Classical Mechanics : : Homework 10

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Problem 01

The equations of torque are

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1$$

Without losing generality let us assume $I_1 < I_2 < I_3$. So Dzhanibekov Effect, or Tennis Racket Theorem, says that I_2 moment of inertia is the one that is along the unstable principle axis.

Unstability along I_2

Let's look at an initial Ω rotating only along $(0, 1, 0)$ (the unstable axis). Note that we are working in the basis where the unit vectors are the principle axis.

Let's have a slight perturbation along for angular velocity $(1, 0, 0)$ that is δ which says

$$(0, \Omega, 0) \to (\delta, \Omega, 0)$$

The external torques are zero. Then using the above perturbation to find the rate of change of ω for the initial moment

$$\tau_1 = 0 \implies \dot{\omega}_1 = 0$$
$$\tau_2 = 0 \implies \dot{\omega}_2 = 0$$
$$\tau_3 = 0 = I_3\dot{\omega}_3 + (I_2 - I_1)\Omega\delta \implies \dot{\omega}_3 = -\frac{I_2 - I_1}{I_3}\Omega\delta$$

Hence after some infinitesimal time Δt we get ω_3 to be

$$\omega_3 \approx -\frac{I_2 - I_1}{I_3}\Omega\delta\Delta t$$

Now the Euler's equations are (external torque is still 0)

$$\dot{\omega}_1 = -\frac{I_3 - I_2}{I_1}\left(-\frac{I_2 - I_1}{I_3}\Omega\delta\Delta t\right)\Omega = \frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\Omega^2\delta\Delta t > 0$$
$$\dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\delta\left(-\frac{I_2 - I_1}{I_3}\Omega\delta\Delta t\right) = -\frac{(I_3 - I_1)(I_2 - I_1)}{I_3 I_2}\Omega\delta^2\Delta t$$

Being a little bit more sloppy with the mathematics we have

$$
\begin{aligned}
\ddot{\omega}_1 &= \frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\Omega^2\delta \\
\ddot{\omega}_2 &= -\frac{(I_3 - I_1)(I_2 - I_1)}{I_3 I_2}\Omega\delta^2
\end{aligned}
$$

We can see above that ω_1's second derivative grows rapidly for a small perturbation δ. This is an exponentially growing motion, instead of an oscillatory one. So the overall direction of rotation does not stay long $(0, \Omega, 0)$ for the whole motion, implying $(0, 1, 0)$ direction of rotation to be unstable.

We picked this to be the direction of I_2 when $I_1 < I_2 < I_3$. So this principal axis is unstable.

This exact same calculation for $(0, \Omega, \delta)$ perturbation will give the exact same exponential growth along perturbation.

Stability along I_1 (or I_3)

Let's start from $\vec{\omega} = (\Omega, 0, 0)$ and then introduce a general perturbation $(\Omega, \epsilon_2, \epsilon_3)$

$$
\begin{aligned}
0 &= \dot{\omega}_1 + \frac{I_3 - I_2}{I_1}\epsilon_2\epsilon_3 \\
0 &= \dot{\epsilon}_2 + \frac{I_1 - I_3}{I_2}\Omega\epsilon_3 \\
0 &= \dot{\epsilon}_3 + \frac{I_2 - I_1}{I_3}\Omega\epsilon_2
\end{aligned}
$$

Fix the signs if necessary

$$
\begin{aligned}
0 &= \dot{\omega}_1 + \frac{I_3 - I_2}{I_1}\epsilon_2\epsilon_3 \\
0 &= \dot{\epsilon}_2 - \frac{I_3 - I_1}{I_2}\Omega\epsilon_3 \\
0 &= \dot{\epsilon}_3 + \frac{I_2 - I_1}{I_3}\Omega\epsilon_2 \\
&\implies \text{Contract the notations} \\
0 &= \dot{\omega}_1 + A\epsilon_2\epsilon_3 \\
0 &= \dot{\epsilon}_2 - B\Omega\epsilon_3 \\
0 &= \dot{\epsilon}_3 + C\Omega\epsilon_2
\end{aligned}
$$

Let's look at the perturbation's differential equations

$$
\begin{aligned}
0 &= \dot{\epsilon}_2 - B\Omega\epsilon_3 \\
0 &= \dot{\epsilon}_3 + C\Omega\epsilon_2 \\
&\implies \text{Take on derivative} \\
0 &= \ddot{\epsilon}_2 - B\Omega\dot{\epsilon}_3 = \ddot{\epsilon}_2 + BC\Omega\epsilon_2 = 0 \\
0 &= \ddot{\epsilon}_3 + C\Omega\dot{\epsilon}_2 = \ddot{\epsilon}_3 + BC\Omega\epsilon_3 = 0
\end{aligned}
$$

We get simple harmonic equations for the perturbations. So they don't make any noticeable difference other than just small precession like motion.

