Quantum Mechanics : : Homework 07

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Problem 1

When we say that the matrices are written in $|\uparrow\rangle_z$ and $\langle\uparrow|_z$ then

$$\hat{S} = \begin{pmatrix} \langle\uparrow| S |\uparrow\rangle & \langle\uparrow| S |\downarrow\rangle \\ \langle\downarrow| S |\uparrow\rangle & \langle\downarrow| S |\downarrow\rangle \end{pmatrix}$$

(a)

$$(S_x - mI)|m\rangle \implies \frac{\hbar}{2}(\sigma_1 - \lambda I)|\lambda\rangle = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Solving the determinant equals 0 for the operator we get $\lambda = 1, -1$ or $m = \frac{\hbar}{2}, -\frac{\hbar}{2}$ The eigenvectors are (I did the computation on paper, check appendix)

$$|\lambda\rangle_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly

$$(\sigma_2 - \lambda I)|\lambda\rangle = \begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Solving for the determinant of operator gives us $\lambda = 1, 1$ again and the associated eigenvectors are

$$|\lambda\rangle_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

(b)

At first let's solve for $\langle\psi|S_i|\psi\rangle$

$$|\psi\rangle = \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z \implies \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \langle\psi| S |\psi\rangle &= \sum_i \langle\psi| S |\lambda_i\rangle \langle\lambda_i|\psi\rangle \\ &= \sum_j \sum_i \langle\lambda_j| S |\lambda_i\rangle \langle\psi|\lambda_j\rangle \langle\lambda_i|\psi\rangle \\ &= \frac{\hbar}{2} \sum_j \sum_i \langle\lambda_j| \sigma |\lambda_i\rangle \langle\psi|\lambda_j\rangle \langle\lambda_i|\psi\rangle \end{aligned}$$

In problem 2 we use a notation where

$$\langle\psi| S_i |\psi\rangle = (\vec{n})_i$$

$$
\begin{aligned}
(\vec{n})_p = \langle\psi|\,\hat{\sigma}_p\,|\psi\rangle &= \sum_j\sum_i \langle\psi_j|\,\sigma_p\,|\psi_i\rangle\,\langle\psi|\psi_j\rangle\,\langle\psi_i|\psi\rangle \\
&= \langle\psi|\uparrow\rangle\,\langle\uparrow|\,\sigma_p\,|\uparrow\rangle\,\langle\uparrow|\psi\rangle \\
&+ \langle\psi|\uparrow\rangle\,\langle\uparrow|\,\sigma_p\,|\downarrow\rangle\,\langle\downarrow|\psi\rangle \\
&+ \langle\psi|\downarrow\rangle\,\langle\downarrow|\,\sigma_p\,|\uparrow\rangle\,\langle\uparrow|\psi\rangle \\
&+ \langle\psi|\downarrow\rangle\,\langle\downarrow|\,\sigma_p\,|\downarrow\rangle\,\langle\downarrow|\psi\rangle
\end{aligned}
$$

For each pauli matrices we can grind the above summation and by doing the whole thing I get

$$
\begin{aligned}
\langle\psi|\sigma_1|\psi\rangle &= (\vec{n})_x = \alpha^*\beta + \alpha\beta^* \\
\langle\psi|\sigma_2|\psi\rangle &= (\vec{n})_y = i\,(\alpha\beta^* - \alpha^*\beta) \\
\langle\psi|\sigma_3|\psi\rangle &= (\vec{n})_z = |\alpha|^2 - |\beta|^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\hbar}{2}\,\langle\psi|\sigma_1|\psi\rangle &= \langle\psi|S_x|\psi\rangle \\
\frac{\hbar}{2}\,\langle\psi|\sigma_2|\psi\rangle &= \langle\psi|S_y|\psi\rangle \\
\frac{\hbar}{2}\,\langle\psi|\sigma_3|\psi\rangle &= \langle\psi|S_z|\psi\rangle
\end{aligned}
$$

