Quantum Mechanics : : Homework 0X

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Problem 01

(a)

For the first particle in position $\theta = \Omega t$.

$$\begin{aligned}
\vec{F}_{\text{cor}}(\theta) &= -2m\vec{\omega} \times \vec{v}(t) = -2m\vec{\omega} \times (\vec{\Omega} \times \vec{R}(t)) \\
&= -2m(\omega\hat{z}) \times (\Omega\vec{y} \times [R\cos\theta\hat{x} + R\sin\theta\hat{z}]) \\
&= -2m\omega\hat{z} \times (\Omega R\cos\theta(-\hat{z}) + \Omega R\sin\theta(-\hat{x})) \\
&= 2m\omega\hat{z} \times (\Omega R\cos\theta\hat{z} + \Omega R\sin\theta\hat{x}) \\
&= 2m\omega\Omega R(-\hat{y}) \\
&= -2m\omega\Omega R\sin(\theta)\hat{y}
\end{aligned}$$

For the second particle is $\vec{F}_{\text{cor}}(\theta + \pi)$. $\vec{F}_{\text{cor}} = 2m\omega\Omega R\sin(\omega)\hat{y}$

(b)

$$\begin{aligned}
\vec{\tau} &= \vec{R}_1 \times \vec{F}_1 + \vec{R}_2 \times \vec{F}_2 \\
&= (R\cos\theta\hat{x} + R\sin\theta\hat{z}) \times (-2m\omega\Omega R\sin\theta\hat{y}) + (-R\cos\theta\hat{x} - R\sin\theta\hat{z}) \times (2m\omega\Omega R\sin\theta\hat{y}) \\
&= \left[(-2m\omega\Omega R^2)(\cos\theta\hat{x} \times \sin\theta\hat{y} + \sin\theta\hat{z} \times \sin\theta\hat{y})\right] \\
&= 2\left[(-2m\omega\Omega R^2)\left(-\cos\theta\sin\theta\hat{z} + \sin^2\theta\hat{x}\right)\right] \\
&= 4m\omega\Omega R^2\left(\cos\theta\sin\theta\hat{z} - \sin^2\theta\hat{x}\right)
\end{aligned}$$

(c)

$$
\begin{aligned}
\vec{\tau} &= 4m\omega\Omega R^2\left(\cos\theta\sin\theta\hat{z} - \sin^2\theta\hat{x}\right) \\
\langle\vec{\tau}\rangle &= \frac{\int_0^T \mathrm{d}t\, 4m\omega\Omega R^2\left(\cos(\Omega t)\sin(\Omega t)\hat{z} - \sin^2(\Omega t)\hat{x}\right)}{\int_0^T \mathrm{d}t} \\
&= \frac{\int_0^T \mathrm{d}t\, 4m\omega\Omega R^2\left(\cos(\Omega t)\sin(\Omega t)\hat{z} - \sin^2(\Omega t)\hat{x}\right)}{\int_0^T \mathrm{d}t} && (T = 2\pi/\Omega) \\
&= 4m\omega\Omega R^2\frac{\left[\hat{z}\int_0^T \mathrm{d}t\, \cos(\Omega t)\sin(\Omega t) - \hat{x}\int_0^T \mathrm{d}t\, \sin^2(\Omega t)\right]}{2\pi/\Omega} \\
&= 4m\omega\Omega R^2\left(-\frac{1}{2}\hat{x}\right) \\
&= -2m\omega\Omega R^2\hat{x}
\end{aligned}
$$

(d)

