Quantum Mechanics : : Homework 09

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Problem 01

$$\begin{aligned}\langle x|\hat{P}^2|\psi\rangle &= \langle x|\hat{P}^2|x'\rangle\langle x'|\psi\rangle \\ &= \langle x|\hat{P}^2|x'\rangle\psi(x') \\ &= (-i\hbar)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) \\ &= -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)\end{aligned}$$

$$\begin{aligned}\hat{H}|\psi\rangle = E|\psi\rangle \implies \hat{H}|\psi\rangle &= 0 \\ \left(\frac{\hat{P}^2}{2m}+\frac{V_0}{a}\hat{X}\right)|\psi\rangle &= 0 \\ \left(\frac{\hat{P}^2}{2m}+\frac{V_0}{a}\hat{X}\right)|x\rangle\langle x|\psi\rangle &= 0 \\ \left(\frac{\hat{P}^2}{2m}|x\rangle+\frac{V_0}{a}x\right)\psi_0(x) &= 0 \\ \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{V_0}{a}x\right)\psi_0(x) &= 0 \\ \implies -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_0(x)}{\mathrm{d}x^2}+\frac{V_0}{a}x\psi_0(x) &= 0\end{aligned}$$

(a)

The zero energy eigenstate is given by

$$\psi_0(x) = \frac{c}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}p\, e^{ipx/\hbar}\exp\left(i\frac{a}{\hbar V_0}\left(\frac{p^3}{6m}\right)\right)$$

Complex Conjugate

$$\begin{aligned}\psi_0(x)^* &= \frac{c^*}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}p\, e^{-ipx/\hbar}\exp\left(-i\frac{a}{\hbar V_0}\left(\frac{p^3}{6m}\right)\right) \\ &= \frac{c^*}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}p\, e^{i(-p)x/\hbar}\exp\left(i\frac{a}{\hbar V_0}\left(\frac{(-p)^3}{6m}\right)\right) \\ &= -\frac{c^*}{\sqrt{2\pi\hbar}}\int_{\infty}^{-\infty}\mathrm{d}u\, e^{iux/\hbar}\exp\left(i\frac{a}{\hbar V_0}\left(\frac{u^3}{6m}\right)\right) && (-p := u) \\ &= \frac{c^*}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{d}u\, e^{iux/\hbar}\exp\left(i\frac{a}{\hbar V_0}\left(\frac{u^3}{6m}\right)\right)\end{aligned}$$

Taking the complex conjugate we can see that the integral is the same with a new dummy variable u from p. For the integral, the original and it's complex conjugate being the same means the integral itself is purely real.

For the normalization constant c, when we compute $\psi^*\nabla\psi$, then we get a factor $c^*c = |c|^2$ which is purely real. Hence $\psi^*\nabla\psi$ has practically no imaginary part meaning $J(x) = 0$.

ψ is a standing wave without time dependence so they are constant in time.

(b)

$$
\begin{aligned}
\frac{\mathrm{d}\langle X\rangle}{\mathrm{d}t} &= \frac{\langle P\rangle}{m} \\
\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}t} &= -\frac{i}{\hbar}\langle[\hat{H},\hat{P}]\rangle = i\hbar\frac{V_0}{a}\left(-\frac{i}{\hbar}\right) = -\frac{V_0}{a} \\
\implies \langle P\rangle &= p_0 - \frac{V_0}{a}t \\
\implies \langle X\rangle &= \int \mathrm{d}t\,\langle P\rangle\frac{1}{m} = x_0 + \frac{p_0}{m}t - \frac{V_0}{2am}t^2
\end{aligned}
$$

Problem 02

If $k = \sqrt{2\mu E/\hbar^2}$ then the wave function is given by

$$\psi_{E,m}(r,\phi) = Ae^{im\phi}J_m(kr)$$

The current density is given by

$$\vec{J} = \frac{\hbar}{\mu}\mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right)$$

