Quantum Mechanics : : Homework 06

October 12, 2024

Ahmed Saad Sabit, Rice University

Problem 01

The eigenstates for a well of dimension L is given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3\ldots \quad \text{and} \quad E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}$$

where the first excited eigenstate is referring to

$$\psi_2^i(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \quad \text{and} \quad E_2 = 4\frac{\hbar^2 \pi^2}{2mL^2}$$

The eigenstates of a well of dimension $2L$ is given by

$$\psi_n(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \quad n = 1, 2, 3\ldots \quad \text{and} \quad E_n = \frac{n^2}{4} \frac{\hbar^2 \pi^2}{2mL^2}$$

I am going to do some guesses from „*The Adiabatic Theorem (11.5.2)*" of Griffiths QM (3rd Ed.). Call the post quench wave-function to be $\Psi(x)$ and the state as $|\Psi\rangle$. The Hamiltonian change happens at $t = 0$. At $t = 0+$ (infinitesimally after the change) the wavefunction hasn't really been notified of the evident expansion, hence $t = 0$ state is still $|\psi_2^i\rangle$. For this

$$|\Psi(0)\rangle = |\psi_2^i\rangle$$

The time evolution would be simply

$$|\Psi(0)\rangle = \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} |\psi_n^f\rangle\langle\psi_n^f|\psi_2^i\rangle$$

This looks atrocious so replacing $\psi_n^f = E_n$ energy eigenstates (after quench),

$$\begin{aligned}
|\Psi(0)\rangle &= \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} |E_n\rangle\langle E_n|\psi_2^i\rangle \\
&= \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} |E_n\rangle \left(\int_0^{2L} \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \mathrm{d}x\right) \\
\langle E_n|\Psi(0)\rangle &= \left(\int_0^{L} \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \mathrm{d}x\right)
\end{aligned}$$

(The initial wave function is only defined for $0 < x < L$ and $\psi(x) = 0$ outside this region)

$|\langle E_n|\Psi(0)\rangle|^2$ is the probability of finding the particle in n-th state.

(a)

So for $n = 1$ ground state after quench,

$$\langle E_1|\Psi(0)\rangle^2 = \left(\int_0^{L} \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{2L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \mathrm{d}x\right)^2 = 0.58$$

b

$n = 2$

$$\langle E_2 | \Psi(0) \rangle^2 = \left(\int_0^L \sqrt{\frac{1}{L}} \sin\left(\frac{2\pi x}{2L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \, \mathrm{d}x \right)^2 = 0$$

Problem 2

(a)

I do the computation on paper (see appendix) to get

$$\hat{J}_a |m\rangle_a = m|m\rangle_a \implies (\hat{J}_a - mI)|m\rangle_a = 0$$

$$|m=1\rangle_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \qquad (\hat{J}_x, m = 1)$$

$$|m=0\rangle_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (\hat{J}_x, m = 0)$$

$$|m=-1\rangle_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \qquad (\hat{J}_x, m = -1)$$

$$|m=1\rangle_y = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \qquad (\hat{J}_y, m = 1)$$

$$|m=0\rangle_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (\hat{J}_y, m = 0)$$

$$|m=-1\rangle_y = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \qquad (\hat{J}_y, m = -1)$$

(b)

Hand computation attached in appendix

$$\begin{aligned} {}_x\langle m=1|m=1\rangle_y &= -\frac{i}{2} \\ {}_x\langle m=1|m=0\rangle_y &= \frac{1}{\sqrt{2}} \\ {}_x\langle m=1|m=-1\rangle_y &= -\frac{i}{2} \end{aligned}$$

(c)

$$e^{i\phi}\hat{R}_z(\theta = \frac{\pi}{2})|m=-1\rangle_x = e^{i\phi}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ i\\ 1\end{bmatrix}$$
$$= e^{i\phi}\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\ 0\\ 1\end{bmatrix} = |m=-1\rangle_y \implies \phi = 0$$

Counter clockwise rotation about z-axis of $|m=-1\rangle)x$ provided that $\phi = 0$.