Problem 02

(a)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{A}\cdot\vec{B}\right)_{\text{fix}} &= \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\cdot\vec{B}\right)_{\text{fix}} + \left(\vec{A}\cdot\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)_{\text{fix}} \\
\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{fix}} &= \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\omega}\times\vec{A} \\
\left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)_{\text{fix}} &= \left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\omega}\times\vec{B}
\end{aligned}
$$

$$
\begin{aligned}
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{A}\cdot\vec{B}\right)_{\text{fix}} = \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\cdot\vec{B}\right)_{\text{fix}} + \left(\vec{A}\cdot\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)_{\text{fix}} &= \left[\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\omega}\times\vec{A}\right]\cdot\vec{B} + \left[\left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\omega}\times\vec{B}\right]\cdot\vec{A} \\
&= \left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)_{\text{rot}}\cdot\vec{B} + \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_{\text{rot}}\cdot\vec{A} + \left[(\vec{\omega}\times\vec{A})\cdot\vec{B} + (\vec{\omega}\times\vec{B})\cdot\vec{A}\right] \\
&\qquad \text{(check appendix for why third term is zero)} \\
&= \left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)_{\text{rot}}\cdot\vec{B} + \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_{\text{rot}}\cdot\vec{A} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{A}\cdot\vec{B}\right)_{\text{rot}}
\end{aligned}
$$

(b)

Recycling what we had above, using $\vec{C} = \vec{A} \times \vec{B}$

$$\begin{aligned}
\left(\frac{d\vec{C}}{dt}\right)_{\text{fixed}} &= \left(\left(\frac{d\vec{A}}{dt}\right)_{\text{rotating}} + \vec{\omega} \times \vec{A}\right) \times \vec{B} + \vec{A} \times \left(\left(\frac{d\vec{B}}{dt}\right)_{\text{rotating}} + \vec{\omega} \times \vec{B}\right). \\
&= \left(\frac{d\vec{A}}{dt}\right)_{\text{rotating}} \times \vec{B} + (\vec{\omega} \times \vec{A}) \times \vec{B} + \vec{A} \times \left(\frac{d\vec{B}}{dt}\right)_{\text{rotating}} + \vec{A} \times (\vec{\omega} \times \vec{B}) \\
&= \left[\left(\frac{d\vec{A}}{dt}\right)_{\text{rotating}} \times \vec{B} + \vec{A} \times \left(\frac{d\vec{B}}{dt}\right)_{\text{rotating}}\right] + \left[(\vec{\omega} \times \vec{A}) \times \vec{B} + \vec{A} \times (\vec{\omega} \times \vec{B})\right] \\
&= \left(\frac{d\vec{A}}{dt}\right)_{\text{rotating}} \times \vec{B} + \vec{A} \times \left(\frac{d\vec{B}}{dt}\right)_{\text{rotating}} + \vec{\omega} \times (\vec{A} \times \vec{B}) \qquad \text{(check appendix for proof)} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{A} \times \vec{B}\right)_{\text{rotating}} + \vec{\omega} \times (\vec{A} \times \vec{B}) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{C}\right)_{\text{rotating}} + \vec{\omega} \times \vec{C}
\end{aligned}$$

So vectors abide by the laws of rotation.

Problem 03

In a steady rotational frame, intuitively speaking rough - the position dependent force is *Centrifugal Force* and velocity dependent forces are *Coriolis Force*.

(a)

Consider no force of magnetic field now. Then all the fictitious forces

$$\vec{F} = m\ddot{\vec{r}} = -2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Now including magnetic force

$$\boxed{\vec{F} = m\ddot{\vec{r}} = -2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - q\vec{v} \times \vec{B}}$$

(b)

Substituting $q = 2m\omega/B$ yields

$$\begin{aligned}
\vec{F} &= -2m\omega v\left[\hat{z}\times\hat{v}\right] - m\omega^2 r\left[\hat{z}\times(\hat{z}\times\hat{r})\right] - \left(\frac{2m\omega}{B}\right)vB\left[\hat{v}\times\hat{z}\right] \\
&= -2m\omega v\left[\hat{z}\times\hat{v}\right] - m\omega^2 r\left[\hat{z}\times(\hat{z}\times\hat{r})\right] - 2m\omega v\left[\hat{v}\times\hat{z}\right] \\
&= -m\omega^2 r\left[\hat{z}\times(\hat{z}\times\hat{r})\right] \\
&= m\omega^2 r\left[(\hat{z}\cdot\hat{z})\hat{r}\right] \\
&= m\omega^2\vec{r}
\end{aligned}$$

This leaves us with

$$\ddot{\vec{r}} + \omega^2\vec{r} = 0$$

This is a simple harmonic equation, Yippie! Please note that this equation holds in the rotational frame.