Firstly computing the gradient

$$
\begin{aligned}
\vec{\nabla}\psi &= \vec{n}_r\frac{\partial}{\partial r}\left(Ae^{im\phi}J_m(kr)\right) + +\vec{n}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\left(Ae^{im\phi}J_m(kr)\right) \\
&= \vec{n}_r Ae^{im\phi}\frac{\partial J_m(kr)}{\partial r} + \vec{n}_\phi\frac{Ae^{im\phi}J_m(kr)}{r}(im)
\end{aligned}
$$

Computing the complex conjugate now

$$\psi^* = A^*e^{-im\phi}J_m(kr)$$

$$
\begin{aligned}
\psi^*\vec{\nabla}\psi &= \vec{n}_r AA^*J_m(kr)\frac{\partial J_m(kr)}{\partial r} + \vec{n}_\phi\frac{AA^*J_m^2(kr)}{r}(im) \\
&= \underbrace{\vec{n}_r\frac{|A|^2}{2}\frac{\partial^2}{\partial r^2}[J_m^2(kr)]}_{\text{purely real}} + \vec{n}_\phi\,(i)\,\frac{m|\psi_{E,m}|^2}{r} \\
\mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right) &= \vec{n}_\phi\frac{m}{r}|\psi_{E,m}|^2
\end{aligned}
$$

From this it shows,

$$\vec{J}(r,\phi) = \vec{n}_\phi\frac{m\hbar}{\mu r}|\psi_{E,m}(r)|^2 = \vec{n}_\phi\frac{m\hbar}{\mu r}\rho(r)$$

Note that $\psi(r,\phi)$ becomes only a function of r when it's magnitude is taken because of the $e^{im\phi}$ term vanishing while taking the complex conjugate.

Interpretation:

Classical particle angular momentum when it rotates along z axis

$$L_z = mr^2\omega = mvr$$

$$v = \frac{L_z}{\mu r}$$

Quantum Mechanically we can treat $L_z = \hbar m$ so we have got a somewhat analogous classical particle

$$v = \frac{m\hbar}{\mu r}$$

Problem 03

(a)

$$
\begin{aligned}
\psi(x,t) &= \langle x|\psi(t)\rangle \\
&= \langle x|\hat{U}(t)|\psi(0)\rangle \\
&= \langle x|\hat{U}(t)\hat{I}|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \ \langle x|\hat{U}(t)|x'\rangle \langle x'|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \, U(x,x';t)\psi_0(x') \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \ \left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2}\right)\psi_0(x') \\
&= \left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2}\right)\psi_0(x') \\
&= \left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2}\right)\left[\left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\exp\left(-\frac{(x'-x_0)^2}{2\Delta_0^2}\right)\right] \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2}\right)\left[\exp\left(-\frac{(x'-x_0)^2}{2\Delta_0^2}\right)\right] \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2}\right)\left[\exp\left(-\frac{(x'-x_0)^2}{2\Delta_0^2}\right)\right] \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x-x')^2}{2\lambda(t)\Delta_0^2} - \frac{(x'-x_0)^2}{2\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x^2+x'^2-2xx')}{2\lambda(t)\Delta_0^2} - \frac{\lambda(t)(x'^2+x_0^2-2x_0x')}{2\lambda(t)\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{i(x^2+x'^2-2xx')-\lambda(t)(x'^2+x_0^2-2x_0x')}{2\lambda(t)\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{ix^2+ix'^2-2ixx'-\lambda(t)x'^2-\lambda(t)x_0^2+\lambda(t)2x_0x'}{2\lambda(t)\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{ix^2-\lambda(t)x_0^2+ix'^2-\lambda(t)x'^2-2ixx'+\lambda(t)2x_0x'}{2\lambda(t)\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(\frac{\left[ix^2-\lambda(t)x_0^2\right]+\left[i-\lambda(t)\right]x'^2+\left[-2ix+\lambda(t)2x_0\right]x'}{2\lambda(t)\Delta_0^2}\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(c+ax'^2+bx'\right) \Longleftarrow \begin{cases} a &= \left[i-\lambda(t)\right]/2\lambda(t)\Delta_0^2 \\ b &= \left[-2ix+\lambda(t)2x_0\right]/2\lambda(t)\Delta_0^2 \\ c &= \left[ix^2-\lambda(t)x_0^2\right]/2\lambda(t)\Delta_0^2 \end{cases} \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp\left(-b^2/4a\right)\exp(c)\exp\left(a\left(x'+(b/2a)\right)^2\right) \\
&= \left(\frac{1}{\pi\Delta_0^2}\right)^{1/4}\left(\frac{1}{2\pi i\lambda(t)\Delta_0^2}\right)^{1/2} \exp\left(-b^2/4a\right)\exp(c)\sqrt{\frac{\pi}{-a}} \\
&= \text{paper and handheld basic algebra where I substitute the coefficients} \\
&= \frac{1}{(\sqrt{\pi}\Delta_0[1+i\lambda(t)])^{\frac{1}{2}}}\exp\left(-\frac{(x-x_0)^2}{2\Delta_0^2(1+i\lambda(t))}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\psi_0(p) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \psi_0(x) e^{-ipx/\hbar} \\
&= \frac{1}{(\pi\Delta_0^2)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\Delta_0^2} - i\frac{px}{\hbar}}\, \mathrm{d}x
\end{aligned}
$$

