

# Quantum Mechanics : : Homework 0X

November 23, 2024

Ahmed Saad Sabit, Rice University

## Problem 01

---

(a)

I always feel weird to use  $\hat{P}$  operator

$$\begin{aligned}\langle \vec{r} | \hat{P}_x | E, m \rangle &= -i\hbar \frac{d\Psi_{E,m}(r)}{dr} \\ \langle \vec{r} | \hat{X}^2 | E, m \rangle &= \langle \vec{r} | \hat{X}^2 | \vec{r} \rangle \langle \vec{r} | E, m \rangle \\ &= x^2 \Psi_{E,m}(r)\end{aligned}$$

Hence

$$\begin{aligned}\langle \vec{r} | \hat{H} | E, m \rangle &= E \Psi_{E,m}(r) = \frac{-\hbar^2}{2\mu} (\nabla^2 \Psi_{E,m}(r)) + \frac{\mu\omega^2}{2} (x^2 + y^2) \Psi_{E,m}(r) \\ &= \frac{-\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{m^2}{r^2} + \frac{1}{r} \frac{d}{dr} \right) \Psi_{E,m}(r) + \frac{\mu\omega^2}{2} r^2 \Psi_{E,m}(r) \\ &= \frac{-\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{m^2}{r^2} + \frac{1}{r} \frac{d}{dr} - \frac{2\mu}{\hbar} \frac{\mu\omega^2}{2} r^2 \right) \Psi_{E,m}(r) \\ &\Rightarrow \frac{-\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{m^2}{r^2} + \frac{1}{r} \frac{d}{dr} - \frac{\mu^2\omega^2}{\hbar} r^2 + \frac{2\mu}{\hbar^2} E \right) \Psi_{E,m}(r) = 0 \\ &= -\omega\hbar b^2 \left( \frac{d^2}{dr^2} - \frac{m^2}{r^2} + \frac{1}{r} \frac{d}{dr} - \frac{\mu^2\omega^2}{\hbar} r^2 + \frac{2\mu}{\hbar^2} E \right) \Psi_{E,m}(r) = 0 \\ &= \left( \frac{d^2}{dy^2} - \frac{m^2}{y^2} + \frac{1}{y} \frac{d}{dy} - y^2 + \frac{2\mu}{\hbar^2} \frac{\hbar}{\mu\omega} E \right) \Psi_{E,m}(r) = 0 \\ &= \left( \frac{d^2}{dy^2} - \frac{m^2}{y^2} + \frac{1}{y} \frac{d}{dy} - y^2 + \frac{2}{\hbar\omega} E \right) \Psi_{E,m}(r) = 0\end{aligned}$$

$$\boxed{\left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} - y^2 + 2\varepsilon \right] \psi_{\varepsilon,m}(y) = 0}$$

(b)

$$\left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} + 2\varepsilon \right] y^\alpha = 0$$

$$\begin{aligned}
\alpha(\alpha-1)y^{\alpha-2} + \alpha y^{\alpha-2} - m^2 y^{\alpha-2} + 2\varepsilon y^\alpha &= 0 \\
\alpha(\alpha-1) + \alpha - m^2 + 2\varepsilon y^{\alpha-\alpha+2} &= 0 \\
\alpha^2 - m^2 + 2\varepsilon y^2 = 0 &\implies \lim_{y \rightarrow 0} \alpha^2 - m^2 + 2\varepsilon y^2 = 0 \\
&\implies \alpha = |m|
\end{aligned}$$

$$\boxed{\lim_{y \rightarrow 0} \psi_{\varepsilon, m}(y) = y^{|m|}}$$

(c)

