Derivation of eccentricity vector

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The specific angular momentum $\vec{h} = \vec{r} \times \dot{\vec{r}}$ We denote $\vec{r}$ as

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

As the vector $\vec{h}$ always lies in the plane perpendicular to $\vec{r}$, $\vec{h} \times \dot{\vec{r}}$ can be found to be

$$\vec{h} \times \dot{\vec{r}} = -|h| \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix}$$

We want to add another vector with it, so that the sum produces a constant vector. That is, we want to find a vector $\vec{A}$ so that

$$\frac{d}{dt}\left(\vec{A} - |h| \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} \right) = 0$$

$$\Rightarrow \frac{d\vec{A}}{dt} = |h| \begin{pmatrix} \ddot{y} \\ \ddot{x} \end{pmatrix}$$

$$\Rightarrow \frac{d\vec{A}}{dt} = \frac{|h|(\ddot{x}^2 + \ddot{y}^2)}{(\ddot{x}^2 + \ddot{y}^2)} \begin{pmatrix} \ddot{y} \\ \ddot{x} \end{pmatrix}$$

The term

$$\frac{1}{(\ddot{x}^2 + \ddot{y}^2)} \begin{pmatrix} \ddot{y} \\ \ddot{x} \end{pmatrix}$$

is a unit vector pointing perpendicular to $\ddot{\vec{r}}$ and $\vec{h}$. Under influence of a central force, this is a unit vector pointing towards $\hat{\theta}$. We can also write the $(\ddot{x}^2 + \ddot{y}^2)$ in the numerator as $\frac{GM}{r^2}$ where $r^2 = \vec{r} \cdot \vec{r}$. Thus,

$$\Rightarrow \frac{d\vec{A}}{dt} = \frac{|h|GM}{r^2}\hat{\theta}$$

$\vec{h}$ can also be written as $\omega_\theta r^2$ where ω_θ is the angular velocity of the vector $\vec{r}$. Substituting this,

$$\Rightarrow \frac{d\vec{A}}{dt} = GM\omega_\theta\hat{\theta}$$

As $\hat{\theta}$ is the direction of change of $\vec{r}$ and ω_θ is its rotation rate, $\omega_\theta\hat{\theta}$ is the change rate of $\hat{r}$.Thus, $\vec{A} = GM\hat{r}$.

Thus, the vector

$$\vec{h} \times \dot{\vec{r}} + GM\hat{r}$$

is a constant. The eccentricity vector,

$$\vec{e} = \frac{\vec{h} \times \dot{\vec{r}}}{GM} + \hat{r}$$