Quantum Mechanics : : Homework 04

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Problem 01

a

I do the matrix multiplication by hand.

$$
\begin{aligned}
\hat{\sigma}_1\hat{\sigma}_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\hat{\sigma}_3 \\
\hat{\sigma}_2\hat{\sigma}_1 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\hat{\sigma}_3 \\
\hat{\sigma}_2\hat{\sigma}_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\hat{\sigma}_1 \\
\hat{\sigma}_3\hat{\sigma}_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\hat{\sigma}_1 \\
\hat{\sigma}_3\hat{\sigma}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\hat{\sigma}_2 \\
\hat{\sigma}_1\hat{\sigma}_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\hat{\sigma}_2
\end{aligned}
$$

The cross product

$$
\begin{aligned}
\hat{\vec{\sigma}} \times \hat{\vec{\sigma}} &= \begin{vmatrix} \vec{n}_x & \vec{n}_y & \vec{n}_z \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix} \\
&= \vec{n}_x\left(\sigma_2\sigma_3 - \sigma_3\sigma_2\right) + \vec{n}_y\left(\sigma_3\sigma_1 - \sigma_1\sigma_3\right) + \vec{n}_z\left(\sigma_1\sigma_2 - \sigma_2\sigma_1\right) \\
&= \vec{n}_x(2i\sigma_1) + \vec{n}_y\left(2i\sigma_2\right) + \vec{n}_z\left(2i\sigma_3\right) \\
&= 2i\left(\sigma_1\vec{n}_x + \sigma_2\vec{n}_y + \sigma_3\vec{n}_z\right) \\
&= 2i\hat{\vec{\sigma}}
\end{aligned}
$$

Note this also is alluring to the Levi-Civita Symbol because

$$
\text{cyclic } i \to j \to k \to i \to j \to k \implies \quad \sigma_i\sigma_j = i\epsilon_{ij}\sigma_k \qquad \text{(where } \{i, j, k\} \in \{1, 2, 3\}\text{)}
$$

b

$$
\left(\vec{U} \cdot \hat{\vec{\sigma}}\right)\left(\vec{V} \cdot \hat{\vec{\sigma}}\right) = \left(U_1\sigma_1 + U_2\sigma_2 + U_3\sigma_3\right)\left(V_1\sigma_1 + V_2\sigma_2 + V_3\sigma_3\right)
$$

$$
\begin{aligned}
\left(\vec{U}\cdot\hat{\vec{\sigma}}\right)\left(\vec{V}\cdot\hat{\vec{\sigma}}\right) &= \left(\sum_{i=1}^{3} U_i\sigma_i\right)\left(\sum_{j=1}^{3} V_j\sigma_j\right) \\
&= \sum_{i,j=1}^{3} U_iV_j\sigma_i\sigma_j \\
&= \sum_{i,j=1,i\neq j}^{3} U_iV_j\sigma_i\sigma_j + \sum_{n=1}^{3} U_nV_n\sigma_n\sigma_n \\
&= \sum_{i,j=1,i\neq j}^{3} iU_iV_j\epsilon_{ij}\sigma_k + \sum_{n=1}^{3} U_nV_n\sigma_n\sigma_n \\
&= [i(U_1V_2-U_2V_1)\sigma_3 + i(U_2U_3-U_3U_2)\sigma_1 + i(U_3U_1-U_1U_3)\sigma_2] + (U_1V_1+U_2V_2+U_3V_3)\hat{I} \\
&= [\vec{U}\times\vec{V}]_3 i\sigma_3 + [\vec{U}\times\vec{V}]_1 i\sigma_1 + [\vec{U}\times\vec{V}]_2 i\sigma_2 + (\vec{U}\cdot\vec{V})\hat{I} \\
&= i\left[\vec{U}\times\vec{V}\right]\cdot\hat{\vec{\sigma}} + \left(\vec{U}\cdot\vec{V}\right)\hat{I}
\end{aligned}
$$

Problem 02

a

Considering the simplest basis in column vector forms

$$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

We want to compute the following operator $\hat{H}$

$$\hat{H} = E_0\left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|\right) - J\left(|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 1| + |1\rangle\langle 3|\right)$$

Matrices for the first term

$$\begin{aligned}
|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| &= \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix} \\
&= \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} \\
&= \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \hat{I}
\end{aligned}$$