$$\begin{aligned}
&\left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} + 2\varepsilon \right] y^{|m|} f(y) = 0 \\
&\frac{d}{dy} \left( |m| y^{|m|-1} f(y) + y^{|m|} f'(y) \right) + \frac{1}{y} \left( |m| y^{|m|-1} f(y) + y^{|m|} f'(y) \right) - \frac{m^2}{y^2} y^{|m|} f(y) + 2\varepsilon y^{|m|} f(y) = 0 \\
&\frac{d}{dy} \left( |m| y^{|m|-1} f(y) + y^{|m|} f'(y) \right) + \left( |m| y^{|m|-2} f(y) + y^{|m|-1} f'(y) \right) - m^2 y^{|m|-2} f(y) + 2\varepsilon y^{|m|} f(y) = 0 \\
&\left( |m|(|m|-1) y^{|m|-2} f(y) + |m| y^{|m|-1} f'(y) + |m| y^{|m|-1} f'(y) + y^{|m|} f''(y) \right) + \\
&\quad + \left( |m| y^{|m|-2} f(y) + y^{|m|-1} f'(y) \right) - m^2 y^{|m|-2} f(y) + 2\varepsilon y^{|m|} f(y) = 0 \\
&y^{|m|-2} f(y) \left[ |m|(|m|-1) + |m| - m^2 \right] + y^{|m|-1} f'(y) \left[ 2|m| + 1 \right] + y^{|m|} \left[ f''(y) + 2\varepsilon f(y) \right] = 0 \\
&y^{|m|-1} f'(y) \left[ 2|m| + 1 \right] + y^{|m|} \left[ f''(y) + 2\varepsilon f(y) \right] = 0 \\
&f'(y) \left[ 2|m| + 1 \right] + y \left[ f''(y) + 2\varepsilon f(y) \right] = 0 \\
&f'(y) \frac{[2|m| + 1]}{y} + \left[ f''(y) + 2\varepsilon f(y) \right] = 0 \\
&\frac{d}{dy} \left( e^{-y^2/2} u(y) \right) \frac{[2|m| + 1]}{y} + \left[ \frac{d^2}{dy^2} \left( e^{-y^2/2} u(y) \right) + 2\varepsilon \left( e^{-y^2/2} u(y) \right) \right] = 0 \\
&\quad (-yu(y) + u'(y)) e^{-y^2/2} \frac{[2|m| + 1]}{y} + \\
&+ \left[ (-u(y) - yu'(y) + u''(y) + y^2 u(y) - yu'(y)) e^{-y^2/2} + 2\varepsilon \left( e^{-y^2/2} u(y) \right) \right] = 0 \\
&\quad (-yu(y) + u'(y)) \frac{[2|m| + 1]}{y} + \\
&+ \left[ (-u(y) - 2yu'(y) + u''(y) + y^2 u(y)) + 2\varepsilon u(y) \right] = 0 \\
&u''(y) + \left[ \frac{2|m| + 1}{y} - 2y \right] u' + (2\varepsilon - 1 - 2|m| - 1)u = 0
\end{aligned}$$

$$\boxed{u''(y) + \left[ \frac{2|m| + 1}{y} - 2y \right] u' + (2\varepsilon - 2|m| - 2)u = 0}$$

(c)

$$\begin{aligned}
u(y) &= \sum_{p=0}^{\infty} c_p y^p \\
u'(y) &= \sum_{p=1}^{\infty} p c_p y^{p-1} = \sum_{p=0}^{\infty} (p+1) c_{p+1} y^p \\
u''(y) &= \sum_{p=2}^{\infty} p(p-1) c_p y^{p-2} = \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} y^p \\
\\
u''(y) + \left[ \frac{2|m|+1}{y} - 2y \right] u' + (2\varepsilon - 2|m| - 2)u &= 0 \\
\\
0 &= \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} y^p + \left[ \frac{2|m|+1}{y} - 2y \right] \sum_{p=0}^{\infty} (p+1) c_{p+1} y^p + (2\varepsilon - 2|m| - 2) \sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} y^p + \underbrace{\left[ (2|m|+1) \sum_{p=-1}^{\infty} (p+2) c_{p+2} y^p - 2 \sum_{p=0}^{\infty} (p) c_p y^p \right]}_{\text{term of interest where } c_1 = 0} + (2\varepsilon - 2|m| - 2) \sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} y^p + (2|m|+1) \sum_{p=0}^{\infty} (p+2) c_{p+2} y^p - 2 \sum_{p=0}^{\infty} (p) c_p y^p + (2\varepsilon - 2|m| - 2) \sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty} y^p \left[ (p+2)(p+1) c_{p+2} + (2|m|+1)(p+2) c_{p+2} - 2p c_p + (2\varepsilon - 2|m| - 2) c_p \right] \\
&= \sum_{p=0}^{\infty} y^p \left[ \underbrace{[(p+2)(p+1) + (2|m|+1)(p+2)]}_{\text{two terms above}} c_{p+2} + [-2p + (2\varepsilon - 2|m| - 2)] c_p \right]
\end{aligned}$$

