Honors Linear Algebra : : Homework 07

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Problem 01

Let $v \in V$ and $v = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$ so the matrix of v is

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Similarly consider $u \in V$ and $u = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$, it's matrix is

$$\mathcal{M}(u) = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

Additivity: From vector summation we directly know,

$$v + u = (b_1 + d_1)\vec{v}_1 + \cdots + (b_n + d_n)\vec{v}_n$$

The matrix representation of $v + u$ is thus,

$$\mathcal{M}(v + u) = \begin{pmatrix} b_1 + d_1 \\ b_2 + d_2 \\ \vdots \\ b_n + d_n \end{pmatrix}$$

Now let's consider matrix addition of $\mathcal{M}(u) + \mathcal{M}(v)$,

$$\mathcal{M}(v) + \mathcal{M}(u) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} b_1 + d_1 \\ b_2 + d_2 \\ \vdots \\ b_n + d_n \end{pmatrix}$$

So apparently,

$$\mathcal{M}(u + v) = \mathcal{M}(u) + \mathcal{M}(v)$$

Multiplicity: Given $\alpha \in \mathbb{F}$ and for the vector

$$v = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$$

$$\alpha v = \alpha b_1\vec{v}_1 + \cdots + \alpha b_n\vec{v}_n$$

The matrix representation is going to be

$$\mathcal{M}(\alpha v) = \begin{pmatrix} \alpha b_1 \\ \alpha b_2 \\ \vdots \\ \alpha b_n \end{pmatrix}$$

Now let's consider the following,

$$\alpha\mathcal{M}(v) = \alpha \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

From scalar to matrix multiplication we can see,

$$\alpha\mathcal{M}(v) = \begin{pmatrix} \alpha b_1 \\ \alpha b_2 \\ \vdots \\ \alpha b_n \end{pmatrix}$$

So turns out

$$\mathcal{M}(\alpha v) = \alpha\mathcal{M}(v)$$

From $\alpha = 0$ case we can show 0 gets mapped to 0 trivially. So the Linearity is Proven.

Problem 02

Consider the i-th vector u_i such that

$$u_i = \lambda v_i$$

Here v_i is a basis vector of V. i ranges from $0, 1, \ldots, n$. Now, for $u_1, \ldots, u_n$ to be a basis vector we need,

$$c_1 u_1 + \ldots c_n u_n = 0$$

If and only if $c_i = 0$ for all i. But as we had defined u_i

$$c_1 (\lambda v_1) + \ldots + (\lambda v_n) = 0$$

$$\lambda (c_1 v_1 + \ldots + c_n v_n) = 0$$

Because $\lambda \neq 0$ the only way this system is zero is if $c_i = 0$, as v_i each are linearly independent basis of V. So the only possible way for this system of equation to hold is for $c_i = 0$, hence,

$$c_1 u_1 + \ldots c_n u_n = 0$$

is linearly independent. Which means $\lambda v_1, \ldots, \lambda v_n$ is a basis.

Problem 03

The given matrix

$$\mathcal{M}(I_V, (\lambda\vec{v}_1, \ldots, \lambda\vec{v}_n), (\vec{v}_1, \ldots, \vec{v}_n))$$

is a matrix of a linear map I_V from the basis $\lambda\vec{v}_1, \ldots, \lambda\vec{v}_n$ to the basis $\vec{v}_1, \ldots, \vec{v}_n$. For now let's call $\{\lambda\vec{v}_i\}$ as $\{\vec{u}_i\}$ for all i,

$$\mathcal{M}(I_V, (\vec{u}_1, \ldots, \vec{u}_n), (\vec{v}_1, \ldots, \vec{v}_n))$$

So a vector $\vec{t} = t_1\vec{u}_1 + \ldots + t_n\vec{u}_n$ transforms into $\vec{t} = t_1\lambda\vec{v}_1 + \ldots + t_n\lambda\vec{v}_n$ where the basis is $\vec{v}_i$. In matrix notation,

$$\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \to \begin{pmatrix} \lambda t_1 \\ \vdots \\ \lambda t_n \end{pmatrix}$$