Now we will compute $\langle\psi|S_i^2|\psi\rangle$

$$
S_i^2 = S_iS_i = \frac{\hbar^2}{2^2}\sigma_i\sigma_i = \frac{\hbar^2}{2^2}\hat{I}
$$

Hence for normalized ψ

$$
\langle\psi|S_i^2|\psi\rangle = \frac{\hbar^2}{2^2}\,\langle\psi|\psi\rangle = \frac{\hbar^2}{2^2}
$$

$$
\begin{aligned}
\Delta S_x = \sqrt{\langle\psi|\,S_x^2\,|\psi\rangle - (\langle\psi|\,S_x\,|\psi\rangle)^2} &= \sqrt{\frac{\hbar^2}{2^2} - \frac{\hbar^2}{2^2}(\vec{n})_x^2} \\
&= \frac{\hbar}{2}\sqrt{1-(\alpha^*\beta+\alpha\beta^*)^2} \\
\Delta S_y = \sqrt{\langle\psi|\,S_y^2\,|\psi\rangle - (\langle\psi|\,S_y\,|\psi\rangle)^2} &= \sqrt{\frac{\hbar^2}{2^2} - \frac{\hbar^2}{2^2}(\vec{n})_y^2} \\
&= \frac{\hbar}{2}\sqrt{1+(-\alpha^*\beta+\alpha\beta^*)^2}
\end{aligned}
$$

(b): Vanishing for Eigenstate case

Eigenstates for S_x are

$$
\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
$$

The entries are all real.

$$
(\alpha,\beta) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \implies (\vec{n})_x = \alpha\beta + \alpha\beta = 1
$$

$$(\alpha, \beta) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \implies (\vec{n})_x = \alpha\beta + \alpha\beta = -1$$

Eigenstates for S_y are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}$$

$$(\alpha, \beta) = \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) \implies (\vec{n})_y/i = -\alpha^*\beta + \alpha\beta^* = -i$$

$$(\alpha, \beta) = \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) \implies (\vec{n})_y/i = -\alpha^*\beta + \alpha\beta^* = i$$

So we get $(n)_x^2 = 1$ and $(n)_y^2/i = -1$

$$\begin{aligned}
\Delta S_x = \sqrt{\langle\psi| S_x^2 |\psi\rangle - (\langle\psi| S_x |\psi\rangle)^2} &= \sqrt{\frac{\hbar^2}{2^2} - \frac{\hbar^2}{2^2}(\vec{n})_x^2} \\
&= \frac{\hbar}{2}\sqrt{1-1} = 0 \\
\Delta S_y = \sqrt{\langle\psi| S_y^2 |\psi\rangle - (\langle\psi| S_y |\psi\rangle)^2} &= \sqrt{\frac{\hbar^2}{2^2} - \frac{\hbar^2}{2^2}(\vec{n})_y^2} \\
&= \frac{\hbar}{2}\sqrt{1-1} = 0
\end{aligned}$$

So the eigenvector cases zero out.

Problem 2

(a)

For $\psi = \alpha |\uparrow\rangle_z + \beta |\downarrow\rangle_z$

$$\begin{aligned}
(\vec{n})_p = \langle\psi| \hat{\sigma}_p |\psi\rangle = \sum_j \sum_i & \langle\psi_j| \sigma_p |\psi_i\rangle \langle\psi|\psi_j\rangle \langle\psi_i|\psi\rangle \\
= & \langle\psi| \uparrow\rangle \langle\uparrow| \sigma_p |\uparrow\rangle \langle\uparrow |\psi\rangle \\
+ & \langle\psi| \uparrow\rangle \langle\uparrow| \sigma_p |\downarrow\rangle \langle\downarrow |\psi\rangle \\
+ & \langle\psi| \downarrow\rangle \langle\downarrow| \sigma_p |\uparrow\rangle \langle\uparrow |\psi\rangle \\
+ & \langle\psi| \downarrow\rangle \langle\downarrow| \sigma_p |\downarrow\rangle \langle\downarrow |\psi\rangle
\end{aligned}$$