Requirement to keep everything 0 (also note  $p = 2k$ )

$$\begin{aligned}
c_{p+2} &= \frac{2p - (2\varepsilon - 2|m| - 2)}{(p+2)(p+1) + (2|m|+1)(p+2)} \\
&= \frac{2p - (2\varepsilon - 2|m| - 2)}{p^2 + p + 2p + 2 + 2|m|p + 4|m| + p + 2} \\
&= \frac{2p - (2\varepsilon - 2|m| - 2)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&= 2 \frac{p - (\varepsilon - |m| - 1)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
\text{for } \varepsilon = 2n + |m| + 1 &\implies 2 \frac{p - (2n + |m| + 1 - |m| - 1)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&= 2 \frac{p - 2n}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&\implies c_{2n+2} = 0 \iff p = 2n
\end{aligned}$$

(c)

$$E(n_x, n_y) = \hbar\omega \left( n_x + \frac{1}{2} + n_y + \frac{1}{2} \right)$$

$n_y \setminus n_x$	0	1	2	3	4	5	6	7	8	9	10
0	$1\hbar\omega$	$2\hbar\omega$	$3\hbar\omega$	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$
1	$2\hbar\omega$	$3\hbar\omega$	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$
2	$3\hbar\omega$	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$
3	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$
4	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$
5	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$
6	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$
7	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$
8	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$
9	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$
10	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$

This is clear as day to notice that for  $\Pi\hbar\omega$  there are  $\Pi$  entries. We can just count diagonally.

$$E(n, m) = \hbar\omega (2n + |m| + 1)$$

$n \setminus  m $	0	1	2	3	4	5	6	7	8	9	10
0	$1\hbar\omega$	$2\hbar\omega$	$3\hbar\omega$	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$
1	$3\hbar\omega$	$4\hbar\omega$	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$
2	$5\hbar\omega$	$6\hbar\omega$	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$
3	$7\hbar\omega$	$8\hbar\omega$	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$
4	$9\hbar\omega$	$10\hbar\omega$	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$
5	$11\hbar\omega$	$12\hbar\omega$	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$
6	$13\hbar\omega$	$14\hbar\omega$	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$	$22\hbar\omega$	$23\hbar\omega$
7	$15\hbar\omega$	$16\hbar\omega$	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$	$22\hbar\omega$	$23\hbar\omega$	$24\hbar\omega$	$25\hbar\omega$
8	$17\hbar\omega$	$18\hbar\omega$	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$	$22\hbar\omega$	$23\hbar\omega$	$24\hbar\omega$	$25\hbar\omega$	$26\hbar\omega$	$27\hbar\omega$
9	$19\hbar\omega$	$20\hbar\omega$	$21\hbar\omega$	$22\hbar\omega$	$23\hbar\omega$	$24\hbar\omega$	$25\hbar\omega$	$26\hbar\omega$	$27\hbar\omega$	$28\hbar\omega$	$29\hbar\omega$
10	$21\hbar\omega$	$22\hbar\omega$	$23\hbar\omega$	$24\hbar\omega$	$25\hbar\omega$	$26\hbar\omega$	$27\hbar\omega$	$28\hbar\omega$	$29\hbar\omega$	$30\hbar\omega$	$31\hbar\omega$

**For  $\hbar\omega$ :** In this case there is one entry for  $m = 0, n = 0$ .

**For  $2\hbar\omega$ :** In this case there is one entry for  $m = 1, n = 0$  but we have to be careful because  $m = -1, n = 0$  is also another separate case, hence keeping 2 entries.