Consider the matrix,

$$\lambda \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

This validly gives us the transformation,

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} \lambda t_1 \\ \lambda t_2 \\ \vdots \\ \lambda t_n \end{pmatrix}$$

Hence,

$$\mathcal{M}(I_V, (\lambda\vec{v}_1, \ldots, \lambda\vec{v}_n), (\vec{v}_1, \ldots, \vec{v}_n)) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

To solve the opposite direction as stated in the problem, from 3.82 in LADR we know that

$$\mathcal{M}(I_V, (\vec{v}_1, \ldots, \vec{v}_n), (\lambda\vec{v}_1, \ldots, \lambda\vec{v}_n)) = \text{Inverse of } \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

As this is a diagonal matrix, our life is easier,

$$\mathcal{M}(I_V, (\vec{v}_1, \ldots, \vec{v}_n), (\lambda\vec{v}_1, \ldots, \lambda\vec{v}_n)) = \begin{pmatrix} \frac{1}{\lambda} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda} \end{pmatrix}$$

This can be easily justifiable because if we start with λt_i components for i-th basis, we get t_i which is an inverse transform. For example, consider a vector $\vec{t} = t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$, after transformation to new basis, $\vec{t} = (t_1/\lambda)\lambda\vec{v}_1 + \cdots + (t_n/\lambda)\vec{v}_n$, so the transformation is,

$$\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \to \begin{pmatrix} \frac{t_1}{\lambda} \\ \vdots \\ \frac{t_n}{\lambda} \end{pmatrix}$$

From the matrix multiplication, it's apparent that,

$$\begin{pmatrix} \frac{1}{\lambda} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} \frac{t_1}{\lambda} \\ \vdots \\ \frac{t_n}{\lambda} \end{pmatrix}$$

Problem 04

Aid for my small brain Just to get a sense of the problem let's try the $m = 3$ case. Then the basis are

$$w_1, w_2, w_3$$

The new basis are

$$w_1', w_2', w_3'$$

The relation of one to the other basis

$$\begin{aligned} w_1' &= T_{11}w_1 + T_{21}w_2 + T_{31}w_3 \\ w_2' &= T_{12}w_1 + T_{22}w_2 + T_{32}w_3 \\ w_3' &= T_{13}w_1 + T_{23}w_2 + T_{33}w_3 \end{aligned}$$

This numbering looks a little bit weird to me. Decomposing above into a matrix form (unnecessary)

$$\begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix}$$

$$\begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} \left(w_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left(w_1' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + w_2' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + w_3' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

The given shift to w_k' is

$$w_k' = B_{1k}w_1 + B_{2k}w_2 + \cdots + B_{mk}w_m$$

This w_k' can be thought of as $T(w_k') = w_k'$ which maps to itself but changes the basis.

$$T(w_k') = B_{1k}w_1 + B_{2k}w_2 + \cdots + B_{mk}w_m = w_k'$$

From the given information we can build a matrix

$$B = \begin{pmatrix} B_{11} & \cdots & B_{m1} \\ \vdots & \ddots & \vdots \\ B_{1m} & \cdots & B_{mm} \end{pmatrix}$$

Here $k = 1, \ldots, m$. The map T is inverse in W and B is the change of basis linear map from $w_1', \ldots, w_m'$ to $w_1, \ldots, w_m$. We can say,

$$B = \mathcal{M}\left(I_W, (w_1', \ldots, w_m'), (w_1, \ldots, w_m)\right)$$

By 3.82 LADR, B is invertible with inverse B^{-1} where

$$B^{-1} = \mathcal{M}\left(I_W, (w_1, \ldots, w_m), (w_1', \ldots, w_m')\right)$$

Problem 05

From V basis $v_1, \ldots, v_n$ we can consider another basis of V that such

$$v_k' = A_{1k}v_1 + \cdots + A_{nk}v_n$$

For $A_{jk} \in \mathbb{F}$ and $j, k = 1, \ldots, n$. This can be assembled into a matrix $A \in \mathbb{F}^{n,n}$. If $T \in \mathcal{L}(V, W)$, we shall show that,

$$\mathcal{M}(T, (v_1', \ldots, v_n'), (w_1', \ldots, w_m')) = B^{-1}\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))A$$