Matrices for the second term

$$\begin{aligned}
&|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 1| + |1\rangle\langle 3| \\
&= \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} \\
&= \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}
\end{aligned}$$

Put them all together

$$\begin{aligned}
\hat{H} &= E_0\left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|\right) - J\left(|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 1| + |1\rangle\langle 3|\right) \\
&= \begin{bmatrix}E_0 & -J & -J\\-J & E_0 & -J\\-J & -J & E_0\end{bmatrix}
\end{aligned}$$

b (i)

The $|E_1\rangle$ in column vector representation

$$|E_1\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right) = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Computing $\hat{H}|E_1\rangle$

$$
\begin{aligned}
\hat{H}|E_1\rangle &= \begin{bmatrix} E_0 & -J & -J \\ -J & E_0 & -J \\ -J & -J & E_0 \end{bmatrix} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} E_0 - J - J \\ -J + E_0 - J \\ -J - J + E_0 \end{bmatrix} \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} E_0 - 2J \\ E_0 - 2J \\ E_0 - 2J \end{bmatrix} \\
&= \frac{E_0 - 2J}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= (E_0 - 2J)\,|E_1\rangle
\end{aligned}
$$

We see the eigen-equation for the Hamiltonian

$$
\hat{H}|E_1\rangle = E_1|E_1\rangle \implies E_1 = E_0 - 2J
$$

The special property can be realized by seeing that,

$$
\begin{bmatrix} E_0 & -J & -J \\ -J & E_0 & -J \\ -J & -J & E_0 \end{bmatrix} = (E_0 + J) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (-J) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
$$

It's very obvious that $|E_1\rangle$ is an eigenvector for both of the upper matrices. And if $\hat{H} = \hat{A} + \hat{B}$ then if $|E_1\rangle$ is eigenvector for both $\hat{A}, \hat{B}$, then

$$
\hat{H}|E_1\rangle = \hat{A}|E_1\rangle + \hat{B}|E_1\rangle = (a+b)|E_1\rangle = E_1|E_1\rangle
$$

Just from looking at the matrices it's obvious that $a = E_0 + J$ and $b = -3J$ so $E_1 = E_0 - 2J$

b (ii) and (iii)

We algebraically initialize the elements of the matrix to minimize the computational effort by hand. Keep in mind that $\hat{H} - E\hat{I} = \hat{0} = \frac{1}{J}\hat{H} - \frac{1}{J}E\hat{I}$ hence

$$
\begin{aligned}
\frac{1}{J}\hat{H} &= \begin{bmatrix} \frac{E_0}{J} & -1 & -1 \\ -1 & \frac{E_0}{J} & -1 \\ -1 & -1 & \frac{E_0}{J} \end{bmatrix} \\
\left(\frac{1}{J}\hat{H}\right) - \frac{E}{J}\hat{I} &= \begin{bmatrix} \frac{E_0 - E}{J} & -1 & -1 \\ -1 & \frac{E_0 - E}{J} & -1 \\ -1 & -1 & \frac{E_0 - E}{J} \end{bmatrix} \\
&= (-1) \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} = 0 \qquad (\lambda = (E - E_0)/J) \\
&\implies \det \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} = 0 \\
&\implies \lambda^3 + 3\lambda + 2 = 0 \\
&\implies (\lambda - 1)^2(\lambda + 2) = 0 \\
&\implies (\lambda^2 - 2\lambda + 1)(\lambda + 2) \\
&\implies \lambda = 1, 1, -2
\end{aligned}
$$

As stated in the problem statement, $0 = (\lambda - \lambda_1)(\lambda^2 + b\lambda + c) \implies 0 = (\lambda - (-2))(\lambda^2 + (-2)\lambda + 1)$ giving us $(b, c) = (-2, 1)$

$$\text{For } \lambda = -2 \text{ the eigen-energy is } \frac{E - E_0}{J} = -2 \implies E = E_0 - 2J$$

$$\text{For } \lambda = 1 \text{ the eigen-energy is } \frac{E - E_0}{J} = 1 \implies E = E_0 + J$$

