$$v_m = \frac{2v_0}{1+\gamma}$$

Now as the dumbell moves off, it shall experiance a SHM in the dumbell frame of center. The oscillation will be,

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{2k}{m}}$$

Where the μ is the reduced mass, the reason of this is found in a problem in Energy chapter.
As there is SHM, there is also a linear motion of the dumbell. The speed of dumbell be v_d, so,

$$x = A\cos(\omega t + \phi) + v_d t$$

This x is the position of the leftside of the dumbell, look when $t = 0$, we have the position equal to zero (setting coordinate at hitting point), so, the correct ϕ is $\phi = \frac{-\pi}{2}$. Having the negative saves some fact we shall see next. Hence,

$$x = v_d t - A\sin(\omega t)$$

The first derivative finds the speed of the dumbell,

$$\dot{x} = v_d + \omega A\cos(\omega t)$$

For time $t = 0$ we know,

$$v_d + \omega A = \frac{2v_0}{1+\gamma}$$

For the mass at the rightside, we have that the motion opposite to the left one, so,

$$\dot{x} = v_d - \omega A\cos(\omega t)$$

But it has speed 0 at $t = 0$, so,

$$v_d = \omega A$$

We have after solving the above equations from both the m's,

$$v_d = \omega A = \frac{v_0}{1+\gamma}$$

The position has to be same X if both of them wants collide again. This requires us,

$$\frac{v_0}{1+\gamma}\left(t + \frac{1}{\omega}\sin\omega t\right) = \frac{\gamma - 1}{\gamma + 1}v_0$$

If you are not idiot as me and finally achieve the solution, we find that,

$$\frac{-\sin\omega t}{\omega t} = \gamma$$

We now recall,

$$\gamma = \frac{m}{M}$$

So, for minimum M, γ has to be max.
The $-\sin\omega t$ is max 1 at $\frac{3\pi}{2}$, so, using that,

$$\boxed{\frac{3\pi m}{2}}$$

I still wonder the day I finally don't screw things up, this is hell very frustrating doing mistake in mass number regularly.

3.4 Gravitomechanics

[2]

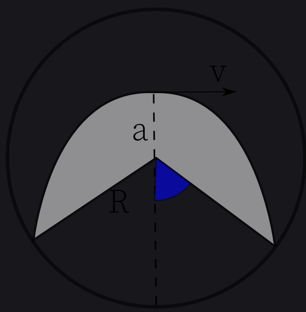


Figure 3.4.1

Problem 55. Consider a trajcetory that is *Parabolic* and the mass is a comet that is in the Earth Sun orbital plane. Now, if they have the mass M_s and m, then what will be the maximum time for the comet, possible to stay in the orbit?

Solution. (Waqar Mirza) Let the comet pass through the perigee. The distance of focus (Sun) and perigee be a. Now we need to look for a. We shall have the total energy $E = 0$ and L as the angular momemtum.

$$\frac{1}{2}v^2 - \frac{GM}{a} = 0$$

$$\left(\frac{L^2}{m^2a^2}\right) = \frac{2GM}{a}$$

With the above two equation we can find,

$$a = \frac{L^2}{2GMm^2}$$

Or, converse is also true,

$$L = m\sqrt{2GMa}$$

We shall now concentrate on the time taken. From the definition of angular momemtum,

$$L = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

So, the integral,

$$\int_0^t \mathrm{d}t = \left(\frac{2m}{L}\right)\left(\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,\mathrm{d}\theta\right)$$

Above, the term $\int_{\theta_1}^{\theta_2} r^2\,\mathrm{d}\theta$ is merely the area of the region of orbit, bounded by the points where the comet orbit cut Earth orbit, and focus. We call that, because, if we assume a line r emerging from the focus to any point in the trajectory, the height gained for an infinitesimal movement is $r\mathrm{d}\theta$, so the area is half times base

[2]Just a fancy name from me.

times hight, follow, $\frac{1}{2}r(rd\theta)$, taking the integral around theta determines the full region. But, finally, if we call the area A, then the time period is,

$$t = \frac{2mA}{L}$$

Solving for the area, we find that,

$$A = \frac{2}{3}\sqrt{a(R-a)}(2a+R)$$

Plugging onto the equations,

$$t = \frac{2}{3}\sqrt{\frac{2}{GM}}\sqrt{R-a}\left(\frac{a}{R}+\frac{1}{2}\right)$$

Maximizing this function is just maximizing the function,

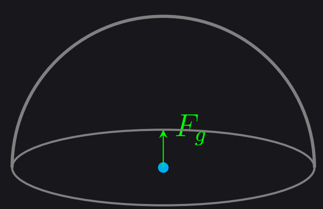
$$f(x) = (1-x)(x+\frac{1}{2})^2$$

We can see that maximum occurs at $x = \frac{1}{2}$. So the maximum time,

$$\boxed{T_{max} = \frac{4}{3}\sqrt{\frac{R^3}{GM}}}$$

Problem 56. One object is in orbit, $h = 100\ km$ above the ground. Another same type of object is falling from a height $h = 100\ km$ from the same point from where the other object in orbit starts its journey. The object falls through a tunnel that has cut the earth at the other side too. What will be the time taken by each to reach the other end?

Solution. The solution is weird, I were told that the orbit objects paths projection on the tunnel is the path taken by the falling object, in real time. So, they are ought to reach at the same time. The half of the orbital period is going to be the time of flight, or fall. And that is just Keplar's law's simple application.



Problem 57 (Force by Hemisphere). Let there be a hemispherical shell of σ mass per unit area, and a particle of mass m resting in the center of it. What will be the net

gravitational force on the particle?

Solution. From Gopal Goel I have learned this neat trick that works out the equation without considering the integration anywhere. First of all, N3L (Newton's Third Law) tells that the force by the shell on the m is equal to the force of m to the shell.
The hemisphere is a nice case, reason is all the part of the hemisphere is equidistant R from the m. Hence, every small point lies R away from m.
If we consider a small mass $\mathrm{d}M = \sigma \mathrm{d}A$ of any random point of the shell, the force is,

$$\mathrm{d}F = G\frac{\sigma \mathrm{d}A}{R^2} = \left(\frac{Gm}{R^2}\right)\sigma \mathrm{d}A$$

We can say that,

$$\mathrm{d}F = \mathrm{d}M\,(g)$$

But, we have to note that all the horizontal components of the force will be zeroed because of symmetry. There will only be force along one direction. If the position of $\sigma \mathrm{d}A$ makes θ with the vertical (respect to the m), then the vertical force is,

$$\mathrm{d}F_{vert} = g\,(\sigma \mathrm{d}A\cos\theta)$$

We have to add up all the contribution to come to a point. But notice what $\mathrm{d}A\cos\theta$ mean, it is the projection of the small area on the horizontal surface, for the whole hemisphere, that is just a circle of R radius. Hence,

$$\int \cos\theta\,\mathrm{d}A = \pi R^2$$

This equation is not exactly true, anyways makes sense that we have to actually locate the projection of the area on surface. That will let us have,

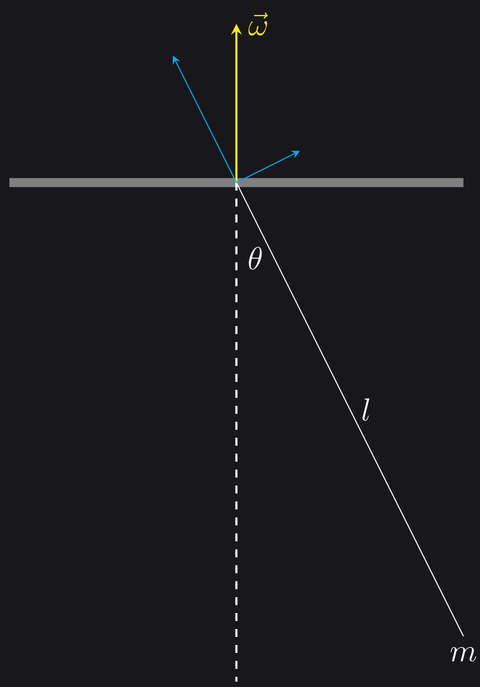
$$F = g\pi R^2\sigma = \frac{Gm}{R^2}\sigma R^2$$

Hence,

$$F = Gm\sigma$$

As the discussion goes, this also gives some idea about shell theorem, we discussed in Electrodynamics part.

3.5 Angular and Rotational Mechanics



Problem 58 (Rodendulum). A stick of length l, mass m, and uniform mass is pivoted at the top end and is let rotating around the vertical axis making an angle θ. Find the angular frequency ω.

Solution. The most basic method is to go to Principal axis, the symmetric axes. Now, along the rod let us call x axis and vertical to it be y. Now, it is sure that

$$I_x = 0$$

And the vertical certifies,

$$I_y = \frac{1}{3}ml^2$$

Now the angular momentum is necessary, we know,

$$\vec{L} = \sum_{i=1}^{3} I_i\omega_i \ \hat{e}_i$$

The $\vec{\omega}$ vector is vertical, and it has a projection on the y axis of rod. So, that is the only component of angular momentum,

$$L = \frac{1}{3}ml^2\omega$$

Now, the $\vec{L}$ will rotate with ω. And it has a projection on the horizontal. That horizontal projection rotates with ω, thus it changes with,

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\omega} \times \vec{L}$$

And here,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau = L\omega\cos\theta$$

There is this torque. And if finding the force that brings the torque, it is just $m\vec{g}$. So, we can tell, after finding the torque,

$$\tau = \frac{1}{3}ml^2\omega^2 \sin\theta\cos\theta = \frac{1}{2}mgl\sin\theta$$

So we are left with this,

$$\omega = \sqrt{\frac{3g}{2l\cos\theta}}$$

Problem 59 (Sliding Lolipop, Easy Method). We have a lolipop made of solid sphere of m mass and radius r, which is radially put with a massless stick. The free stick is pivoted on the ground, and everything is frictionless. The sphere is "made" sliding on the ground in rotation Ω. Pivot point to sphere contact point is R. Find the normal force using **Newton's 2nd Law**

Solution. Just looking at the system, it has a mass force

$$m\vec{g}$$

and it points down. The rotation is allowed by the fact that there is a radial pull with the force

$$F = m\Omega^2 R$$

And it points horizontally. This force will make a Tension on the rod. That rod tension makes some angle poiting downwards. And that force's down-pointing vertical, contributing to the Normal force, that is,

$$F\tan\theta = m\Omega^2 R\tan\theta = m\Omega^2 r$$

So, total normal force,

$$N = mg + m\Omega^2 r$$

I must make a TiKZ.

Problem 60 (Sliding Lolipop, Rotation Nerd Method). We have a lolipop made of solid sphere of m mass and radius r, which is radially put with a massless stick. The free stick is pivoted on the ground, and everything is frictionless. The sphere is "made" sliding on the ground in rotation ω. Pivot point to sphere contact point is R. Find the normal force using **Angular Momentum** because you have became a rotation nerd.

Solution. I made the mistake to initially make the assumption the sphere to be a point mass. But it isn't. Looking carefully, some component of $\vec{\omega}$ will be perpendicular to the rod and some component will be along the axis. Telling that the angle made by rod and horizontal is θ, angular momentum perpendicular to the rod L_1, and pointing along the rod is L_2. And moment of inertia through an axis going through the center of sphere is I_c.

$$
\begin{aligned}
L_1 &= \left(I_c + m(r^2 + R^2)\right)(\omega \cos\theta) \\
L_2 &= I_c \omega \sin\theta
\end{aligned}
$$

First of the equation from the Parallel axis theorem, that the net moment of inertia is found form the Center of Mass MOI added with mr^2 where r is the distance from COM. The horizontal projection of $\vec{L}$ is considerable because only this is the one that changes, hence subject to torque, and it is,

$$
\begin{aligned}
L &= L_1 \sin\theta - L_2 \cos\theta \\
&= \left(I_c + m(r^2 + R^2)\right)(\omega \cos\theta)\sin\theta - I_c \omega \sin\theta \cos\theta \\
&= m(r^2 + R^2)\omega \ \sin\theta \cos\theta
\end{aligned}
$$

Here,

$$\tau = \omega L$$

A torque must exist that will keep the system rotating, this force can only be $N - mg$ where N is the sought normal force and thus,

$$\tau = (N - mg)R = m(r^2 + R^2)\omega^2 \sin\theta \cos\theta$$

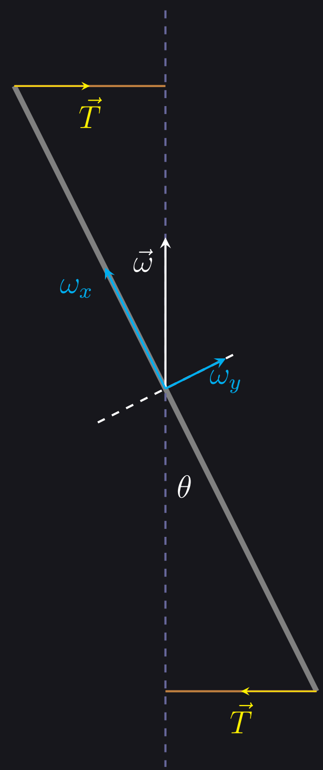
Using the fact that,

$$\cos\theta = \frac{R}{\sqrt{r^2 + R^2}} \qquad \sin\theta = \frac{r}{\sqrt{r^2 + R^2}}$$

$$(N - mg)R = m(r^2 + R^2)\omega^2 \frac{R}{\sqrt{r^2 + R^2}} \frac{r}{\sqrt{r^2 + R^2}}$$

So, we can solve for the N,

$$N = mg + m\omega^2 r$$



Problem 61. Here is this stick that is rotating with an angle θ with the vertical. It intersects with the vertical rotation axis at the center, rod length is l. The angular speed is ω and rod has the mass m. The ends are fixed with strings and find the tension in the string.

Solution. The ω has a component vertical to the rod and along the rod, the one vertical to the rod is the only one that is not zero. So, the angular momentum is,

$$L = \frac{1}{12} m l^2 \omega \sin\theta$$

This is rotating with ω speed, the horizontal component is changing, so,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{1}{12} m l^2 \omega^2 \sin\theta \cos\theta$$

Now, there must be some source of torque, and it is going to be the *Tension of the strings*, so, the $\vec{T}$ force will be perpendicular to the rod, and hence,

$$\tau = 2\left(\frac{l}{2} T \cos\theta\right)$$

And equating with the $\frac{\mathrm{d}L}{\mathrm{d}t}$, we find the answer,

$$\boxed{T = \frac{1}{12} m l \omega^2 \sin\theta}$$

Problem 62 (`Roll off`). There are two wheel that has been fixed with an axle, and the system is above a frictionless surface. The mass of each of the wheel is m and there is a rotation of ω in each of the wheels. Consider the MOI of the wheel I along axle axis.

They are then given a rotation Ω making axis perp to the axle rod. The length of the axle rod is l. Find out the maximum speed Ω for which one the wheel lifts off.

