Quantum Mechanics Homework 01

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Problem 1 (a)

Studying the problem

The initial configuration of the string

$$q(x,0) = f(x) = \begin{cases} \frac{2h}{L}x & 0 \le x \le \frac{L}{2} \\ 2h - \frac{2h}{L}x & \frac{L}{2} \le x \le L \end{cases}$$

$$\frac{\partial q(x,t)}{\partial t}\bigg|_{t=0} = \sum_{n=1}^{\infty} d_n \Omega_n \phi_n(x) = g(x) = 0 \implies \boxed{d_n = 0}$$

The general solution to the string equation (assumed solution is separable between time and position)

$$q(x,t) = \sum_{n=1}^{\infty} [c_n \cos(\Omega_n t) + d_n \sin(\Omega_n t)]\phi_n(x)$$

For $t = 0$ we get,

$$q(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

We are interested on finding the general solution of $q(x,t)$ that will hold for the future given this initial condition. The variables of our equation are obviously x, t and what we need to find out is c_n, d_n. The next sub-section will find out a solution for c_n (d_n is trivially zero given zero initial velocity).

Solving for c_n

Let us do the following computation now. Let us multiply both sides of the above equation with $\phi_p(x)$ where p represents the p-th term while we take a summation over the index of n.

$$f(x)\phi_p(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)\phi_p(x)$$

Just so that we can invoke the inner product between orthonormal bases, we can take an integral with the following way

$$\begin{aligned} \int_0^L \mathrm{d}x\, f(x)\phi_p(x) &= \int_0^L \mathrm{d}x \left(\sum_{n=1}^{\infty} c_n \phi_n(x)\phi_p(x) \right) \\ &= \sum_{n=1}^{\infty} c_n \int_0^L \mathrm{d}x\, \phi_n(x)\phi_p(x) \\ &= \sum_{n=1}^{\infty} c_n \delta_{np} \frac{L}{2} \\ &= c_p \frac{L}{2} \end{aligned}$$

This above gives us the p-th term

$$c_p = \frac{2}{L}\int_0^L \mathrm{d}x\, f(x)\phi_p(x)$$

Using the explicit equation for the bases and also looking at the piecewise function, we can write,

$$\begin{aligned}
c_p &= \frac{2}{L}\int_0^L \mathrm{d}x\, f(x)\sin\left(\frac{p\pi x}{L}\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{p\pi x}{L}\right) + \int_{\frac{L}{2}}^{L} f(x)\sin\left(\frac{p\pi x}{L}\right)\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2h}{L}x\sin\left(\frac{p\pi x}{L}\right) + \int_{\frac{L}{2}}^{L} \left(2h - \frac{2h}{L}x\right)\sin\left(\frac{p\pi x}{L}\right)\right) \\
&= \frac{2}{L}\left(\frac{hL}{\pi^2 p^2}\left[2\sin\left(p\frac{\pi}{2}\right) - \pi p\cos\left(p\frac{\pi}{2}\right)\right] - \frac{hL}{\pi^2 p^2}\left[2\sin\left(\pi p\right) - 2\sin\left(p\frac{\pi}{2}\right) - \pi p\cos\left(p\frac{\pi}{2}\right)\right]\right) \\
&= \frac{8h}{\pi^2 p^2}\sin\left(p\frac{\pi}{2}\right)\left[1 - \cos\left(p\frac{\pi}{2}\right)\right]
\end{aligned}$$

Hence if I write this huge mess properly

$$\boxed{c_n = \frac{8h}{\pi^2 p^2}\sin\left(\frac{p\pi}{2}\right)\left[1 - \cos\left(\frac{p\pi}{2}\right)\right]}$$

Discussion on odd and even modes

Problem 1 (b)

Studying the Problem

Initially the string is tight and straight hence

$$q(x,0) = \sum_{n=1}^{\infty} c_n\phi_n(x) = f(x) = 0 \implies \boxed{c_n = 0}$$

$$\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} = g(x) = v_0\theta\left(a - \left|x - \frac{L}{2}\right|\right)$$

Where $\theta(x)$ is a Heaviside step function (outputs 1 whenever input is 0 or positive). I am not going to waste my and graders time by re-writing everything I wrote above, the procedure we are going to follow is same as above.

Computation of d_n

$$\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} d_n\Omega_n\phi_n(x) \\
\int_0^L \mathrm{d}x\, g(x)\phi_p(x) &= \sum_{n=1}^{\infty}\int_0^L \mathrm{d}x\, d_n\Omega_n\phi_n(x)\phi_p(x) \\
\int_{L/2-a}^{L/2+a} v_0\phi_p(x) &= d_p\Omega_p\frac{L}{2}
\end{aligned}$$

Problem 3

$$\begin{aligned}
|v+w|^2 &= \langle v+w|v+w\rangle \\
&= \langle v|v+w\rangle + \langle w|v+w\rangle \\
&= \langle v|v\rangle + \langle v|w\rangle + \langle w|v\rangle + \langle w|w\rangle \\
&= |v|^2 + |w|^2 + \langle v|w\rangle + \langle v|w\rangle^* \\
&= |v|^2 + |w|^2 + 2\mathrm{Re}\left(\langle v|w\rangle\right) \\
&\leq |v|^2 + |w|^2 + 2|\langle v|w\rangle| \\
&\leq |v|^2 + |w|^2 + 2|v||w| \\
&\leq \left(|v| + |w|\right)^2
\end{aligned}$$

This shows that

$$|v+w| \leq |v| + |w|$$

Problem 4(a)

I will write $\hat{\sigma}^n$ as simply σ^n for this problem. I did the multiplication by hand.

$$(\sigma^1)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^2)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^3)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We had been already defined