This above equation is almost like the gaussian we did in the class. On paper I do the following rough work to substitute the integral and we get,

$$
\boxed{\psi_0(p) = \left(\frac{2\Delta_0^2}{\pi}\right)^{1/4} e^{-ipx_0/\hbar} e^{-\frac{\Delta_0^2 p^2}{2\hbar^2}}}
$$

$$
\langle x|\psi\rangle = \psi(x,t) = \left(\frac{2\Delta_0^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipx/\hbar} e^{-ip^2 t/2m\hbar} e^{-\Delta_0^2 p^2/2\hbar^2} e^{-ipx_0/\hbar}
$$

$$
\begin{aligned}
a &= \frac{\Delta_0^2}{2\hbar^2} + i\frac{t}{2m\hbar} \\
b &= i\frac{x - x_0}{\hbar}
\end{aligned}
$$

This exponent can be treated as a, b so that $-ap^2 + bp$ and using the integral we used in class

$$
\begin{aligned}
\psi(x,t) &= \left(\frac{2\Delta_0^2}{\pi}\right)^{1/4} \frac{1}{2\pi\hbar} \sqrt{\frac{\pi}{a}} e^{b^2/4a} \\
\psi(x,t) &= \frac{1}{\pi\Delta_0^2(1 + i\lambda(t))^{1/2}} e^{-\frac{(x-x_0)^2}{2\Delta_0^2(1+i\lambda(t))}} \\
&= \frac{1}{(\sqrt{\pi}\Delta_0[1 + i\lambda(t)])^{\frac{1}{2}}} \exp\left(-\frac{(x-x_0)^2}{2\Delta_0^2(1 + i\lambda(t))}\right)
\end{aligned}
$$

Problem 04

(a)

$$
\begin{aligned}
\psi_{\mathrm{II}} &= Ce^{-k_2 x} + De^{k_2 x} \\
\psi_{\mathrm{II}}^* &= C^* e^{-k_2 x} + D^* e^{k_2 x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}} &= -k_2 Ce^{-k_2 x} + k_2 De^{k_2 x} \\
\psi_{\mathrm{II}} \frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}} &= \left(C^* e^{-k_2 x} + D^* e^{k_2 x}\right)\left(-k_2 Ce^{-k_2 x} + k_2 De^{k_2 x}\right) \\
&= \underbrace{-k_2 CC^* e^{-2k_2 x} + k_2 DD^* e^{2k_2 x}}_{\text{purely real}} + k_2 C^* D - k_2 CD^* \\
\mathrm{Im}\left(\psi_{\mathrm{II}} \frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}}\right) &= k_2\left(C^* D - CD^*\right) \\
J_{\mathrm{II}}(x) = \frac{\hbar}{m}\mathrm{Im}\left(\psi_{\mathrm{II}} \frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}}\right) &= \frac{k_2\hbar}{m}\left(C^* D - CD^*\right)
\end{aligned}
$$

$$\boxed{J_{\mathrm{II}}(x) = \frac{k_2\hbar}{m}\left(C^*D - CD^*\right)}$$

Analysis: The current is steady, and it does not depend on position so it's uniform. Depending on C, D it is possibly non-zero.