Solution. Each of the wheel has an angular momentum $I\omega$, thus total angular momentum associated to ω is,

$$L = 2I\omega$$

Now, from intuition, this is clear that only the angular momentum with ω is changing because of Ω rotation. The angular momentum of Ω is constant, and thus it cannot keep any torque with it.
And from the statement "lift off" this is sure that there must be some weird torque that brings about the lifting. And that torque must be with that $2I\omega$ term.
As we know, for a vector $\vec{a}$ changing has,

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{a}}{\mathrm{d}t}\right)_{rotating frame} + \omega \times \vec{a}$$

So, here the $L = 2I\omega$ is rotating with Ω, thus,

$$\tau = 2I\omega\Omega$$

Also, there is no change of $2I\omega$ in the rotating frame, so $\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}_{rot} = 0$. This torque can only come if there is some mysterious force, $\vec{F}$,

$$\tau = 2\left(\frac{l}{2}F\right) = lF$$

This force works on the two wheels on two sides. Assuming the $2I\omega$ points in the right, the right side has the F pointing downwards. But on the left F works upwards.
And this is the end, if the left side F is greater than the mass of the wheel on the left, the wheel must lift off pivoting the right wheel. So, we can finally tell that,

$$l(mg) = 2I\omega\Omega$$

And solving this,

$$\boxed{\Omega = \frac{mgl}{2I\omega}}$$

3.6 Parabolic Projectiles

So recently (28 Nov of 2020) I have been looking some problems which have different set up but same solution if we use the solution of another problem.

It has been quite fascinating to work with this, so it is necessary for me to include them here.

The general idea is that the equation of the trajectory of a projectile is given by,

$$y = x \tan\theta_0 - \frac{gx^2}{2(v_0\cos\theta)^2}$$

Which certifies that the trajectory is strictly a "Parabola". And thus, the parabolic trajectory must have "Focus". Again, Parabola has some fundamental definition of being the "locus" of the points that are at equal distance from a line called *Directrix* and a focus point F.

Problem 63 (Focus of the Projectile)**.** For a projectile with initial speed v_0, find the distance of the focus point F from the launching point O.

There are two method of doing the problem, one is invoking the geometry at the launching point and relating it to the vertical line and focus point, another method is to use the definition of parabola using the directrix. Both are interesting, but I prefer geometric one as it seems to be more powerful in terms of intuition.

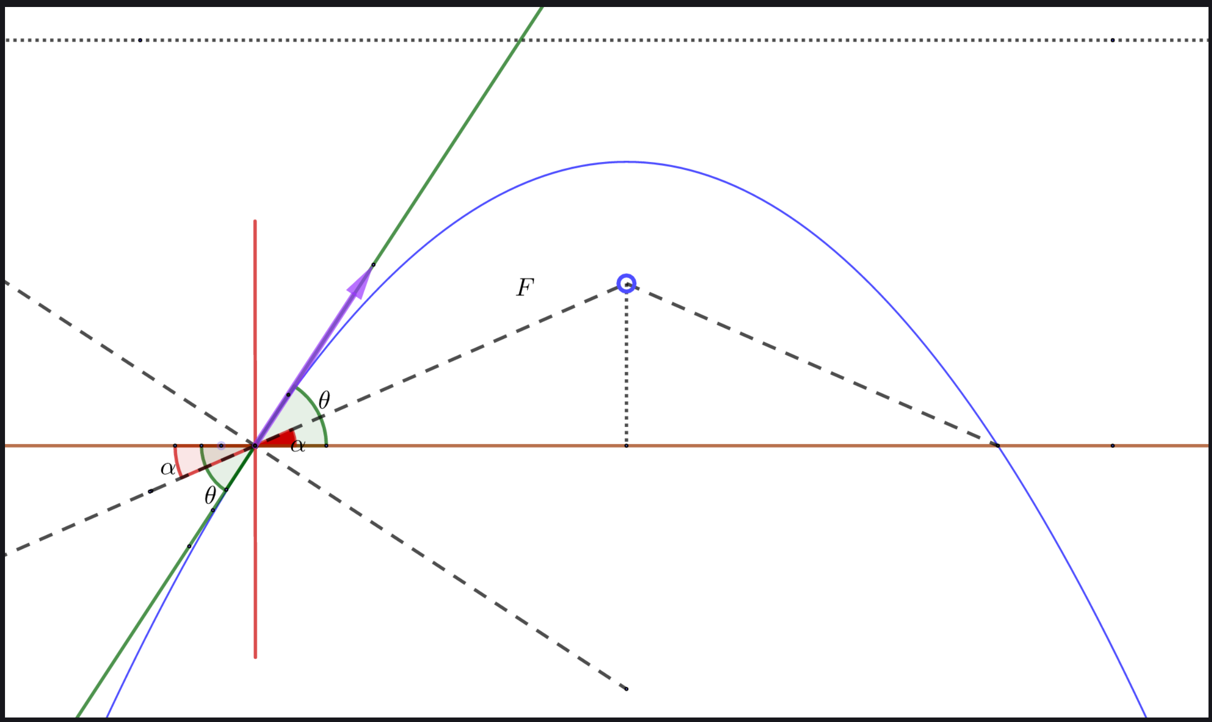


Figure 3.6.1

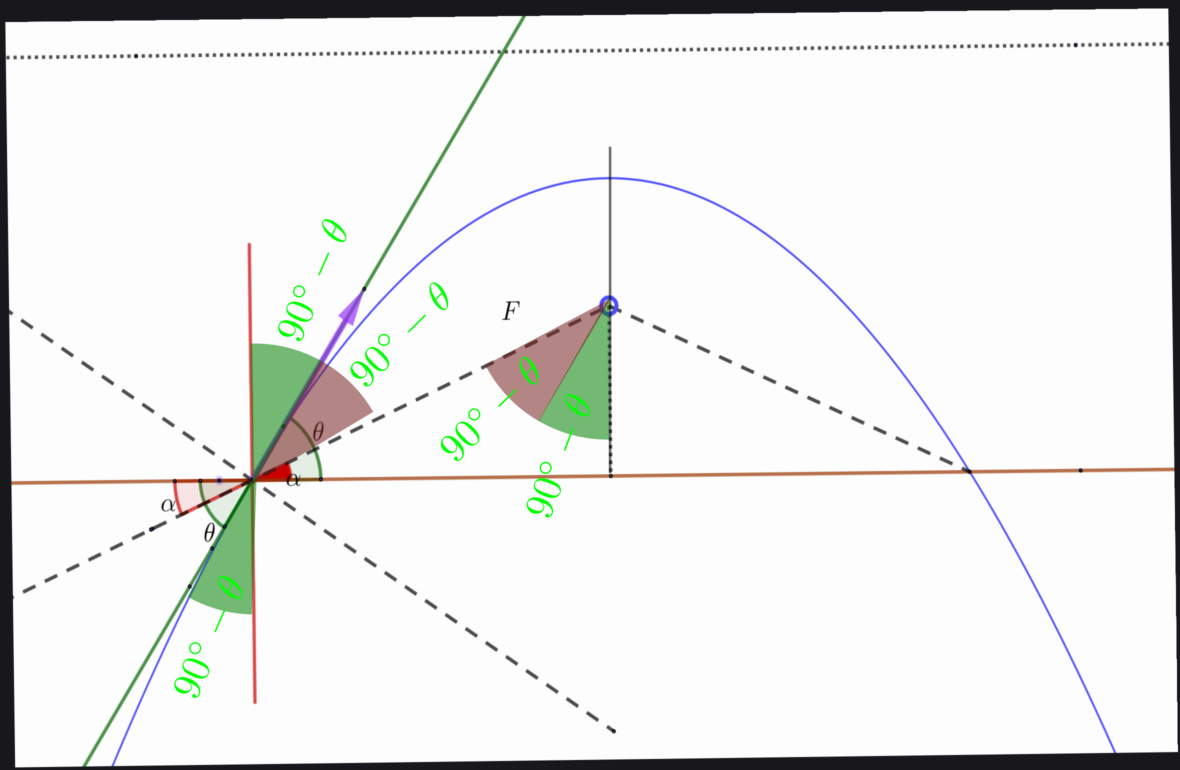


Figure 3.6.2

Solution. We launch the mass at θ angle, at speed v_0. Now, there exists a triangle with corners launching point, focus, and vertically below focus. The end of triangle is called α.
Now looking at the dotted line that is perpendicular to the initial velocity reminds us that parabolic mirror focuses all the light to the focus. Thus an incoming light, vertically coming towards the mirror wil be reflected towards the focus.
Here the red vertical line crossing point O should reflect to focal point F if it was a mirror. By law of reflection, the angle made by red line with the dotted line (perp to $\vec{v}_0$) is equal to the dotted line with OF line. Using this we can come to the second diagram of this problem. There, the dark green angle region in the lower left is equal to $90° - \theta$. This is made possible with the law of reflection rule we made.
And this dark green region has a opposite angle. But, this has a neighbour dark magenta that is also $90° - \theta$. Together they make an alternate angle at the focus point region as in the diagram.
Finally, using the triangle consisting focus, O tells us that,

$$180° = \alpha + 90° + 180° - 2\theta$$

This leaves us with,

$$\alpha = 2\theta - 90°$$

And the other angle with focus is,

$$180° - 2\theta$$

This is clear that the orthogonal projection of the line OF is half of the range of projectile. The range is,

$$R = \frac{2v_0^2 \sin\theta\cos\theta}{g}$$

Half of the projectile with necessary simplification,

$$\frac{R}{2} = \frac{v_0^2 \sin 2\theta}{2g}$$

Now,

$$OF \ \sin(180° - 2\theta) = OF \ \sin 2\theta = \frac{v_0^2 \sin 2\theta}{2g}$$

So, we finally find that ,

$$OF = \frac{v_0^2}{2g}$$

Theorem 5 (Focus of Projectile) — The Focus F of a projectile is located,

$$r = \frac{v_0^2}{2g}$$

from the launching point at an angle,

$$\alpha = 2\theta - 90^\circ$$

if launching angle is θ. Also note that the horizontal projection of the r line is half of the "Projectile Range". So, this also helps finding the range of the projectile.

The distance is not dependent on θ, this make the line a kind of vector $\vec{r}$ which can rotate with varied launching angle θ, without affecting the line length. We also note that projection of the $\vec{r}$ is the half of range. But when will the vector $\vec{r}$ make maximum projection on the ground?

It would when the whole vector is parallel to the ground, or I like to say, lying on the ground. Then the $\alpha = 0$, this tells us,

$$0 = 2\theta - 90^\circ \qquad \rightarrow \qquad \theta = 45^\circ$$

Which actually is the condition for a projectile range to be maximum!

Theorem 6 (Range from Focus line) — The range of a projectile is maximum along the focus line. (Needs more attention to make sense).

We are at the point to actually understand what is a parabola.

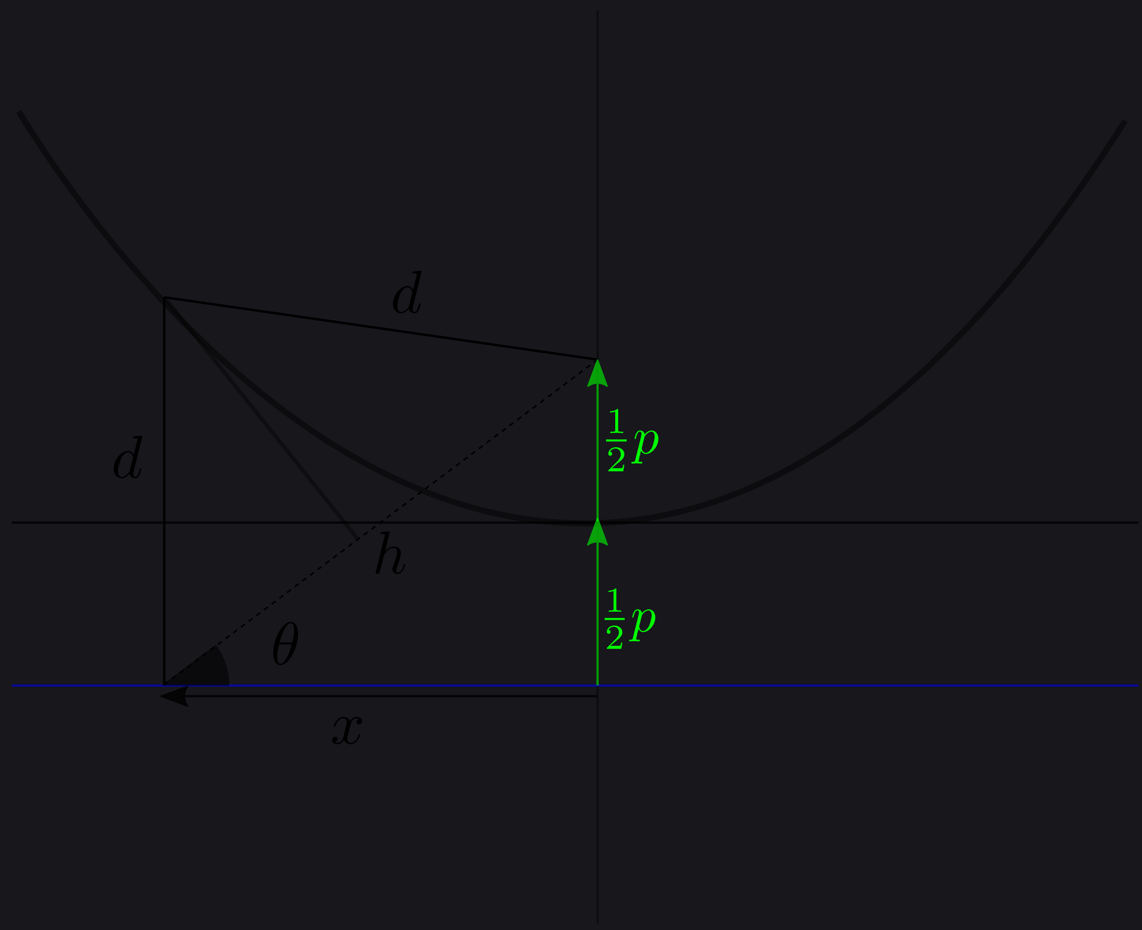


Figure 3.6.3

The lower bluish line is the so called "Directrix line". The focus is selected at any point at p height. Now, there exists a point that is d distance from the directrix and also d distance from the focus. This point lies in the parabola. Tracing out all the points that have this equality of distance gives us a parabola. It happens to be that the lowest point is $\frac{1}{2}p$ above directrix, and as rule says, $\frac{1}{2}p$ from focus too.

Can we find a rule to relate d and x as a function? We can try.