**For  $3\hbar\omega$  :** For this we are dealing with  $m = 2, 0$  for respective  $n$ , and there is another term for  $m = -2$ . Hence keeping 3 entries.

We can go on like this for ever and count that for  $\Pi\hbar\omega$  we have  $\Pi$  number of entries.

## Problem 02

---

(a)

$$\hat{x} = \frac{1}{b} \hat{X}$$

$$\hat{p} = \frac{b}{\hbar} \hat{P}$$

$$\hat{a} |n=0\rangle = 0$$

$$\frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) |n=0\rangle = 0$$

$$\frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) \hat{I} |n=0\rangle = 0$$

$$\frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) |p\rangle \langle p|n=0\rangle = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \langle p'| (\hat{x} + i\hat{p}) |p\rangle \langle p|n=0\rangle = 0$$

$$\frac{1}{\sqrt{2}} \left[ \langle p'| \hat{x} |p\rangle + i \langle p'| \hat{p} |p\rangle \right] \langle p|n=0\rangle = 0$$

$$\frac{1}{\sqrt{2}} \left[ \langle p'| \hat{x} |p\rangle \psi_0(p) + i \langle p'| \hat{p} |p\rangle \psi_0(p) \right] = 0$$

$$i\hbar \frac{1}{b} \frac{d}{dp} \psi_0(p) + \frac{b}{\hbar} ip \psi_0(p) = 0$$

$$\Rightarrow \frac{\hbar}{b} \frac{d}{dp} \psi_0(p) + \frac{b}{\hbar} p \psi_0(p) = 0$$

$$\frac{\hbar^2}{b^2} \frac{d}{dp} \psi_0(p) + p \psi_0(p) = 0$$

$$\frac{\hbar^2}{b^2} \frac{d\psi_0(p)}{\psi_0(p)} = -p dp$$

$$\frac{\hbar^2}{b^2} \ln \left( \frac{\psi_0(p)}{\psi_0(0)} \right) = -\frac{p^2}{2}$$

$$\ln \left( \frac{\psi_0(p)}{\psi_0(0)} \right) = -\frac{b^2}{2\hbar^2} p^2$$

$$\ln \left( \frac{\psi_0(p)}{\psi_0(0)} \right) = -\frac{\hbar}{2m\omega\hbar^2} p^2$$

$$\ln \left( \frac{\psi_0(p)}{\psi_0(0)} \right) = -\frac{p^2}{2\omega m\hbar}$$

$$\psi_0(p) = \psi_0(0) e^{-p^2/2m\omega\hbar}$$

$$|\psi_0(p)|^2 = \psi_0^2(0) \int_{-\infty}^{\infty} dp e^{-p^2/m\omega\hbar} = \psi_0^2(0) \frac{\sqrt{2\pi}}{\sqrt{2/m\omega\hbar}} = \psi_0^2(0) \sqrt{m\omega\pi\hbar}$$

$$\Rightarrow |\psi_0(p)|^2 = \psi_0^2(0) \sqrt{m\omega\pi\hbar}$$

$$\Rightarrow \Psi_0(p) = \frac{e^{-p^2/2m\omega\hbar}}{(m\omega\pi\hbar)^{1/4}}$$