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All of these result matrix above is the identity matrix that validates

$$(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \sigma^0$$

Problem 4(b)

We are required to solve for $A^{\mu,\nu}$ where

$$A^{\mu,\nu} = \sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu$$

Note that it is obvious

$$A^{\mu,\nu} = A^{\nu,\mu}$$

Computing each of the matrix multiplications, and also referring to previous computations

$$A^{k,k} = A^{1,1} = A^{2,2} = A^{3,3} = 2(\sigma^k)^2 = 2\sigma^0 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
A^{1,2} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^{2,3} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^{1,3} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

Different indexes cause a zero-matrix, and similar causes a double of identity matrix. From here we can easily figure out that

$$A^{\mu,\nu} = 2\sigma^0 \delta_{\mu,\nu}$$

Problem 4(c)

I am going to borrow the computations I did last problem

$$\mathsf{Tr}[\sigma^1\sigma^1] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^2\sigma^2] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^3\sigma^3] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^1\sigma^2] = \mathsf{Tr} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^2\sigma^1]$$

$$\mathsf{Tr}[\sigma^2\sigma^3] = \mathsf{Tr} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^3\sigma^2]$$

$$\mathsf{Tr}[\sigma^1\sigma^3] = \mathsf{Tr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^3\sigma^1]$$

From this we can see that same-index gives 2 and different gives 0. From this it's obvious

$$\mathsf{Tr}[\sigma^\mu\sigma^\nu] = 2\delta_{\mu\nu}$$

Problem 4(d)

Expanding the equation of the operator

$$\hat{V} = \sum_{i=1}^{3} V_i \sigma^i = V_1\sigma^1 + V_2\sigma^2 + V_3\sigma^3$$

Multiply σ^p where $p \in \{1, 2, 3\}$

$$\hat{V}\sigma^p = V_1\sigma^1\sigma^p + V_2\sigma^2\sigma^p + V_3\sigma^3\sigma^p$$

Taking the trace and using the property $\mathsf{Tr}(A+B) = \mathsf{Tr}(A) + \mathsf{Tr}(B)$

$$\mathsf{Tr}(\hat{V}\sigma^p) = V_1\,(2\delta_{1p}) + V_2\,(2\delta_{2p}) + V_3\,(2\delta_{3p})$$

From this using the definition of the $\delta_{\mu,p}$ we can simply write,

$$V_p = \frac{1}{2}\mathsf{Tr}(\hat{V}\sigma^p)$$

Using the form

$$\hat{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we can find the coefficients V_p

$$\hat{V}\sigma^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies V_0 = \frac{a+d}{2}$$

$$\hat{V}\sigma^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \implies V_1 = \frac{b+c}{2}$$

$$\hat{V}\sigma^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} ib & -ia \\ id & -ic \end{pmatrix} \implies V_2 = i\frac{b-c}{2}$$

$$\hat{V}\sigma^3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \implies V_3 = \frac{a-d}{2}$$

So our representation is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{ib-ic}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I've checked this in Wolfram Alpha and it seems to work.

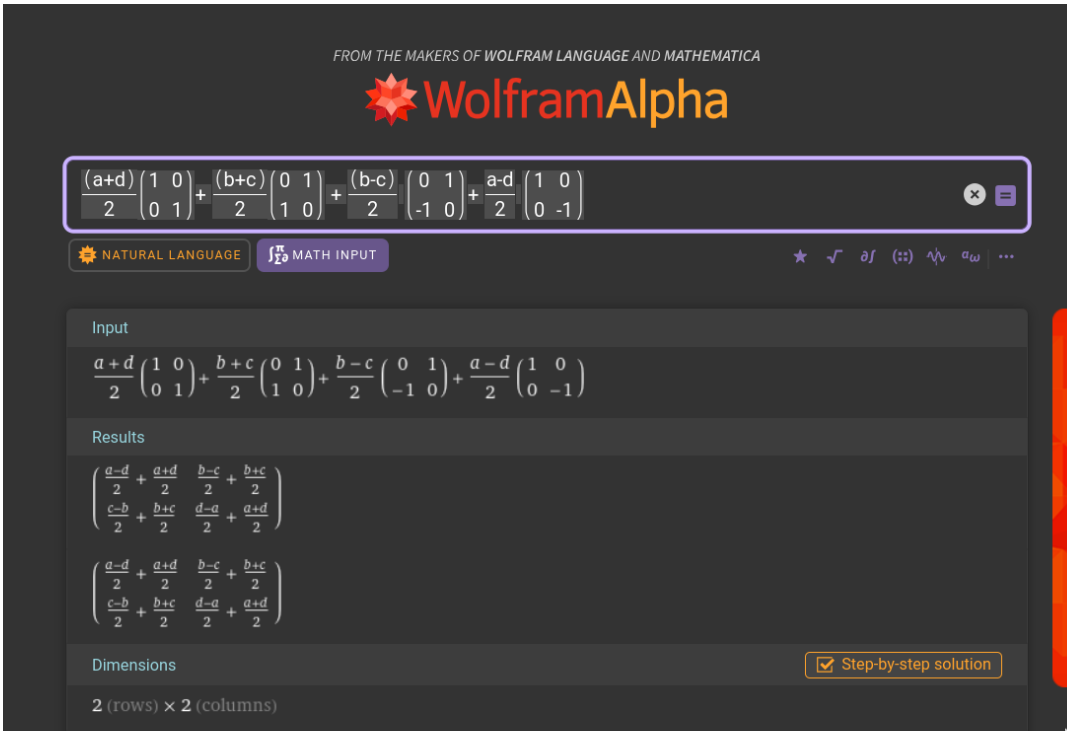

Figure 1: ss/pauli-basis.png