March 7, 2024

Ahmed Saad Sabit, Rice University

Eigenvalues and Eigenvectors

We are mainly dealing with $T \in \mathcal{L}(V)$ where T is an operator.

Definition 1. Invariant Subspace $U \subset V$ subspace such that $Tu \in U$ for all $u \in U$. $T(U) \subset U$ where $T(U)$ is range$T$. Trivial examples,

$$V, \{0\}, \text{range} T, \text{null } T$$

Suppose $u \in V$ $u \neq 0$ and then of course u defines 1-dim subspace of V,

$$\lambda u : \lambda \in \mathbb{F}$$

Then T will have this as the invariant subspace $Tu = \lambda u$.

In this case we can say λ is an eigenvalue of T, and $Tu = \lambda u$ $u \neq 0$. And u is called an Eigenvector of T corresponding to the Eigenvalues.

Suppose $\dim V$ is finite.

$$T \in \mathcal{L}(V), \quad \lambda \in \mathbb{F}$$

Then these conditions are equivalent

- λ is an eigenvalue of T
- $T - \lambda I$ is not injective.
- $T - \lambda I$ is not surjective.
- $T - \lambda I$ is not bijective.

An example is $T \in \mathcal{L}(\mathbb{F}^2)$ defined by $T(x_1, x_2) = (-x_2, x_1)$. Look for eigenvalue of T.

$T(x_1, x_2)$ is $= \lambda(x_1, x_2)$. That means

$$(-x_2, x_1) = \lambda(x_1, x_2)$$

Looking at the equations,

$$-x_2 = \lambda x_1$$

$$x_1 = \lambda x_2$$

From here $x_1 = \lambda(-\lambda x_1) = -\lambda^2 x_1$. For $\mathbb{F} = \mathbb{R}$ then either $\lambda = 0$ or $x_1, x_2 = 0, 0$. So T has no eigenvalue.

But if $\mathbb{F} = \mathbb{C}$ then $x \neq 0$ if and only if $\lambda^2 = -1$ so $\lambda = \pm i$.

Linearly Independent Eigenvectors

$v_1, \ldots, v_n$ be eigenvectors of T such that their corresponding eigenvalues are distinct. Then $v_1, \ldots, v_n$ are linearly independent.

$$Tv_1 = \lambda_1 v_1$$

$$Tv_2 = \lambda_2 v_2$$

$$Tv_n = \lambda_n v_n$$

To prove this, we use contradiction.

> **Proof.** Some non-trivial linear combination of these vectors of the vector is 0. We want to write the shortest equation of m length (not n) that will give us a nice 0.
>
> $$a_1 v_1 + \ldots + a_m v_m = 0$$
>
> And $a_j \neq 0$. Operate with T,
>
> $$a_1 \lambda_1 v_1 + \ldots + a_m \lambda_m v_m = 0$$
>
> Now this is also valid, which is a separate equation,
>
> $$a_1 \lambda_m v_1 + \ldots + \ldots + a_m \lambda_m v_m = 0$$
>
> But we can subtract the two from here,
>
> $$a_1(\lambda_1 - \lambda_m) v_1 + \ldots a_m(\lambda_1 - \lambda_m) v_m = 0$$
>
> Then we have another shorter equation. □

If $T \in \mathcal{L}(V)$ then null T and range T are both invariant under the action of T.

Problem 08

$P^2 = P$ the find λ. So,

$$Pu = \lambda u$$

$$PPu = \lambda Pu \implies p^2 u = \lambda^2 u$$

So

$$(P^2 - \lambda P)u = 0$$

$$(P - \lambda P)u = 0$$

$$(1 - \lambda)Pu = 0$$

$$(1 - \lambda)\lambda u = 0$$

$$(1 - \lambda)\lambda = 0$$