Notice that d builds a isosceles triangle, where the h is line joining the x on directrix and focus. We will say that $y = 0$ at the directrix line. So,

$$h^2 = x^2 + p^2$$

But the triangle invokes,

$$2d\cos(90^\circ - \theta) = 2d\sin\theta = h$$

But $\sin\theta = \frac{p}{h}$, so,

$$2d\frac{p}{h} = h$$

And,

$$2dp = x^2 + p^2$$

We have found a required equation, but let us change our frame's height, so that the minima of the parabola is located at (0,0), so,

$$y = d - \frac{1}{2}p \qquad \rightarrow \qquad d = y + \frac{1}{2}p$$

So,

$$2\left(y + \frac{1}{2}p\right)p = x^2 + p^2$$

Giving us,

$$x^2 = 2py$$

So, all parabolic equation has,

$$y = \frac{x^2}{2p}$$

form, recalling p is distance of directrix from focus.

Problem 64 (Again, with Locus rule). For a projectile with initial speed v_0, find the distance of the focus point F from the launching point O. Now use the locus rule of parabola we just derived.

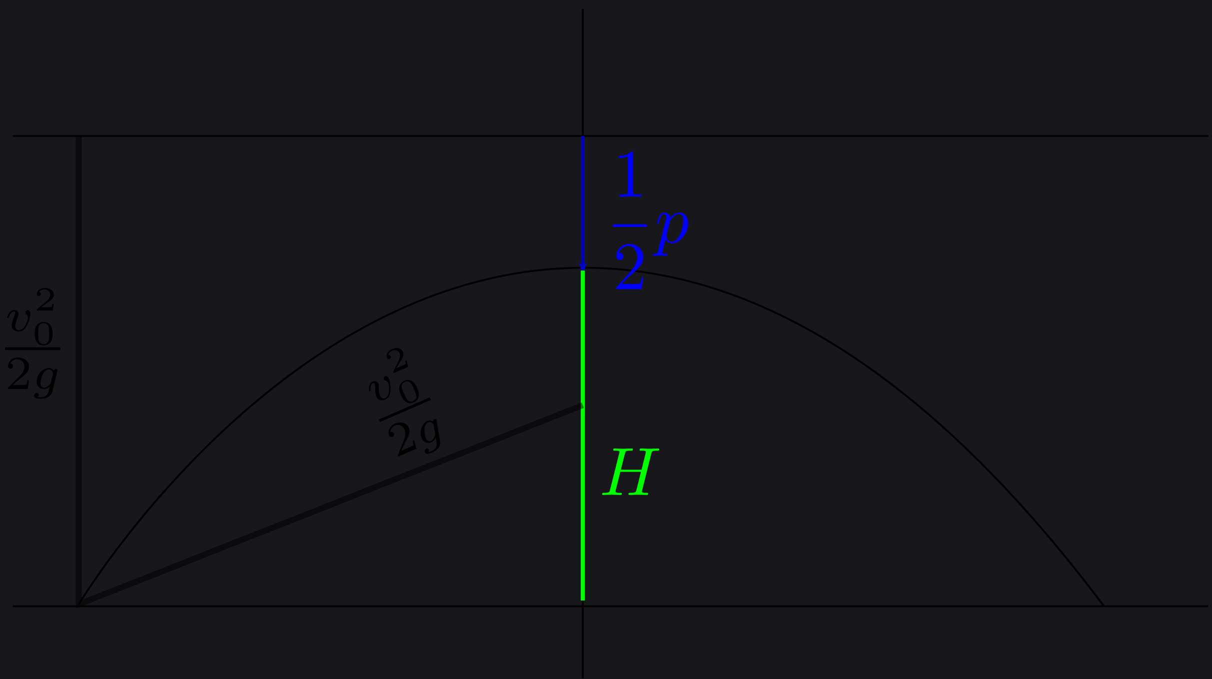


Figure 3.6.4

Solution. The max height possible,

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

We can see that the distance of focus is going to be $p + H$ from the launching point. The half of range of the projectile,

$$\frac{R}{2} = \frac{v_0^2 \sin\theta\cos\theta}{g}$$

As we know,

$$\frac{x^2}{2p} = y$$

Looking at the left vertical bar, making our head upside down, with apparently $y = 0$ as the tip H above, then $R/2$ aside, the height of parabola is going to be H. Using this,

$$\left(\frac{v_0^2 \sin\theta\cos\theta}{g}\right)\frac{1}{2p} = \frac{v_0^2 \sin^2\theta}{2g}$$

We found,

$$p = \frac{v_0^2 \cos^2\theta}{2g}$$

Hence,

$$d = H + p = \frac{v_0^2 \sin^2\theta}{2g} + \frac{v_0^2 \cos^2\theta}{2g} = \frac{v_0^2}{2g}$$

So, we have got another type of solution,

$$\boxed{r = \frac{v_0^2}{2g}}$$

Problem 65 (Above the roof!). There is this roof with dimension in the figure, taken from J. Kalda's Mech, and we are to find the least possible speed to topple over the roof thrown from anywhere left from the ground.

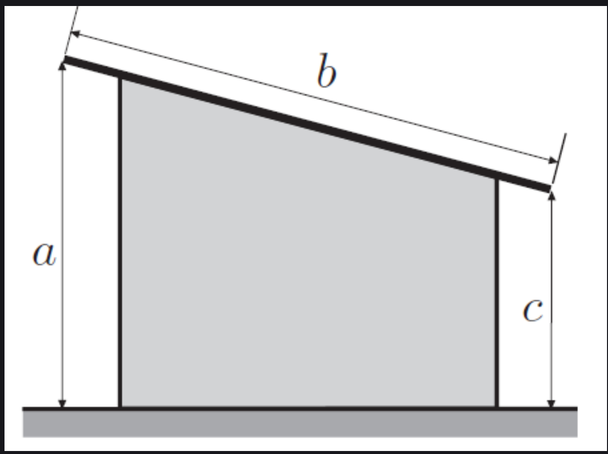


Figure 3.6.5

Solution. The projection of the $\vec{r}$ focus vector is the range of the particle in air. If the projectile has the focus vector $\vec{r}$ such a way that it lies on the required surface (the inclined roof), the projection is max and thus the maximum distance is being covered for the speed, and if this equals to the roof dimensions, we have the optimal case.
Let at the left corner of the roof, the speed of the particle be v, and we know, if the speed it is launched from ground is v_0

$$v^2 = v_0^2 - 2ag$$

Then the focus vector line is,

$$\frac{v^2}{2g} = \frac{v_0^2 - 2ag}{2g}$$

And when the particle falls and touches the right corner, the speed is,

$$v_{right}^2 = v_0^2 + 2g(a - c)$$

When falling, the v_{right} also has a focus line, this will meet the focus line of the left corner speed focus line given the trajectory is optimal. Refer to the diagram I made in paint.
The total length of the lines is going to be b, and thus,

$$\frac{v^2}{2g} + \frac{v^2 + 2g(a - c)}{2g} = b$$

Now plug that $v^2 = v_0^2 - 2ag$, and we solve this for v_0 and get,

$$\boxed{v_0^2 = g(a + b + c)}$$

I don't know why this depends on the perimeter of the roof

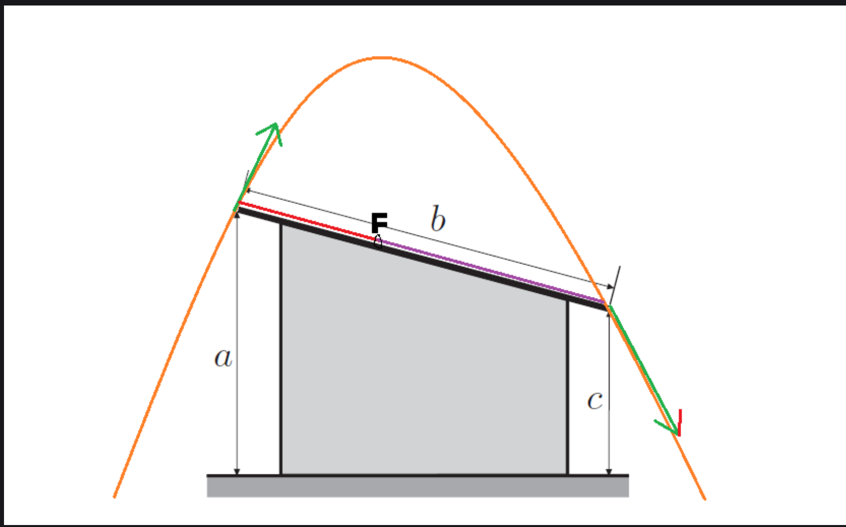


Figure 3.6.6

Problem 66 (Physicist Grasshopper). A wooden log of radius R lies in front of a Physicist Grasshoper back to home from college. The grasshoper calculates the minimum speed he needs to hop and get to the other point of the log and successfully does the stunt. What should be his minimal speed to do this?

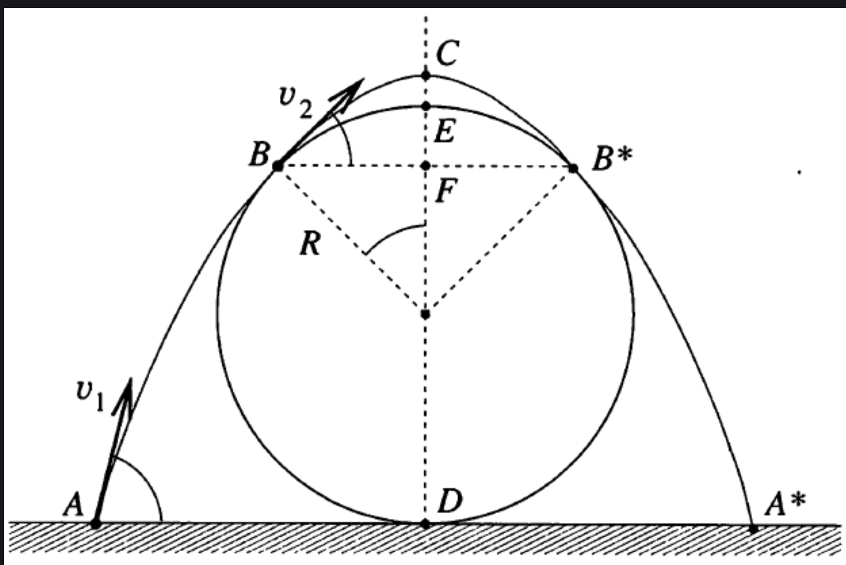


Figure 3.6.7

> **Solution.** When trajectory will surely touch the log if the projectile is optimal. When it does so, then that speed when touching should make the max range, that will cause the grasshopper to reach otherside with least speed with max possible distance. And because the line made of touching points on log is horizontal, of

course the velocity vector will make 45° with the horizontal.

$$v_2^2 = v_1^2 - 2gR\,(1+\sin 45) = v_1^2 - 2gR\left(1+\frac{1}{\sqrt{2}}\right) = v_1^2 - gR(2-\sqrt{2})$$

And for the range, we can think that the focus line is horizontal, thus,

$$\frac{v_2^2}{2g} = 2R\cos 45^\circ = \sqrt{2}R$$

This tells us that,

$$v_1^2 = \sqrt{2}gR + gR\left(2+\sqrt{2}\right)$$

So, the final answer is,

$$\boxed{v_0^2 = 2gR(1+\sqrt{2})}$$

Problem 67. There is a projectile that has been thrown at a speed v and angle α angle. Now if the situation is that the projectile motion occurs at $x-y$ plane and the viewer views it from the $\hat{z}$ direction, the projectile has a form of parabola as given by the equation of parabola we notice.

But what if the viewer makes an angle θ with the $\hat{z}$ axis and views the projectile? Find the apparent shape of the projectile from the mentionend viewpoint.

Chapter 4

Electrodynamics

4.1 Electric Fields and Gauss's Law

Theorem 7 — The Electric Field inside a shell with uniform charge distribution is zero.

Well, this is not only about Electric field inside a conductor. This theorem is easily proved using experiments, but, this is only satisfied if the Electric field maintains inverse r squared dependence. Motivated by Benjamin Franklin, and done by Joseph Pristley, this experiment is also an experimental proof of this distance relationship.

Theorem 8 — The Electric Field from *point sources* are in $\frac{1}{r^2}$ form

Proof. Consider an arbitrary point P in the sphere. Now, let at one side, there be a small patch of the shell, that has an area, say, A_1, keep in mind this should be small. Now, let the solid angle made be, Ω. Now, this Ω can also extend to opposite side, hence, if the distances to patches is shown be d then,

$$A_1 = \Omega d_1^2 \qquad A_2 = \Omega d_2^2$$

So, the charge is $A\sigma$, hence, ratio of charges,

$$\frac{q_1}{q_2} = \frac{d_1^2}{d_2^2}$$

Now, the electric force on a test charge j is

$$F = jk\frac{q}{d^2}$$

and ratio of the forces by the two patches,

$$\frac{F_1}{F_2} = \frac{q_1}{q_2}\frac{d_2^2}{d_1^2} = \frac{d_1^2}{d_2^2}\frac{d_2^2}{d_1^2} = 1$$

So the net force is balanced off, thus there is no remaining effective field.

Problem 68. Find the Field of a charged cone that has slant hight L and the tip half angle θ. Find the field on the tip if the charge density is σ.

Solution. I will just write what to do, the approach considering the geometry around the L is more easy to deal with, we can break the cone into a set of infinitesimal rings and each of them have area,

$$\mathrm{d}A = 2\pi(l\sin\theta)\mathrm{d}l$$

the charge is $\sigma \mathrm{d}A$, now, the Electric Field,

$$\mathrm{d}E_{\perp} = \mathrm{d}E \cos\theta = \frac{k\sigma 2\pi \sin\theta l \mathrm{d}l}{l^2} \cos\theta$$

Integration gives us,

$$E = \sigma k\pi \sin 2\theta \ln \frac{L}{L_0}$$

This blows up at the tip if $L_0 = 0$.

Problem 69. What geometry of the cone irrespective of it's size will determine that the *Electric Field* is maximized?

Solution. We have found that, $E = \sigma k\pi \sin 2\theta \ln \frac{L}{L_0}$, now here, only θ is the geometric property that doesn't need to be variant under zooming in or out. So, the max the $\sin 2\theta$ can be is 1, so,

$$\sin 2\theta = 1 = \sin \frac{\pi}{2}$$

So very surely, $\theta = \frac{\pi}{4} = 45°$.

Theorem 9 — There is a cylinder, that is extremely long. The ends are closed. It carries a charge distribution on it's surface. But irrespective of what charge it can have, the net *Electric Field* at the center of the cap is zero

Instead of directly putting a solution, I will take the long method and do this in two part. The proof is followed by the next problem.

Problem 70. There is an open ended cylinder of radius R that has one of it's end ending at infinity. It is a shell and carries a charge surface density σ. Find out what will be the immediate Electric Field E at the center of the open end?

Usually there is two prefered ways of doing the math.

1. Cut the Cylinder to infinite amounts of Rings.

2. Cut the Cylinder to infinite amounts of Strips that are parallel to cylinder axis.

Both of them are great, I took the ring one.

