Honors Multivariable Calculus : : Homework 04

January 31, 2024

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1 Problem

$f^{-1}(U)$ being open means that there exists a point $\vec{a}$ in $f^{-1}(U)$ such that it is a limit point. Being a limit point, the set of points $\vec{x}$ around $\vec{a}$ within $r > 0$ distance all are member of $f^{-1}(U)$.

This set be called $B_r(\vec{a})$ such that

$$B_r(\vec{a}) = \{\vec{x} \in f^{-1}(U) : |\vec{x} - \vec{a}| < r > 0\}$$

So $B_r(\vec{a}) \subset f^{-1}(U)$ signifies that $\vec{a}$ is a limit point.

Having the input set and output set both being open it is given that $f(B_r(\vec{a}))$ will also be an open set.

$$f(B_r(\vec{a})) \subset B_\epsilon(f(\vec{a})) \subset U$$

But if there is a set $B_\epsilon(f(\vec{a}))$ around $f(\vec{a})$ then that just simply means that there exists continuous points around $f(\vec{a})$. That means $f(\vec{x}) - f(\vec{a})$ norms are within ϵ. This is a condition of continuity hence proves this function f is continuous.

2 Problem

$$h'(t) = \lim_{h\to 0} \frac{f(t+h)\cdot g(t+h) - f(t)\cdot g(t)}{h}$$

We can do this,

$$h'(t) = \lim_{h\to 0} \frac{f(t+h)\cdot g(t+h) + [f(t+h)\cdot g(t) - f(t+h)\cdot g(t)] - f(t)\cdot g(t)}{h}$$

$$h'(t) = \lim_{h\to 0} \frac{[f(t+h)\cdot g(t+h) - f(t+h)\cdot g(t)] + [f(t+h)\cdot g(t) - f(t)\cdot g(t)]}{h}$$

$$h'(t) = \lim_{h\to 0} \frac{[f(t+h)\cdot g(t+h) - f(t+h)\cdot g(t)] + [f(t+h)\cdot g(t) - f(t)\cdot g(t)]}{h}$$

$$h'(t) = \lim_{h\to 0} \frac{f(t+h)\cdot [g(t+h) - g(t)] + [f(t+h) - f(t)]g(t)}{h}$$

Taking the limits,

$$h'(t) = f(t)\cdot g'(t) + f'(t)\cdot g(t)$$

Addition is commutative had been considered prior.

3 Problem

Consider the distance vector $\vec{d}(t) = \vec{f}(t) - \vec{a}$, and it's norm is minimized when the distance is minimal. For ease of computation, if $|\vec{d}(t)|$ is minimal, then so as $|\vec{d}(t)|^2$. Using this,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\vec{d}(t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{d}(t)\cdot\vec{d}(t)\right) = 2\frac{\mathrm{d}}{\mathrm{d}t}\vec{d}(t)\cdot\vec{d}(t) = 0$$

We know that the tangent of $\vec{f}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{d}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{f}(t) + 0$$

Because we have seen $\frac{\mathrm{d}\vec{f}(t)}{\mathrm{d}t}\cdot\vec{d}(t) = 0$, so we have orthogonality of the two vectors.

4 Problem

Defining $h'(t) = f(t) \times g(t)$

$$h'(t) = \lim_{h\to 0}\frac{f(t+h)\times g(t+h) - f(t)\times g(t)}{h}$$

We can do this,

$$h'(t) = \lim_{h\to 0}\frac{f(t+h)\times g(t+h) + [f(t+h)\times g(t) - f(t+h)\times g(t)] - f(t)\times g(t)}{h}$$

$$h'(t) = \lim_{h\to 0}\frac{[f(t+h)\times g(t+h) - f(t+h)\times g(t)] + [f(t+h)\times g(t) - f(t)\times g(t)]}{h}$$

$$h'(t) = \lim_{h\to 0}\frac{[f(t+h)\times g(t+h) - f(t+h)\times g(t)] + [f(t+h)\times g(t) - f(t)\times g(t)]}{h}$$

$$h'(t) = \lim_{h\to 0}\frac{f(t+h)\times[g(t+h) - g(t)] + [f(t+h) - f(t)]g(t)}{h}$$

Taking the limits,

$$h'(t) = f(t)\times g'(t) + f'(t)\times g(t)$$

Addition is commutative had been considered prior.

5 Problem

(a)

The angle between $\vec{p}(t)$ and $\vec{v}(t)$ is θ. Then the perpendicular component of $\vec{v}$ over $\vec{p}$ would be $|\vec{v}|\sin\theta$. The cross product norm is basically the area of this triangle the two vectors make. In δt time, the area swept is hence an infinitesimal triangle

$$\text{area} = \delta A = \frac{1}{2}|\vec{p}(t)||\vec{v}(t)|\sin\theta\delta t$$

From here we can do the sacred δt division on both sides to get,

$$\frac{\delta A}{\delta t} = \text{Area Rate} = \frac{1}{2}|\vec{p}(t)\times\vec{v}(t)|$$

(b)

Equal areas swept in equal times basically means that the area rate is a constant, and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right) = 0$$

(c)

As a Physics major I will give a shoutout to Angular Momentum being conserved. Also, I want to mention the scalar multiple λ which shows $\vec{a}(t) = \lambda\vec{r}$ is not a constant and itself is a function of time.

$$\lambda = \frac{GMm}{|\vec{r}(t)|^2}$$

I was about to make a grave mistake by using λ constant.

Now, this basically means,

$$\vec{r}(t) \times \vec{a}(t) = 0$$

We can do some circus with the vectors and derivatives,

$$\vec{r}(t) \times \vec{a}(t) = \vec{r}(t) \times \frac{\mathrm{d}\vec{v}(t)}{\mathrm{d}t} = 0$$

Notice that the cross product above can be found through the following process,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{r}(t) \times \vec{v}(t)\right) = \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \times \vec{v}(t) + \vec{r}(t) \times \frac{\mathrm{d}\vec{v}(t)}{\mathrm{d}t} = \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t) = 0$$

This just says $\partial_t(\vec{r}(t) \times \vec{v}(t)) = 0$ Which basically validates the area sweep remains constant, hence proving Kepler's Second Law. I cannot curb my urge to mention $\vec{r} \times \vec{v}$ is angular momentum.

6 Problem

$f : \mathbb{R} \to \mathbb{R}^m$ hence,

$$f = (f_1, \ldots, f_m)$$

Mean value theorem individually for the i-th function

$$f_i'(c_i) = \frac{f_i(b) - f_i(a)}{b - a}$$

A single point c exists if for all i we have the same c_i. Let's try a parametrized form where we start from a and end at b such that for $t \in [0, 1]$ we have a line

$$L(t) = tb + (1 - t)a$$

Thus defining

$$\lambda(t) = f(L(t))$$

We have

$$\frac{\lambda(1) - \lambda(0)}{1 - 0} = \lambda'(\tau)$$

The chain rule implies that from $\lambda'(t)$ we have $\nabla f(L(\tau)) \cdot (b - a)$. If we define c to be $L(\tau)$ then $c \in L(\tau)$, and since $\lambda(0) = a$ and $\lambda(1) = b$ then we are confirmed that a single point c exists such that

$$f(b) - f(a) = \nabla f(c) \cdot (b - a)$$

So mean value theorem extends for other dimensions too.