February 28, 2025

Majorana Fermion in TFI Model

Technical Description

We start by defining a Hamiltonian

$$\hat{H} = -J \sum_j \left(\hat{Z}_j \hat{Z}_{j+1} + g \hat{X}_j \right) \tag{1}$$

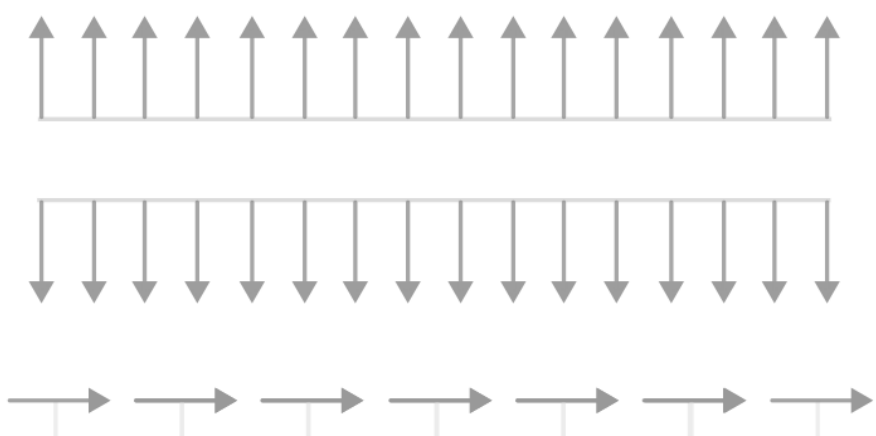
where g is a parameter that we are going to tweak. For instance, for $g \to 0$

$$\hat{H} \approx -J \sum_j \hat{Z}_j \hat{Z}_{j+1}$$

and for $g \to \infty$

$$\hat{H} \approx -gJ \sum_j \hat{X}_j.$$

The limit behavior can help us make guesses about the Eigenstates of the Hamiltonian.

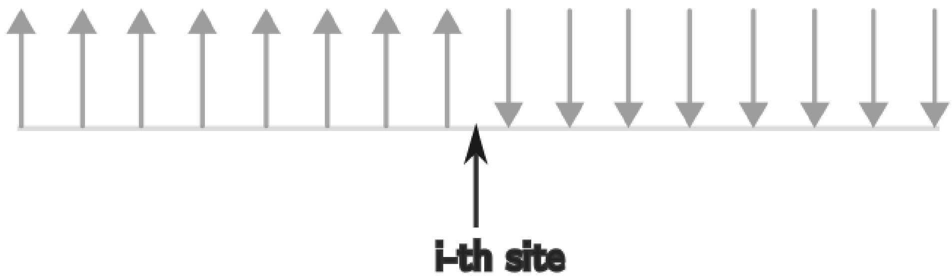


Figuur 1: Eigenstates for $g \to 0$ and $g \to \infty$

Excitations are domain walls. We can define an excitation operator by

$$\hat{\tau}_i^z \left|\uparrow\uparrow\uparrow \cdots\right\rangle = \prod_{j > \bar{i}} \hat{x}_j \left|\uparrow\uparrow\uparrow \cdots\right\rangle \tag{2}$$

where $\bar{i} = i + \frac{1}{2}$.



Figuur 2: Equation 2 visualization.

Defining two operators from intuition

Let's create two definitions inspired by above

$$\hat{\chi}_j = \hat{z}_j \hat{\tau}^z_{j+\frac{1}{2}} = z_j \prod_{i>j} x_i \tag{3}$$

$$\hat{\tilde{\chi}}_j = \hat{y}_j \hat{\tau}^z_{j+\frac{1}{2}} = \underbrace{\hat{y}_j \hat{x}_j}_{-i\hat{z}_j} \hat{x}_j \cdots \hat{x}_j \prod_{i>j} \hat{x}_j = -iz_j \sum_{i \geq j} \hat{x}_i \tag{4}$$

Intuition for this is:

1. $g \ll 1$, then $\langle \hat{z}_j \rangle \approx 1$, so $\hat{x}_j \sim \langle \hat{z}_j \rangle \hat{\tau}^z_{j+\frac{1}{2}} \sim \hat{\tau}^z_{j+\frac{1}{2}}$ creates a domain wall.
2. $g \gg 1$, then $\hat{\tau}^z_j \approx 1$, so $\hat{x}_j$ and $\hat{\tilde{x}}_j$ flip $\hat{z}_j \to -\hat{z}_j$.

Algebra of $\hat{\chi}, \hat{\tilde{\chi}}$

For cleanliness we will drop the hats $\hat{\chi} \to \chi$ for this subsection.

$$\begin{aligned} \chi_j^T &= \chi_j \\ \tilde{\chi}_j &= \tilde{\chi}_j \\ \underbrace{\chi_i \chi_j}_{\text{WLOG, } i<j} &= z_i \sum_{i'>i} x_i \cdot z_j \sum_{j'>j} x_i = -z_j z_i \cdot \tau_j^z \tau_i^z = -\chi_j \chi_i \end{aligned}$$

Emergence of Fermions

For all $i \neq j$ here

$$\{\chi_i, \chi_j\} = 0 \quad \text{and} \quad \{\tilde{\chi}_i, \tilde{\chi}_j\} = 0$$

With the inclusion of the $i = j$ case

$$\{\chi_i, \chi_j\} = 2\delta_{ij}$$

This is algebra of Majoranas!