Problem 3

(a)

I did the first problem by hand and the second one step by step used a matrix calculator in Wolfram Alpha, what I got is

$$\hat{S}_x \implies U^T S_x U = \hbar\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}}\\ 0 & 1 & 0\end{bmatrix}$$
$$= \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

Similarly

$$\hat{S}_y \implies U^T S_y U = \hbar\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}}\\ 0 & 1 & 0\end{bmatrix}$$
$$= \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}$$

(b)

In this basis

$$\hat{S}_z = \hbar\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

here the eigenstate of $|m\rangle_z$ corresponds to an eigenvalue $m_z\hbar$. For $m = 1$

$$(\hat{S}_z - \hbar\hat{I})|m=1\rangle_z = 0 \implies \begin{bmatrix}0 & 0 & 0\\ 0 & -\hbar & 0\\ 0 & 0 & -2\hbar\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \implies |m=1\rangle_z = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

For $m = -1$

$$(\hat{S}_z + \hbar\hat{I})|m=1\rangle_z = 0 \implies \begin{bmatrix}2\hbar & 0 & 0\\ 0 & \hbar & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \implies |m=-1\rangle_z = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

Note that I computed the trivial system of equation computation by head.

Now

$$
\begin{aligned}
\hat{S}_+|m=1\rangle_z &= \hbar \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\hat{S}_+|m=1\rangle_z &= \hbar \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

This is easy to eye ball.

(c)

We require to solve the eigenvalue of $\hat{S}_x$, for $m = 1$

$$
\begin{aligned}
(\hat{S}_x - \hbar\hat{I})|m=1\rangle_x = |0\rangle &\implies \begin{bmatrix} -\hbar & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\hbar & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\hbar \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies |m=1\rangle_x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \\
(\hat{S}_x + \hbar\hat{I})|m=-1\rangle_x = |0\rangle &\implies \begin{bmatrix} \hbar & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & \hbar & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & \hbar \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies |m=-1\rangle_x = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

System of equation I got here was done on paper.

$$
\begin{aligned}
\hat{S}_+|m=1\rangle_x &\implies \hbar \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \neq |0\rangle \\
\hat{S}_-|m=-1\rangle_x &\implies \hbar \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} = \hbar \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -1 \end{bmatrix} \neq |0\rangle
\end{aligned}
$$

None is annihilated.

Problem 4

a

Using a matrix calculator

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}$$
$$\hat{S}_y^2 = \frac{\hbar^2}{2}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}$$
$$\hat{S}_y^3 = \frac{\hbar^3}{4}\begin{bmatrix}0 & -2i & 0\\ 2i & 0 & -2i\\ 0 & 2i & 0\end{bmatrix} = \hat{S}_y\frac{\hbar^3}{2}$$
$$\hat{S}_y^4 = \frac{h^4}{2}\hat{S}_y^2$$

$$\begin{aligned}\mathrm{e}^{-\frac{i}{\hbar}\hat{S}_y\theta} &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - i\frac{\theta}{\hbar}S_y - \frac{\theta^2}{2!\hbar}S_y^2 + \frac{i\theta^3}{3!\hbar^3}S_y^3 + \frac{\theta^4}{4!\hbar^4}S_y^4 + \cdots \\ &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - i\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}\frac{\theta}{\sqrt{2}} - \frac{\theta^3}{3!\sqrt{2}}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} + \frac{\theta^4}{4!2}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix} + \cdots\end{aligned}$$

$$\frac{1+\cos\theta}{2} = 1 - \frac{\theta^2}{2!2} + \frac{\theta^4}{4!2} + \cdots$$
$$\frac{1}{\sqrt{2}}\sin\theta = \text{Taylor expansion of sine}\cdot\frac{1}{\sqrt{2}}$$

Carefully inspecting the notation we can see that

$$R(\theta) = \begin{bmatrix}\frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2}\end{bmatrix}$$

$$|m_{\vec{n}} = 1\rangle = \begin{bmatrix}\frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2}\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix}$$

(b)