For each pauli matrices we can grind the above summation and by doing the whole thing I get

$$\begin{aligned}
(\vec{n})_x &= \alpha^*\beta + \alpha\beta^* \\
(\vec{n})_y &= i\,(\alpha\beta^* - \alpha^*\beta) \\
(\vec{n})_z &= |\alpha|^2 - |\beta|^2
\end{aligned}$$

$$
\begin{aligned}
(\vec{n})_x^2 + (\vec{n})_y^2 + (\vec{n})_z^2 &= (\alpha^*\beta)^2 + (\alpha\beta^*)^2 + 2(|\alpha|^2|\beta|^2) \\
&+ -(\alpha^*\beta)^2 - (\alpha\beta^*)^2 + 2(|\alpha|^2|\beta|^2) \\
&+ (|\alpha|^2)^2 + (|\beta|^2)^2 - 2(|\alpha|^2|\beta|^2) \\
&= (|\alpha|^2 + |\beta|^2)^2 \\
&= \langle\psi|\psi\rangle^2 \\
&= 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
|\psi\rangle\langle\psi| &= [\alpha|\uparrow\rangle + \beta|\downarrow\rangle][\alpha^*\langle\uparrow| + \beta^*\langle\downarrow|] \\
&= \alpha\alpha^*|\uparrow\rangle\langle\uparrow| + \alpha\beta^*|\uparrow\rangle\langle\downarrow| + \alpha^*\beta|\downarrow\rangle\langle\uparrow| + \beta\beta^*|\downarrow\rangle\langle\downarrow|
\end{aligned}
$$

$$
\begin{aligned}
\vec{n}\cdot\hat{\vec{S}} &= (\vec{n})_x\hat{S}_1 + (\vec{n})_y\hat{S}_2 + (\vec{n})_z\hat{S}_3 \\
&= \langle\psi|\sigma_1|\psi\rangle\hat{S}_1 + \langle\psi|\sigma_2|\psi\rangle\hat{S}_2 + \langle\psi|\sigma_3|\psi\rangle\hat{S}_3 \\
&= \frac{\hbar}{2}\left(\langle\psi|\sigma_1|\psi\rangle\sigma_1 + \langle\psi|\sigma_2|\psi\rangle\sigma_2 + \langle\psi|\sigma_3|\psi\rangle\sigma_3\right) \\
\vec{n}\cdot\hat{\vec{S}}|\psi\rangle &= \frac{\hbar}{2}\left(\langle\psi|\sigma_1|\psi\rangle\sigma_1 + \langle\psi|\sigma_2|\psi\rangle\sigma_2 + \langle\psi|\sigma_3|\psi\rangle\sigma_3\right)|\psi\rangle \\
&= \frac{\hbar}{2}\left(\langle\psi|\sigma_1|\psi\rangle\sigma_1|\psi\rangle + \langle\psi|\sigma_2|\psi\rangle\sigma_2|\psi\rangle + \langle\psi|\sigma_3|\psi\rangle\sigma_3|\psi\rangle\right) \\
\langle\psi|\vec{n}\cdot\hat{\vec{S}}|\psi\rangle &= \frac{\hbar}{2}\left(\langle\psi|\sigma_1|\psi\rangle\langle\psi|\sigma_1|\psi\rangle + \langle\psi|\sigma_2|\psi\rangle\langle\psi|\sigma_2|\psi\rangle + \langle\psi|\sigma_3|\psi\rangle\langle\psi|\sigma_3|\psi\rangle\right) \\
&= \frac{\hbar}{2}\left((\vec{n})_x^2 + (\vec{n})_y^2 + (\vec{n})_z^2\right) \\
&= \frac{\hbar}{2}
\end{aligned}
$$

