Computational Complex Analysis : : Homework 03

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1 Problem

The first few derivatives of the function are, evaluated at $z = 0$,

$$f^{(1)} = \frac{d}{dz}\frac{1}{(1-z)^3} = \frac{3}{(1-z)^4} = 3 = \frac{3!}{2!}$$

$$f^{(2)} = \frac{d}{dz}\frac{3}{(1-z)^4} = \frac{12}{(1-z)^5} = 12 = \frac{4!}{2!}$$

$$f^{(3)} = \frac{d}{dz}\frac{12}{(1-z)^5} = \frac{60}{(1-z)^6} = \frac{5!}{2!}$$

We can see the pattern, so using the McLaurin Series,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

This gives us,

$$\boxed{\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \frac{(2+n)!}{2!\,n!}}$$

2 Problem

Like the previous problem on the paper I computed the a_n each,

$$f^{(1)} = \frac{6z}{(3-z)^3}$$

$$f^{(2)} = \frac{6(3+2z)}{(3-z)^4}$$

$$f^{(3)} = \frac{36(z+3)}{(3-z)^5}$$

Using a calculator,

$$a_0, a_1, a_2, a_3, \ldots = 0, 0, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{160}{81}, \frac{400}{81} \ldots$$

We get,

$$\boxed{\left(\frac{z}{3-z}\right)^2 = \frac{1}{9}z^2 + \frac{2}{27}z^3 + \frac{1}{27}z^4 + \frac{4}{243}z^5 + \frac{5}{729}z^6 + \cdots}$$

3 Problem

We take repeated derivatives and find out a_n each one by one, doing the calculation like above with the help of a calculator,

$$\text{series}(a_n) = 0, 1, 2, 2, 0, -4, -8, \dots$$

From that we get,

$$\boxed{\sin(z)e^z = z + z^2 + \frac{1}{3}z^2 - \frac{1}{30}z^5 - \frac{1}{90}z^6 + \dots}$$

4 Problem

Each derivative considered $z = 0$ for $e^z + e^{\omega z} + e^{\omega^2 z}/3$ gives

$$f^{(1)} = \frac{1 + \omega + \omega^2}{3}$$
$$f^{(2)} = \frac{1 + \omega^2 + \omega^4}{3}$$
$$f^{(3)} = \frac{1 + \omega^3 + \omega^6}{3}$$
$$f^{(4)} = \frac{1 + \omega^4 + \omega^8}{3}$$

Here $\omega = e^{\frac{2\pi i}{3}}$ so

$$\omega^{(3k)} = e^{2\pi k i} = 1$$

And noticing,

$$(1 + \omega + \omega^2)^2 = \omega^4 + 2\omega^3 + 3\omega^2 + 2\omega + 1 = 0$$

We get something interesting,

$$\omega^4 + 3\omega^2 + 1 + 2\omega^3 + 2\omega = 0$$

Breaking down the $3\omega^2$ and using $\omega^3 = 1$

$$\omega^4 + \omega^2 + 1 + 2 + 2\omega^2 + 2\omega = 0$$

We get,

$$\omega^4 + \omega^2 + 1 = 0$$

It can be shown that further terms are also 0. From here we have all the rest of the terms to be 0. Hence the maclaurin series,

$$\frac{e^z + e^{\omega z} + e^{\omega^2 z}}{3} = \frac{1}{0!} = \boxed{1}$$

5 Problem

Let's define $q = z - \pi i$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi i)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \right) q^n$$

This is quite similar to the maclaurin series of e^x, and considering $x = -p$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{p^n}{n!}$$

So what we have up there is simply,

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \right) q^n = e^{-q} = e^{\pi i - z} = \boxed{-e^{-z}}$$