The eigenvalues are

$$\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = 1$$

And corresponding eigen-energy

$$E_1 = E_0 - 2J, \quad E_2 = E_0 + J, \quad E_3 = E_0 + J$$

Problem 03

a

$$\hat{T}|n\rangle = |n+1\rangle \implies \langle n|T^\dagger = \langle n+1|$$
$$\text{now, } \langle n|T^\dagger|n+1\rangle = \langle n+1||n+1\rangle = 1$$
$$\text{or, } \langle n|T^\dagger T|n\rangle = 1$$
$$\text{and, } \langle n|T^\dagger|k+1\rangle = \langle n+1||k+1\rangle = 0 \qquad (n \neq k)$$
$$\implies T^\dagger T = \hat{I}$$

b

To compute $\left[\hat{H}, \hat{T}\right]$ we compute the two operator terms individually

$$\hat{H}\hat{T} = \left(\sum_{n=-\infty}^{\infty} \left[E_0|n\rangle\langle n| + J\left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right)\right]\right)\hat{T}$$

$$\hat{H}\hat{T}|n\rangle = \left(\sum_{n=-\infty}^{\infty} \left[E_0|n\rangle\langle n| + J\left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right)\right]\right)|n+1\rangle$$
$$= \Big[E_0|n\rangle\langle n| + J\Big(|n+1\rangle\langle n| + |n\rangle\langle n+1|\Big)\Big]|n+1\rangle + \Big[E_0|n+1\rangle\langle n+1| + J\Big(|n+2\rangle\langle n+1| + |n+1\rangle\langle n+2|\Big)\Big]|n+1\rangle$$
$$= J|n\rangle + E_0|n+1\rangle + J\left(|n+2\rangle\right)$$

$$\hat{T}\hat{H}|n\rangle = \hat{T}\Bigg(\Big[E_0|n\rangle\langle n| + J\Big(|n+1\rangle\langle n| + |n\rangle\langle n+1|\Big)\Big]|n\rangle + \Big[E_0|n-1\rangle\langle n-1| + J\Big(|n\rangle\langle n-1| + |n-1\rangle\langle n|\Big)\Big]|n\rangle\Bigg)$$
$$= \hat{T}\big(E_0|n\rangle + J|n+1\rangle + J|n-1\rangle\big)$$
$$= E_0|n+1\rangle + J|n+2\rangle + J|n\rangle = \hat{H}\hat{T}|n\rangle$$

$$\therefore \left(\hat{H}\hat{T} - \hat{T}\hat{H}\right)|n\rangle = |0\rangle \implies \left[\hat{H}, \hat{T}\right] = 0$$

c

Using the form given for the energy, and the eigen-equation for $\hat{T}$, we determine the general formula for $\psi_{E,n}$ in terms of $\psi_{E,0}$

$$\begin{aligned}
|E\rangle &= \sum_{n=-\infty}^{\infty} |n\rangle \psi_{E,n} \\
T|E\rangle &= \sum_{n=-\infty}^{\infty} T|n\rangle \psi_{E,n} \\
e^{-i\phi}|E\rangle &= \sum_{n=-\infty}^{\infty} |n+1\rangle \psi_{E,n} \\
\sum_{n=-\infty}^{\infty} e^{-i\phi}|n\rangle \psi_{E,n} &= \sum_{n=-\infty}^{\infty} |n+1\rangle \psi_{E,n} \\
\implies e^{-i\phi}|n+1\rangle \psi_{E,n+1} &= |n+1\rangle \psi_{E,n} \\
\therefore \; e^{-i\phi} &= \frac{\psi_{E,n}}{\psi_{E,n+1}}
\end{aligned}$$

The inductive relation between two coefficient is

$$\psi_{E,n+1} = e^{i\phi}\psi_{E,n} \implies \psi_{E,1} = e^{i\phi}\psi_{E,0} \implies \psi_{E,2} = e^{i\phi}\psi_{E,1} = e^{2i\phi}\psi_{E,0}$$
$$\implies \psi_{E,n} = e^{in\phi}\psi_{E,0}$$

