Honors Linear Algebra : : Class 17

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Ahmed Saad Sabit, Rice University

5B: Minimal Polynomial

5.22 Existence, Uniqueness, and degree of Minimal Polynomial

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Definition 5.21 and 5.22.

$dimV = 1$ case.

$T \in \mathcal{L}(V)$ and $V = \mathbb{F} \cdot \vec{v}$. Now,

$$T(w) = T(av) = aT(v) = abv$$

So,

$$T = bI$$

and a monic polynomial, $p(z) = z + \text{ const }$ So,

$$p(T) = T + \text{ const } I$$

Zero if $\text{ const } = b$ so,

$$p(z) = z - b$$

Is the needed polynomial.

To prove uniqueness,

$$p(T) = 0$$

$$r(T) = 0$$

Now consider the two minimal polynomials (with leading coefficient 1 and minimum degree), the degree of polynomial $p - r$is $<$ degree of p. So,

$$p(z) = z^m + \cdots$$

$$r(t) = z^m + \cdots$$

$$(p - r)T = 0$$

5.27 : Eigenvalues are the zeros of minimal polynomial.

λ is a eigenvalue of T if and only if the minimal polynomial p for T satisfies $p(\lambda) = 0$. $\Longleftarrow$ reasoning, $p(\lambda) = 0$ so divisible by $z - \lambda$.

$$p(z) = (z - \lambda)q(z)$$

$$p(T) = (T - \lambda I)q(z) = 0$$

choose $q(v) \neq 0$, then,

$$p(T)v = (T - \lambda I)q(T)v = 0$$

Eigenvalue of T.

$\implies$ λ is eigenvalue of T. Then,

$$Tv = \lambda v$$

for an eigenvector $\neq 0$ then,

$$p(T) = 0$$

and $p(T)v = 0$.

5.29: Use polynomial algebra to try to divide q by p.

Exercise 6.

applied T to the equation again. That gave

$$T^2 + I = 0$$

Maybe minimal polynomial has lesser degree, then,

$$z + C = 0$$

Then,

$$T + cI = 0$$

$$T = -cI$$

this is not the case, and the $z + C$ is not the right one. So the correct minimal for this is,

$$z^2 + 1$$