(b)

The wavefunctions for each region are

$$\psi_{\mathrm{I}}(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$
$$\psi_{\mathrm{II}}(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$
$$\psi_{\mathrm{III}}(x) = Ee^{ik_1x} \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

To compute the probability current

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{I}}(x) = ik_1Ae^{ik_1x} - ik_1Be^{-ik_1x} \qquad \psi_{\mathrm{I}}^* = A^*e^{-ik_1x} + B^*e^{ik_1x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}}(x) = ik_2Ce^{ik_2x} - ik_2De^{-ik_2x} \qquad \psi_{\mathrm{II}}^* = C^*e^{-ik_2x} + D^*e^{ik_2x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{III}}(x) = ik_1Ee^{ik_1x} \qquad \psi_{\mathrm{III}}^* = E^*e^{-ik_1x}$$

$$\begin{aligned}
\psi_{\mathrm{I}}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{I}}(x) &= ik_1AA^* - ik_1BB^* - ik_1A^*Be^{-2ik_1x} + ik_1AB^*e^{2ik_1x} \\
&= ik_1(|A|^2 - |B|^2) - ik_1A^*Be^{-2ik_1x} + \left(ik_1A^*Be^{-2ik_1x}\right)^* \\
\implies \mathrm{Im}\left(\psi_{\mathrm{I}}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{I}}(x)\right) &= k_1\left(|A|^2 - |B|^2\right) \\
\implies \mathrm{Im}\left(\psi_{\mathrm{II}}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{II}}(x)\right) &= k_2\left(|C|^2 - |D|^2\right) \\
\implies \mathrm{Im}\left(\psi_{\mathrm{III}}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\mathrm{III}}(x)\right) &= k_2|E|^2
\end{aligned}$$

Finalizing our results for the probability current where going towards positive x is considered as positive current

$$\begin{aligned}
J_{\mathrm{I}} &= \frac{\hbar k_1}{m}(|A|^2 - |B|^2) \\
J_{\mathrm{II}} &= \frac{\hbar k_2}{m}(|C|^2 - |D|^2) \\
J_{\mathrm{III}} &= \frac{\hbar k_1}{m}|E|^2
\end{aligned}$$

Invoking continuity through defining $\phi = e^{-ik_1L/2}$ and $\theta = e^{-ik_2L/2}$

$$\psi_{\mathrm{I}}(-L/2) = \psi_{\mathrm{II}}(-L/2) \quad \text{and} \quad \frac{\mathrm{d}\psi_{\mathrm{I}}}{\mathrm{d}x}(-L/2) = \frac{\mathrm{d}\psi_{\mathrm{II}}}{\mathrm{d}x}(-L/2)$$

$$A\phi^* + B\phi = C\theta^* + D\theta$$

$$ik_1 A\phi^* - ik_1 B\phi = ik_2 C\theta^* - ik_2 D\theta \qquad \Longrightarrow \begin{bmatrix} \phi^* & \phi \\ ik_1\phi^* & -ik_1\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \theta^* & \theta \\ ik_2\theta^* & -ik_2\theta \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} \theta^* & \theta \\ ik_2\theta^* & -ik_2\theta \end{bmatrix}^{-1} \begin{bmatrix} \phi^* & \phi \\ ik_1\phi^* & -ik_1\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

Very similarly,

$$\begin{bmatrix} \theta & \theta^* \\ ik_2\theta & -ik_1\theta^* \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \phi & \phi^* \\ ik_1\phi & -ik_2\phi^* \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \phi & \phi^* \\ ik_1\phi & -ik_2\phi^* \end{bmatrix}^{-1} \begin{bmatrix} \theta & \theta^* \\ ik_2\theta & -ik_1\theta^* \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$\begin{bmatrix} E \\ 0 \end{bmatrix}$ can be expressed in terms of $\begin{bmatrix} A \\ B \end{bmatrix}$.

