Quantum Mechanics Homework 05

Ahmed Saad Sabit

Problem 01

(a)

$$
\begin{aligned}
I(\alpha)I(\alpha) &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha \frac{x^2}{2}} \int_{-\infty}^{\infty} \mathrm{d}y\, e^{-\alpha \frac{y^2}{2}} \\
I^2(\alpha) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, e^{-\alpha \frac{x^2+y^2}{2}} \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} \mathrm{d}r\, \mathrm{d}\theta\, \frac{\partial(r,\theta)}{\partial(x,y)} e^{-\alpha \frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}} \\
&= 2\pi \int_{0}^{\infty} \mathrm{d}r\, r e^{-\alpha \frac{r^2}{2}} \\
&= 2\pi \int_{0}^{\infty} \mathrm{d}\left(\frac{r^2}{2}\right) e^{-\alpha \frac{r^2}{2}} \\
&= 2\pi \int_{0}^{\infty} \mathrm{d}u\, e^{-\alpha u} \\
&= 2\pi \frac{1}{-\alpha}[-1 + e^{-\alpha(\infty)}] \\
&= \frac{2\pi}{\alpha} \\
\implies I(\alpha) &= \sqrt{\frac{2\pi}{\alpha}}
\end{aligned}
$$

(b)

First find the integral probability distribution interpretation for the $\hat{X}^2$ for a state ψ

$$
\begin{aligned}
\langle \hat{X}^2 \rangle &\implies \langle \psi | \hat{X}\hat{X} | \psi \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \langle \psi | \hat{X}\hat{X} | x \rangle \langle x | \psi \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \langle \psi | \hat{X} | x \rangle x \langle x | \psi \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \langle \psi | x \rangle x^2 \langle x | \psi \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 \psi^*(x)\psi(x) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 |\psi(x)|^2 \\
&= \frac{1}{(\pi\Delta^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-(1/\Delta^2)x^2}
\end{aligned}
$$

Now we require to solve the integral

$$
\begin{aligned}
I(\alpha) &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha \frac{x^2}{2}} \\
\frac{\mathrm{d}}{\mathrm{d}\alpha} I(\alpha) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}\alpha} e^{-\alpha \frac{x^2}{2}} \\
\frac{\mathrm{d}}{\mathrm{d}\alpha} \sqrt{\frac{2\pi}{\alpha}} &= -\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-\alpha \frac{x^2}{2}} \\
\sqrt{2\pi}\left(-\frac{1}{2}\right)\frac{1}{\sqrt{\alpha^3}} &= -\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-\alpha \frac{x^2}{2}} \\
\sqrt{\frac{2\pi}{\alpha^3}} &= \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-\alpha \frac{x^2}{2}}
\end{aligned}
$$

Drawing the coefficients together

$$
\frac{\alpha}{2} = \frac{1}{\Delta^2} \implies \alpha = \frac{2}{\Delta^2} \implies \alpha^3 = \frac{8}{\Delta^6}
$$

Hence the integral with $\alpha \to \Delta$

$$
\int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-\alpha \frac{x^2}{2}} = \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-(1/\Delta^2)x^2} = \sqrt{2\pi\alpha^{-3}} = \sqrt{2\pi\Delta^6/8} = \sqrt{\pi(\Delta^3)^2/2^2} = \frac{\Delta^3}{2}\sqrt{\pi}
$$

Put the remaining pieces together

$$
\begin{aligned}
\langle \hat{X}^2 \rangle &= \frac{1}{\sqrt{\pi}\Delta} \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-(1/\Delta^2)x^2} \\
&= \frac{1}{\sqrt{\pi}\Delta}\left(\frac{\Delta^3}{2}\sqrt{\pi}\right) \\
&= \frac{\Delta^2}{2}
\end{aligned}
$$

Problem 03

(a)

Consider the wave equation for $|x| \leq \frac{L}{2}$.

$$
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = \left(B\left[\left(\frac{1+s}{2}\right)\left(-k_2^2\cos(k_2 x)\right) + \left(\frac{1-s}{2}\right)\left(-k_2^2\sin(k_2 x)\right)\right]\right) = -k_2^2\psi(x)
$$

Putting it to Schrodinger's equation equation gives

$$
-k_2^2\psi(x) = -\frac{2m}{\hbar^2}[E - V_0]\psi(x) \implies k_2^2 = \frac{2m}{\hbar^2}[E - V_0]
$$