The general form of energy is then

$$|E\rangle = \psi_{E,0} \sum_{n=-\infty}^{\infty} e^{in\phi}|n\rangle$$

$$\begin{aligned}
\hat{H}|E_n\rangle &= \psi_{E,n}\hat{H}|n\rangle = \psi_{E,n}\Big(E_0|n\rangle + J|n+1\rangle + J|n-1\rangle\Big) \\
\hat{H}|E\rangle = \hat{H}\left(\sum_{n=-\infty}^{\infty} \psi_{E,n}|n\rangle\right) &= \sum_{n=-\infty}^{\infty} \psi_{E,n}\Big(E_0|n\rangle + J|n+1\rangle + J|n-1\rangle\Big) \\
\langle n'|\hat{H}|E\rangle &= \sum_{n=-\infty}^{\infty} \psi_{E,n}\Big(E_0\langle n'|n\rangle + J\langle n'|n+1\rangle + J\langle n'|n-1\rangle\Big) \\
&= E_0\psi_{E,n'} + J\psi_{E,n'-1} + J\psi_{E,n'+1} \\
&= E_0\psi_{E,n'} + J\frac{\psi_{E,n'}}{e^{i\phi}} + Je^{i\phi}\psi_{E,n'} \\
&= \psi_{E,n'}(E_0 + Je^{-i\phi} + Je^{i\phi}) \\
&= \psi_{E,n'}(E_0 + 2J\cos(\phi))
\end{aligned}$$

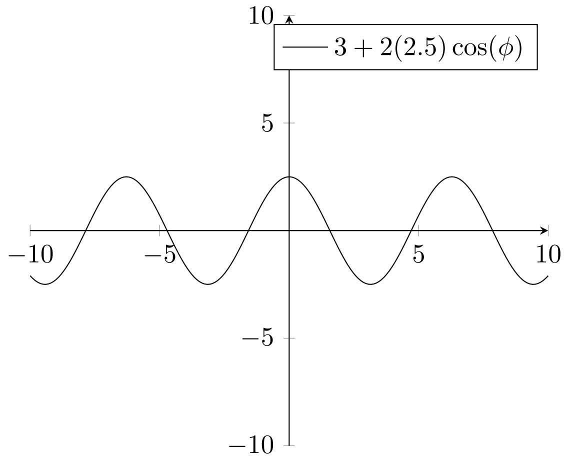

Figure 1: Simple pgfplot for aesthetic purposes $E = 3, J = 2.5$

$$
\begin{aligned}
[E_0 + 2J\cos(\phi)]_{\max} &= E_0 + 2J \\
[E_0 + 2J\cos(\phi)]_{\min} &= E_0 - 2J
\end{aligned}
$$

Problem 04

$$
\begin{aligned}
\langle 1|\hat{X}|1\rangle &= \langle 1|\hat{I}\hat{X}\hat{I}|1\rangle \\
&= \langle 1| \left(\int_0^L \mathrm{d}x\, |x\rangle\langle x|\right) \hat{X} \left(\int_0^L \mathrm{d}x'\, |x'\rangle\langle x'|\right) |1\rangle \\
&= \left(\int_0^L \mathrm{d}x\, \langle 1|x\rangle\langle x|\right) \hat{X} \left(\int_0^L \mathrm{d}x'\, |x'\rangle\langle x'|1\rangle\right) \\
&= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}x'\, \langle 1|x\rangle\langle x'|1\rangle\langle x|\hat{X}|x'\rangle \\
&= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}x'\, \langle 1|x\rangle\langle x'|1\rangle x'\delta(x-x') \\
&= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}x' \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x'\right)\right) x'\delta(x-x') \\
&= \int_0^L \mathrm{d}x' \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x'\right)\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x'\right)\right) x'\delta(x-x') \\
&= \frac{2}{L}\int_0^L \mathrm{d}x'\, x' \sin^2\left(\frac{\pi}{L}x'\right) \\
&= \frac{2}{L}\left(\frac{L}{2}\right)^2 \\
&= \frac{L}{2}
\end{aligned}
$$

$$
\begin{aligned}
\langle 1|\hat{X}|2\rangle &= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}x' \, \langle 1|x\rangle\langle x'|2\rangle\langle x|\hat{X}|x'\rangle \\
&= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}x' \left(\sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right)\right)\left(\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L}x'\right)\right) x'\delta(x-x') \\
&= \frac{2}{L}\int_0^L \mathrm{d}x' \, x' \sin\left(\frac{\pi}{L}x'\right) \sin\left(\frac{2\pi}{L}x'\right) \\
&= \frac{2}{L}\left(-\frac{8L^2}{9\pi^2}\right) \\
&= -\frac{16L}{9\pi^2}
\end{aligned}
$$