$$\begin{bmatrix} \theta^* & \theta \\ ik_2\theta^* & -ik_2\theta \end{bmatrix}^{-1} \begin{bmatrix} \phi^* & \phi \\ ik_1\phi^* & -ik_1\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} \phi & \phi^* \\ ik_1\phi & -ik_2\phi^* \end{bmatrix}^{-1} \begin{bmatrix} \theta & \theta^* \\ ik_2\theta & -ik_1\theta^* \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} \phi & \phi^* \\ ik_1\phi & -ik_2\phi^* \end{bmatrix}^{-1} \begin{bmatrix} \theta & \theta^* \\ ik_2\theta & -ik_1\theta^* \end{bmatrix} \begin{bmatrix} \theta^* & \theta \\ ik_2\theta^* & -ik_2\theta \end{bmatrix}^{-1} \begin{bmatrix} \phi^* & \phi \\ ik_1\phi^* & -ik_1\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

```
In[*]:= ϕ = Exp[I*k1*L/2]
```

Out[*]= $e^{\frac{i\, k1\, L}{2}}$

```
In[*]:= θ = Exp[I*k2*L/2]
```

Out[*]= $e^{\frac{i\, k2\, L}{2}}$

In[*]:= M1 = $\begin{pmatrix} \phi\wedge(-1) & \phi \\ I*k1*\phi\wedge(-1) & -I*k1*\phi \end{pmatrix}$

Out[*]= $\left\{\left\{e^{-\frac{1}{2} i\, k1\, L},\ e^{\frac{i\, k1\, L}{2}}\right\},\ \left\{i\, e^{-\frac{1}{2} i\, k1\, L}\, k1,\ -i\, e^{\frac{i\, k1\, L}{2}}\, k1\right\}\right\}$

In[*]:= M2 = $\begin{pmatrix} \theta\wedge(-1) & \theta \\ I*k2*\theta\wedge(-1) & -I*k2*\theta \end{pmatrix}$

Out[*]= $\left\{\left\{e^{-\frac{1}{2} i\, k2\, L},\ e^{\frac{i\, k2\, L}{2}}\right\},\ \left\{i\, e^{-\frac{1}{2} i\, k2\, L}\, k2,\ -i\, e^{\frac{i\, k2\, L}{2}}\, k2\right\}\right\}$

In[*]:= M3 = $\begin{pmatrix} \theta & \theta\wedge(-1) \\ I*k2*\theta & -I*k2*\theta\wedge(-1) \end{pmatrix}$

Out[*]= $\left\{\left\{e^{\frac{i\, k2\, L}{2}},\ e^{-\frac{1}{2} i\, k2\, L}\right\},\ \left\{i\, e^{\frac{i\, k2\, L}{2}}\, k2,\ -i\, e^{-\frac{1}{2} i\, k2\, L}\, k2\right\}\right\}$

In[*]:= M4 = $\begin{pmatrix} \phi & \phi\wedge(-1) \\ I*k1*\phi & -I*k1*\phi\wedge(-1) \end{pmatrix}$

Out[*]= $\left\{\left\{e^{\frac{i\, k1\, L}{2}},\ e^{-\frac{1}{2} i\, k1\, L}\right\},\ \left\{i\, e^{\frac{i\, k1\, L}{2}}\, k1,\ -i\, e^{-\frac{1}{2} i\, k1\, L}\, k1\right\}\right\}$

In[*]:= E1 = $\begin{pmatrix} F \\ 0 \end{pmatrix}$

Out[*]= {{F}, {0}}

```
In[*]:= FullSimplify[Inverse[M1].M2.Inverse[M3].M4.E1]
```


In[*]:= M4 = $\begin{pmatrix} \phi & \phi\wedge(-1) \\ I*k1*\phi & -I*k1*\phi\wedge(-1) \end{pmatrix}$

Out[*]= $\left\{\left\{e^{\frac{i\, k1\, L}{2}},\ e^{-\frac{1}{2} i\, k1\, L}\right\},\ \left\{i\, e^{\frac{i\, k1\, L}{2}}\, k1,\ -i\, e^{-\frac{1}{2} i\, k1\, L}\, k1\right\}\right\}$