Similarly consider the wave equation for $x > L/2$

$$
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = (-k_1)^2 A e^{-k_1 x} = k_1^2\psi(x)
$$

Putting it to Schrodinger's equation (note here $V(x) = 0$)

$$k_1^2\psi(x) = -\frac{2m}{\hbar^2}[E]\psi(x)$$

For this

$$k_1^2 + k_2^2 = \frac{2m}{\hbar^2}[-E + E - V_0] = \frac{2mV_0}{\hbar^2} = q^2$$

(b)

At boundary of a side of the well, we require

$$\psi_{\text{in}}(x) = \psi_{\text{out}}(x) \quad \text{and} \quad \frac{\mathrm{d}\psi_{\text{in}}(x)}{\mathrm{d}x} = \frac{\mathrm{d}\psi_{\text{out}}(x)}{\mathrm{d}x}$$

Even parity case $s = 1$

Note that for the following computations $x = L/2$

$$\psi_{\text{in}}(x) = B\cos(k_2 x)$$

The wave solution for $x > L/2$

$$\psi_{\text{out}}(x) = Ae^{-k_1 x}$$

First condition

$$B\cos k_2 x = Ae^{-k_1 x}$$

Second condition

$$-k_2 B\sin k_2 x = -k_1 Ae^{-k_1 x}$$

This is true for boundary $x = L/2$ The two equations relating A, B as required in the problem

$$\boxed{Ae^{-k_1 L/2} = B\cos k_2 L/2}$$

$$\boxed{Ak_1 e^{-k_1 L/2} = Bk_2 \sin k_2 L/2}$$

Dividing these two equations

$$k_1 = k_2 \tan(k_2 L/2)$$

From previous computation of Schrodinger's equation inside the well we have

$$k_2^2 = \frac{2m}{\hbar^2}[E - V_0] = q^2\frac{E - V_0}{V_0} \implies k_2 = \sqrt{\frac{2m}{\hbar^2}[E - V_0]}$$

$$q^2 = \frac{2m}{\hbar^2}V_0 \implies \frac{q^2}{V_0} = \frac{2m}{\hbar}$$

$$k_1^2 + k_2^2 = q^2 \implies k_1 = \sqrt{q^2 - k_2^2} \implies \sqrt{q^2 - \frac{2m}{\hbar^2}[E - V_0]} \implies k_1^2 = \frac{2m}{\hbar^2}E = q^2\frac{E}{V_0}$$

The equation for Energy that we can get from this is

$$\frac{k_1}{k_2} = \tan\left(\frac{k_2 L}{2}\right)$$
$$\frac{E}{E - V_0} = \tan^2\left(\frac{k_2 L}{2}\right)$$
$$\frac{E}{E - V_0} = \tan^2\left(\frac{qL}{2}\sqrt{\frac{E - V_0}{V_0}}\right)$$

$$\boxed{\frac{E}{E - V_0} = \tan^2\left(\frac{qL}{2}\sqrt{\frac{E - V_0}{V_0}}\right)}$$

Parity of $s = -1$ case

We get

$$\psi_{\text{in}}(x) = B\sin(k_2 x) \quad \text{and} \quad \frac{\mathrm{d}\psi_{\text{in}}(x)}{\mathrm{d}x} = k_2 B\cos(k_2 x)$$
$$\psi_{\text{out}}(x) = -Ae^{k_1 x} \quad \text{and} \quad \frac{\mathrm{d}\psi_{\text{out}}}{\mathrm{d}x} = -k_1 A e^{k_1 x}$$

We have the equation relating A, B at $x = -L/2$

$$-Ae^{k_1 x} = B\sin(k_2 x)$$
$$-k_1 A e^{k_1 x} = k_2 B\cos(k_2 x) \implies \frac{1}{k_1} = \frac{\tan(k_2 x)}{k_2} \implies k_1 = k_2\cot(k_2 x)$$
$$k_1 = k_2(-\cot(k_2 L/2))$$

We can avoid the whole computation by simply replacing the tan of previous equation with $-\cot$

$$\boxed{\frac{E}{E - V_0} = \cot^2\left(\frac{qL}{2}\sqrt{\frac{E - V_0}{V_0}}\right)}$$

(c)

$$\tan\left(q\frac{L}{2}\sqrt{\frac{E - V_0}{V_0}}\right) = \tan\left(\frac{k_2 L}{2}\right)$$