Given the amplitude of the perturbation being small, we can simply see that $\dot{\omega}_1 \approx 0$. For this we can say that I_1 is a stable principle axis direction.

Because of the symmetry of the equations, just flipping the notations will immediately prove the exact same reasoning for I_3 too. So we proved that Mr. Dzanibekov was right (or Tennis Racket Theorem is true).

Problem 02

(a)

For the hollow sphere, the inertia tensor is not going to have any off diagonal elements because of the symmetry (symmetry being any principle axis has the same moment of inertia given the origin is in the center of the ball). Also it's going to be a multiple of the Identity Matrix because of the spherical symmetry.

Through any axis crossing the center, the moment of inertia can be easily computed.

$$I_{ij} = \int_V \rho \left(r^2 \delta_{ij} - x_i x_j \right) \, dV$$

For our case I_{11}, I_{22}, I_{33} are the same thing. And any off diagonals are zero.

$$\rho = \frac{M}{\frac{4}{3}\pi(a^3 - b^3)}$$

$$I = \int_V \rho r^2 \sin^2 \theta \, dV$$

For the sphere

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

Take the integral

$$I = \rho \int_b^a \int_0^\pi \int_0^{2\pi} r^4 \sin^3 \theta \, d\phi \, d\theta \, dr$$

ϕ symmetry

$$\int_0^{2\pi} 1 \, d\phi = 2\pi$$

Latitude integral

$$\int_0^\pi \sin^3 \theta \, d\theta = \int_0^\pi (1 - \cos^2 \theta) \sin \theta \, d\theta = \left[-\frac{\cos \theta}{3} + \frac{\cos^3 \theta}{3} \right]_0^\pi = \frac{4}{3}$$

Radius from b to a

$$\int_b^a r^4 \, dr = \frac{1}{5} \left[r^5 \right]_b^a = \frac{1}{5}(a^5 - b^5)$$

Put everything together

$$I = \rho \cdot 2\pi \cdot \frac{4}{3} \cdot \frac{1}{5}(a^5 - b^5) = \frac{8\pi}{15} \rho (a^5 - b^5)$$

$$I = \frac{8\pi}{15} \cdot \frac{M}{\frac{4}{3}\pi(a^3 - b^3)} \cdot (a^5 - b^5)$$

$$I = \frac{2M}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}$$

So the tensor is

$$\hat{I} = \frac{2M}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

Angular impulse applied by cue (r distance from center)

$$\tau\Delta t = rF\Delta t = r\Delta p$$

Angular impulse received by ball

$$\Delta L = L - 0 = I\omega$$

They should be equal

$$r\Delta p = I\omega \tag{1}$$

Linear momentum gained is Δp. Invoke no slip condition then

$$\omega = \frac{v}{a} = \frac{\Delta p}{ma}$$

Use this with (1) then

$$r\Delta p = I\frac{\Delta p}{ma} \implies r = \frac{I}{ma}$$

What we get for H is

$$H = r + a = \frac{I}{ma} + a$$

In this case it will be

$$H = \frac{2}{5a}\frac{a^5 - b^5}{a^3 - b^3} + a$$

(c)

$b = 0$ **Solid Sphere**

$$H = \frac{2}{5a}a^2 + a = \frac{2}{5}a + a = \frac{7}{5}a$$

$b \to a$ **Thin Sphere**

$$\begin{aligned}
H &= \frac{2}{5a}\frac{a^5 - b^5}{a^3 - b^3} + a \\
&= \frac{2}{5a}\frac{a^5 - (a-\varepsilon)^5}{a^3 - (a-\varepsilon)^3} + a \\
&= \frac{2}{5a}\frac{a^5 - a^5(1-\frac{\varepsilon}{a})^5}{a^3 - a^3(1-\frac{\varepsilon}{a})^3} + a \\
&\approx \frac{2}{5a}\frac{a^5 - a^5(1-5\frac{\varepsilon}{a})}{a^3 - a^3(1-3\frac{\varepsilon}{a})} + a \\
&= \frac{2}{5a}\frac{a^5(1-1+5\frac{\varepsilon}{a})}{a^3(1-1+3\frac{\varepsilon}{a})} + a \\
&= \frac{2}{3a}a^2 + a \\
&= \frac{2}{3}a + a = \frac{5}{3}a
\end{aligned}$$

Problem 03