In[*]:= E1 = $\begin{pmatrix} F \\ 0 \end{pmatrix}$

Out[*]= {{F}, {0}}

```
In[*]:= FullSimplify[Inverse[M1].M2.Inverse[M3].M4.E1]
```


Out[*]= $\left\{\left\{\frac{e^{i\,(k1-k2)\,L}\, F\left(-e^{2\, i\, k2\, L}\,(k1-k2)^2+(k1+k2)^2\right)}{4\, k1\, k2}\right\},\ \left\{-\frac{i\, F\,(k1-k2)\,(k1+k2)\,\mathrm{Sin}[k2\, L]}{2\, k1\, k2}\right\}\right\}$

In[*]:= R = FullSimplify$\left[\left(\left(-\frac{i\, F\,(k1-k2)\,(k1+k2)\,\mathrm{Sin}[k2\, L]}{2\, k1\, k2}\right)\left(\frac{i\, F\,(k1-k2)\,(k1+k2)\,\mathrm{Sin}[k2\, L]}{2\, k1\, k2}\right)\right)\Big/\left(\left(\frac{e^{i\,(k1-k2)\,L}\, F\left(-e^{2\, i\, k2\, L}\,(k1-k2)^2+(k1+k2)^2\right)}{4\, k1\, k2}\right)\left(\frac{e^{-i\,(k1-k2)\,L}\, F\left(-e^{-2\, i\, k2\, L}\,(k1-k2)^2+(k1+k2)^2\right)}{4\, k1\, k2}\right)\right)\right]$

Out[*]= $\frac{\left(k1^2-k2^2\right)^2 \mathrm{Sin}[k2\, L]^2}{4\, k1^2\, k2^2\, \mathrm{Cos}[k2\, L]^2+\left(k1^2+k2^2\right)^2 \mathrm{Sin}[k2\, L]^2}$

In[*]:= T = FullSimplify$\left[1\Big/\left(\left(\frac{e^{i\,(k1-k2)\,L}\left(-e^{2\, i\, k2\, L}\,(k1-k2)^2+(k1+k2)^2\right)}{4\, k1\, k2}\right)\left(\frac{e^{-i\,(k1-k2)\,L}\left(-e^{-2\, i\, k2\, L}\,(k1-k2)^2+(k1+k2)^2\right)}{4\, k1\, k2}\right)\right)\right]$

Out[*]= $\frac{1}{\mathrm{Cos}[k2\, L]^2+\frac{\left(k1^2+k2^2\right)^2 \mathrm{Sin}[k2\, L]^2}{4\, k1^2\, k2^2}}$

```
In[*]:= FullSimplify[T + R]
```

Out[*]= 1

Figure 1: ./ss/9/2.png

Exactly putting the computation into wolfram mathematica

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} E\dfrac{e^{1+i(k_1-k_2)L}\left((k_1+k_2)^2-(k_1-k_2)^2e^{2ik_2L}\right)}{4k_1k_2} \\ \\ -iE\dfrac{(k_1-k_2)(k_1+k_2)\sin(k_2L)}{2k_1k_2} \end{bmatrix}$$

$$\begin{aligned} \frac{B^2}{A^2} &= \frac{J_B}{J_A} = \frac{(k_1^2-k_2^2)\sin^2(k_2L)}{4k_1^2k_2^2\cos^2(k_2L)+(k_1^2+k_2^2)^2\sin^2(k_2L)} \\ \frac{E^2}{A^2} &= \frac{J_E}{J_A} = \frac{1}{\cos^2(k_2L)+\frac{(k_1^2+k_2^2)^2\sin^2(k_2L)}{4k_1^2k_2^2}} \end{aligned}$$

b - i

$$T+R = \frac{J_E}{J_A}+\frac{J_B}{J_A} = \frac{1}{\cos^2(k_2L)+\frac{(k_1^2+k_2^2)\sin^2(k_2L)}{4k_1^2k_2^2}} + \frac{(k_1^2-k_2^2)\sin^2(k_2L)}{4k_1^2k_2^2\cos^2(k_2L)+(k_1^2+k_2^2)\sin^2(k_2L)} = 1$$