We had shown that

$$B = \mathcal{M}(I_W, (w'_1, \ldots, w'_m), (w_1, \ldots, w_m))$$

And in similar way we can say that

$$A = \mathcal{M}\left(I_V, (v'_1, \ldots, v'_n), (v_1, \ldots, v_n)\right)$$

By using 3.81 LADR we have

$$\begin{aligned} B^{-1}\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))A = \\ = \left(\mathcal{M}\left(I_W, (w_1, \ldots, w_m), (w'_1, \ldots, w'_m)\right)\right) * \left(\mathcal{M}\left(I_V, (v_1, \ldots, v_n), (w_1, \ldots, w_m)\right) A\right) \\ = \left[\mathcal{M}(I_W T, (v_1, \ldots, v_n), (w'_1, \ldots, w'_m)) * \mathcal{M}\left(I_V, (v'_1, \ldots, v'_n), (v_1, \ldots, v_n)\right)\right] \\ = \mathcal{M}\left(I_W T I_V, (v'_1, \ldots, v'_n), (w'_1, \ldots, w'_m)\right) \end{aligned}$$

By the definitions of I_V and I_W, we have

$$T = I_W T$$

$$T = T I_V$$

Thus,

$$T = I_W T I_V$$

This follow that,

$$\mathcal{M}\left(I_W T I_V, (v'_1, \ldots, v'_n), (w'_1, \ldots, w'_m)\right) = \mathcal{M}\left(T, (v'_1, \ldots, v'_n), (w'_1, \ldots, w'_m)\right)$$

Hence forth as desired we get,

$$\mathcal{M}\left(T, (v'_1, \ldots, v'_n), (w'_1, \ldots, w'_m)\right) = B^{-1}\mathcal{M}\left(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)\right) A$$

Problem 06

Consider $A = \mathcal{M}(T)$. Then if $A \in \mathbb{F}^{m,n}$ then A is a m-by-n matrix. The linear map $T \in \mathcal{L}(V, W)$ and basis of V is $v_1, \ldots, v_n$ (notice the n) and W is $w_1, \ldots, w_m$ (notice the m). This also means

$$\dim V = n \quad \dim W = m$$

A being invertible means that T transformation also has an inverse. But T is only inverse if T is **injective** and **surjective**.

From 3.22 LADR we know that *linear map to lower dimensional space is not injective* so a condition on V, W is,

$$\dim V \geq \dim W \implies n \geq m$$

From 3.24 LADR we know that *linear map to higher dimensional space is not surjective* so another condition is,

$$\dim V \leq \dim W \implies n \leq m$$

The only way both of the condition is true is if $n = m$. Hence $m = n$ proven for $A \in \mathbb{F}^{m,n}$

Problem 07

Let's consider the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If this 2-by-2 matrix is invertible, then there exists another matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let's compute the right hand side, then we get,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

Look at the $2, 2$ entry of the matrix multiplication, and for the matrix to be invertible we need,

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But this is impossible because the $2, 2$ entry $0 \neq 1$. So the considered matrix is not invertible.

Problem 08

($\Longrightarrow$) ST is invertible.

Let R be the inverse. This satisfies $R(ST) = I$. Now suppose $v \in \text{null } T$. This means $Tv = 0$. Then

$$\begin{aligned} v &= Iv \\ &= R(ST)v \\ &= RS(0) \\ &= R(0) \\ &= 0 \end{aligned}$$

Thus $v \subset \{0\}$. This is the only possible way for T to have a null. Hence T is injective. We also know from 3.65 LADR that T is injective hence means T is also invertible.