(b)

$$\begin{aligned}\psi_0(x) &= \langle x|n=0\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle \langle p|n=0\rangle dp = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \Psi_0(p) dp \\&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \frac{e^{-p^2/2m\omega\hbar}}{(m\omega\pi\hbar)^{1/4}} dp \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \int_{-\infty}^{\infty} e^{ipx/\hbar - p^2/2m\omega\hbar} dp \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \int_{-\infty}^{\infty} e^{\beta p - \frac{\alpha}{2} p^2} dp \iff \begin{cases} \alpha &= \frac{1}{m\omega\hbar} \\ \beta &= \frac{ix}{\hbar} \end{cases} \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left( \sqrt{\frac{2\pi}{\alpha}} e^{\beta^2/2\alpha} \right) \iff \begin{cases} \alpha &= \frac{1}{m\omega\hbar} \\ \beta &= \frac{ix}{\hbar} \end{cases} \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left( \sqrt{2\pi m\omega\hbar} e^{-x^2(m\omega\hbar)/2\hbar^2} \right) \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left( \sqrt{2\pi m\omega\hbar} e^{-x^2(m\omega)/2\hbar} \right) \\&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left( \sqrt{2\pi m\omega\hbar} e^{-x^2/2b} \right) \\&= \frac{\sqrt{m\omega}}{(m\omega\pi\hbar)^{1/4}} \left( e^{-x^2/2b} \right) \\&= \frac{(m\omega)^{1/4}}{(\pi\hbar)^{1/4}} \left( e^{-x^2/2b} \right) \\&= \frac{1}{\pi^{1/4} b^{1/2}} e^{-x^2/2b}\end{aligned}$$

## Problem 03

---

(a)

For  $n = 1$  :

$$\begin{aligned}\left(y - \frac{d}{dy}\right) e^{-y^2/2} &= y e^{-y^2/2} - \frac{d}{dy} e^{-y^2/2} \\ &= y e^{-y^2/2} + y e^{-y^2/2} \\ &= 2y e^{-y^2/2}\end{aligned}$$

For  $n = 2$  :

$$\begin{aligned}\left(y - \frac{d}{dy}\right)^2 e^{-y^2/2} &= \left(y - \frac{d}{dy}\right) \left(y - \frac{d}{dy}\right) e^{-y^2/2} \\ &= \left(y - \frac{d}{dy}\right) 2y e^{-y^2/2} \\ &= 2y^2 e^{-y^2/2} - \frac{d}{dy} (2y e^{-y^2/2}) \\ &= 2y^2 e^{-y^2/2} - (2e^{-y^2/2} - 2y^2 e^{-y^2/2}) \\ &= 4y^2 e^{-y^2/2} - 2e^{-y^2/2} \\ &= (4y^2 - 2) e^{-y^2/2}\end{aligned}$$

$$\begin{aligned}\psi_1(x) &= \frac{2y e^{-y^2/2}}{\sqrt{2b\sqrt{\pi}}} \\ \psi_2(x) &= \frac{(4y^2 - 2) e^{-y^2/2}}{2\sqrt{2b\sqrt{\pi}}}\end{aligned}\quad (y = x/b)$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty} dx e^{-\alpha x^2/2} &= \sqrt{\frac{2\pi}{a}} \\ \frac{d}{d\alpha} \int_{-\infty}^{\infty} dx e^{-\alpha x^2/2} &= \frac{d}{d\alpha} \sqrt{\frac{2\pi}{a}} \\ \int_{-\infty}^{\infty} dx \frac{d}{d\alpha} e^{-\alpha x^2/2} &= \sqrt{2\pi} \left(-\frac{1}{2} a^{-\frac{3}{2}}\right) \\ -\frac{1}{2} \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2/2} &= \frac{\sqrt{2\pi}}{a^{\frac{3}{2}}} \left(-\frac{1}{2}\right) \\ \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2/2} &= \frac{\sqrt{2\pi}}{a^{3/2}}\end{aligned}$$



(c)