Solution. The electric field of a charged ring is,

$$\vec{E} = \frac{xQ}{4\pi\varepsilon_0(R^2 + x^2)^{3/2}\hat{x}}$$

As there is charge in this ring, the area of a slit is $\mathrm{d}A = 2\pi R\ \mathrm{d}x$, charge is $\mathrm{d}\sigma = 2\pi R\ \mathrm{d}x\sigma$, and this goes to the above equation. I shall take the equation in terms of θ, as that makes life easier, but the final answer that we get is,

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

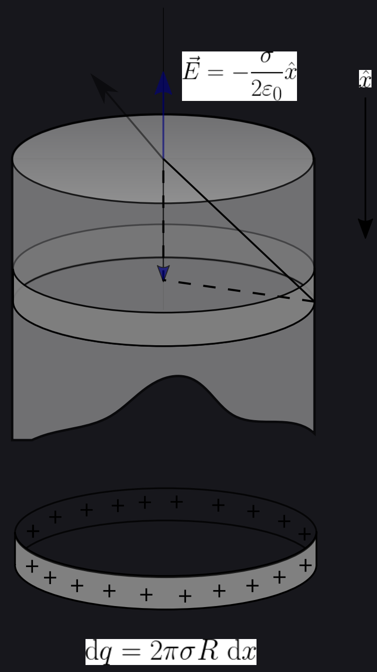


Figure 4.1.1

So, the next part of the last theorem's proof follows that, if there is a cap, and it has a charge density of σ, the same, then the field just above the center of the cap is $\frac{\sigma}{2\varepsilon}$. Now case happens is that the magnitude of the cylinder and the cap is equal, and they point in opposite, so, for this reason, the field is zeroed.

Problem 71. Two very long, parallel, thin, insulating rods are uniformly charged with equal but opposite electric charge densities. What is the shape of the *Electric Field lines*?

Solution. Let us remind ourselves that the Electric Field above a charged rod is given in the form,

$$\vec{E} = \frac{\alpha}{r}\hat{r}$$

Where I don't care about the α, don't need to at the moment. Now let us consider a random point P. At there, the fields take ratio,

$$\frac{|E_1|}{|E_2|} = \frac{r_2}{r_1}$$

So, we have this inverse proportionality. Now let us check the diagram and give attention to the triangle made by the field. Notice I drew the resultant $\vec{E} = \vec{E_1} + \vec{E_2}$ a bit small. Because of the last proportionality, we can see that, the vector line E_1 can correspond to the line r_2 and the line E_2 can correspond to r_1. This just proves that the *triangle made by the $\vec{E}$ is equal to the triangle made by $\vec{r}$ position vectors.* Hence by idea and intuition, superposing the field triangle and making it similar to the postion vector triangle, we build a angle equality.

But those who know the next theroem on Circle Tangents and angles can assume the solution instantly. The theorem states that the Electric Field lines must be a Circle

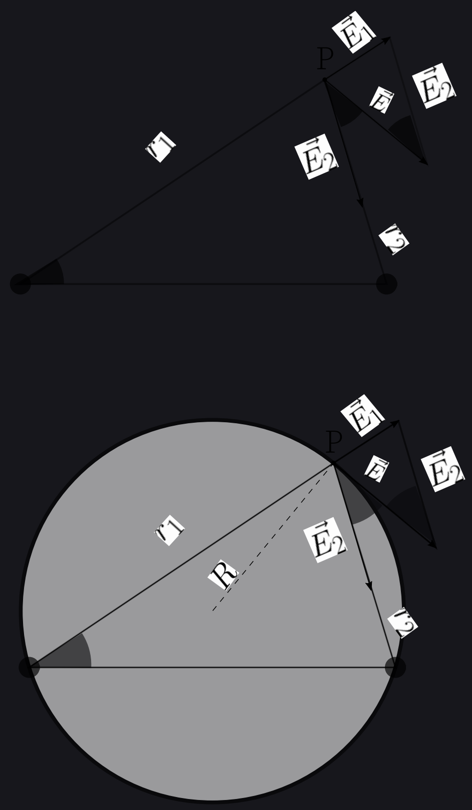


Figure 4.1.2

Theorem 10 — (`Geometry of Circle, Tangent, Angle`) If any circle has a tangent , and there is some triangle inside the circle, meeting one point of the tangent, as in the figure, then the two black angles must be equal.
In the problem we shall use the converse, if these angles match up, then surely they are a part of a circle.

Proof. Just look at the visual proof I made and adjusted below.

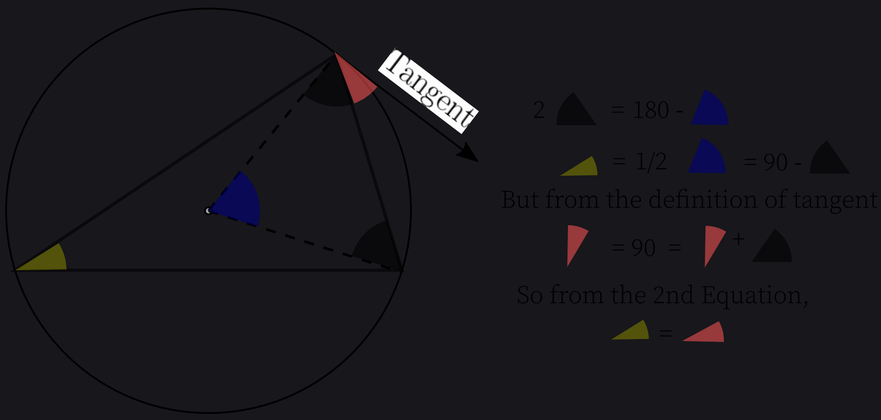


Figure 4.1.3

Problem 72. There is a charge Q and a distance away is a square plate that has side length a. If a perendicular from the surface of the square plate is drawn from one of the corners, it will intersect the charge. Find the flux through the square plate.

Solution. It can be easy to end up with difficult integral so easily, but as Jaan Kalda said, to exploit if there is any visual symmetry, we can do the following. As in the figure, we can put the charge with 24[a] such plates forming a cube, with the charge at the center. Now just using symmetry arguemnts, total flux is,

$$\Phi = \frac{Q}{\varepsilon_0}$$

And also because of symmetry, each of the 24 plates should have the same share of flux, so within a single plate, there will be a flux,

$$\boxed{\phi = \frac{\Phi}{24} = \frac{Q}{24\varepsilon_0}}$$

[a] The reasoning I made is that if there is a cube, then it should have 6 faces, but each face having 4 parts, gives total $6 \times 4 = 24$ plates

Figure 4.1.4

Problem 73. There is a large sphere conductor of radius R and a small conductor disk with radius r, now, Q_1 charge is in the large sphere and we touch the small disk to it and touch it to the ground to gradually, decrease the charge in the large sphere. What amount of times this should be done to make the charge in the sphere Q_2?

Solution. When the small disk after being charged makes contact with the ground, it must leave all of it's charge instantly.
As the dimension is small we can neglect edge effects and assume, *when the disk makes contact with the sphere, it literally copies one part of the sphere, as it is small.* Now, the charge density of the surface decreases with the amount of attempts made (charging the disk and touching it to the ground.) So, at very first with 0 attempts, the charge on sphere is,

$$Q_1$$

Then, when contact of the sphere and disk is made, *the surface charge density of both objects become the same, followed by the previous resoning.* So, when making the 1st attempt,

$$\sigma = \frac{Q_1}{4\pi R^2} \qquad \Delta q = \sigma \pi r^2 = Q_1 \frac{r^2}{4R^2}$$

So, charge on the sphere after first attempt,

$$Q_1' = Q_1 - Q_1 \frac{r^2}{4R^2} = Q_1 \left(1 - \frac{r^2}{4R^2}\right)$$

When the attempt is done for the second time,

$$Q_1'' = Q_1 \left(1 - \frac{r^2}{4R^2}\right) - Q_1 \left(1 - \frac{r^2}{4R^2}\right) \frac{r^2}{4R^2} = Q_1 \left(1 - \frac{r^2}{4R^2}\right)^2$$

So the power above the $1 - \frac{r^2}{4R^2}$ term is the number of attempts, for this,

$$Q_2 = Q_1 \left(1 - \frac{r^2}{4R^2}\right)^m$$

Solving this and making use of the fact the m must be an integer we tell that, $m \in \mathbb{Z}^+$,

$$\boxed{m = \left\lfloor \frac{\log\left(\frac{Q_2}{Q_1}\right)}{\log\left(1 - \frac{r^2}{4R^2}\right)} \right\rfloor}$$

Small approximations over the logarithm can be made.

Problem 74. What is the E field arbitrarily above a string that has λ linear charge density at distance r?

Solution. The idea is that the arbitrarity can be broken to two parts that are right angle triangles. Well that is actually nice simplification.
We are at one ends above where we want to measure the field. The angle made from that point to other end of tread is θ_0. For a charge density (linear) λ, we can tell that,

$$\mathrm{d}E_{\perp} = \mathrm{d}E\cos\theta = \frac{k\lambda\,\mathrm{d}x}{r^2}\cos\theta$$

That is made by a small charge $\mathrm{d}q = \lambda\ dx$.

$$\mathrm{d}x = \frac{r\mathrm{d}\theta}{\cos\theta} \qquad r = \frac{h}{\cos\theta}$$
$$\therefore \mathrm{d}x = \frac{h\ d\theta}{\cos^2\theta} \qquad r^2 = \frac{h^2}{\cos^2\theta}$$

Put whatever wherever necessary,

$$\mathrm{d}E_{\perp} = \frac{k\lambda h\mathrm{d}\theta\ \cos^2\theta}{\cos^2\theta h^2}(\cos\theta)$$

That is,

$$\int \mathrm{d}E_{\perp} = \frac{k\lambda}{h}\int_0^{\theta_0}\cos\theta\,\mathrm{d}\theta$$

Finally,

$$E_{\perp} = \frac{k\lambda}{h}\sin\theta_0$$

We have to concentrate on the field parallel to the string now. If you have followed till here, it is easy to notice,

$$\mathrm{d}E_{=} = \frac{k\lambda}{h}(\sin\theta)\mathrm{d}\theta$$

The integration from 0 to θ_0 gives,

$$E_{=} = \frac{k\lambda}{h}(1-\cos\theta_0)$$

So, there will be a component of the field that will be perpendicular to the wire and another will be parallel to the wire, so getting both of them,

$$E_{\perp} = \frac{k\lambda}{h}\sin\theta_0$$
$$E_{=} = \frac{k\lambda}{h}(1-\cos\theta_0)$$

From the same above solution we can get the infinite wire solution.

Problem 75. There is a uniformly charged square plate of side d with total charge Q. A point charge q is placed symmetrically above the plate $d/2$ distance above. How large is the force acting on the small charge?

Solution. The flux of the charge q at any point on the plate,

$$\mathrm{d}\Phi = E\,\mathrm{d}A = \frac{kq}{h^2 + x^2 + y^2}\mathrm{d}x\;\mathrm{d}y\cos\theta$$

The force of $\mathrm{d}x\;\mathrm{d}y$ at q is,

$$\mathrm{d}F = q\mathrm{d}E = q\frac{k\sigma\,\mathrm{d}x\;\mathrm{d}y}{h^2 + x^2 + y^2}\cos\theta$$

This is extremely clear that,

$$\mathrm{d}F = \sigma\,\mathrm{d}\Phi$$

The flux is found from the symmetry consideration as the problem of charge aside a plate, but here, the charge can be thought of a charge in the middle of the cube, so the flux is distributed equally to all the plates, a single plate should have,

$$\Phi = \frac{q}{6\varepsilon_0}$$

Integrating,

$$F = \sigma\Phi \qquad \to \qquad F = \sigma\frac{q}{6\varepsilon_0}$$

The σ can be used to,

$$\boxed{F = \frac{Qq}{6\varepsilon_0 d^2}}$$

Problem 76 (The Charged rod Pendulum). There is an infinite charged plane with a surface charge density of σ and there is a rod near it, in equilibrium. The rod is pivoted in its center and it is divided by two parts, one half of the part left to the center is negetively charged and other side is positively charged. The rod is of length d. The linear charge density of one end is λ and another side $-\lambda$.
Find the period of oscillation if it does so.

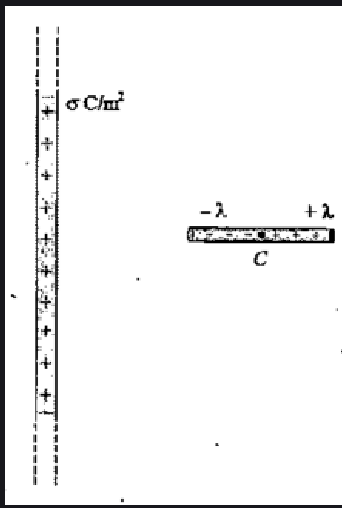


Figure 4.1.5: C is the Pivot.

This problem was asked in a question, the response I made elaborated the method how usually such a problem is done. I include it below, as a note, as it only is just what we are supposed to think. I will not change the format though.

Note. What I get from the Question is that it is in an Equilibrium, and the condition of Equilibrium is that if you shift the object a little bit from the equilibrium, then there will be some other force that will bring the object back where it started. But if it does so, then there will be some kinetic energy to the body when it returns to its position, this will overthrow the object from there. Hence this thing you have in equilibrium will oscillate back and forth, think about the pendulum, a similar case.

Now, when you have this equillibrium, then certainly net force acting on the system is zero. But if you displace it by a small amount, say x, then there will be another force proportional to the x and bring the object back where it was.

In this problem, you need to find the force that acts on the rod if you displace it a little. As there is a pivot, this force won't be able to move the small rod AB away, it can only rotate it. So there will be a torque.

1) Set up your equation to find the net torque it you displace the AB a little angle θ around, then the opposing force will create another torque that will bring the θ to be zeroed.

2) So, you can tell, $\tau = -k\theta$.

3) And if you can tell that $\tau = I\alpha = \ddot{\theta}$, where I is moment of inertia and α is the angular acceleration.

4) From SHM idea the period is found.

Now we are saved by the fact the plane is infinite (thank almighty god), so the field from the surface is going to be uniform whatever the distance you are from, the field would be $E = \frac{\sigma}{2\varepsilon_0}$, and thus, you can use the idea a dipole to find the torque. As much as I remember, the field, if uniform, can cause a dipole to have a torque $\tau = \vec{E} \times \vec{p}$ (this equation might be wrong, please check on google or UniPhy6). The dipole moment is just a set up if two opposite charge is separated by d distance, then the $\vec{p} = qd\hat{}p$, where the $\hat{p}$ is directed 1 unit vector if you project that vector going from the negative to the positive charge.

Problem 77 (Patience). Three small positively charged pearls lie one at each vertex of a triangle. Their masses are m_1 , m_2, m_3 and their charges are Q_1,, Q_2, Q_3, respectively. When the pearls are released from rest, each moves along a different straight line, the three motions taking place in a vacuum with negligible effects due to gravity. What special conditions has to be satisfied for this to happen?

