Honors Multivariable Calculus : : Homework 03

January 25, 2024

Ahmed Saad Sabit, Rice University

1 Problem

(a) By definition a ball is

$$B_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{a}| < r\}$$

Here $r > 0$. Let there be a point $\vec{y}$ such that $\vec{y} \in B_r(\vec{a})$. This means $|\vec{y} - \vec{a}| < r$. We can consider a ball around $\vec{y}$,

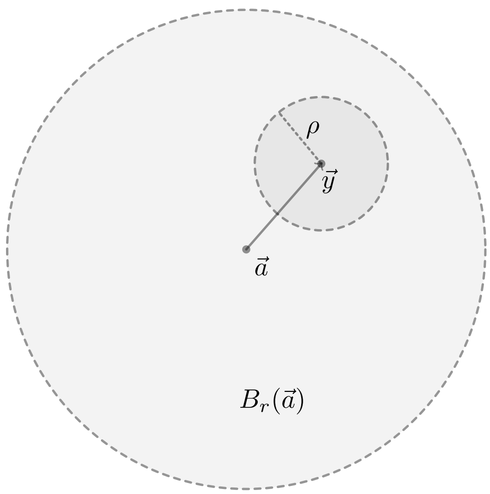

Figure 1: Proof of a ball being open set

of radius. The ball around $\vec{y}$,

$$B_\rho(\vec{y}) = \{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{y}| < \rho\}$$

If $B_\rho(\vec{y})$ exists inside $B_r(\vec{a})$ for $\rho > 0$ then $\vec{y}$ must be an interior point. $\vec{y}$ can be in general, any point that is a member of $B_r(\vec{a})$, hence, proving all points being interior, and hence, $B_r(\vec{a})$ being an open set.

Now, we can choose ρ to be,

$$\rho = r - |\vec{y} - \vec{a}|$$

If $\vec{y} \in B_r(\vec{a})$, then $|\vec{y} - \vec{a}| < r$ for all cases, hence $\rho > 0$. Thus, there always exists ρ radius ball around a member point in the set $B_r(\vec{a})$ that is a member of the set. This hence proves all points are interior points, hence the set is open.

(b) The complement of the open set is a closed set. Consider the complement of $\overline{B_r}(\vec{a})$

$$\mathbb{R}^n \setminus \overline{B_r}(\vec{x}) = \{\vec{x} \in R^n : |\vec{x} - \vec{a}| > r\}$$

We can consider a point $\vec{y}$ outside of the $\overline{B_r}(\vec{a})$ such that the ball around it has radius ρ,

$$\rho = |\vec{y} - \vec{a}| - r$$

From conditions, $|\vec{y} - \vec{a}| > r$ for all cases if it wants to be member of the complement set. Hence, $\rho > 0$ always exists, hence a ball always exists for the complement set that does not have any member point from the $\overline{B_r}(\vec{a})$, hence the complement $\mathbb{R}^n \setminus \overline{B_r}(\vec{a})$ is always an open set. Which by definition means the $\overline{B_r}(\vec{a})$ is a closed set.

2 Problem

(a)

> Definition 1. For a set $D \in \mathbb{R}^n$, we say that a point $\vec{a} \in \mathbb{R}^n$ is a **Limit Point** of D, if, for every $r > 0$, there is some point $\vec{x} \in D$ such that $\vec{x} \neq \vec{a}$ and $|\vec{x} - \vec{a}| < r$.

Being a bit loose with tools we use, this is simply a ball, we define a ball like the last problem.

Let's pick $\vec{x} \in A$. The ball around it is a set $B_r(\vec{x})$. This $\vec{x}$ follows a few conditions revolving around its periphery r. Let $r > 0$, then

- if $B_r(\vec{x}) \in A$ it is an interior point by definition. And hence, also a limit point by ball definition.
- if $B_r(\vec{x}) \notin A$ then it's not a limit point of A. We have nothing to do with this.
- if $B_r(\vec{x})$ has some member points $\vec{y}$ such that $\vec{y} \in A$, and some member points $\vec{y}' \notin A$, for all $r > 0$, the by definition this belongs to the boundary point definition for A.

$\vec{x}$ can either be in A, or either be in $\mathbb{R}^n \setminus A$, or either in both. And we have found each cases separately, hence proving limit point of A is either in A or at it's boundary.

(b) Consider the set $\mathbb{R}^n \setminus A$, and A. The points that are not the boundary points are,

$$\vec{x} \in \mathbb{R}^n \setminus A : \vec{x} \notin A$$

$$\vec{x} \notin \mathbb{R}^n \setminus A : \vec{x} \in A$$

Consider this $\vec{x}$ to be somewhere outside ∂A. Consider a boundary point $\vec{c} \in \partial A$. The set P be such that $\vec{c} \notin P$. Let's pick ρ such that,

$$\rho = \min(|\vec{x} - \vec{c}_1|, |\vec{x} - \vec{c}_2|, \dots)$$

Now considering the ball R around $\vec{x}$,

$$B_R(\vec{x}) = \{\vec{x} \in \mathbb{R}^n : R < \rho\}$$

Given $\vec{x} \notin \partial A$, $\rho > 0$ for all case. Hence, $B_R(\vec{x})$ always exists with points, hence proving $\vec{x}$ to be limit point. $B_R(\vec{x})$ can exist for any $\vec{x} \in \mathbb{R}^n \setminus \partial A$, for this, the rest of the area being an open set, ∂A is closed.

