Deriving the Gravitational Potential from Scratch

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0.1 Finding the Equation of Potential Energy

Force is given by,

$$\vec{F} = m\vec{a}$$

Now, force can also be shown as the rate of change of momentum,

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

Let's assume that m mass is constant (in most cases it does), so,

$$\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

We can write the acceleration $\frac{\mathrm{d}v}{\mathrm{d}t}$ so that,

$$\frac{\mathrm{d}v}{\mathrm{d}t} \times 1 = \frac{\mathrm{d}v}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

This is just chain rule in calculus, but you don't need to worry about it, what we did above is kind of correct. Now why did we do it? I will tell.

You know work is defined by the formula,

$$W = \vec{F} \cdot \vec{x}$$

This equation is true if F is a constant, but if F is not a constant, then, we need to integrate,

$$W = \int \vec{F} \cdot \mathrm{d}\vec{x}$$

The force equation can be bought to this format, so, now we can say,

$$F = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Solve this,

$$F \ \mathrm{d}x = mv \ \mathrm{d}v$$

Integrate this,

$$\int \vec{F} \cdot \mathrm{d}\vec{x} = \int mv \ \mathrm{d}v$$

$$\int_{x_0}^{x} \vec{F} \ \mathrm{d}\vec{x} = \frac{1}{2}mv^2 + C$$

Here we are trying to calculate the energy needed to force an object from x_0 to x distance. We have a constant of integration C, now, it turns out that,

$$C = \frac{1}{2}mv^2 - \int_{x_0}^{x} \vec{F}\ \mathrm{d}\vec{x}$$

We know that $\frac{mv^2}{2}$ is kinetic energy, but what about this $-\int F\mathrm{d}x$? We can define a new function,

$$V = \int \vec{F}\cdot \mathrm{d}\vec{x}$$

This tells us,

$$C = \frac{1}{2}mv^2 + V$$

This above experssion is always a constant, which is just the **Conservation of Energy**. So, F is a function of x, this can be also said that, $F = F(x)$, where F depends on x. Because $F(x)$ is being integrated, V is also a function of x, so,

$$V(x) = -\int \vec{F}\cdot \mathrm{d}\vec{x}$$

This gives us the complete equation of the *Conservation of Energy*,

$$\boxed{\frac{1}{2}mv^2 + V(x) = E}$$

Where E is constant, the total energy.

0.2 Solve Potential Energy for Gravity

Gravitational potential energy is,

$$F = G\frac{m_1 m_2}{x^2}$$

Where x is the distance between the two bodies.

Let's think about the case of Earth, and a mass that is x distance from it's center. Draw a nice picture, Now, $\mathrm{d}\vec{x}$ is a small increment in the distance, so obviously it will point away from the earth. But the $\vec{F}$ must point towards earth because gravity is always attractive. Hence, the dot product $\vec{F}\cdot \mathrm{d}\vec{x}$ is $-F\ \mathrm{d}x$, solving the rest of them,

$$\begin{aligned} V(x) &= -\int -F\ \mathrm{d}x \\ &= \int G\frac{Mm}{x^2}\mathrm{d}x \end{aligned}$$

This integration yields,

$$V(x) = -\frac{GMm}{x} + C$$

At infinity, where $x \to \infty$, then, $V(\infty) = 0$, thus, $C = 0$, we don't need to worry about it, we get,

$$\boxed{V(r) = -\frac{GMm}{r}}$$

0.3 About the Minus sign

Let this mass m be at a distance say $10\ m$, let's push that against the gravity and put it at $20\ m$, then from common sense, as you are working against the force of gravity, your work done would be greater than 0, it will be positive. Let's check this,

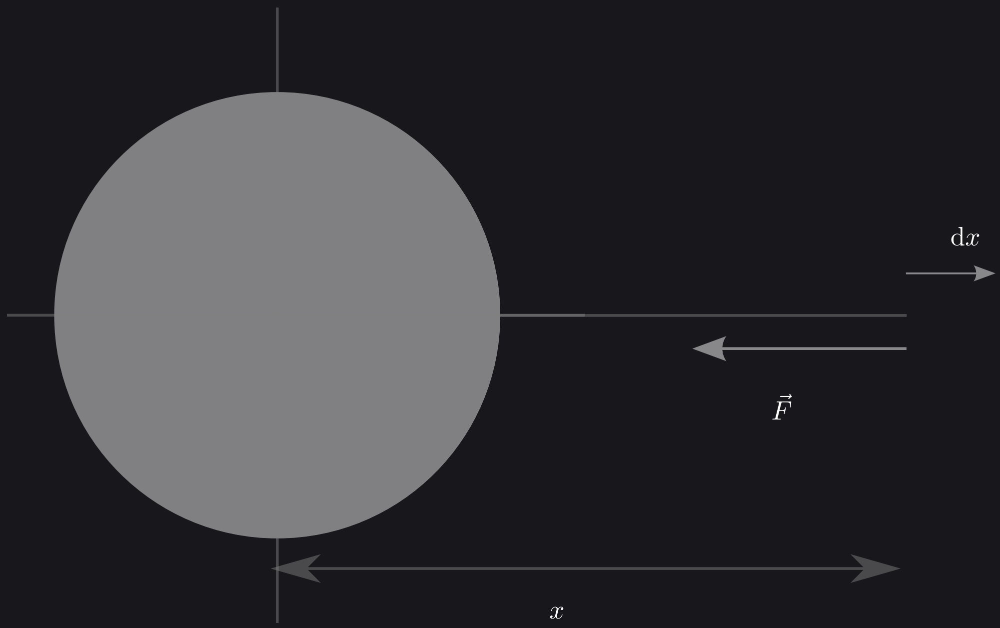

Figure 0.2.1: Diagram of the System

the change in the potential energy means that work has been done, you have to move the mass slowly so that it doesn't gain much kinetic energy, we only want to work with potential energy. So, let' move the mass from 10 to 20, and check the change in potential energy,

$$\Delta V = (V_{\text{final}} - V_{\text{initial}}) = V(20) - V(10) = -GMm\left(\frac{1}{20} - \frac{1}{10}\right) = -GMm\left(-\frac{1}{20}\right)$$
$$= \frac{GMm}{20}$$

Check that this experssion is positive, so the thing is logical and makes sense.

Do this quick problem and check the theory yourself. This will clarify your thoughts.

Problem 1. There is a mass m located at a distance r_1 from the center of a planet of mass M. If this mass comes close to the planet, and comes at a distance r_2 from it (remind $r_2 < r_1$), then what is the change in potential energy? Solve it algebraically.