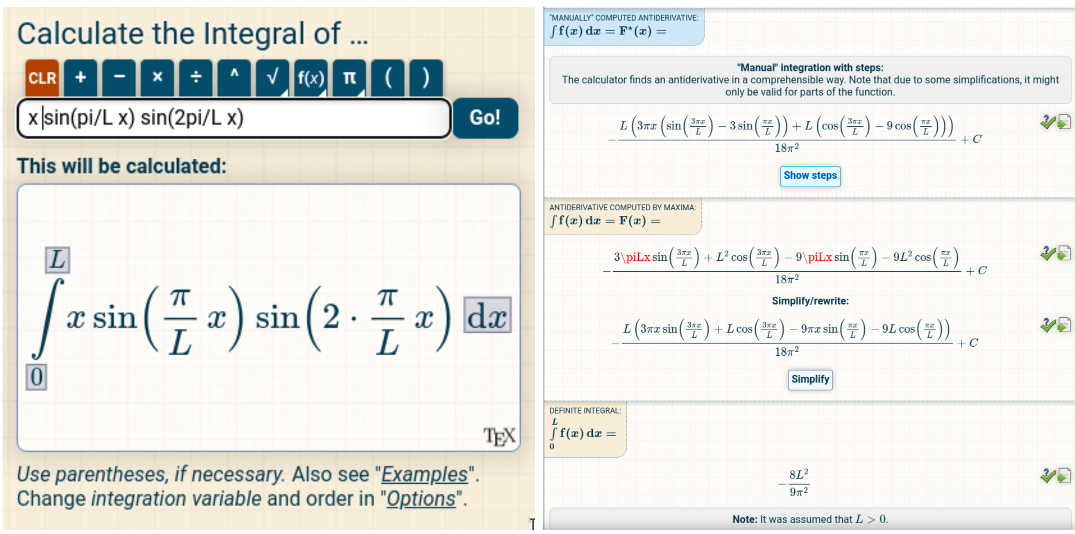

Figure 2: ss/inti02.png

$$
\begin{aligned}
\langle 1|\hat{K}|1\rangle &= \int_0^L \mathrm{d}x' \, \langle 1|\hat{K}|x'\rangle\langle x'|1\rangle \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \langle 1|x\rangle\langle x|\hat{K}|x'\rangle\langle x'|1\rangle \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \langle 1|x\rangle\langle x|\hat{K}|x'\rangle\langle x'|1\rangle \\
&= \frac{2}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \sin\left(\frac{\pi}{L}x\right) \langle x|\hat{K}|x'\rangle \sin\left(\frac{\pi}{L}x'\right) \\
&= \frac{2}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \sin\left(\frac{\pi}{L}x\right) \delta(x-x') \left(-i\frac{\mathrm{d}}{\mathrm{d}x'}\right) \sin\left(\frac{\pi}{L}x'\right) \\
&= \frac{2}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \sin\left(\frac{\pi}{L}x\right) \delta(x-x') \, (-i) \left(\frac{\pi}{L}\right) \cos(\frac{\pi}{L}x') \\
&= -i\frac{2\pi}{L^2}\int_0^L \mathrm{d}x' \, \sin\left(\frac{\pi}{L}x'\right) \cos\left(\frac{\pi}{L}x'\right) \\
&= 0
\end{aligned}
$$