Problem 04

(a)

$$\begin{aligned}
\psi(p) = \langle p|\psi\rangle &= \langle p|\hat{I}|\psi\rangle \\
&= \langle p| \int_{-\infty}^{\infty} \mathrm{d}x |x\rangle\langle x|\psi\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \langle p|x\rangle\langle x|\psi\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ipx/\hbar}\psi(x)
\end{aligned}$$

(b)

Wave Mechanical Fourier Transform

$$\begin{aligned}
&\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0\frac{x}{a}\right]\psi(x) = E\psi(x) \\
&\implies \left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0\frac{x}{a}\right]\psi(x)e^{-ipx/\hbar} = E\psi(x)e^{-ipx/\hbar} \\
&\implies \int_{-\infty}^{\infty} \mathrm{d}x \, \left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0\frac{x}{a}\right]\psi(x)e^{-ipx/\hbar} = \int_{-\infty}^{\infty} \mathrm{d}x \, E\psi(x)e^{-ipx/\hbar} \\
&\implies \int_{-\infty}^{\infty} \mathrm{d}x \, \left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\psi(x)e^{-ipx/\hbar} + \int_{-\infty}^{\infty} \mathrm{d}x \, \left[V_0\frac{x}{a}\right]\psi(x)e^{-ipx/\hbar} = \int_{-\infty}^{\infty} \mathrm{d}x \, E\psi(x)e^{-ipx/\hbar} \\
&\implies \left(-\frac{\hbar^2}{2m}\right)\left[\underbrace{\left.\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}e^{-ipx/\hbar}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\mathrm{d}\psi(x)}{\mathrm{d}x}(-ip/\hbar)e^{-ipx/\hbar}}_{\text{Integration by Parts done TWICE}}\right] + \frac{V_0}{a}\int_{-\infty}^{\infty} \mathrm{d}x \, x\psi(x)e^{-ipx/\hbar} = \int_{-\infty}^{\infty} \mathrm{d}x \, E\psi(x)e^{-ipx/\hbar} \\
&\implies \frac{p^2}{2m}\psi(p) + \frac{V_0}{a}\left(-\frac{\hbar}{i}\right)\frac{\mathrm{d}}{\mathrm{d}p}\int_{-\infty}^{\infty} \mathrm{d}x \, \psi(x)e^{-ipx/\hbar} = E\int_{-\infty}^{\infty} \mathrm{d}x \, \psi(x)e^{-ipx/\hbar} \\
&\implies \left[\frac{p^2}{2m}\right]\psi(p) + \frac{V_0}{a}\left(-\frac{\hbar}{i}\right)\frac{\mathrm{d}}{\mathrm{d}p}\psi(p) = E\psi(p)
\end{aligned}$$

Matrix Mechanical Wave Transform

$$\begin{aligned}
\hat{H}|E\rangle = E|E\rangle &\implies \langle x|\hat{H}|E\rangle = E\langle x|E\rangle \\
&\implies \langle x|\frac{\hat{P}^2}{2m} + \frac{V_0}{a}\hat{X}|E\rangle = E\langle x|E\rangle
\end{aligned}$$

We know know that

$$\frac{1}{2m}\langle x|\hat{p}\hat{p}|\Psi\rangle = \frac{1}{2m}\frac{\hbar}{i}\frac{d}{dx}\langle x|\hat{p}|\Psi\rangle = \frac{1}{2m}\frac{\hbar}{i}\frac{d}{dx}\frac{\hbar d}{dx}\langle x|\Psi\rangle = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|\Psi\rangle$$
$$\frac{V_0}{a}\langle x|\hat{X}|\Psi\rangle = \frac{V_0}{a}x\langle x|\Psi\rangle$$

In momentum basis,

$$\begin{aligned}\hat{H}|E\rangle = E|E\rangle &\implies \langle p|\hat{H}|E\rangle = E\langle p|E\rangle \\ &\implies \langle p|\frac{\hat{P}^2}{2m} + \frac{V_0}{a}\hat{X}|E\rangle = E\langle p|E\rangle\end{aligned}$$

Computing each terms

$$\begin{aligned}&\langle p|\frac{\hat{P}^2}{2m}|E\rangle = \frac{p^2}{2m}\langle p|E\rangle \\ &\frac{V_0}{a}\langle p|\hat{X}|E\rangle = \frac{V_0}{a}\left(-\frac{\hbar}{i}\right)\frac{\mathrm{d}}{\mathrm{d}p}\langle p|E\rangle \qquad \implies \left[\frac{p^2}{2m} - \frac{\hbar}{i}\frac{V_0}{a}\frac{\mathrm{d}}{\mathrm{d}p}\right]\Psi(p) = E\Psi(p)\end{aligned}$$