Now let's show S is invertible. Let there be another vector u such that $u \in \text{null } S$. Because we know T is invertible, define $u^* = T^{-1}(u)$. Then,

$$\begin{aligned} u^* &= Iu^* \\ &= R(ST)u^* \\ &= RS(u) \\ &= R(0) \\ &= 0 \end{aligned}$$

This says $u^* = 0$, and hence $u^* = T^{-1}(0) = 0$. We proved $u = 0$, so S is injective hence also invertible.

($\Longleftarrow$) S and T are invertible.

S^{-1} and T^{-1} exist. Then let's try the following,

$$\begin{aligned} (T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}IT \\ &= T^{-1}T \\ &= I \end{aligned}$$

And,

$$
\begin{aligned}
(ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\
&= SIS^{-1} \\
&= SS^{-1} \\
&= I
\end{aligned}
$$

Problem 09

The system of equation can be easily written in terms of matrix multiplication with vector,

$$
\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}
$$

Condensed form,

$$\mathcal{A}\vec{x} = \vec{c}$$

Matrix $\mathcal{A}$ is $\mathcal{A} \in \mathbb{F}^{n,n}$. It can be thought of a linear map $T : V \to W$ where $\dim V = \dim W = n$.

(a) $\implies$ (b)

(a) mentions that $\vec{x} = \vec{0}$ is the only possible solution for $\vec{c} = \vec{0}$. This means from the fundamental theorem of Linear Algebra,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$n = \dim \text{null } T + n \implies \dim \text{null } T = 0$$

Where $\dim(\text{range } T) = \dim W = n$. More importantly $\dim \text{null } T = 0$ means that T is *injective*. From 3.65 LADR we know that Injectivity is same as Surjectivity for finite dimension case and hence invertibility.

$\mathcal{A}^{-1}$ exists as per given conditions. So,

$$\mathcal{A}\vec{x} = \vec{c}$$

Here we can do the following,

$$\mathcal{A}^{-1}(\mathcal{A}\vec{x}) = \mathcal{A}^{-1}\vec{c}$$

$$I\vec{x} = \mathcal{A}^{-1}\vec{c}$$

Which means,

$$\vec{x} = \mathcal{A}^{-1}\vec{c}$$

From injectivity we know that $\vec{x}$ is unique and we are also guarenteed $\vec{x}$ exists. $\vec{x} = (x_1, x_2, \ldots, x_n)$ is the solution to this system.

(b) $\implies$ (a)

For every $\vec{c} \in W$ we have a solution $\vec{x} \in V$. If we consider a linear map $T \in \mathcal{L}(V, W)$ such that

$$T(\vec{x}) = \vec{c}$$

Every $\vec{c} \in W$ has a solution, which means that range $T = W$. This is the definition of surjectivity. From 3.65 LADR we know that Surjectivity implies Injectivity and hence Invertibility.

This map being injective implies that

$$T(\vec{x}) = 0 \implies \vec{x} = 0$$

So $T(\vec{x}) = \vec{c}$ where $\vec{c} = 0$ means $\vec{x} = 0$ and that is the only solution.

Problem 10

Suppose one vector space V_r in $\Pi = V_1 \times V_2 \times \cdots \times V_m$ is infinite-dimensional where Π itself is finite dimensional. From definition of products,

$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\}$$

Solution 01 using a member vector

Let's consider a member element $\mathbf{q} \in \Pi$. It can be written in the form,

$$\mathbf{q} = (\vec{q_1}, \ldots, \vec{q_m})$$

Where $\vec{q}_k \in V_k$. We had defined V_r to be the infinite dimensional vector space. Hence,

$$\vec{q_r} = (f_1, f_2, \ldots, f_\infty)$$

Where $f_i \in \mathbb{F}$. $\vec{q_r}$ requires infinite number of basis vectors because,

$$\vec{q_r} = f_1(1, 0, \ldots) + f_2(0, 1, \ldots) + \cdots$$

So if $\mathbf{q} = (\vec{q}_1, \ldots, \vec{q}_r, \ldots, \vec{q}_m)$ has to span all of Π it needs to go through all multiples of all possible basis vectors of $\vec{q}_r$. But $\vec{q}_r$ having infinite basis yields $\mathbf{q}$ to have infinite dimension too.