So we have

$$
\begin{aligned}
\langle\psi|\vec{n}\cdot\hat{\vec{S}}|\psi\rangle &= \frac{\hbar}{2}\langle\psi|\psi\rangle \\
\langle\psi|\left(\vec{n}\cdot\hat{\vec{S}}|\psi\rangle\right) &= \langle\psi|\left(\frac{\hbar}{2}|\psi\rangle\right) \\
\text{uniqueness} \implies \vec{n}\cdot\hat{\vec{S}}|\psi\rangle &= \frac{\hbar}{2}|\psi\rangle
\end{aligned}
$$

Quick proof for uniqueness of ket's

Say $\langle a|b\rangle = \langle a|c\rangle$ and $|b\rangle \neq |c\rangle$. Then by linearity

$$0 = \langle a|b\rangle - \langle a|c\rangle = \langle a|\left(|b\rangle - |c\rangle\right)$$

The first condition $\langle a|b\rangle = \langle a|c\rangle$ breaks if $|b\rangle \neq |c\rangle \implies |b\rangle - |c\rangle \neq 0$ hence for the first condition to hold it's required that $|b\rangle = |c\rangle$.

For above solution consider $|b\rangle = \vec{n}\cdot\hat{\vec{S}}|\psi\rangle$ and $|c\rangle = (\hbar/2)|\psi\rangle$.

Problem 3

(a)

$$
\begin{aligned}
\exp\left(-i\frac{\sigma_2}{2}\theta_1\right) &= \cos(\theta_1/2)\hat{I} - i\sigma_2\sin(\theta_1/2) \\
&= \cos(\theta_1/2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}(-i\sin(\theta_1/2)) \\
&= \cos(\theta_1/2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}(\sin(\theta_1/2)) \\
\\
&= \begin{bmatrix}\cos(\theta_1/2) & -\sin(\theta_1/2)\\ \sin(\theta_1/2) & \cos(\theta_1/2)\end{bmatrix}
\end{aligned}
$$

Note next computation is done by θ_1, we will fix it to be θ_2 later.

$$
\begin{aligned}
\exp\left(-i\frac{\sigma_3}{2}\theta_1\right) &= \cos(\theta_1/2)\hat{I} - i\sigma_3\sin(\theta_1/2) \\
&= \cos(\theta_1/2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}(-i\sin(\theta_1/2)) \\
&= \cos(\theta_1/2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}(\sin(\theta_1/2)) \\
\\
&= \begin{bmatrix}\cos(\theta_1/2) - i\sin(\theta_1/2) & 0\\ 0 & \cos(\theta_1/2) + i\sin(\theta_1/2)\end{bmatrix}
\end{aligned}
$$

Finalizing what we have gotten

$$
\begin{aligned}
\exp\left(-i\frac{\sigma_2}{2}\theta_1\right) &= \begin{bmatrix}\cos(\theta_1/2) & -\sin(\theta_1/2)\\ \sin(\theta_1/2) & \cos(\theta_1/2)\end{bmatrix} \\
\exp\left(-i\frac{\sigma_3}{2}\theta_2\right) &= \begin{bmatrix}\cos(\theta_2/2) - i\sin(\theta_2/2) & 0\\ 0 & \cos(\theta_2/2) + i\sin(\theta_1/2)\end{bmatrix} \\
\exp\left(-i\frac{\sigma_2}{2}\theta_3\right) &= \begin{bmatrix}\cos(\theta_3/2) & -\sin(\theta_3/2)\\ \sin(\theta_3/2) & \cos(\theta_3/2)\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(-i\frac{\sigma_2}{2}\theta_3\right)\exp\left(-i\frac{\sigma_3}{2}\theta_2\right)\exp\left(-i\frac{\sigma_2}{2}\theta_1\right) \\
&= \begin{bmatrix}\cos(\theta_3/2) & -\sin(\theta_3/2)\\ \sin(\theta_3/2) & \cos(\theta_3/2)\end{bmatrix}\begin{bmatrix}\cos(\theta_2/2) - i\sin(\theta_2/2) & 0\\ 0 & \cos(\theta_2/2) + i\sin(\theta_1/2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1/2) & -\sin(\theta_1/2)\\ \sin(\theta_1/2) & \cos(\theta_1/2)\end{bmatrix}
\end{aligned}
$$