Jordan-Wigner Transformation

$$\hat{c}_j = \frac{1}{2}\left(\hat{\chi}_j - i\hat{\tilde{\chi}}_j\right) \tag{5}$$

$$\hat{c}_j^T = \frac{1}{2}\left(\hat{\chi}_j + i\hat{\tilde{\chi}}_j\right) \tag{6}$$

There are some helpful properties which are

$$\{\hat{c}_i, \hat{c}_j^T\} = \delta_{ij}$$
$$\hat{c}_i^2 = 0$$

Let's apply this to the operator $\hat{x}_j$,

$$\hat{x}_j = -i\hat{y}_j\hat{z}_j = -i\hat{y}_j\hat{\tau}^z_{j+\frac{1}{2}} \cdot \hat{z}_j\hat{\tau}^z_{j+\frac{1}{2}} = -i\hat{\tilde{\chi}}_j\hat{\chi}_j$$

$$\hat{x}_j = 1 - 2\hat{c}_j^T\hat{c}_j = (-1)^{\hat{c}_j^T\hat{c}_j}$$

The last equation above is the Fermion Parity that can be easily shown by the definitions of Jordan-Wigener Transformation. It does the following procedure on a ket

$$\hat{c}_j^T\hat{c}_j \left|\rightarrow\right\rangle_j = 0$$

$$\hat{c}_j^T\hat{c}_j \left|\leftarrow\right\rangle_j = 1$$

This procedure on a ket can help us count the number of spin flips and domain walls through the following two equations respectively,

$$\text{Number of spin flips } = \text{Number of Fermions } = -\sum_j (-1)^{n_j} = +i\sum \hat{\tilde{\chi}}_j\hat{\chi}_j \text{ for } g \gg 1$$

$$\text{Number of Domain Walls } = \sum_j \hat{z}_j\hat{z}_{j+1} = \sum_j i\hat{\tilde{\chi}}\hat{\chi}_{j+1} \text{ for } g \ll 1$$

Re-writing the Hamiltonian

We can do the following re-write with the transformation

$$\hat{\mathcal{H}}_{\text{TFI}} = -J\sum_j (\underbrace{i\hat{\tilde{\chi}}_{j+1}\hat{\chi}_j}_{\text{hopping}} + \underbrace{gi\hat{\chi}_j\hat{\tilde{\chi}}_j}_{\text{chem. potential}}) = -J\sum_j \left[\underbrace{c_j^T c_{j+1} + c_j^T c_{j+1}^T + \text{h.c.}}_{\text{conserves the number of fermions, i.e. } \mathbb{Z}_2 \text{ symmetry}} - 2gc_j^T c_j + g\right]$$

Dual Fermions

$$\gamma_{j+\frac{1}{2}} = -\tilde{\chi}_{j+1} = iz_{j+1} \cdot \tau^z_{j+\frac{3}{2}} \tag{7}$$
$$\tilde{\gamma}_{j+\frac{1}{2}} = \chi_j = z_j \cdot \tau^z_{j+\frac{1}{2}} \tag{8}$$
$$\tag{9}$$

This maps

$$\mathcal{H}_{\text{TFI}} = -J \sum_{\bar{j}} \left(+gi\tilde{\gamma}_{\bar{j}+1}\gamma_{\bar{j}} + i\tilde{\gamma}_{\bar{j}}\gamma_{\bar{j}} \right)$$

Let's look at the phases now. For $g \ll 1$ we get

$$\mathcal{H}_{g\to 0} = -J \sum_{\bar{j}} (-1)^{d^+_{\bar{j}} d_{\bar{j}}} \qquad (d_{\bar{j}} = (1/2)(\gamma_{\bar{j}} - i\tilde{\gamma}_{\bar{j}}))$$

that's where we see new Fermions. The ground state

$$|\text{ground state as } g \to 0\rangle = |\tilde{n}_{\bar{j}} = 0\rangle \to \text{vacuum of fermions by } d_{\bar{j}}$$

$$d_{\bar{j}}|n_{\bar{j}} = 0\rangle = 0 \qquad (\text{for all } j)$$

$d^+_{\bar{j}} d_{\bar{j}}$ is supposed to count the number of Domain Walls but there are none!

For $g \gg 1$ we get

$$\mathcal{H}_{g\to\infty} = -Jg \sum_j (-1)^{c^T_j c_j}$$

The ground state

$$|\text{ground state as } g \to \infty\rangle = |n_j = 0\rangle$$

is a vacuum of c_j fermions. Again, $c^T_j c_j$ counts the number of spin flips but there are none!

Relation to Kitaev Chains

$$\mathcal{H}_{\text{Kitaev}} = \sum_j -\frac{(-|\Delta| - +)}{2} i\tilde{\chi}_{j+1}\chi_j - \frac{(-|\Delta| + +)}{2} i\chi_j\tilde{\chi}_{j+1} - \frac{i\mu}{2}\tilde{\chi}_j\chi_j$$

Choosing $|\Delta| \to +$ maps onto the TFI model with $g \to \frac{\mu}{2}$. In particular when $\mu \to 0$ ($g \to 0$), the Kitaev Chain realizes an SPT phase with two unpaired Majorana modes at its ends. This corresponds to $\mathbb{Z}_2$ topological spin chain in a non-trivial SPT phase.