Details on shape: Equilibrium is established at $r = 0$ hence establishing the center of the turntable to be the center. Two component solution

$$\begin{aligned}
\ddot{x} + \omega^2 x = 0 &\implies x = x_0\sin(\omega t + \phi_x) \\
\ddot{y} + \omega^2 y = 0 &\implies y = y_0\sin(\omega t + \phi_y)
\end{aligned}$$

This is the very beautiful *Lissayous Curves!*

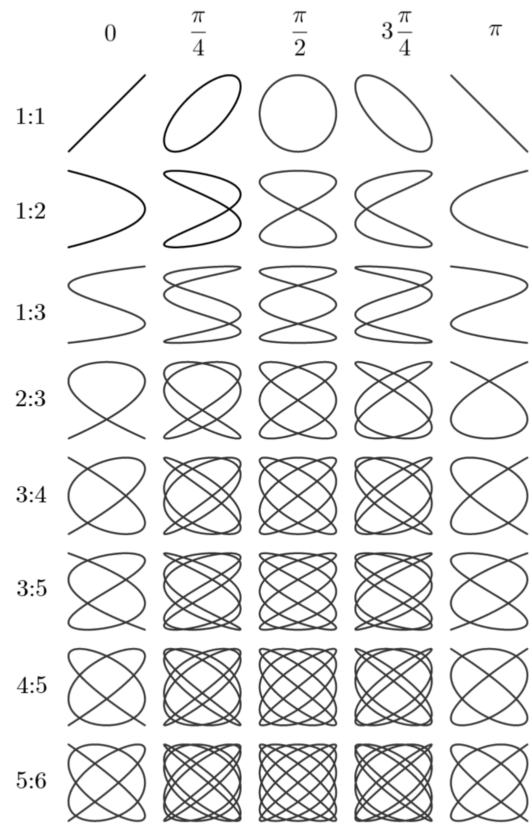

Figure 1: Ratio of $x_0 : y_0$ versus the phase difference $|\phi_x - \phi_y|$

(c)

For half as large q we end up with

$$
\begin{aligned}
\vec{F} &= -2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - q\vec{v} \times \vec{B} \\
&= -2m\omega v \left[\hat{z} \times \hat{v}\right] - m\omega^2 r \left[\hat{z} \times (\hat{z} \times \hat{r})\right] - (q)\, vB \left[\hat{v} \times \hat{z}\right] \\
&= -2m\omega v \left[\hat{z} \times \hat{v}\right] - m\omega^2 r \left[\hat{z} \times (\hat{z} \times \hat{r})\right] - \frac{1}{2}\left(\frac{2m\omega}{B}\right) vB \left[\hat{v} \times \hat{z}\right] \\
&= -2m\omega v \left([\hat{z} \times \hat{v}] + \frac{1}{2}[\hat{v} \times \hat{z}]\right) - m\omega^2 \vec{r} \\
&= -2m\omega v \left([\hat{z} \times \hat{v}] - \frac{1}{2}[\hat{z} \times \hat{v}]\right) - m\omega^2 \vec{r} \\
&= -2m\omega v \left(\frac{1}{2}[\hat{z} \times \hat{v}]\right) - m\omega^2 \vec{r} \\
&= -m\omega v[\hat{z} \times \hat{v}] - m\omega^2 \vec{r} \\
\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= -\omega v\,(\hat{z} \times \hat{v}) - \omega^2 \vec{r} && \text{(NOTE: rotating frame derivative)} \\
\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= -\vec{\omega} \times \vec{v} - \omega^2 \vec{r}
\end{aligned}
$$

