Honors Multivariable Calculus : : Homework 13

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Problem 01

The force is given by $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ and the curve C follows the path $\vec{p} : [a, b] \to \mathbb{R}^n$. $[a, b]$, as a Physics major, to me is time. We are to compute the line integral of $\vec{F}$ on C.

This integral is given by

$$\int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b \vec{F}(\vec{p}(t)) \cdot \vec{p}'(t) \, \mathrm{d}t$$

With the help of Newton's Second Law

$$\int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b m\vec{p}''(t) \cdot \vec{p}'(t) \, \mathrm{d}t$$

If $\vec{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))$, then

$$\int_a^b m\vec{p}''(t) \cdot \vec{p}'(t) \, \mathrm{d}t = \sum_{i=1}^{n} \int_a^b m p_i''(t) p_i'(t) \, \mathrm{d}t$$

This has reduced into a single variable integral problem. We can use integration by parts now. The formula is,

$$\int_a^b u \, \mathrm{d}v = uv]_a^b - \int_a^b v \, \mathrm{d}u$$

With the "substitution" of $\mathrm{d}v = p_i''(t)\mathrm{d}t$ and $u = p_i'(t)$

$$\int_a^b m p_i''(t) p_i'(t) \mathrm{d}t = m(p_i'(t))^2]_a^b - \int_a^b m p_i'(t) p_i''(t) \, \mathrm{d}t$$

Which happens to be a relatively simple solution,

$$\int_a^b m p_i''(t) p_i'(t) \, \mathrm{d}t = \frac{m p_i'(b)^2}{2} - \frac{m p_i'(a)^2}{2}$$

Now to take the summation, we will use the Generalized Pythogoras Theorem for general n dimension. In Physics sense, if the basis is orthonormal, then speed $s(a), s(b)$ is

$$\sum_{i=1}^{n} p_i'(a)^2 = s^2(a)$$

$$\sum_{i=1}^{n} p_i'(b)^2 = s^2(b)$$

Hence

$$\sum_{i=1}^{n} \int_a^b m p_i''(t) p_i'(t) \, \mathrm{d}t = \sum_{i=1}^{n} \frac{m p_i'(b)^2}{2} - \sum_{i=1}^{n} \frac{m p_i'(a)^2}{2} = \frac{m s^2(b)}{2} - \frac{m s^2(a)}{2}$$

Problem 02

The vector field is given by the equation

$$\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$$

where $\vec{F} \in \mathbb{R}^2$ and $F_i \in \mathbb{R}$ where $i = 1, 2$. The point $\vec{a}$ is defined by,

$$\vec{a} = (a,b) \in \mathbb{R}^2$$

As given in the problem

$$c = (F_2)_x(\vec{a}) - (F_1)_y(\vec{a})$$

Because F_1, F_2 are $\mathbb{R}^2 \to \mathbb{R}$ I can write the following notation too

$$c = \frac{\partial F_2}{\partial x}(\vec{a}) - \frac{\partial F_1}{\partial y}(\vec{a})$$

We are supposed to compute the following

$$\int_C \vec{F}$$

where C is defined by the path $(a,b) - (a+t,b) - (a+t,b+t) - (a,b+t)$, a simple square on $\mathbb{R}^2$. Let the paths $P_1 + P_2 + P_3 + P_4 = C$ hence

$$\int_C \vec{F} = \int_{P_1} \vec{F} + \int_{P_2} \vec{F} + \int_{P_3} \vec{F} + \int_{P_4} \vec{F}$$

Let's begin working on $\int_{P_1} \vec{F}$. This is a simple path integral of the vector function $\vec{F}$ on the "straight line" that connects (a,b) with $(a+t,b)$. Define this path with the parametric $\vec{p}_1 : [0,t] \to \mathbb{R}^2 : \vec{p}_1(\sigma) = (a+\sigma, b)$.

I will do this for all paths individually, **please be aware of the direction of σ for the path**

$$\begin{aligned}
&\vec{p}_1 : [0,t] \to \mathbb{R}^2 \quad \vec{p}_1 = (a+\sigma, b) && \sigma \in [0,t] \text{ increasing} \\
&\vec{p}_2 : [0,t] \to \mathbb{R}^2 \quad \vec{p}_2 = (a+t, b+\sigma) && \sigma \in [0,t] \text{ increasing} \\
&\vec{p}_3 : [0,t] \to \mathbb{R}^2 \quad \vec{p}_3 = (a+\sigma, b+t) && \sigma \in [0,t] \text{ decreasing} \\
&\vec{p}_4 : [0,t] \to \mathbb{R}^2 \quad \vec{p}_4 = (a, b+\sigma) && \sigma \in [0,t] \text{ decreasing}
\end{aligned}$$

