April 22, 2024

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The Set $\mathbb{R}$

I like to think $\mathbb{R}$ as a "bag full of all the numbers I can imagine", which includes negative numbers because why not? Guess 69, it's in $\mathbb{R}$. Guess 23001.1930, it's also in $\mathbb{R}$. Guess $\sin 20°$, it's in $\mathbb{R}$. If any number a is a member of $\mathbb{R}$ we say that

$$a \in \mathbb{R}$$

which means a is an element of $\mathbb{R}$. This could be any number. But to do interesting things with these numbers we have to define "mathematical operations", for instance addition, multiplication and their respective inverses (subtraction and division). We can define a definitions for "distance" between two elements of $\mathbb{R}$. When we have defined the well set of rules and operations on $\mathbb{R}$, we have a **space** on $\mathbb{R}$. I will expand on this on the next subsection.

> Definition 1. The *Set* of all real numbers is defined to be $\mathbb{R}$.

Please note that for this course we will be using Base-10 number system with the common sensical decimals.[1]

The Space of $\mathbb{R}$

I will first talk rigorously about this "space" thing. Spaces are basically like all the "objects" (set) and "protocols or rules" (structure) you need to play chess. The chessboard, pieces and maybe a timer make up the "set" of your "space". The rules of time limit, the way to move each knight, pawn and rook etc. form the protocol, or "structure" of the "space". We need to talk about "spaces" because we need to, roughly, define numbers and addition-multiplications in the first place.

We will be interested on the *set* $\mathbb{R}^n$ of numbers, whose elements are n-tuples (as you should know from linear algebra but I will still define them in next section). The *structure* (or mathematical operations) we will define on this will complete the space and we name it "vector space".

Later we will include additional rules that will help us turn these vectors spaces into a Euclidean Space. These additional rules are basically Dot Products, just so that we can have a geometric representations of angles and stuffs. For now I will define it for one-dimension $\mathbb{R} = \mathbb{R}^1$ case.

> Definition 2. A *One Dimensional Vector Space* is the Set $\mathbf{V}$ such that $\mathbf{V} \subset \mathbb{R}$ defined with two mappings[a]
>
> $$+ : \mathbf{V} \otimes \mathbf{V} \to \mathbf{V} \qquad \text{(Scalar Addition)}$$
> $$\times : \mathbb{R} \otimes \mathbf{V} \to \mathbf{V} \qquad \text{(Scalar Multiplication)}$$
>
> Considering $x, y, z \in \mathbf{V}$ and $a, b \in \mathbb{R}$,[b] the eight properties that elucidate the two mappings we instantiated above are as follows.
>
> 1. $x + (y + z) = (x + y) + z$

[1] $1, 2, 3, \ldots$

2. $x + y = y + x$

3. $x + 0 = x$

4. $x + (-x) = 0$

5. $(a \times b) \times x = a \times (b \times x)$

6. $(a + b) \times x = a \times x + b \times x$

7. $a \times (x + y) = a \times x + a \times y$

8. $1 \times x = x$

Additionally for Scalar Multiplication,

$$a \times b = b \times a$$

TODO: Comments needed on this.

[a] note that the $\otimes$ symbol just means a mathematical operation.

[b] To avoid confusion, I want to mean that a, b might real numbers not necessarily members of $\mathbf{V}$

Usually while doing maths we can avoid $\times$ sign for scalar multiplication, even if it's between a Scalar and a Vector. I am going to define 3 new rules imposed on this Vector Space to turn it into an Euclidean Space. But please note that the concept of *inner products* make more sense in n dimensions while we are working with $\mathbb{R}^n$. For now, just focus on the distance $d(x, y)$.

Definition 3. A *One Dimensional Euclidean Space* is a Vector Space on the set $\mathbf{V} \subset \mathbb{R}$ with three additional rules. Consider $x, y, z \in \mathbf{V}$ and $a, b \in \mathbb{R}$. Firstly, concept of **Inner Product** that satisfies

1. $\langle x, x \rangle > 0$ if $x \neq 0$.

2. $\langle x, y \rangle = \langle y, x \rangle$.