Find the angle of the trinagle formed by the pearls at the beginnning of their motions, if the charge-to-mass rations of the three pearls are in proportion,

$$\frac{Q_1}{m_1} : \frac{Q_2}{m_2} : \frac{Q_3}{m_3} = 1 : 2 : 3$$

4.2 Potentials and Vector Calculus

Problem 78. There is a solid charge of radius R and the constant charge density ρ. Now, find the electric field inside the ball with the function of distance r from the center.

> **Solution.** We can just use the Gauss's law in symmetry and find that,
>
> $$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{4\pi r^3\rho}{3r^2}$$
>
> So, we can tell that,
>
> $$\boxed{E_r = \frac{\rho r}{3\varepsilon_0}}$$

Problem 79. Find the potential inside the ball of charge density ρ and find that in the region inside and outside the ball.

> **Solution.** We shall calculate the potential at r outside the ball, with the potential of the infinity, in respect. So,
>
> $$\Phi = -\int_{\infty}^{r} E_r(r)\,\mathrm{d}r$$
>
> And ...

Problem 80. We have a circular disk of raduis a and it has a uniform charge distributed over the surface in σ. Find the potential at the rim of the disk.

> **Solution.** What we need to do is, to find the potential by taking an integral over the slice of the disk's potential. So, we do it in two parts.
> **The Single Slice.**
>
> $$\mathrm{d}q_1 = \sigma \times r\mathrm{d}\theta = \sigma r\mathrm{d}\theta\mathrm{d}r$$
>
> The potential will be,
>
> $$\mathrm{d}\Phi = k\int \frac{1}{r}\,\mathrm{d}q = k\int_0^R \frac{\sigma r\mathrm{d}\theta}{r}\,\mathrm{d}r = Rk\sigma\mathrm{d}\theta$$
>
> **For the whole disk**
> We notice that,
>
> $$R = 2a\cos\theta$$
> $$\mathrm{d}\Phi = 2a\cos\theta k\sigma\mathrm{d}\theta$$

We can dare to take the integral, the limits must be seen carefully, it is from $\frac{\pi}{2}$ to $\frac{-\pi}{2}$

$$Ff = 2ak\sigma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, \mathrm{d}\theta$$

The final result is,

$$\boxed{\Phi = \frac{a\sigma}{\pi\varepsilon_0}}$$

Problem 81. Derive the equation of the potential at a distance r from the center of a Dipole of moment $\vec{p}$, that makes angle θ with the upper direction of the dipole.

Solution. We can use the geometry to see that there will be an excess of distance or a lessen of distnace from that particular point provided that we are observing the matter from a far postion. So, we can see, that,

$$\Phi(r,\theta) = \frac{kq}{r - \frac{l\cos\theta}{2}} - \frac{kq}{r + \frac{l\cos\theta}{2}} = \frac{kq}{r}\left(\frac{1}{1 - \frac{l\cos\theta}{2r}} - \frac{1}{1 + \frac{l\cos\theta}{2r}}\right)$$

Now let us recall some Taylor expansion, recall that,

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon$$

So, we can write the above symmetrical mess as,

$$\Phi(r,\theta) = \frac{kq}{r}\left[\left(1 + \frac{l\cos\theta}{2r}\right) - \left(1 - \frac{l\cos\theta}{2r}\right)\right] = \frac{kql\cos\theta}{r^2}$$

And this using the fact that, $p = ql$,

$$\boxed{\Phi(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

In vector form, this is essentially,

$$\boxed{\Phi = \frac{\vec{p}\times\hat{r}}{4\pi\varepsilon_0 r^2}}$$

Problem 82. If $\int_P \vec{E}\cdot \mathrm{d}\vec{s}$ is 0 for any closed path, then prove that the integral between two such point is independent.

Solution. Let us have two points Π, and two different paths Γ. Now, let us take a closed integral with the path Γ_0 so that, the path goes through the point Π_1 and Π_2, so, we shall have two parts in the path, one path Γ_1 that moves from Π_2 to Π_1

and Γ_2 that moves from Π_1 to Π_2. So,

$$\int_{\Gamma_0} \vec{E} \cdot \mathrm{d}\vec{s} = \int_{\Gamma_1} \vec{E}\,\mathrm{d}\vec{s} + \int_{\Gamma_2} \vec{E} \cdot \mathrm{d}\vec{s} = 0$$

So, we have that,

$$\int_{\Gamma_1} \vec{E} \cdot \mathrm{d}\vec{s} = -\int_{\Gamma_2} \vec{E} \cdot \mathrm{d}\vec{s}$$

But if we go a little bit more clear about the points, we have that,

$$\int_{\Pi_2,\Gamma_1}^{\Pi_1} \vec{E} \cdot \mathrm{d}\vec{s} = -\int_{\Pi_1,\Gamma_2}^{\Pi_2} \vec{E} \cdot \mathrm{d}\vec{s} = \int_{\Pi_2,\Gamma_2}^{\Pi_1} \vec{E} \cdot \mathrm{d}\vec{s}$$

So, between poits Π_1 and Π_2, any path ranging from Γ can be suitable.

Problem 83. A solid sphere has radius R and uniform volume charge density ρ. Find the potential of the cneter evaluating the volume integration equation for the Potential.

Solution. The equation is, $\Phi = \int_V \frac{\rho}{4\pi\varepsilon_0}\,\mathrm{d}V$. In my attempt I find the small volume in some ancient manner,

$$\begin{aligned}
\mathrm{d}V &= \frac{4}{3}\pi(r+\mathrm{d}r)^3 - \frac{4}{3}\pi(r)^3 \\
&= \frac{4\pi}{3}(3r\mathrm{d}r(r+\mathrm{d}r)) \\
&= 4\pi r^2\mathrm{d}r
\end{aligned}$$

Charge in the shell, $\mathrm{d}q = \rho 4\pi r^2\mathrm{d}r$. Integrating from 0 to R is,

$$\Phi_{center} = \int_0^R \frac{\rho 4\pi r^2}{4\pi\varepsilon_0 r}\,\mathrm{d}r = \frac{R^2\rho}{2\varepsilon}$$

(I must find out why the integration helps the shell to find the potential Φ at the center.)

The first problem is actually a big problem with a few sets, and it is quite educative, thanks to Purcell and Morin.

Problem 84 (`Spherical Charge Storage`)**.** We have a sphere of radius R and it is a shell (though it should not matter). We put Q charge in it. Now if it is grouded with the ground, what will be the initial current flowing if the wire we use has resistance $\mathcal{R}$?

Solution. The solution requires us to think, what will happen first? There must be a current as the question asks, but why? Because there is a potential difference, the word ground is just the experimental jargon of the word that is elaborated as

the zero potential.
To find the potential, we need to bring a charge from very far $V = 0$ point to near the sphere, the symmetry from the Gauss's Law helps us to know that,

$$V = k\frac{Q}{r}$$

In case we bring r distance to the sphere relative to center. Ohm's law enables us to use $V = I\mathcal{R}$ and say that,

$$\boxed{I_0 = \frac{kQ}{\mathcal{R}R}}$$

The result is much more useful than it really seems above, see, that this current I is also equal to the rate of change of the charge in the sphere. This is common sense, but it leads us to tell that,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{kQ}{\mathcal{R}R}$$

So, solving this, we find that, the answer can be written in the form,

$$Q = Q_0 e^{-\mathcal{R}Ct}$$

Where C is just acting as a constant, but guess what, this actually means that this sphere is actually a capacitor, a thing that can keep charges and burst them when needed. Hence, a metal sphere is a capacitor.

Theorem 11 (Metal Sphere and Capacitance) — Metal Sphere can store charge and discharge just like a capacitor. The equivalence is given by,

$$C = 4\pi R\varepsilon_0$$

Problem 85 (`Image Charges for Grounded Spherical Shell: 1`). A point charge $-q$ is located at $x = a$, and a point charge Q is located at $x = A$. Show that the locus of points with $\phi = 0$ is a circle in the xy plane (and hence a spherical shell in space).

Solution. The basic solution lies in considering the equation of a circle in Cartesian Plane Coordinates rather than the usual (also traditional) trigonometric approaches I use often. For reference, the equation of a circle is,

$$(x - h)^2 + (y - k)^2 = r^2$$

Where the circle is located at $(x, y) = (h, k)$. Or else just by expanding,

$$x^2 + y^2 - 2xh - 2yk + h^2 + k^2 = r^2$$

So, at first we need to know the points with a specific potential ϕ. At any point that is located at x, y, the potential is,

$$\phi = \frac{-q}{\sqrt{(x-a)^2 + y^2}} + \frac{Q}{\sqrt{(A-x)^2 + y^2}}$$

When potential of some points are zero, we get that,

$$\frac{q}{\sqrt{(x-a)^2 + y^2}} = \frac{Q}{\sqrt{(A-x)^2 + y^2}}$$

$$(x-a)^2 + y^2 = \frac{q^2}{Q^2}\left((x-A)^2 + y^2\right)$$

Replacing q^2/Q^2 with c

$$x^2 + y^2 - 2ax + a^2 = c\left(x^2 + y^2 - 2Ax + A^2\right)$$

$$\left(x^2 + y^2\right)(1-c) - 2x(a - cA) + \left(a^2 - cA^2\right) = 0$$

$$x^2 + y^2 - 2x\frac{a - cA}{1-c} + \frac{a^2 - cA^2}{1-c} = 0$$

The circle equation can be forced to come as there is no coefficient with the $x^2 + y^2$ terms.

$$x^2 + y^2 - 2x\frac{a - cA}{1-c} = \frac{cA^2 - a^2}{1-c}$$

And introducing the circle equation, we have an analog, $h = \frac{a-cA}{1-c}$,

$$x^2 + y^2 - 2x\left(\frac{a-cA}{1-c}\right) + \left(\frac{a-cA}{1-c}\right)^2 = \frac{cA^2 - a^2}{1-c} + \left(\frac{a-cA}{1-c}\right)^2$$

The circle equation is ready, we just need to look for the radius r^2 now,

$$\begin{aligned}
r^2 &= \frac{cA^2 - a^2}{1-c} + \left(\frac{a-cA}{1-c}\right)^2 \\
&= \frac{(1-c)(cA^2 - a^2) + a^2 + c^2A^2 - 2caA}{(1-c)^2} \\
&= \frac{cA^2 - a^2 - c^2A^2 + ca^2 + a^2 + c^2A^2 - 2caA}{(1-c)^2} \\
&= \frac{cA^2 + ca^2 - 2caA}{(1-c)^2} \\
&= (A-a)^2\frac{c}{(1-c)^2}
\end{aligned}$$

We have came to the result that,

$$r^2 = (A-a)^2 \frac{c}{(1-c)^2}$$

Followed by the circle equation, we can finally conclude that the center of the $\phi = 0$ locus is located at,

$$\boxed{x_0 = \frac{a - A\frac{q^2}{Q^2}}{1 - \frac{q^2}{Q^2}}}$$

And the radius of the circle is at,

$$\boxed{r = (A-a)\frac{\frac{q}{Q}}{1 - \frac{q^2}{Q^2}}}$$

Problem 86. What must be the relation among q, Q, a and A so that the center of the circle is located at $x = 0$?

> **Solution.** Getting back to the problem, we will invoke the x coordinate is going to be $k = 0$. Thus,
>
> $$\frac{a - A\frac{q^2}{Q^2}}{1 - \frac{q^2}{Q^2}} = 0$$
>
> Solving this a little bit, and setting the numerator to zero (as denominator can't make zero using division),
>
> $$\begin{aligned} \frac{a - A\frac{q^2}{Q^2}}{1 - \frac{q^2}{Q^2}} &= 0 \\ aQ^2 - q^2 A &= 0 \\ aQ^2 &= q^2 A \end{aligned}$$
>
> Or just to say,
>
> $$a = cA$$
>
> The result is suddenly quite simple, the condition is that,
>
> $$\boxed{a = A\frac{q^2}{Q^2}}$$

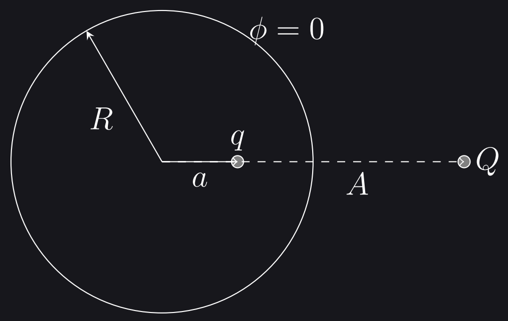


Figure 4.2.1: A Perfectly scaled diagram of the system we've been analyzing so far.

Problem 87. Explain why the previous results imply that: if a charge Q is is *externally* located at $A > R$, from the center of the *GROUNDED Sphere* with radius R, then the external field is just same as the case, of a, point charge $q = \frac{QR}{A}$ (magnitude) is located at a distance $a = \frac{R^2}{A}$ from the center of the shell.

Solution. To get the meaning of using the previous analysis becomes vivid when we actually look at the word "Grounded". It implies that the sphere shell we are dealing with is just $\phi = 0$ locus as we previously found. The $\hat{z}$ axis is not need to be cared because symmetry actually takes nice care of it. Using this result where circle is at $x = 0$, (the condition $a = cA$), we can solve for the radius,

$$r = (A - a)\frac{\frac{q}{Q}}{1 - \frac{q^2}{Q^2}} = A\left(1 - \frac{q^2}{Q^2}\right)\frac{\frac{q}{Q}}{1 - \frac{q^2}{Q^2}} = A\frac{q}{Q}$$

Using this equation, directly we come to the point that,

$$Q\frac{r}{A} = q$$

The radius of the equipotential $\phi = 0$ and this is also same as the grounded sphere, hence,

$$r = R$$

For which,

$$\boxed{q = \frac{QR}{A}}$$

That finally declares the validity.
Now to look for the position of the charge, we see that,

$$r^2 = A^2\frac{q^2}{Q^2} = A^2\frac{a}{A} = aA$$

Thus,

$$\boxed{a = \frac{r^2}{A} = \frac{R^2}{A}}$$

And this also solves the requirement.

Problem 88 (In·nite Grid for Resistor). There is an infinite lattice of resistor wiht each end having resistance R. What is the resistance between two adjacent nodes?