Consider the functions without the normalization constants in front

$$\Psi_0(x) = e^{-y^2/2}$$

$$\Psi_1(x) = ye^{-y^2/2}$$

$$\Psi_2(x) = (4y^2 - 2)e^{-y^2/2}$$

$$\begin{aligned}\langle 0|2\rangle &\Rightarrow \int_{-\infty}^{\infty} dy (4y^2 - 2)e^{-y^2} \\ &= \int_{-\infty}^{\infty} dy 4y^2 e^{-y^2} - \int_{-\infty}^{\infty} dy 2e^{-y^2} \\ &= \sqrt{2\pi} \left( \frac{4}{2^{3/2}} - \frac{2}{2^{1/2}} \right) \\ &= \sqrt{2\pi} \left( \frac{2}{2^{1/2}} - \frac{2}{2^{1/2}} \right) = 0\end{aligned}$$

$$\begin{aligned}\langle 1|1\rangle &= \int_{-\infty}^{\infty} dx y^2 e^{-y^2} \\ &= b \int_{-\infty}^{\infty} dy y^2 e^{-y^2} \\ &= b \frac{\sqrt{2\pi}}{2^{3/2}} \\ &= b \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2}^3} = b \frac{\sqrt{\pi}}{2}\end{aligned}$$

Coefficients of  $\Psi_1(x)$  are

$$C = \frac{2}{\sqrt{2b\sqrt{\pi}}} \Rightarrow C^2 = \frac{2}{b\sqrt{\pi}}$$

Putting that in place

$$\langle 1|1\rangle = C^2 \int_{-\infty}^{\infty} dx y^2 e^{-y^2} = \frac{2}{b\sqrt{\pi}} b \frac{\sqrt{\pi}}{2} = 1$$

## Problem 04

---

(a)

$$\begin{aligned}
 \hat{X} &= \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^T) & \hat{P} &= \frac{i}{\sqrt{2}}(\hat{a}^T - \hat{a}) \\
 \hat{X}\hat{X} &= \frac{1}{2}(\hat{a} + \hat{a}^T)(\hat{a} + \hat{a}^T) & \hat{P}\hat{P} &= -\frac{1}{2}(\hat{a}^T - \hat{a})(\hat{a}^T - \hat{a}) \\
 &= \frac{1}{2}(\hat{a}\hat{a} + \hat{a}^T\hat{a}^T + \hat{a}^T\hat{a} + \hat{a}\hat{a}^T) & &= -\frac{1}{2}(\hat{a}\hat{a} + \hat{a}^T\hat{a}^T - \hat{a}^T\hat{a} - \hat{a}\hat{a}^T)
 \end{aligned}$$

$$\begin{aligned}
 \langle n | \hat{X} | n \rangle &= \frac{1}{\sqrt{2}} \langle n | \hat{a} + \hat{a}^T | n \rangle \\
 &= \frac{1}{\sqrt{2}} (\langle n | \hat{a} | n \rangle + \langle n | \hat{a}^T | n \rangle) \\
 &= \frac{1}{\sqrt{2}} (\sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle) = 0 \\
 \langle n | \hat{P} | n \rangle &= \frac{i}{\sqrt{2}} \langle n | \hat{a}^T - \hat{a} | n \rangle \\
 &= \frac{i}{\sqrt{2}} (\langle n | \hat{a}^T | n \rangle - \langle n | \hat{a} | n \rangle) = 0
 \end{aligned}$$

$$\begin{aligned}
 \langle n | \hat{X}^2 | n \rangle &= \frac{1}{2} (\langle n | \hat{a}\hat{a} | n \rangle + \langle n | \hat{a}^T\hat{a}^T | n \rangle + \langle n | \hat{a}\hat{a}^T | n \rangle + \langle n | \hat{a}^T\hat{a} | n \rangle) \\
 &= \frac{1}{2} (0 + 0 + (n+1) + n) \\
 &= \frac{1}{2} (2n+1) \\
 &= n + \frac{1}{2} \\
 \langle n | \hat{P}^2 | n \rangle &= -\frac{1}{2} (\langle n | \hat{a}\hat{a} | n \rangle + \langle n | \hat{a}^T\hat{a}^T | n \rangle - \langle n | \hat{a}\hat{a}^T | n \rangle - \langle n | \hat{a}^T\hat{a} | n \rangle) \\
 &= -\frac{1}{2} (0 + 0 - ((n+1) + n)) \\
 &= n + \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \Delta X &= \sqrt{\frac{\hbar}{m\omega}} \sqrt{\langle n | \hat{X}^2 | n \rangle - \langle n | \hat{X} | n \rangle^2} = \sqrt{\frac{\hbar}{m\omega}} \sqrt{n + \frac{1}{2}} \\
 \Delta P &= \hbar \sqrt{\frac{m\omega}{\hbar}} \sqrt{\langle n | \hat{P}^2 | n \rangle - \langle n | \hat{P} | n \rangle^2} = \hbar \sqrt{\frac{m\omega}{\hbar}} \sqrt{n + \frac{1}{2}}
 \end{aligned}$$