b - ii

$$\begin{aligned} T &= \frac{1}{\cos^2(k_2L)+\frac{(k_1^2+k_2^2)^2\sin^2(k_2L)}{4k_1^2k_2^2}} \\ &\to \frac{1}{\cos^2(k_1L)+\frac{(k_1^2+k_1^2)^2\sin^2(k_1L)}{4k_1^2k_1^2}} = \frac{1}{\cos^2(k_1L)+\sin^2(k_1L)} = 1 \end{aligned}$$

b - iii

$$T = \frac{1}{\cos^2(0)+\frac{(k_1^2+k_2^2)^2\sin^2(0)}{4k_1^2k_2^2}} = \frac{1}{1+0} = 1$$

b - iv

$$\cos^2(x) \approx \left(1-\frac{x^2}{2}+\cdots\right)\left(1-\frac{x^2}{2}+\cdots\right) = \left(1-\frac{x^2}{2}-\frac{x^2}{2}+\cdots\right) = 1-x^2+\cdots$$

$$\begin{aligned} \lim_{k_2\to 0} T &\approx \frac{1}{1-k_2^2L^2+\frac{k_1^4k_2^2L^2}{4k_1^2k_2^2}} && \text{(I am only keeping } k_2^2 \text{ terms)} \\ &= \frac{1}{1+\frac{k_1^2L^2}{4}} \end{aligned}$$

Problem 05

Re-written form and taking a small integral from $-\varepsilon$ to ε

$$
\begin{aligned}
\frac{\hbar^2\lambda}{m}\delta(x)\psi(x) - E\psi(x) &= \frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) \\
\int_{-\varepsilon}^{\varepsilon}\left[\frac{\hbar^2\lambda}{m}\delta(x-0)\psi(x) - E\psi(x)\right]\mathrm{d}x &= \frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)\,\mathrm{d}x \\
\frac{\hbar^2\lambda}{m}\psi(0) &= \frac{\hbar^2}{2m}\left[\frac{\mathrm{d}\psi(0)_+}{\mathrm{d}x} - \frac{\mathrm{d}\psi(0)_-}{\mathrm{d}x}\right] \\
2\lambda\psi(0) &= \frac{\mathrm{d}\psi(0)_+}{\mathrm{d}x} - \frac{\mathrm{d}\psi(0)_-}{\mathrm{d}x}
\end{aligned}
$$

This gives us the discontinuity at the $x = 0$ position.

Now using the equality of wave function $\psi(0)_< = \psi(0)_>$ on both of the sides of barrier we get

$$A + B = C$$

And using above discontinuity, realizing for $x \leq 0$ (hence also $x = 0$) the wave function is given by $\psi_<(x)$,

$$2\lambda\,(A+B) = ikC - ik\,(A-B) \implies 2\lambda C = ikC - ik\,(A-B)$$

Solving some algebra

$$
\begin{aligned}
2\lambda C &= ik(C - A + B) \\
2\lambda C &= ik(A + B - A + B) \\
C &= \frac{ik}{\lambda}B
\end{aligned}
$$

$$
\begin{aligned}
\implies A + B = C &= \frac{ik}{\lambda}B \\
1 + \frac{B}{A} &= \frac{ik}{\lambda}\frac{B}{A} \\
1 &= \left(\frac{ik}{\lambda} - 1\right)\frac{B}{A} \\
\frac{B}{A} &= \frac{1}{\dfrac{ik}{\lambda} - 1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{C}{A} &= \frac{ik}{\lambda}\frac{B}{A} \\
\frac{C}{A} &= \frac{\frac{ik}{\lambda}}{\frac{ik}{\lambda} - 1}
\end{aligned}
$$

To solve for following

$$
\begin{aligned}
\frac{B}{A} &= \frac{1}{\dfrac{ik}{\lambda} - 1} \\
\frac{C}{A} &= \frac{\frac{ik}{\lambda}}{\frac{ik}{\lambda} - 1} \\
R &= \frac{\lambda^2}{k^2 + \lambda^2} \\
T &= \frac{k^2}{\lambda^2 + k^2} \\
R + T &= 1
\end{aligned}
$$