Solution. Let us imagine we are running total I current into one of the node of the lattice. There are 4 ways for the current to go away hence each path has current $\frac{I}{4}$ going outwards. Because there is some radial symettry, the current gets out at infinity.
Now imagine superposing another system, where we are taking away I current from a node adjacent to the node we are putting current in. CUrrent flow is like water flow, we can superimpose one configuration on the other to get a net effect. It is just like putting one electric field on the other and adding them at every point to get a net field (like a dipole).
If you try to analyse how the current is flowing overall, you can notice that the current flow will also have dipole-dipole feeling.
We don't at all put our attention what is the current flow at every path; that is useless; and that is why I spent so much time wasting behind it. Learn from my mistakes.
If we try to imagine (and do it correctly), the current flowing by the middle path, connecting the current source nad si nk directily, is,

$$\frac{I}{2}$$

The resistance of the path is R. Meanwhile, if the potential differrence between the node is ϕ, and equivalent resistance is R_{eq}, then,

$$R_{eq} = \frac{\phi}{I}$$

The unknown is ϕ. But we know, for the current flowing along the path joining the source and si nk node is $I/2$. Resistnace is R, thus,

$$\phi = \frac{I}{2}R$$

The potential difference of the two point is directly found by looking at a segment of the infinite circuit. Now,

$$R_{eq} = \frac{IR}{2I}$$

Thus the answer that is bit ridicouling,

$$R_{eq} = \frac{R}{2}$$

4.3 Electric Circuits

I tend to make this section the day I would have custody of a printer and also the wish to solve –

I am unable to think very clearly because there is some function/arrangement outside and they arrangement people are playing music in loud speakers in high volume I am uable to think or type clearly. The "l" key is not working as It should.

I have to find various problems solution I have written in the aops forum not here. Then copy paste them from here (there, my head is not active at 100 percent.

Theorem 12 (The Basic Circuit Rules) —

1. The sum of electrical currents into a node of a circuit (a node is a point) is Zero.
2. Along a closed loop of an electrical circuit, the sum of voltage drops and ups across the circuit elements sum up to zero. *The analogy is taking a path integral and ending up at the point where we started. The path integral rule..*

Theorem 13 (Ohm's Law) — The potential difference across an object in a circuit is,

$$V = IR$$

Where the I is the current R is the Resistance.

Problem 89. The follow circuit is a bridge one. Solve the equivalent resistance for the two ends

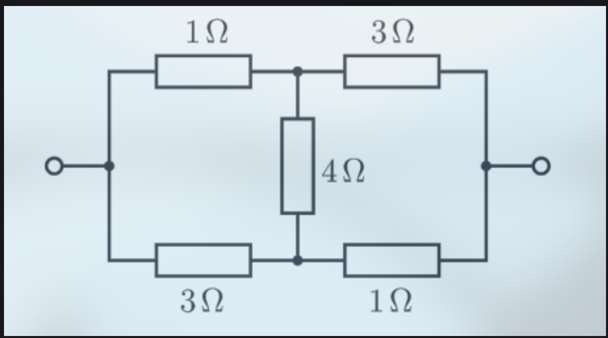


Figure 4.3.1

Solution. For a Δ circuit nodes A, B, C, and resistances R_{AB}, R_{BC}, R_{CA}. Now, for a Y circuit, R_A, R_B, R_C, the transformation,

$$R_A = \frac{R_{AB} \cdot R_{AC}}{R_{AB} + R_{BC} + R_{CA}}$$

This gives us, in Ω [ommited till the end],

$$R_A = 1/2$$

$$R_B = 3/2$$
$$R_C = 3/8$$

Now we shall have a Parallel Circuit at the right side, the 2 resistors above and below form a series, For the [b]Upper[/b] series,

$$1/2 + 3 = 7/2$$

And for the [b]Lower[/b]

$$3/2 + 1 = 5/2$$

Equivalent resistance of the parallel circuit is thus,

$$\frac{7/2 \cdot 5/2}{7/2 + 5/2} = 35/24$$

This resistance is in series with the 3/8 resistance, so finally the equivalent resistance R is,

$$R = 35/24 + 3/8 = 11/6$$

Problem 90. To prove that the maximal current in a circuit is achieved when the Load resistance is equal to the Internal Resistance of the battery.

Solution. Let the battery EMF be ε. It has an internal resistance of r and thus the power taken by a load of R is,

$$P = I^2 R = \left(\frac{\varepsilon^2}{(R+r)^2}\right) R = \frac{\varepsilon^2 R}{(R+r)^2}$$

Now, we have this $P = P(R)$, so for some special load resistance value $\tilde{R}$, the Power output is going to be the maximum. Now for sake of making the math easy, we shall consider the equation of $1/P$, now if P is maximized, then $1/P$ will be minimized.

$$\frac{1}{P} = \frac{(R+r)^2}{\varepsilon^2 R}$$

For $1/P$ being the minimum,

$$\frac{\mathrm{d}}{\mathrm{d}R}\frac{1}{P} = \frac{1}{\varepsilon^2}\left(1 - \frac{r}{R^2}\right) = 0$$

This equation can only be solved if we note that $R = r$. Thus, the maximum Power to the load is possible when the Load Resistance is equal to the Internal Resistance, so,

$$\tilde{R} = r$$

Infinite circuit problem heavily relies on the use of Superposition.

Problem 91 (In·nite Grid). There is an infinite grid of wires with each side having 1Ω resistance, now what is the equivalent resistance between two adjacent nodes?

Solution. We use the idea of the Superposition from the very beginning, think of setups of current flow that can add up to give the require setup.
The required setup is to input current into node A and take it off from the adjacent B node.
The current flow of this required setup is easily achieved, think that current is only flown into by node A and drawn out from infinity (very far).
And think the opposite from node B, current is drawn out from B and put into from infinity ∞.
Superpose these two setups, this will result I current into A and drawn out from B. The current flow when current only input to A or only drawn out of B has rotational symmetry. And the superposition full fills the necessity accordingly.
The current that flows between node A and B is $2 \times \frac{I}{4}$. The resistance between A and B is $R = 1\Omega$. Hence the potential difference, is,

$$\Delta\Phi = \frac{I}{2}R = \frac{I}{2}$$

As $R = 1\Omega$. So, net current flow being I, the net potential difference $\Delta\Phi$ is also,

$$\Delta\Omega = IR_{equivalent}$$

And both potential difference between A and B is same in two cases, so,

$$IR_e = \frac{I}{2}R$$

This solves that the equivalent resistance is just half,

$$\boxed{R_e = \frac{R}{2}}$$

Numerically here answer for $R = 1\Omega$ is $\frac{1}{2}$

Problem 92. There is a Three Dimensional Cubic Lattice and each of the side of the lattice has resistance 1Ω, if the lattice is infinite sized, what is the equivalent resistance between two adjacent nodes?

Solution. We drive a current I into A. This splits into 6 currents of $\frac{I}{6}$. We then take out a current I at B. This splits into 6 current of $\frac{I}{6}$. So the net current is $\frac{I}{3}$. So $V = \frac{IR}{3}$ and thus $R_{eq} = \frac{V}{I} = \frac{R}{3}$.

4.4 Alternating Currents

I am going to write this, as I review through the Jaan Kalda Booklet for Alternating Current, after I am done, I hope to look at the Universtiy Physics textbook for it.

There can be Voltage Sources in a Circuit, where the *Source* will give always a constant potential difference across it,

Or, there can be a Current Source that will always give influx to a constant current in a circuit.

Now, we can just stay aloof from putting constant current or voltage (for voltage, I mean a potential difference), and put current that vary overtime.

We can just put current of voltage whose value go up and down. Say in a cyclic manner. Like, oscillation,

$$I = I_0 \sin(\omega t + \phi)$$

$$V = V_0 \sin(\omega t + \phi)$$

We better introduce the Euler's Formula to make things better and simple,

$$e^{i\psi} = \cos\psi + i\sin\psi$$

Hence, the current can be written as,

$$I = I_0 e^{i(\omega t + \phi)} = (I_0 e^{i\phi})(e^{i\omega t}) = I_c e^{i\omega t}$$

Here, I have introduced a thing called I_c that is a special complex amplitude. And very surely we never need to be too mindful about it because at the end of the day, the real part of this is always going to be just I_0.

Now the calculation of Power is little bit different, because it is always real and having both I and V as complex, it's calculation is a little more complex too, but it is not that bad though.

Oscilation will cause the magnitude of power to vary, we are always interested in it's average, $\langle P \rangle$.

Defining a function $\mathcal{R}$ which is to yield the real value of anything complex, we can say,

$$\langle P \rangle = \langle \mathcal{R} I_c e^{i\omega t} \cdot \mathcal{R} V_c e^{i\omega t} \rangle$$

Now what is the real part? There is a way to do it.

Theorem 14 — The real part of any complex number z is,

$$\mathcal{R}z = \frac{z+z^*}{2}$$

Proof.

$$z = a + ib$$

Hence, if we take the complex conjugate $i \to -i$, then,

$$(a+ib) + (a-ib) = z + z^* = 2a$$

Thus, taking half shall yield us the real part, a.

So,

$$P = \left\langle \frac{I_c e^{i\omega t} + I_c^* e^{-i\omega t}}{2} \cdot \frac{V_c e^{i\omega t} + V_c^* e^{-i\omega t}}{2} \right\rangle$$

Now solving a little more,

$$\frac{1}{4}\left\langle \left[IVe^{2i\omega t} + I^*V^*e^{-2i\omega t} + IV^*e^0 + I * Ve^0 \right] \right\rangle$$

Now, the full period average of $e^{2\omega it}$ is just 0. Hence,

$$P = \frac{1}{4}\left\langle I_0 e^{i\phi_1} V_0 e^{-i\phi_2} + I_0 e^{-i\phi_2} V_0 e^{i\phi_1} \right\rangle$$

Solves to,

$$P = \frac{1}{2} I_0 V_0 \cos(\phi_1 - \phi_2)$$

For the RMS value,

$$P = I_{rms} V_{rms} \cos(\phi_1 - \phi_2)$$

Solving from this, we can also say,

$$P = |I|^2 \mathcal{R}Z$$

Where Z is net Impedance.

Problem 93. Consider a soldering gun with nominal power $P = 30\ W$ and nominal voltage $V = 220\ V$, frequency of AC is $\nu = 50\ Hz$. Which capacitance needs to be connected in series to the iron in order to reduce the power down to $P_1 = 20\ W$?

Solution. Let the 30 W power be P_1 and the 20 W power be P_2. The nominal voltage is always $V_0 = 220\ V$, which will be the same input for both with and

without the capacitor, connected with the Soldering Gun.
Now we shall assume the impedance of the Soldering Gun will be completely real without any imaginary parts. Or to say simply the Soldering Gun has only Ohmic Resistance and nothing crazy (initial condition of my head was this iron must have some sort of inductance, which is just an unnecessary detail we should ignore now).
Now, the ohmic resistance is R from,

$$P_1 = \frac{V_0^2}{R} \qquad \rightarrow \qquad R = \frac{V_0^2}{P_1}$$

Let us connect the capacitor, so, it will make a complex impedance,

$$Z = R + \frac{1}{i\omega C}$$

The power will yield us an equation where we can solve for C,

$$P_2 = |I|^2 \mathbb{R} Z$$

Solving this,

$$P_2 = \frac{V_0^2}{|Z|^2}\mathbb{R}Z = \frac{V_0^2}{R^2 - \frac{1}{\omega^2 C^2}} R$$

And inputting the equation of $R = \frac{V_0^2}{P_1}$.

$$C = \frac{1}{V_0^2 \times 2\pi\nu}\sqrt{\frac{P_2}{\frac{1}{P_1} - \frac{P_2}{P_1^2}}}$$

Problem 94 (Electric Circuit Pr 94). Find L and R in terms of R_1, R_2, R_C, L, if the reading of the voltmeter is 0.

Solution. We to see that the pair L, R_1 are connected in parallel with R_2, Z , where Z is the combined net impedance of the resistor R_C and C capacitor. There is this Voltmeter connecting the joints of L, R_1 and $R2, Z$.
We then can clearly see 3 equations,

$$Z = \frac{R_C \frac{1}{i\omega C}}{R_C + \frac{1}{i\omega C}}$$

From Kirchoff's Potential Rule,

$$I_1 = \frac{V}{R + i\omega L + R_1}$$

$$I_2 = \frac{V}{R_2 + Z}$$

Now from the reading of Potential, this is clear that,

$$(V - I_1 \omega i L - I_1 R) - (V - I_2 R_2) = 0$$

Solving this, we can find L and R, only their real parts matter, so,

$$\boxed{L = R_1 R_2 C}$$

$$\boxed{R = R_1 R_2 / R}$$

Problem 95 (High Pass and Low Pass ·lter). There is a voltage source, a resistor R, inductor L and capacitance C, find the $\frac{v_{out}}{v}$ for v_{out} which is the potential across the capacitor, for frequency very low. This case it's a low pass filter.

Solution. The potential accross the capacitor is, in complex impedances,

$$v_{out} = IZ_C$$

For the overall complex voltage v, we have,

$$v = IZ = I\left(Z_{net}\right)$$

Here, Z_{net} is,

$$Z_{net} = R + i\omega Ll + \frac{1}{i\omega C}$$

From this,

$$Z_{net} = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

As you know,

$$\frac{1}{i} = -i$$

Now, the value we need,

$$\left|\frac{v_{out}}{v}\right| = \frac{|Z_c|}{|Z|}$$

Here,

$$|Z_c| = \sqrt{\frac{1}{\omega^2 C^2}}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

We have,

$$\frac{v_{out}}{v} = \sqrt{\frac{\frac{1}{\omega^2 C^2}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Too small ω means,

$$= \sqrt{\frac{1}{(\omega^2C^2)\left(\frac{1}{\omega^2C^2}\right)}}$$

Because $\frac{1}{\omega^2}$ is too big and R is small relative to it, the thing becomes,

$$v_{out}/v = \frac{1}{\omega^2C^2}$$

Problem 96 (For High Pass Filter). Find the v_{out}/v for a ω being very high, here v_{out} is the potential across the Resistor plus Inductor.

Solution. The v_{out} is,

$$v_{out} = I\left(R + i\omega L\right)$$

Thus,

$$v_{out}/v = |z_{RL}|/|z|$$

We get,

$$= \sqrt{\frac{R^2 + \omega^2L^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Using approxes,

$$\frac{v_{out}}{v} = 1$$

For ω being too large.
But if the ω is too low,

$$\frac{v_{out}}{v} = \omega RC$$

Chapter 5

Magnetism

5.1 Ordinary Magnetism

Magnetism is the wonderful relativistic correction of the Electric component. There are some very common charachter that is follows for the usual case. I like to think that as the Pure Magnetism problems, and problem that involve Electricity and Magnetism together. The union of two ideas.

Theorem 15 (Charges moving in Magnetic fields) — Whatever the field might be, whatever is the motion of charge, if the charge velocity vector tends to be perpendicular to the Magnetic field, then surely the charge should take a *Circular* path, given by the fact,

$$\vec{F} = q\vec{v} \times \vec{B}$$

Hence,

$$qvB = m\frac{v^2}{R} \qquad \rightarrow \qquad R = \frac{mv}{qB}$$

Problem 97 (Estonian Physics Olympiad, Kristian Kuppart, Taavet Kalda). A particle with a positive charge q and velocity $\vec{v}$ moves towards a rectangular strip so that that its velocity vector forms an angle α with the normal of the strip. The thickness of the strip is d and on it is located a homogeneous $\hat{z}$-directional magnetic field with an induction $\vec{B}$ (from the plane of the paper directed towards us). For what maximal falling angle αmax does the particle still go through the magnetic field? It is known that the particle that enters the strip perpendicularly would go through the strip, but not perpendicularly when it reaches the end.