(b)

$$\Rightarrow \Delta X \Delta P = \hbar \left( n + \frac{1}{2} \right)$$

$$\Delta X \Delta P \geq \frac{\hbar}{2}$$

$$\Delta X \Delta P = \frac{\hbar}{2} \iff n = 0$$

## Problem 05

---

$$\begin{aligned}
 \frac{d}{dt} \langle \Omega \rangle (t) &= \frac{i}{\hbar} \langle [H, \Omega] \rangle (t) \\
 \frac{d}{dt} \langle a \rangle (t) &= \frac{i}{\hbar} \langle [H, a] \rangle (t) \\
 &= \frac{i}{\hbar} \langle \left[ \hbar \omega \left( a^T a + \frac{1}{2} I \right), a \right] \rangle \\
 &= i\omega \langle [a^T a, a] \rangle \\
 &= -i\omega \langle a \rangle \\
 \int \frac{1}{\langle a \rangle} \frac{d}{dt} \langle a \rangle dt &= \int -i\omega dt \\
 \langle a \rangle (t) &= \langle a \rangle (0) e^{-i\omega t}
 \end{aligned}$$

## Scratchwork

---

$$\begin{aligned}
 \frac{d}{dt} \langle \Omega \rangle (t) &= \frac{i}{\hbar} \langle [H, \Omega] \rangle (t) \\
 \frac{d}{dt} \langle a \rangle (t) &= \frac{i}{\hbar} \langle [H, a] \rangle (t) \\
 &= \frac{i}{\hbar} \langle \psi | Ha - aH | \psi \rangle \\
 &= \frac{i}{\hbar} [\langle \psi | Ha | \psi \rangle - \langle \psi | aH | \psi \rangle] \\
 &= \frac{i(\hbar\omega)}{\hbar} \left[ \langle \psi | \left( a^T a + \frac{1}{2} I \right) a | \psi \rangle - \langle \psi | a \left( a^T a + \frac{1}{2} I \right) | \psi \rangle \right] \\
 &= i\omega [\langle \psi | a^T a a | \psi \rangle - \langle \psi | a a^T a | \psi \rangle] \\
 &= i\omega [\langle \psi(n) | a^T a a | \psi(n) \rangle - \langle \psi(n) | a a^T a | \psi(n) \rangle] \quad (\text{rewrite with } n\text{-th energy eigenstate}) \\
 &= i\omega [\sqrt{n-1} \langle \psi(n) | a^T a | \psi(n-1) \rangle - \sqrt{n-1} \langle \psi(n) | a a^T | \psi(n-1) \rangle] \\
 &= i\omega [\sqrt{(n-2)(n-1)} \langle \psi(n) | a^T | \psi(n-2) \rangle - \sqrt{n(n-1)} \langle \psi(n) | a | \psi(n) \rangle] \\
 &= i\omega [\sqrt{(n-2)(n-1)^2} \langle \psi(n) | \psi(n-1) \rangle - \sqrt{n(n-1)^2} \langle \psi(n) | \psi(n-1) \rangle] \\
 &= i\omega [\sqrt{(n-2)(n-1)^2} - \sqrt{n(n-1)^2}] \langle \psi(n) | \psi(n-1) \rangle
 \end{aligned}$$