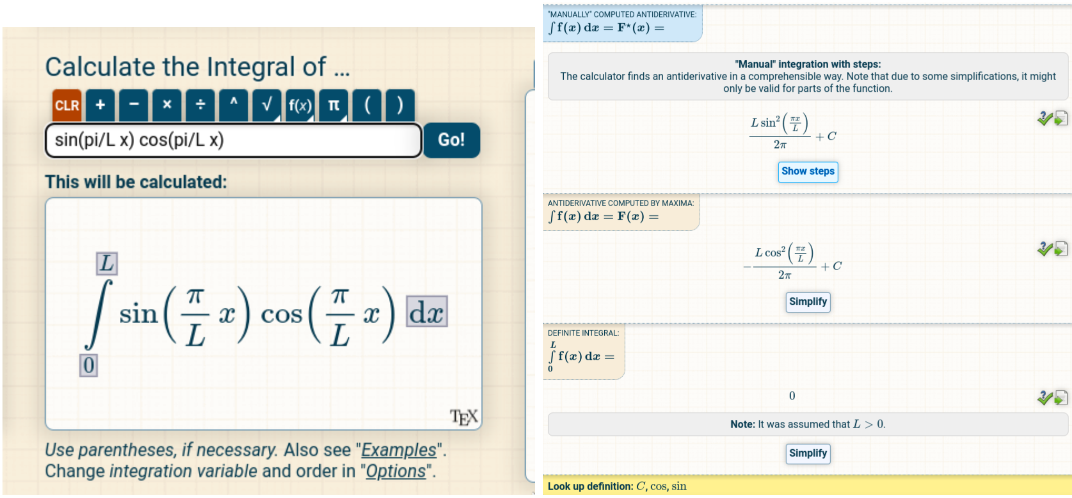

Figure 3: ss/inti04.png

$$
\begin{aligned}
\langle 1|\hat{K}|2\rangle &= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \langle 1|x\rangle\langle x|\hat{K}|x'\rangle\langle x'|2\rangle \\
&= \frac{2}{L}\left(-i\frac{2\pi}{L}\right)\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x \, \sin\left(\frac{\pi}{L}x\right)\delta(x-x')\cos\left(\frac{2\pi}{L}x'\right) \\
&= -i\frac{4\pi}{L^2}\int_0^L \mathrm{d}x' \, \sin\left(\frac{\pi}{L}x'\right)\cos\left(\frac{2\pi}{L}x'\right) \\
&= -i\frac{4\pi}{L^2}\left(-\frac{2L}{3\pi}\right) \\
&= i\frac{8}{3L}
\end{aligned}
$$

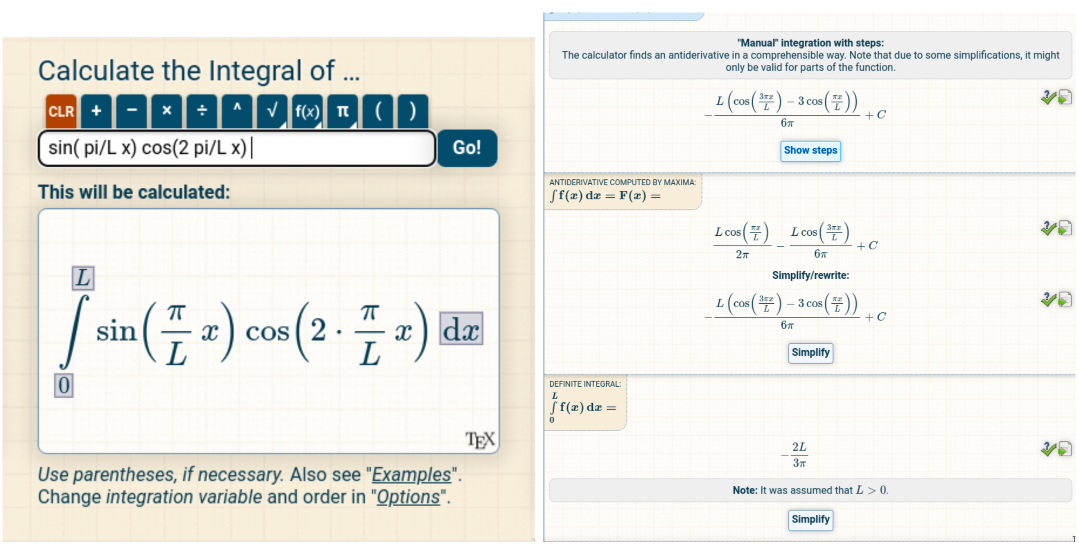

Figure 4: ss/inti07.png

b

$$
\begin{aligned}
\langle k_1|\hat{X}|k_1\rangle &= \langle k_1|\hat{I}\hat{X}\hat{I}|k_1\rangle \\
&= \left(\int_0^L \mathrm{d}x'\, \langle k_1|x'\rangle\langle x'|\right)\hat{X}\left(\int_0^L \mathrm{d}x\, |x\rangle\langle x|k_1\rangle\right) \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, \langle k_1|x'\rangle\langle x'|\hat{X}|x\rangle\langle x|k_1\rangle \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, \langle k_1|x'\rangle x\delta(x'-x)\langle x|k_1\rangle \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, x\left(\frac{1}{\sqrt{L}}e^{-ik_1x'}\right)\left(\frac{1}{\sqrt{L}}e^{ik_1x}\right)\delta(x'-x) \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, xe^{-ik_1x}e^{ik_1x} \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x \\
&= \frac{1}{L}\left(\frac{L^2}{2}\right) \\
&= \frac{L}{2}
\end{aligned}
$$