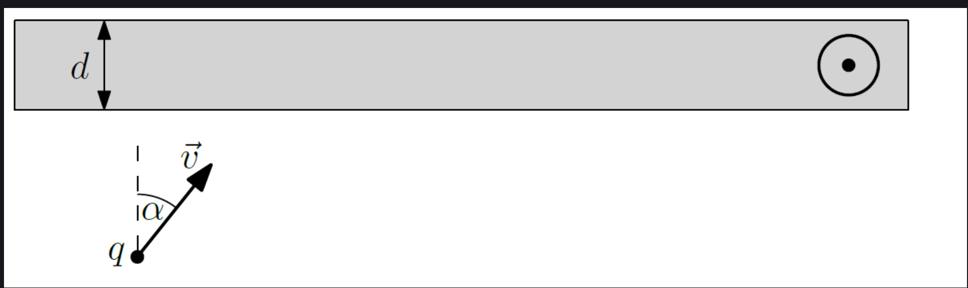


Figure 5.1.1: Finding the maximum angle α_{max}

Solution. The particle will of course rotate in a circle of radius $R = \frac{mv}{qB}$ when it is in the magnetic field region. So, at the limiting case, the particle will get out from the strip parallel to the side. Otherwise, the particle returns to the strip because of the circle.

Note. The center of the circle does not need to be necessarily inside the region though.

From drawing a nice diagram, we can see that,

$$r \sin \alpha_{max} + d = r$$

Instead, you can also find that,

$$r(1 - \cos \alpha_{max}) = d$$

Here, $r = \frac{mv}{qB}$. Solving this, gives us,

$$\boxed{\sin^{-1}\left(1 - \frac{q\ d\ B}{mv}\right)}$$

Or, this solution is also totally feasible,

$$\boxed{\frac{\pi}{2} - \cos^{-1}\left(1 - \frac{q\ d\ B}{mv}\right)}$$

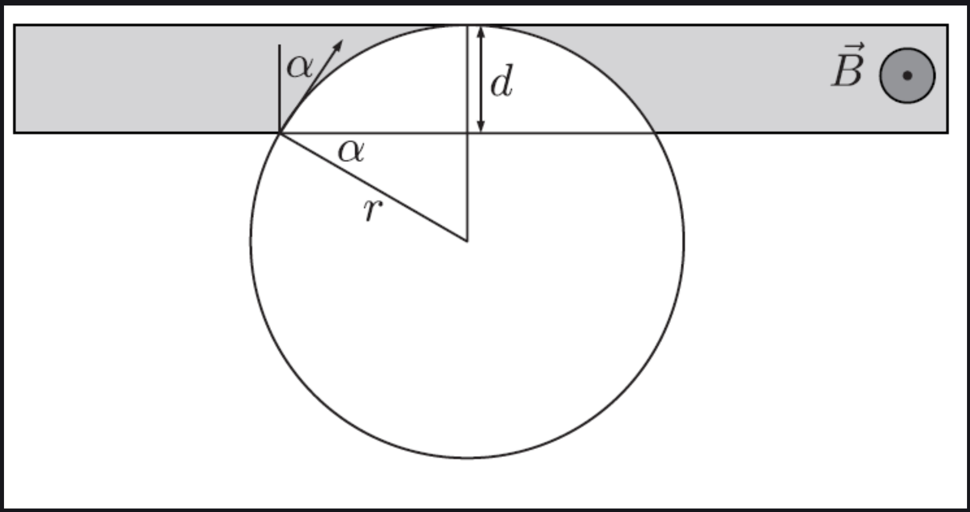


Figure 5.1.2: Notice the Geometry

Problem 98 (Magnetic Paintball, Ashmit Dutta). The system is set up as in the figure, the cylinder of radius r has a uniform magnetic field that is directed upwards (towards us). The particle starts from just below the cylinder, that has this charge q. Take referance from the diagram, the pipe AB host the particle, and for some help in math, the angle made by the perpendicular that crosses the cylinder circle to the point B is $\phi_0 = \frac{\pi}{3}$.

1. If the Magnetic field suddenly comes to zero in a very small time, what would be the speed the charged particle gain?

2. If the Magnetic field reduces by,

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = -k$$

then what will be the final speed attained by the particle, when it gets out of the tube from point B?

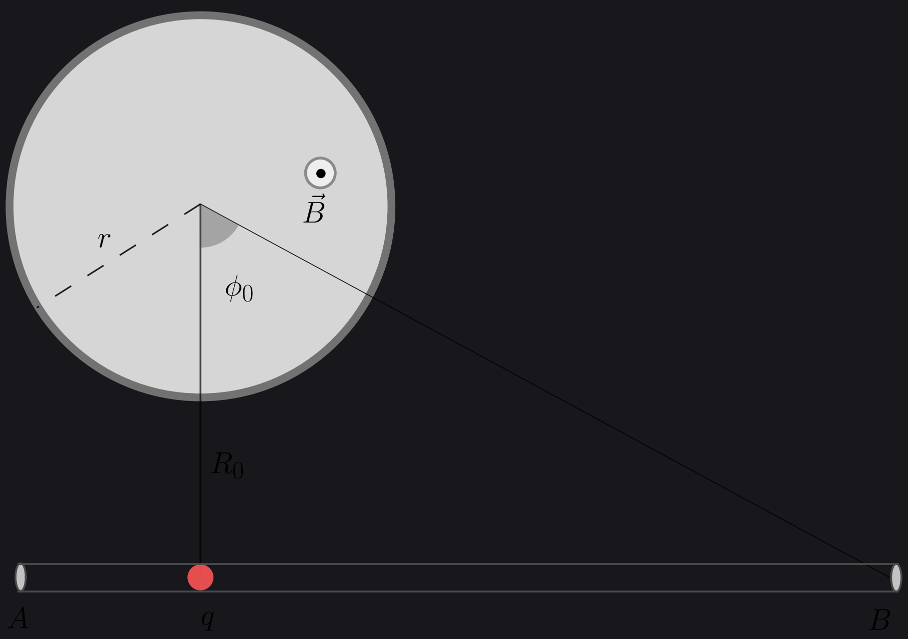


Figure 5.1.3: Refer to the diagram for the variables.

Solution. 1. Draw another circle of radius R_0, with axis to the cylinder center, this loop that we think has a chaning magnetic field inside it. And from Faraday's law, there is supposed to be an EMF from the changing *flux* and that EMF is the reason of an electric field path integral as below,

$$\varepsilon = \oint_L \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

And from expanding the idea using symmetry,

$$E = \frac{\pi r^2}{2\pi R_0} \frac{\mathrm{d}B}{\mathrm{d}t} = \frac{r^2}{2R_0} \frac{\Delta B}{\Delta t}$$

And, using the idea of impulse of force,

$$F\Delta t = m\Delta v$$

So, we can tell from this equation with the Electric force,

$$v = \frac{qE}{m}\Delta t = \frac{qBr^2}{2mR_0}$$

Be careful that the above analysis only holds for Δt sufficiently small, otherwise the Electric force reduces as the particle moves off, or, you can tell, the force causes the R_0 not to stay constant and makes it more general.

Problem 99 (Is Magnetic Paintball good to use?). Make some guesses of the real life numerical values so that we can actually build such a system, then would the speed of the particle make any sense?

Problem 100 (Bar Magnetic Force Law). A pendulum system is made with a bar magnet, hung with a string of length l. We bring another identical magnet is bought near it, aligned with the magnet. So, when we bring the magnet, they are separated by a distance d and the hung magnet is displaced by x. Using this idea, assuming that magnetic force has the form,

$$F_B \propto d^{-n}$$

Find the number n.

The fun fact about this problem is initially I thought this was damn impossible and esoteric, through the method is just very stereotypical. Motivated by the virtual work principle from the Mechanics part.

Chapter 6

Thermal Physics

6.1 Thermal Physics

6.2 General

Problem 101. Show that the depth of a ice plate formed on a pond due to cold weather is proportional to the square root of the time the weather is below 0 degree celsius.

> **Solution.** The heat conductivity equation,
>
> $$\frac{\mathrm{d}H}{\mathrm{d}t} = k\frac{A}{x}(T - T_0)$$
>
> Now the latent heat of forming ice,
>
> $$\Delta Q = (m)\lambda = x(A\rho_{\text{ice}})\lambda$$
>
> Put these two together and we get,
>
> $$\boxed{x_0 = \sqrt{\alpha t_0}}$$
>
> α is some constant you get from solving the system.

6.3 Ideal Gas Equation

Problem 102. A vessel of volume $V = 30\ l$ contains ideal gas at temperature $0°\ C$. After a portion of gas is let out, the pressure changes by $\Delta = 0.78\ atm$. The temperature remains constant, find the mass of the released gas, when in standard conditions, the $\rho = 1.3\ g/l$.

> **Solution.** The Ideal Gas Equation directly comes into the game,
>
> $$\Delta pV = \Delta nRT$$
>
> Here the Δn is for the change in the mole (amount) of gas in the system. Now, we are given the standard condition density that will help us find the mass,
>
> $$p_0 V = \frac{m}{M}RT_0$$
>
> Here m is the mass of the gas and M is the mass of 1 mole of gas. Hence,
>
> $$p_0 = \rho\frac{1}{M}RT_0 \to \rho = \frac{p_0 M}{RT_0} \to M = R\frac{T_0\rho}{p_0}$$

Now, we can directly solve for m,

$$\frac{\Delta p V}{RT} \times M = m$$

Hence,

$$\frac{\Delta p V}{RT} \frac{RT_0 \rho}{p_0} = m$$

In standard condition, $p_0 = 1\ atm = 1 \times 10^5\ Pa$ and $T_0 = 20^\circ\ C = 293\ K$. Putting the numbers in place,

$$m = \frac{\Delta p}{p_0} \frac{T_0}{T} \rho V = 28.34\ atm$$

Problem 103 (`Valve and two Cylinder`). Two identical vessels are connected with a pipe having a valve that lets gas from one side to other if and only if the pressure difference across the valve is $\Delta p \geq 1.1\ atm$. Initially, there is vacuum in one of the vessel and the other vessel is at temperature $T_1 = 27^\circ$ and pressure $p_1 = 1\ atm$. After some time, both of the vessels were heated to the temperature $T_2 = 107^\circ$. What will be the pressure of the vessel now that was in vacuum?

When I first tried to solve the problem, the mistake I did was to try understand what happens when the gas tries to flow when pressure is high across the valve. That mistake taught me a nice lesson, care about the initial and final states when it is suitable, not the middle case when things are not in equilibrium (or moving).

Solution. Initially, the state is,

$$p_1 V = nRT_1$$

After the heating is done, and the gasses have moved from one to another accordingly through the valve, there will be two states of the two vessels.

We can feel from now that the difference between the pressure of the two vessel after heating is surely going to be $\Delta p = 1.1\ atm$. Otherwise the gas would flow until this difference is achieved. To be direct,

$$p_2 = p_3 + \Delta p$$

Here p_2 is the pressure of the vessel that had gas initially, now heated, and p_3 is the pressure after the gas has flowed into it after the heating process. The gas equations,

$$p_2 V = (n - \Delta n)\, RT_2$$

$$p_3 V = (\Delta n)\, RT_2$$

Here the Δn is the amount of gas that has flowed from one vessel to another. Now we can start solving equations,

$$\frac{p_3 V}{RT_2} = \Delta n$$

And from initial case,

$$n = \frac{p_1 V}{RT_1}$$

Hence, the $p_2 V = (n - \Delta n) RT_2$ equation,

$$p_2 V = \left(\frac{p_1 V}{RT_1} - \frac{p_3 V}{RT_2} = \Delta n \right) RT_2$$

And just taking off the factors,

$$p_2 = p_1 \frac{T_2}{T_1} - p_3$$

And as we know, $p_2 = p_3 + \Delta p$,

$$\Delta p + 2p_3 = p_1 \frac{T_2}{T_1}$$

That solves,

$$\boxed{p_3 = \frac{1}{2} \left(p_1 \frac{T_2}{T_1} - \Delta p \right) = 0.083\ atm}$$

The first time I was doiing this it took so much time!

Problem 104. How Degrees of Freedom if the Gas molecules have (in standard conditions) density ρ and sound propagation speed v ?

Problem 105. One mole of a certain ideal gas is contained under weightless piston of a vertical cylinder at temperature T. The space over the piston opens open to the atmosphere. What amount of work has to be performed so that isothermally the gas volume increase n times by slowly raising the piston?

6.4 1st Law of Thermodynamics

Problem 106. Derive the fact that the internal energy of some random gas is $pV/\gamma - 1$, and also show that for general case, $\frac{i}{2}R = C_v$.

So that there is some fundamental taste to what we shall try to find, we better start from the basic idea, though that is not necessary, we can absorb the fact that each degree of freedom will have an energy $\frac{1}{2}kT$ per particle.