3 Problem

Suppose that D is a subset of $\mathbb{R}^n$. Now $f, g : D \to \mathbb{R}$ is continuous for all points. We have to show that $h : D \to \mathbb{R}^2$ given $h(\vec{x}) = (f(\vec{x}), g(\vec{x}))$ is continuous for all points.

Because h is a linear map,

$$|h(f(\vec{x}), g(\vec{x})) - h(f(\vec{a}), g(\vec{a}))| = |h(f(\vec{x}) - f(\vec{a}), g(\vec{x}) - g(\vec{a}))| < \epsilon$$

Because of continuity,

$$|f(\vec{x}) - f(\vec{a})| < \epsilon_f$$
$$|g(\vec{x}) - g(\vec{a})| < \epsilon_g$$

Hence,

$$|h(f(\vec{x}) - f(\vec{a}), g(\vec{x}) - g(\vec{a}))| < |h(\epsilon_f, \epsilon_g)|$$

We can have $|h(\epsilon_f, \epsilon_g)|$ given $\vec{x} \to \vec{a}$ and it's distance norm is smaller than some δ, which it already is.

So h is continuous.

4 Problem

Assume we have a map $f : \mathbb{R}^2 \to \mathbb{R}$. Let this be injective and continuous. If B is a closed disk in $\mathbb{R}^2$, then f to B is one-to-one from B to $f(B)$. $f(B)$ should be a compact connected subset of $\mathbb{R}$, or simply, a segment. Take a point p such that $p \in B$ and $f(p)$ is not an endpoint of segment $f(B)$. Then $f(B \setminus \{p\})$ is not connected while B $\{p\}$ is still connected. Which is a contradiction. My intuitive point is two random points on the line segment probably

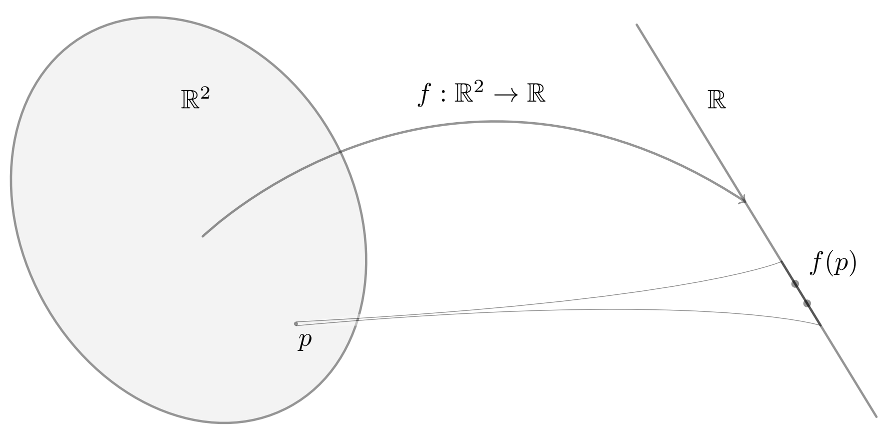

Figure 2: Diagram to illustrate the problem 4

maps to the same point p, which breaks down injective condition.

5 Problem

(a) Let the sets be $U_1, U_2, \ldots$. So, let's consider the first U_1 and U_2. So,

$$x \in U_1 \cup U_2$$

If $x \in U_1$, and U_1 is open, the there for sure exists $r > 0$ such that $B_r(x)$ is a subset of U_1 and that also happens to be a subset of $U_1 \cup U_2$. From definition we know $B_r(x)$ is an open set. Hence $U_1 \cup U_2$ is an open set. Like so we can prove $(U_1 \cup U_2) \cup U_3$ is an open set using the similar method. Hence the series of union is an open set.

(b) Consider each ball $B_r(x)_i$ in every i-th set in the intersection. x by definition is a member common in every set (because of intersection). Because it is open, consider the smallest ball $B_r(x)_{\min}$ that is amongst the sets. Hence, this should be a member of the intersection, because every other else balls are bigger. [Solution inspired from "Understanding Analysis: Abott"]

(c) Not necessarily. One counter example I know of is the open interval $(-\frac{1}{n} \dots \frac{1}{n}) \subset \mathbb{R}$. Taking intersection of all, it happens to be $\{0\}$. Which is closed.

One example I came up with while wandering at the sky is considering an infinite chain of disks with radius r_i that keep decreasing. So, for $n \geq N$, there always exists $r_n < \epsilon$.

These disks who happen to share the same center but keep decreasing to the limit of 0 radius, well, the only thing common with them is $\{0\}$ in the intersection. This is a closed set. This is one counter example so it won't work to say $\forall S$ sets.