The second derivative of a purely rotating vector $\vec{A}$ with $\vec{\Omega}$

$$
\begin{aligned}
\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} &= \vec{\Omega} \times \vec{A} \\
\frac{\mathrm{d}^2\vec{A}}{\mathrm{d}t^2} &= \vec{\Omega} \times \left[\vec{\Omega} \times \vec{A}\right] \\
&= -\Omega^2 \vec{A}
\end{aligned}
$$

This is a derivative in the rotating frame, for an outside observer,

$$
\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \vec{\omega} \times \vec{v} = \left.\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right|_{\text{lab}} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\omega^2 \vec{r}
$$

$(\vec{\omega} \times \vec{A}) \cdot \vec{B} = -\vec{A} \cdot (\vec{\omega} \times \vec{B})$

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z), \quad \vec{A} = (A_x, A_y, A_z), \quad \vec{B} = (B_x, B_y, B_z).$$

$$\vec{\omega} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ A_x & A_y & A_z \end{vmatrix}.$$

$$\vec{\omega} \times \vec{A} = \big(\omega_y A_z - \omega_z A_y\big)\hat{\imath} - \big(\omega_x A_z - \omega_z A_x\big)\hat{\jmath} + \big(\omega_x A_y - \omega_y A_x\big)\hat{k}.$$

$$(\vec{\omega} \times \vec{A}) \cdot \vec{B} = (\omega_y A_z - \omega_z A_y)B_x + (-(\omega_x A_z - \omega_z A_x))B_y + (\omega_x A_y - \omega_y A_x)B_z.$$

$$= \omega_y A_z B_x - \omega_z A_y B_x - \omega_x A_z B_y + \omega_z A_x B_y + \omega_x A_y B_z - \omega_y A_x B_z.$$

$$\vec{\omega} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ B_x & B_y & B_z \end{vmatrix}.$$

$$\vec{\omega} \times \vec{B} = \big(\omega_y B_z - \omega_z B_y\big)\hat{\imath} - \big(\omega_x B_z - \omega_z B_x\big)\hat{\jmath} + \big(\omega_x B_y - \omega_y B_x\big)\hat{k}.$$

$$-\vec{A} \cdot (\vec{\omega} \times \vec{B}) = -\left(A_x(\omega_y B_z - \omega_z B_y) + A_y(-(\omega_x B_z - \omega_z B_x)) + A_z(\omega_x B_y - \omega_y B_x)\right).$$

$$= \omega_y A_z B_x - \omega_z A_y B_x - \omega_x A_z B_y + \omega_z A_x B_y + \omega_x A_y B_z - \omega_y A_x B_z.$$

Comparing the expanded expressions for

$$(\vec{\omega} \times \vec{A}) \cdot \vec{B} \quad \text{and} \quad -\vec{A} \cdot (\vec{\omega} \times \vec{B}),$$

we see they are identical. Therefore

$$(\vec{\omega} \times \vec{A}) \cdot \vec{B} = -\vec{A} \cdot (\vec{\omega} \times \vec{B}).$$

$(\vec{\omega} \times \vec{A}) \times \vec{B} + \vec{A} \times (\vec{\omega} \times \vec{B})$

$$(\vec{\omega} \times \vec{A}) \times \vec{B} = \big((\vec{\omega} \times \vec{A}) \cdot \vec{B}\big)\vec{A} - \big((\vec{\omega} \times \vec{A}) \cdot \vec{A}\big)\vec{B}.$$

$$\vec{A} \times (\vec{\omega} \times \vec{B}) = \big(\vec{A} \cdot \vec{B}\big)\vec{\omega} - \big(\vec{A} \cdot \vec{\omega}\big)\vec{B}.$$

$$(\vec{\omega} \times \vec{A}) \times \vec{B} + \vec{A} \times (\vec{\omega} \times \vec{B}) = \vec{\omega} \times (\vec{A} \times \vec{B}).$$