Solution. I have to regenralize it from the beginning, one of the astounding fact that made me love Thermodynamics, refer to Irodov and all other text book and stuff.
So, from the Equipartition of Energy Theorem, we at first keep in mind that Energy is dependent on the Quadratic (2nd Power) of a Variable. Thus, let,

$$E = \alpha x^2$$

Remember meanwhile,

$$P(\epsilon) = ce^{-\beta\epsilon}$$

If $\epsilon = \epsilon(x)$, then, $P(\epsilon) = P(x)$, a commonsense that don't easily work in time.
x is just some random variable, not necessarily distance or so. Now, the Canonical Probability of Having energy E is proportional to the Boltzmann factor β. Thus, from normalization view, the probability $P(x)$ of the system having energy αx^2 is proportional to the Boltzmann factor $e^{-\beta\alpha x^2}$. Knowing with the constant c, $P(x) = ce^{-\beta\alpha x^2}$ So,

$$\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = 1 = c\int_{-\infty}^{\infty} e^{-\beta\alpha x^2}\,\mathrm{d}x \quad \rightarrow \quad c = \frac{1}{\int_{-\infty}^{\infty} e^{-\beta\alpha x^2}}$$

$$\begin{aligned}\langle E\rangle = \int_{-\infty}^{\infty} E\,P(x)\,\mathrm{d}x =& \frac{\int_{-\infty}^{\infty} \alpha x^2 e^{-\beta\alpha x^2}}{\int_{-\infty}^{\infty} e^{-\beta\alpha x^2}}\\ =& \frac{1}{2\beta}\\ =& \frac{1}{2}kT\end{aligned}$$

So every independent variable of dynamical system (Degree of Freedom) can be assigned a $\frac{1}{2}kT$. This is amazing, for i such freedom we can assign, $\frac{\langle i\rangle}{2}kT$ That's why,

$$\langle E\rangle = \frac{\langle i\rangle}{2}kT$$

So, the Total Energy of N molecules at T is given by, keeping $pV = NkT = nRT$ in mind,

$$E = \frac{\langle i\rangle}{2}NkT = \frac{\langle i\rangle}{2}nRT$$

Now let us move to some constructive work,
Heat Capacity :Isochoric (Molar)
From the 1st Law of Thermodynamics,

$$\Delta E = \Delta Q - \Delta W$$

For the case the W work is done by the system. So,

$$\frac{\langle i \rangle}{2} nR\Delta T = \Delta Q - \Delta W$$

Let us find the *Heat Capacity at constant Volume*, C_v, constant volume tells that $\Delta W = 0$, so,

$$\frac{\langle i \rangle}{2} nR\Delta T = \Delta Q$$
$$\frac{\langle i \rangle}{2} R = \frac{\Delta Q}{n\Delta T}$$
$$\frac{\langle i \rangle}{2} R = C_v$$

So, the heat capacity of isochoric system is $C_v = \frac{\langle i \rangle}{2} R$.
Heat Capacity :Isobaric (Molar)
This case, pressure stays constant and let's us write that, $\Delta W = p\Delta V = nR\Delta T$. So,

$$\frac{\langle i \rangle}{2} nR\Delta T = \Delta Q - nR\Delta T$$

Now a bit simplifying,

$$\frac{\langle i \rangle}{2} nR\Delta T + nR\Delta T = \Delta Q$$
$$\frac{\langle i \rangle}{2} R + R = \frac{\Delta Q}{n\Delta T}$$
$$\frac{\langle i \rangle}{2} R + R = C_p$$

So, the heat capacity of isobaric system is $C_p = \frac{\langle i \rangle}{2} R + R$.
Ratio of Heat Capacity γ:
We know that this adiabatic entity, putting what we learnt

$$\frac{C_p}{C_v} = \gamma$$
$$\frac{\frac{\langle i \rangle}{2} R + R}{\frac{\langle i \rangle}{2} R} = \gamma$$
$$1 + \frac{2}{\langle i \rangle} = \gamma$$
$$\frac{\langle i \rangle}{2} = \frac{1}{\gamma - 1}$$

So, internal energy can be written in this manner,

$$E = \frac{\langle i \rangle}{2} nRT = \frac{\langle i \rangle}{2} pV = \frac{pV}{\gamma - 1}$$

Problem 107 (`Moving gas brakes`). A thermally insulated vessel containing a gas whose molar mass is M and the ratio of specific heats $C_p/C_v = \gamma$ moves with a speed v. What will be the increase in the Temperature of the gas if it suddenly stops?

Solution. Using the conservation of energy, after breaking, all the kinetic energy is gone. But where do they go? For sake of this problem, we let them go to increase the Temperature of the vessel gas, or increase the Thermal Energy (say Internal Energy).

$$\Delta T = \frac{1}{2} mv^2 = \frac{1}{2} nMv^2 = nC_v \Delta T$$

But we can show that,

$$C_v = \frac{R}{\gamma - 1}$$

We write,

$$\frac{nR\Delta T}{\gamma - 1} = \frac{1}{2} nMv^2$$

This solves,

$$\boxed{\Delta T = \frac{Mv^2 (\gamma - 1)}{2R}}$$

6.5 2nd Law of Thermodynamics

Problem 108. In which case will the efficiency of a Carnot Cycle be higher? For increasing the hot thermal reservoir temperature by ΔT or reducing the cooler one by ΔT ?

Solution. As we know that the efficiency is written as,

$$\eta = 1 - \frac{T_c}{T_h}$$

That's why, hot plus delta and cold minus delta,

$$1 - \frac{T_c - \Delta T}{T_h} \quad and \quad 1 - \frac{T_c}{T_h + \Delta T}$$

That happens to be,

$$\frac{T_c - \Delta T - T_h}{T_h} \quad and \quad \frac{T_c - \Delta T - T_h}{T_h + \Delta T}$$

So, the denominator being equal, heating up the hot resorvoir will give lower efficiency.

Problem 109. Find the Entropy of a 1 mole ideal gas.

Solution. We define entropy as,

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T}$$

If we denote that the entropy for an ideal gas is function of,

$$S = S\left(T, V\right),$$

We can say that,

$$\mathrm{d}S = \left(\frac{\partial S}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial S}{\partial V}\right)_T \mathrm{d}V$$

Let us try to integrate the equation ofr S, the first partial derivative can be solved in this way,

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{C_V \mathrm{d}T}{T}$$

Taking the $\mathrm{d}T$ on the other side,

$$\frac{\partial S}{\partial T} = \frac{C_V}{T}$$

Hence, we have,

$$\mathrm{d}S = \frac{C_V}{T}\mathrm{d}T + \left(\frac{\partial S}{\partial V}\right)_T \mathrm{d}V$$

We can also think that by constant volume it is Isochoric process $\Delta W = 0$. So, $\mathrm{d}Q = C_V \mathrm{d}T$. For the next case, we can directly see,

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{p\mathrm{d}V}{T}$$

That solved,

$$\frac{\partial S}{\partial V} = \frac{p}{T}$$

In this case we have constant temperature (isothermal), hence whatever $\mathrm{d}Q$ we apply goes to work and increase volume $p\ \mathrm{d}V$. We couldd have also directly used the Maxwell's Relation for the above case,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

From Ideal Gas Equation,

$$\frac{p}{T} = \frac{R}{V}$$

Hence,

$$dS = \frac{C_V}{T}dT + \frac{R}{V}dV$$

By integrating,

$$\Delta S = C_V \int \frac{1}{T}\,dT + R\int \frac{1}{V}\,dV$$

This ultimately solves to,

$$\boxed{\Delta S = C_V \ln \frac{T}{T_0} + R \ln \frac{V}{V_0} + \text{Constant}}$$

We can also use some basic logarithm rule to bring the T_0 and V_0 terms under the constant, so,

$$\Delta S = C_V \ln T + R \ln V + C$$

Problem 110. Find another equation consisting p and C_p from the above one, that tells the Entropy of a mole of ideal gas.

Solution. We start with two equations,

$$C_V + R = C_p$$

And,

$$pV = RT$$

That is for $n = 1$. Now,

$$\begin{aligned}
\Delta S &= C_V \ln T + R \ln V + C \\
&= C_p \ln T - R \ln T + R \ln \frac{RT}{p} + C \\
&= C_p \ln T - R \ln T + R \ln RT - R \ln P + C \\
&= C_p \ln T + R \ln \frac{RT}{T} - R \ln P + C \\
&= C_p \ln T - R \ln P + C
\end{aligned}$$

So, the final equation is,

$$\boxed{\Delta S = C_p \ln T - R \ln p + \text{Constant}}$$

I refer to the next problem as one of the most important problem kept in this book.

Problem 111 (`Thermal Blocks running Engine`)**.** There are two bodies with temperature independent heat capacity C_h and C_l. These are used as the heat resorvoirs of a Carnot Engine. Derive the expression of the total work that can be attained from the engine.

The temperatures are T_h and T_l respectively.

Solution. Let us think actually what is going to happen, the hot resorvoir T_h will give off some heat. That heat will be transferred to be used as output work by the Engine, the residue $\Delta Q - \Delta W$ will be sent to T_l.
This is a continuous process, and naturally we would like to be infinitesimal in this case. Let some small $\mathrm{d}Q_h$ heat pulled from hot body and push $\mathrm{d}Q_l$ heat in the cooled body.
We shall be consistent whether heat gets out or gets in,

$$\mathrm{d}Q_h = -C_h \, \mathrm{d}T_h$$
$$\mathrm{d}Q_l = C_l \, \mathrm{d}T_l$$

Now, how would the working process of a Carnot Engine look "Thermodynamically"? We can actually use a very nice property of a Carnot engine when it works between two temperatures. It is,

$$\frac{\mathrm{d}Q_h}{T_h} = \frac{\mathrm{d}Q_l}{T_l}$$

Now, we can think in this manner, the work done by the engine is going to be the total heat out from the hot body subtracted by the heat gained by cooler body, the rest of the energy must go to Work. And from the above set of equations, this is clear just by an integration we can find the heat energy transferred.

$$\int_{T_l}^{T_0} \frac{\mathrm{d}Q_l}{T_l} = \int_{T_h}^{T_0} \frac{\mathrm{d}Q_h}{T_h}$$

The engine should stop working when the difference of temperature between the bodies is zero, hence reaching an equilibrium, say T_0. We solve the equation putting what we have from the initial idea,

$$\int_{T_l}^{T_0} \frac{C_l}{T_l} \, \mathrm{d}T = -\int_{T_h}^{T_0} \frac{C_h}{T_h} \, \mathrm{d}T$$

What we have is,

$$C_l \ln \frac{T_0}{T_l} = -C_h \ln \frac{T_0}{T_h}$$
$$C_l \left(\ln T_0 - \ln T_l \right) = -C_h \left(\ln T_0 - \ln T_h \right)$$
$$\ln T_0 \left(C_h + C_l \right) = C_h \ln T_h + C_l \ln T_l$$

By the logarithm rules,

$$\ln T_0^{C_h + C_l} = \ln T_h^{C_h} T_l^{C_l}$$

Which is,

$$T_0^{C_h + C_l} = T_h^{C_h} T_l^{C_l}$$

We have got a final equilibrium temperature, and this is clear that,

$$\Delta Q_h = C_h \left(T_h - T_0\right)$$

$$\Delta Q_l = C_l \left(T_0 - T_l\right)$$

So, wrapping up, total work done,

$$\Delta W = \Delta Q_h - \Delta Q_l = C_h T_h + C_l T_l - \left(C_h + C_l\right) \sqrt[C_h + C_l]{T_h^{C_h} T_l^{C_l}}$$

So,

$$\boxed{\Delta W = C_h T_h + C_l T_l - \left(C_h + C_l\right) \sqrt[C_h + C_l]{T_h^{C_h} T_l^{C_l}}}$$

6.6 Kinetic Theory and looking at each Molecules

Problem 112. Calculate what fraction of Molecules,
a) traverse without collision exceeding the mean free path.
b) mean free path within λ and 2λ.

Solution. We know that,

$$N/N_0 = e^{-x/\lambda}$$

So, the ratio of particle with mean free path infinite, $N/N_0 = 0$. Mean free path till λ at max $1/e$. So, all that fall within $\lambda < x$

$$\Delta N/N_0 = 1/e$$

Now those who are at max 2λ and least λ are,

$$\Delta N/N_0 = \frac{1}{e^{\lambda/\lambda}} - \frac{1}{e^{2\lambda/\lambda}} = \frac{1}{e} - \frac{1}{e^2}$$

The percentage of particles that are free in the boundary of 0 to x without colliding is $e^{-x/\lambda}$ where $\lambda = \frac{1}{\sqrt{2}An}$. We can add them and subtract them logically.
One method that makes sense is just taking the derivative and again taking an integral, the net effect is zero but it helps.

$$\frac{dN}{dx} = -N_0/\lambda e^{-x/\lambda}$$
$$dN = -N_o/\lambda \int_{x=a}^{x=b} e^{-x/\lambda}\, dx$$
$$\Delta N/N_0 = \left[\frac{1}{e^{-x/\lambda}}\right]_a^b$$
$$\Delta N/N_0 = \left[\frac{1}{e^{-b/\lambda}} - \frac{1}{e^{-a/\lambda}}\right]$$

Problem 113. A vessel contains a monoatomic gas at T. Use the Maxwell Boltzmann Speed Distribution to calculate the mean kinetic energy of the molecules. Now do the same for those molecules that are effused into a evacuated box.

Solution. It is quite important to remember that,
The speed distribution for the effused particle is proportional to

$$v^3 e^{-mv^2/2kT}$$

Now,

$$\int_0^\infty f(v)\, \mathrm{d}v = 1$$

That also gives a way to normalize in this manner,

$$\int_0^{\infty} \alpha v^3 e^{-mv^2/2kT}\, dv = 1 \quad \alpha = \frac{1}{\int_0^{\infty} v^3 e^{-mv^2/2kT}\, \mathrm{d}v}$$

The kinetic energy for effused particle shall be,

$$\langle v^2 \rangle = \int_0^{\infty} v^2 f(v)\, dv$$

$$|KinEn| = 1/2m \times \langle v^2 \rangle = 1/2m \frac{\int_0^{\infty} v^5 e^{-mv^2/2kT}\, \mathrm{d}v}{\int_0^{\infty} v^3 e^{-mv^2/2kT}\, \mathrm{d}v}$$

Using integral table,

$$\langle KinEn \rangle = 2kT$$

Now, for the monoatomic, in free space we already know,

$$\langle KinEn \rangle = \frac{3}{2} kT$$

So, energy being conserved should make the both equal, hence,

$$2kT' = \frac{3}{2} kT$$

That gives, $T' = 1.33T$.

Problem 114. A closed vessel is partially filled with liquid mercury. There is a hole above the liquid with area A and it is placed at a region of High Vacuum at T and after 30 days it was found that it was lighter by δm. Estimate the vapour pressure of mercury at T. We know the relative molecular mass of mercury.

Solution. Understand what is happening here. We know that the rate of hitting by molecule per unit area of a gas is Flux Φ. Thus, total mass lost by a whole per unit time is $m\Phi A = m\frac{dN}{dt} = \frac{dM}{dT}$. Because mass of N molecule is Nm.
We know that,

$$\Phi = \frac{p}{\sqrt{2\pi mkT}}$$

Using the idea we can write that the pressure inside the container (that is the molecules average pressure made on the surface of the container by continuous collision, is also vapour pressure),

$$p = \sqrt{\frac{2\pi KT}{m}} \frac{1}{A} \left| \frac{dM}{dt} \right|$$

This can give the Vapour Pressure, that is just simply the existing pressure inside the container by evaporated material.

Problem 115. Show that the time dependence of the pressure inside an oven with a small hole containing hot gas,

$$p(t) = p(0)e^{-t/\tau}$$

If,

$$\tau = \frac{V}{A}\sqrt{\frac{2\pi m}{kT}}$$

Solution. This will be a worth while writing. I started thinking that as there is V in the equation, we can put the raw IGE in the solution. It has magic in it, that it never lets me down.

$$pV = NkT$$

Took a time derivative,

$$\frac{dp}{dt} = \frac{K}{V}\left(T\frac{dn}{dt} + n\frac{dT}{dt}\right)$$

I had to think for quite a whole lot of time what to do. But my brain did spark the fact that we can for now, as it is an oven, has constant temperature, so that some work can be enabled that makes us near to the solution, thus, if $dT/dt = 0$, we have considerable simplification,

$$\frac{dp}{dt} = -\frac{kT}{V}\frac{dN}{dt}$$

Hopefully that dN/dt is just the ΦA, the number of molecules getting out per unit time. This has also enabled me to put in the $\frac{pA}{\sqrt{2\pi mkT}}$, so what I was left with,

$$\frac{dp}{p} = \frac{A}{V}\sqrt{\frac{kT}{2\pi m}}dt$$

Integrated that,

$$\ln p/p_0 = \frac{A}{V}\sqrt{\frac{kT}{2\pi m}}t$$

That if well regarded finds us out that,

$$p = p_0 e^{-t/\tau}$$