To compute the integral over the path, we end up having a single variable parametrization. The path integral, generally

$$\int_P \vec{F} = \int_P \vec{F} \cdot \mathrm{d}\vec{s} = \int_0^t \vec{F}(\vec{p}(\sigma)) \cdot \vec{p}\,'(\sigma)\mathrm{d}\sigma$$

For each line segment

$$\int_{P_1} \vec{F} = \int_0^t \vec{F}(\vec{p}_1(\sigma)) \cdot \vec{p}_1'(\sigma)\mathrm{d}\sigma = \int_0^t \vec{F}(a+\sigma, b) \cdot \begin{pmatrix}1\\0\end{pmatrix} \mathrm{d}\sigma = \int_0^t F_1(a+\sigma, b)\mathrm{d}\sigma$$

$$\int_{P_2} \vec{F} = \int_0^t \vec{F}(\vec{p}_2(\sigma)) \cdot \vec{p}_2'(\sigma)\mathrm{d}\sigma = \int_0^t \vec{F}(a+t, b+\sigma) \cdot \begin{pmatrix}0\\1\end{pmatrix} \mathrm{d}\sigma = \int_0^t F_2(a+t, b+\sigma)\mathrm{d}\sigma$$

$$\int_{P_3} \vec{F} = \int_0^t \vec{F}(\vec{p}_3(\sigma)) \cdot \vec{p}_3'(\sigma)\mathrm{d}\sigma = \int_0^t \vec{F}(a+\sigma, b+t) \cdot \begin{pmatrix}-1\\0\end{pmatrix} \mathrm{d}\sigma = -\int_0^t F_1(a+\sigma, b+t)\mathrm{d}\sigma$$

$$\int_{P_4} \vec{F} = \int_0^t \vec{F}(\vec{p}_4(\sigma)) \cdot \vec{p}_4'(\sigma)\mathrm{d}\sigma = \int_0^t \vec{F}(a, b+\sigma) \cdot \begin{pmatrix}0\\-1\end{pmatrix} \mathrm{d}\sigma = -\int_0^t F_2(a, b+\sigma)\mathrm{d}\sigma$$

The total path integral is then,

$$\int_C \vec{F} = \int_0^t \mathrm{d}\sigma \left(F_1(a+\sigma, b) + F_2(a+t, b+\sigma) - F_1(a+\sigma, b+t) - F_2(a, b+\sigma)\right)$$

$$\int_C \vec{F} = \int_0^t \mathrm{d}\sigma\, (F_1(a+\sigma, b) - F_1(a+\sigma, b+t)) + \int_0^t \mathrm{d}\sigma\, (F_2(a+t, b+\sigma) - F_2(a, b+\sigma))$$
$$\int_C \vec{F} = -\int_0^t \mathrm{d}\sigma\, (F_1(a+\sigma, b+t) - F_1(a+\sigma, b)) + \int_0^t \mathrm{d}\sigma\, (F_2(a+t, b+\sigma) - F_2(a, b+\sigma))$$

Mean value theorem has some forms that we can use here, the single variable case

$$\phi(b) - \phi(a) = (b-a)\frac{\mathrm{d}\phi}{\mathrm{d}t}\, (z \in [a,b])$$

Using this

$$\int_C \vec{F} = -\int_0^t t\frac{\partial F_1}{\partial y}(a+\sigma, b+b_0) + \int_0^t t\frac{\partial F_2}{\partial x}(a+a_0, b+\sigma)$$
$$\int_C \vec{F} = \int_0^t \mathrm{d}\sigma \left(t\frac{\partial F_2}{\partial x}(a+a_0, b+\sigma) - t\frac{\partial F_1}{\partial y}(a+\sigma, b+b_0) \right)$$

Now the helpful mean value theorem is going to be,

$$\int_a^b \mathrm{d}t\, \gamma(t) = (b-a)\gamma(t)$$

Using this on the equation

$$\int_C \vec{F} = t^2\frac{\partial F_2}{\partial x}(a+a_0, b+b_1) - t^2\frac{\partial F_1}{\partial y}(a+a_1, b+b_0)$$

Now what we have is,

$$\frac{\int_C \vec{F}}{t^2} = \frac{\partial F_2}{\partial x}(a+a_0, b+b_1) - \frac{\partial F_1}{\partial y}(a+a_1, b+b_0)$$