3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

We define inner product on $x, y \in \mathbb{R}$

$$\langle x, y \rangle = xy$$

Secondly, from Inner Products we get the concept of a **Norm** which can be thought of as a "function" (we will define this later) that satisfies

1. $|x| > 0$ if $x \neq 0$

2. $|ax| = |a||x|$

3. $|x + y| \leq |x| + |y|$

We define the norm (associated with inner product) on $\mathbb{R}$.[a]

$$|x| = \sqrt{\langle x, x \rangle}$$

Thirdly, from the idea of Norm we can provide a definition of **Distance** between two elements $x, y \in \mathbf{V}$ that satisfies the conditions

1. $d(x, y) > 0$ unless $x = y$

2. $d(x, y) = d(y, x)$

3. $d(x, z) \leq d(x, y) + d(y, z)$

We define the distance on $\mathbb{R}$ between to elements

$$d(x, y) = |x - y|$$

[a] Which is basically the non-negative value of x in one dimensional case. Basically $= \sqrt{x^2} = +x$

The space of $\mathbb{R}^n$

A *space* is anything that has a set of "objects" $\mathbf{V}$ and a mathematical structure defined on it. We are specifically interested on *Euclidean Spaces.*

Definition 4. A *Vector Space* is a set $\mathbf{V} \subset \mathbb{R}^n$ defined with two mappings[a]

$$\begin{aligned} \mathbf{V} \otimes \mathbf{V} &\to \mathbf{V} \qquad \text{(Vector Addition)} \\ \mathbb{R} \otimes \mathbf{V} &\to \mathbf{V} \qquad \text{(Scalar Multiplication)} \end{aligned}$$

Considering $\vec{x}, \vec{y}, \vec{z} \in \mathbf{V}$ and $a, b \in \mathbb{R}$, the eight properties that elucidate the two mappings we instantiated above are as follows.

1. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. $\vec{x} + \vec{0} = \vec{x}$
4. $\vec{x} + (-\vec{x}) = \vec{0}$
5. $(ab)\vec{x} = \vec{a}(b\vec{x})$
6. $(a + b)\vec{x} = a\vec{x} + b\vec{x}$
7. $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$
8. $1\vec{x} = \vec{x}$

[a] note that the $\otimes$ symbol just means a mathematical operation.

Note A *Euclidean Vector Space* is a finite-dimensional **Inner Product Space** over the real numbers.

I prefer a slightly different wording of the exact same thing for simplicity.

Definition 5. A *Euclidean Space* is a Vector Space on the set $\mathbf{V} \subset \mathbb{R}^n$ with an additional rule of **Inner Product**. Consider $\vec{x}, \vec{y}, \vec{z} \in \mathbf{V}$ and $a, b \in \mathbb{R}$. Firstly, concept of **Inner Product** that satisfies

1. $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x} \neq \vec{0}$.
2. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.
3. $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$

We define inner product on $\vec{x}, \vec{y} \in \mathbb{R}$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Secondly, from Inner Products we get the concept of a **Norm** which can be thought of as a "function" (we will define this later) that satisfies

1. $|\vec{x}| > 0$ if $\vec{x} \neq 0$
2. $|a\vec{x}| = |a||\vec{x}|$
3. $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

We define the norm (associated with inner product) on $\mathbb{R}^n$.

$$|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

Thirdly, from the idea of Norm we can provide a definition of **Distance** between two elements $\vec{x}, \vec{y} \in \mathbf{V}$ that satisfies the conditions

1. $d(\vec{x}, \vec{y}) > 0$ unless $\vec{x} = \vec{y}$
2. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
3. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

We define the distance on $\mathbb{R}^n$ between to elements

$$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$$

Introduction

Whatever you see in this particular font is *intuition*. Through *intuition* I mean the "not so correct" way of writing things that will provide you with a rough picture to think about. The slightly incorrect literature is supposed to give you an anchor for imagination - though the rigorous correctness is embodied in the *Definitions, Propositions, Theorems.*

The goal I am trying to achieve with this handout is **putting all the things I've learned in 232 in one single place in a way my malfunctioning brain can understand.** I have particularly struggled through every single classes other than integration because I couldn't convince myself why certain things existed and behaved the way they did. This was remarkably debilitating for my academics because I had taken two other math courses and I had to spent three times the time only working on 232.

Now as I think about it, right before finals, the inability to *chronologically* and *logically* not being able to structure the ideas was the fatal flaw I was dealing with. This note is an attempt to fix that.

Please don't forget to read the footnotes.