$$
\begin{aligned}
\langle k_1|\hat{X}|k_2\rangle &= \left(\int_0^L \mathrm{d}x'\langle k_1|x'\rangle\langle x'|\right)\hat{X}\left(\int_0^L \mathrm{d}x\, |x\rangle\langle x|k_2\rangle\right) \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, \langle k_1|x'\rangle\langle x'|\hat{X}|x\rangle\langle x|k_2\rangle \\
&= \frac{1}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, xe^{-ik_1x'}e^{ik_2x}\delta(x'-x) \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, xe^{i(k_2-k_1)x} \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, xe^{i\left(2\frac{\pi}{L}\right)x} \\
&= \left(\frac{1}{L}\right)\left(-\frac{iL^2}{2\pi}\right) \\
&= -\frac{iL}{2\pi}
\end{aligned}
$$

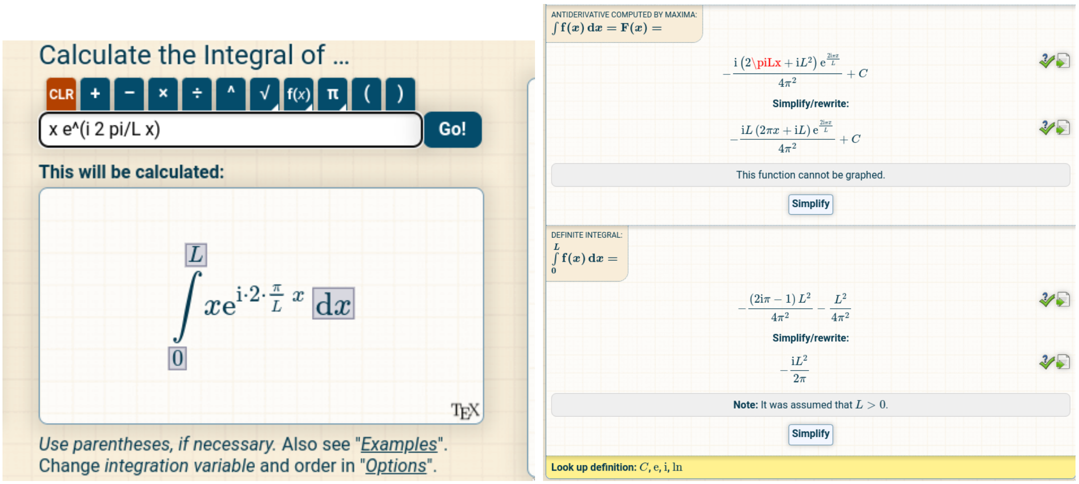

Figure 5: ss/balsal2.png

$$
\begin{aligned}
\langle k_1|\hat{K}|k_1\rangle &= \left(\int_0^L \mathrm{d}x'\, \langle k_1|x'\rangle\langle x'|\right)\hat{K}\left(\int_0^L \mathrm{d}x\, |x\rangle\langle x|k_1\rangle\right) \\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, \langle k_1|x'\rangle\langle x'|\hat{K}|x\rangle\langle x|k_1\rangle \\
&= \frac{1}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, e^{-ik_1x'}\left(-i\delta(x'-x)\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{ik_1x} \\
&= \frac{-i}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, e^{-ik_1x'}\delta(x'-x)ik_1e^{ik_1x} \\
&= \frac{k_1}{L}\int_0^L \mathrm{d}x\, e^{-ik_1x}e^{ik_1x} \\
&= \frac{k_1}{L}\int_0^L \mathrm{d}x \\
&= k_1 \\
&= \frac{2\pi}{L}
\end{aligned}
$$