Because $a_0, a_1, b_0, b_1 \in [0,t]$ and if $t \to 0$ then similarly $a_0, a_1, b_0, b_1 \to 0$

$$\frac{\int_C \vec{F}}{t^2} = ((F_2)_x - (F_1)_y)(\vec{a})$$

Problem 03

Equation of the circle in xy plane is given on Cartesian Coordinates from Polar coordinates through

$$x = R + r\cos\theta$$
$$y = r\sin\theta$$

R is the position of the center of the circle and thus $R = 5$. To complete the circle we require $\theta \in [0, 2\pi]$. The radius of the circle is 2 hence $r = 2$. This creates,

$$x = 5 + 2\cos\theta$$
$$y = 2\sin\theta$$

If we create a sweep of the circle with the axis of y then we create a torus

$$x = (5 + 2\cos\theta)\cos\phi$$
$$y = 2\sin\theta$$
$$z = (5 + 2\cos\theta)\sin\phi$$

With the required bound on ϕ being $\phi \in [0, 2\pi]$.

So the torus can be found using the variables θ, ϕ through the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T(\theta, \phi) = ([(5 + 2\cos\theta)\cos\phi], [2\sin\theta], [(5 + 2\cos\theta)\sin\phi])$$
$$\partial_\theta T = (-2\sin\theta\cos\phi, 2\cos\theta, -2\sin\theta\sin\phi)$$
$$\partial_\phi T = (-(5 + 2\cos\theta)\sin\phi, 0, (5 + 2\cos\theta)\cos\phi)$$

Surface area of this S surface which is the torus can be transformed into another surface D so that

$$\iint_S 1 = \iint_D \left|\frac{\partial T}{\partial\theta} \times \frac{\partial T}{\partial\phi}\right| = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=2\pi} \left|\frac{\partial T}{\partial\theta} \times \frac{\partial T}{\partial\phi}\right| \,\mathrm{d}\theta\,\mathrm{d}\phi$$

For ease of using Wolfram Alpha I will substitute (θ, ϕ) with (x, y).

$$\begin{aligned}\partial_\theta T \times \partial_\phi T = (&10\cos(x)\cos(y) + 4\cos^2(x)\cos(y),\\ &10\cos^2(y)\sin(x) + 4\cos(x)\cos^2(y)\sin(x) + 10\sin(x)\sin^2(y) + 4\cos(x)\sin(x)\sin^2(y),\\ &10\cos(x)\sin(y) + 4\cos^2(x)\sin(y))\\ = ([&\partial_\theta T \times \partial_\phi T]_1, [\partial_\theta T \times \partial_\phi T]_2, [\partial_\theta T \times \partial_\phi T]_3)\end{aligned}$$

$$\begin{aligned}|\partial_\theta T \times \partial_\phi T|^2 =& [\partial_\theta T \times \partial_\phi T]_1^2 + [\partial_\theta T \times \partial_\phi T]_2^2 + [\partial_\theta T \times \partial_\phi T]_3^2\\ =& (10\cos(x)\cos(y) + 4\cos^2(x)\cos(y))^2\\ &+ (10\cos^2(y)\sin(x) + 4\cos(x)\cos^2(y)\sin(x) + 10\sin(x)\sin^2(y) + 4\cos(x)\sin(x)\sin^2(y))^2\\ &+ (10\cos(x)\sin(y) + 4\cos^2(x)\sin(y)))^2\\ =& 4(2\cos(x) + 5)^2\end{aligned}$$

$$|\partial_\theta T \times \partial_\phi T| = 2(5 + 2\cos(\theta))$$

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=2\pi} (10 + 4\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{2\pi} 20\pi\,\mathrm{d}\phi = \boxed{40\pi^2}$$

This happens to align with the google surface area of torus. We have correct solution. Hellya.

Problem 04

a

The field is given by

$$\vec{F}(x,y) = \langle xy, y^2 \rangle$$

Parametric of a circle,

$$p : [0, 2\pi] \to \mathbb{R}^2 \quad \text{where} \quad p(\theta) = (a\cos\theta, a\sin\theta)$$

Path integral,

$$\int_0^{2\pi} \vec{F}(p(\theta))p'(\theta)\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(a^2 \frac{\sin 2\theta}{2}, a^2 \sin^2\theta\right) \cdot (-a\sin\theta, a\cos\theta)\ \mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta\, a^3 \left(-\frac{\sin 2\theta}{2}\sin\theta + \frac{\sin 2\theta}{2}\sin\theta\right) = \boxed{0}$$

b

$$\text{curl}\,\vec{F} \text{ along } \vec{z} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 - x = \boxed{-x}$$

The curl is non-zero. This is not path independent. Though there can be paths that can give 0 path integral.