Requiring basic computational assist from online matrix calculator and mathematica

$$\langle S_x\rangle = \langle m_n = 1|S_x|m_n = 1\rangle = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix} = \hbar\sin\theta$$

$$(\Delta S_x)^2 = \langle m_n = 1|S_x^2 - \langle S_x\rangle^2|m_n = 1\rangle = \frac{\hbar^2}{2}\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix} - \langle m_n = 1|\hbar^2\sin^2\theta|m_n = 1\rangle$$

$$= \frac{\hbar^2}{2}\cos^2\theta$$

$$\langle S_y\rangle = \langle m_n = 1|S_y|m_n = 1\rangle = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix} = 0$$

$$(\Delta S_y)^2 = \langle m_n = 1|S_y^2 - \langle S_y\rangle^2|m_n = 1\rangle = \frac{\hbar^2}{2}\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix}$$

$$= \frac{\hbar^2}{2}$$

$$\langle S_z\rangle = \langle m_n = 1|S_z|m_n = 1\rangle = \hbar^2\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix} = \hbar^2\cos^2\theta$$

$$(\Delta S_z)^2 = \langle m_n = 1|S_z^2 - \langle S_z\rangle^2|m_n = 1\rangle = \hbar^2\begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{bmatrix} - \hbar^2\cos^2\theta$$

$$= \frac{\hbar^2\sin^2\theta}{2}$$

$$\boxed{(\Delta S_x) = \frac{\hbar}{\sqrt{2}}\cos\theta}$$

$$\boxed{(\Delta S_y) = \frac{\hbar}{\sqrt{2}}}$$

$$\boxed{(\Delta S_z) = \frac{\hbar}{\sqrt{2}}\sin\theta}$$

Problem 5

(a)

$$
\begin{aligned}
\hat{T}(R)\Psi(x) &= \exp\left(-\frac{i}{\hbar}R\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\Psi(x) = \exp\left(-R\frac{\partial}{\partial x}\right)\Psi(x) \\
&= \Psi(x) - R\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x) + R^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(x) + \cdots \\
&= \Psi(x-R) && \text{(Similar to expansion of } f(x-a))
\end{aligned}
$$

$$
\begin{aligned}
\langle x|\hat{T}(R)|\Psi\rangle &= \int_{-\infty}^{\infty}\mathrm{d}x'\,\langle x|\hat{T}(R)|x'\rangle\langle x'|\Psi\rangle \\
&= \int_{-\infty}^{\infty}\mathrm{d}x'\langle x|x'\rangle\hat{T}(R)\Psi(x') \\
&= \int_{-\infty}^{\infty}\mathrm{d}x'\,\langle x|x'\rangle\Psi(x'-R) \\
&= \Psi(x-R)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x'|\hat{T}(R)|x\rangle &= \sum_{n=0}^{\infty}\frac{(-R)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\langle x'|x\rangle \\
&= \sum_{n=0}^{\infty}\frac{(-R)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x'-x) \\
&= \delta(x'-(x+R)) \\
&= \langle x'|x+R\rangle \\
&\implies \hat{T}(R)|x\rangle = |x+R\rangle
\end{aligned}
$$

(c)

$$
[\hat{X},\hat{P}] = i\hbar\hat{I}
$$
$$
[\hat{A},\hat{I}] = 0
$$

$$
\begin{aligned}
T^T XT &= \exp\left(\frac{i}{\hbar}R\hat{P}\right)\hat{X}\exp\left(-\frac{i}{\hbar}R\hat{P}\right) \\
&= \hat{X} + \frac{1}{1!}\left[(i/\hbar)R\hat{P},\hat{X}\right] + \frac{1}{2!}\left[(i/\hbar)R\hat{P},[i/\hbar R\hat{P},\hat{X}]\right] + \cdots \\
&= \hat{X} + \frac{1}{1!}\left[(i/\hbar)(-i\hbar)R\hat{I}\right] + \frac{1}{2!}\left[(i/\hbar)R\hat{P},I\right] + \cdots \\
&= \hat{X} + R\hat{I} + 0 + \cdots && \text{(every other terms has a } [\hat{P},[\hat{P},\hat{X}]] \text{ term)} \\
&= \hat{X} + R\hat{I}
\end{aligned}
$$

This consistent because *eq*.18 is translation is position eigenstate while this result we found here is translation in position operator.