$$\begin{aligned}
\langle k_1|\hat{K}|k_2\rangle &= \left(\int_0^L \mathrm{d}x'\,\langle k_1|x'\rangle\langle x'|\right)\hat{K}\left(\int_0^L \mathrm{d}x\,|x\rangle\langle x|k_2\rangle\right)\\
&= \int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\,\langle k_1|x'\rangle\langle x'|\hat{K}|x\rangle\langle x|k_2\rangle\\
&= \frac{1}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, e^{-ik_1x'}\left(-i\delta(x'-x)\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{ik_2x}\\
&= \frac{-i}{L}\int_0^L \mathrm{d}x' \int_0^L \mathrm{d}x\, e^{-ik_1x'}\delta(x'-x)ik_2e^{ik_2x}\\
&= \frac{k_2}{L}\int_0^L \mathrm{d}x\, e^{-ik_1x}e^{ik_2x}\\
&= \frac{k_2}{L}\int_0^L \mathrm{d}x\, e^{i(2\pi/L)x}\\
&= 0
\end{aligned}$$

Problem 05

0.1 a

$$\begin{aligned}
\langle k|\hat{X}|k'\rangle &= \left(\int_{-\infty}^{\infty} \mathrm{d}x\,\langle k|x\rangle\langle x|\right)\hat{X}\left(\int_{-\infty}^{\infty} \mathrm{d}x'\,|x'\rangle\langle x'|k'\rangle\right)\\
&= \int \mathrm{d}x \int \mathrm{d}x'\,\langle k|x\rangle\langle x|\hat{X}|x'\rangle\langle x'|k'\rangle\\
&= \int \mathrm{d}x\,\int \mathrm{d}x'\,\frac{1}{\sqrt{2\pi}}e^{-ikx}\langle x|\hat{X}|x'\rangle\frac{1}{\sqrt{2\pi}}e^{ik'x}\\
&= \int \mathrm{d}x\,\int \mathrm{d}x'\,\frac{1}{\sqrt{2\pi}}e^{-ikx}x'\delta(x-x')\frac{1}{\sqrt{2\pi}}e^{ik'x'}\\
&= \int \mathrm{d}x\,\int \mathrm{d}x'\,\frac{1}{\sqrt{2\pi}}e^{-ikx}\delta(x-x')\left(x'\frac{1}{\sqrt{2\pi}}e^{ik'x'}\right)\\
&= \int \mathrm{d}x'\,\frac{1}{\sqrt{2\pi}}e^{-ikx'}\left(x'\frac{1}{\sqrt{2\pi}}e^{ik'x'}\right)\\
&= \int \mathrm{d}x'\,\frac{1}{\sqrt{2\pi}}e^{-ikx'}\left(x'\frac{1}{\sqrt{2\pi}}e^{ik'x'}\right)\\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^2\int \mathrm{d}x'\,x'e^{i(k'-k)x'}\\
&= (-1)(-i)\left(\frac{1}{\sqrt{2\pi}}\right)^2\int \mathrm{d}x'\,\frac{\mathrm{d}}{\mathrm{d}k}e^{i(k'-k)x'}\\
&= i\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{2\pi}\int_{-\infty}^{-\infty} \mathrm{d}x\,e^{i(k'-k)x'}\right)\\
&= i\frac{\mathrm{d}}{\mathrm{d}k}\delta(k'-k) = i\frac{\mathrm{d}}{\mathrm{d}k}\delta(k-k')
\end{aligned}$$

b

$$
\begin{aligned}
\langle k|\hat{X}|f\rangle &= \int_{-\infty}^{\infty} \mathrm{d}k' \, \langle k|\hat{X}|k'\rangle\langle k'|f\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}k' \, \left(i\frac{\mathrm{d}}{\mathrm{d}k}\delta(k-k')\right)\langle k'|f\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}k' \, \left(i\delta(k-k')\frac{\mathrm{d}}{\mathrm{d}k'}\right)\langle k'|f\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}k' \, \left(i\delta(k-k')\frac{\mathrm{d}}{\mathrm{d}k'}\right)\left(\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-ik'x}}{\sqrt{2\pi}}f(x)\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}k' \, \left(i\delta(k'-k)\frac{\mathrm{d}}{\mathrm{d}k'}\right)\left(\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-ik'x}}{\sqrt{2\pi}}f(x)\right) \\
&= \left(i\frac{\mathrm{d}}{\mathrm{d}k}\right)\left(\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{-ikx}}{\sqrt{2\pi}}f(x)\right) \\
&= i\frac{\mathrm{d}}{\mathrm{d}k}\tilde{f}(k)
\end{aligned}
$$