Solution 02 using a formulation of dimension

What does dimension mean for Product Space? Example.
Consider a simple $\pi = V_1 \times V_2 \times V_3$, then a member of this π is

$$\mathbf{d} = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$$

if each vector spaces V_i are two dimensional,

$$\mathbf{d} = \left(a^x \begin{pmatrix}1\\0\end{pmatrix} + a^y \begin{pmatrix}0\\1\end{pmatrix}, b^x \begin{pmatrix}1\\0\end{pmatrix} + b^y \begin{pmatrix}0\\1\end{pmatrix}, c^x \begin{pmatrix}1\\0\end{pmatrix} + c^y \begin{pmatrix}0\\1\end{pmatrix} \right)$$

Dimension (specifically Hamel Dimension from Wikipedia), the dimension of vector space V is the number of basis of V over it's base field. For π we have total of 6 basis. The set of basis for this system is,

$$\begin{gathered}
\{(1,0),(0,0),(0,0)\},\\
\{(0,1),(0,0),(0,0)\},\\
\{(0,0),(1,0),(0,0)\},\\
\{(0,0),(0,1),(0,0)\},\\
\{(0,0),(0,0),(1,0)\},\\
\{(0,0),(0,0),(0,1)\}
\end{gathered}$$

The representation of $\mathbf{d}$ is,

$$\begin{aligned}
\mathbf{d} =& a^x[(1,0),(0,0),(0,0)]\\
&+a^y[(0,1),(0,0),(0,0)]\\
&+b^x[(0,0),(1,0),(0,0)]\\
&+b^y[(0,0),(0,1),(0,0)]\\
&+c^x[(0,0),(0,0),(1,0)]\\
&+c^y[(0,0),(0,0),(0,1)]
\end{aligned}$$

Obviously the terms $t^z \in \mathbb{F}$ where $t = a, b, c$ and $z = x, y$

Suppose one vector space V_r in $\Pi = V_1 \times V_2 \times \cdots \times V_m$ is infinite-dimensional where Π itself is finite dimensional. From definition of products,

$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\}$$

Basis of Π is,

$$\begin{aligned}
\text{Basis set } \Sigma = & \bigcup_{j=1}^{\dim V_1} \{(\vec{v}_j, 0, \ldots, 0) : v_j \in \text{basis of } V_1\}\\
\cup & \bigcup_{j=1}^{\dim V_2} \{(0, \vec{v}_j, \ldots, 0) : v_j \in \text{basis of } V_2\}\\
\cdots \cup & \bigcup_{j=1}^{\dim V_m} \{(0, 0, \ldots, \vec{v}_j) : v_j \in \text{basis of } V_m\}
\end{aligned}$$

We can count the number of elements we have in the mentioned set above to get the dimension. Turns out for all V_j being finite, we simply have $\dim V_1 + \ldots + \dim V_m$. But as we are considering the infinite dimensional V_r the basis is,

$$\begin{aligned}
\Sigma = &\bigcup_{j=1}^{\dim V_1} \{(\vec{v}_j, 0, \ldots, 0) : v_j \in \text{basis of } V_1\} \\
&\cup \bigcup_{j=1}^{\dim V_2} \{(0, \vec{v}_j, \ldots, 0) : v_j \in \text{basis of } V_2\} \\
&\cdots \cup \bigcup_{j=1}^{\infty} \{(0, 0, \ldots, \vec{v}_j, \ldots, 0) : v_j \in \text{basis of } V_r\} \cup \ldots \\
&\ldots \cup \bigcup_{j=1}^{\dim V_m} \{(0, 0, \ldots, \vec{v}_j) : v_j \in \text{basis of } V_m\}
\end{aligned}$$

Counting this sets gives us $\dim V_1 + \dim V_2 + \ldots + \infty + \ldots + \dim V_m$. So dimension of the product space Π is ∞. This is a contradiction to the definition of what we started with, hence there can be no V_r vector space that is infinite dimensional in Π. So the member vector spaces are all finite.