(c)

The differential equation is

$$\left[-\frac{\hbar V_0}{ia}\right]\frac{\mathrm{d}\psi(p)}{\mathrm{d}p} = \left[E - \frac{p^2}{2m}\right]\psi(p)$$

Not that bad,

$$\begin{aligned}\Lambda\frac{\mathrm{d}\psi(p)}{\psi(p)} &= (E - p^2/2m)\mathrm{d}p \\ \int\frac{\mathrm{d}\psi(p)}{\psi(p)} &= \int\left(\frac{E}{\Lambda} - \frac{p^2}{2m\Lambda}\right)\mathrm{d}p \\ \ln\left(\frac{\psi(p)}{\psi_0(p)}\right) &= \frac{E}{\Lambda}p - \frac{p^3}{6m\Lambda} + C \\ \ln\left(\frac{\psi(p)}{\psi(0)}\right) &= \frac{E}{\Lambda}p - \frac{1}{6m\Lambda}p^3 \\ \psi(p) &= \psi(0)\exp\left(\frac{E}{\Lambda}p - \frac{1}{6m\Lambda}p^3\right) \\ \psi(p) &= \psi(0)\exp\left(\frac{-iaE}{\hbar V_0}p + \frac{ia}{6m\hbar V_0}p^3\right)\end{aligned}$$

$$\boxed{F(p,E) = \exp\left(\frac{-iaE}{\hbar V_0}p + \frac{ia}{6m\hbar V_0}p^3\right)}$$

(d)

$$
\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, \psi(0) \exp\left(\frac{-iaE}{\hbar V_0} p + \frac{ia}{6m\hbar V_0} p^3 \right) e^{ipx/\hbar} \\
&= \frac{\psi_p(0)}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, \exp\left(\frac{-iaE}{\hbar V_0} p + \frac{ix}{\hbar} p + \frac{ia}{6m\hbar V_0} p^3 \right) \\
&= \frac{\psi_p(0)}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, \exp\left(\frac{i}{\hbar} \left[\frac{-aE}{V_0} + x \right] p + \frac{ia}{6m\hbar V_0} p^3 \right) \\
&= \frac{\psi_p(0)}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, \exp\left(\frac{i}{\hbar} \left[x - \frac{aE}{V_0} \right] p + \frac{ia}{6m\hbar V_0} p^3 \right) \\
&\implies \psi(x) = F\left(x - a\frac{E}{V_0} \right)
\end{aligned}
$$

From the computation it is clear that F here is the inverse fourier transform at zero energy.

Problem 05

(a)

$$
\begin{aligned}
G_3 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_3^2 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_3^3 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_3^4 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_3^5 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = G_3
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \\
\sin\theta &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} \cos\theta = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots & -\sin\theta = -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots & 0 \\ \sin\theta = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots & \cos\theta = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \text{ roughly speaking } \sum x^n/n! \begin{bmatrix} 1, 0, -1, 0, 1, \ldots & 0, -1, 0, 1, 0 \ldots & 0 \\ 0, 1, 0, -1, 0, 1 \ldots & 1, 0, -1, 0, 1 \ldots & \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{x^2}{2!} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{x^3}{3!} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots \\
&= \hat{I} + xG_3 + \frac{x^2}{2!}G_3^2 + \frac{x^3}{3!}G_3^3 + \cdots = e^{\theta G_3}
\end{aligned}
$$

(b)

I do the computation by hand on paper.

$$
G_1G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
G_2G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [J_1, J_2] = i^2G_1G_2 - i^2G_2G_1 = G_2G_1 - G_1G_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i^2G_3 = iJ_3
$$

$$
G_2G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

$$
G_3G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad [J_2, J_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = i^2G_1 = iJ_1
$$

$$
G_3G_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
G_1G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad [J_3, J_1] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = i^2G_2 = iJ_2
$$

From the commutator relationship we know that

$$
[J_x, J_y] = -[J_y, J_x] \implies \varepsilon_{x,y,z} = -\varepsilon_{y,x,z}
$$

So hence proving

$$
[J_a, J_b] = i\varepsilon_{abc}J_c
$$