Note of Shame: I have been trying to find a general solution to this for 2 days now. I just realized we can just set $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $\theta_3 = -\frac{\pi}{2}$ just because I didn't read the problem properly. I literally tried to run Rodriguez Rotation equations to get somewhere. Sigh.

I have attached the symbolab crunch for the given $\theta_1, \theta_2, \theta_3$ values and what I have gotten is

$$\begin{aligned}
\begin{bmatrix} \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} + i \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(-i\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\hat{I} - i\sigma_1 \left(-\frac{1}{\sqrt{2}}\right) \\
&= \cos\left(\frac{\pi/2}{2}\right)\hat{I} - i((-\vec{n}_x) \cdot \hat{\vec{\sigma}}) \sin\left(\frac{\pi/2}{2}\right) \\
&= \cos\left(\frac{\phi}{2}\right) - i(\vec{n}_\phi \cdot \hat{\vec{\sigma}}) \sin\left(\frac{\phi}{2}\right) && (\vec{\phi} = \phi\vec{n}_\phi)
\end{aligned}$$

$$\cos\left(\frac{\phi}{2}\right) = \frac{1}{\sqrt{2}} \quad \text{(and)} \quad \sin\left(\frac{\phi}{2}\right) = \frac{1}{\sqrt{2}} \implies \frac{\phi}{2} = \frac{\pi}{4} \implies \phi = \frac{\pi}{2}$$

We got

$$\boxed{\vec{\phi} = -\frac{\pi}{2}\vec{n}_x}$$

$$\begin{pmatrix}\cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right)\\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right) & 0\\ 0 & \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right)\\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)\end{pmatrix}$$

Solution

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & i\frac{\sqrt{2}}{2}\\ i\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

Solution steps

$$\begin{pmatrix}\cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right)\\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right) & 0\\ 0 & \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right)\\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)\end{pmatrix}$$

$$\begin{pmatrix}\cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right)\\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right) & 0\\ 0 & \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}-i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\\ -\frac{1}{2}+i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\end{pmatrix}$$

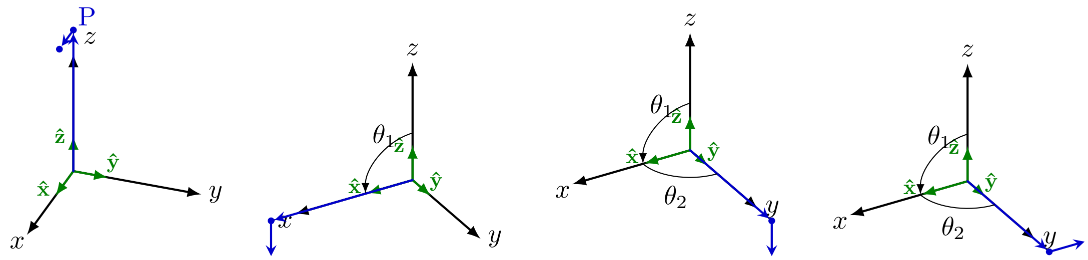
$$=\begin{pmatrix}\frac{1}{2}-i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\\ -\frac{1}{2}+i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right)\\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}-i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\\ -\frac{1}{2}+i\frac{1}{2} & \frac{1}{2}+i\frac{1}{2}\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right)\\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & i\frac{\sqrt{2}}{2}\\ i\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}} & i\frac{\sqrt{2}}{2}\\ i\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

(b): insane place to type 300 additional lines of TiKZ code

This state is basically a **spinor** so I need to show a flagpole diagram to show what happens here.



The vector now lies along $\vec{y}$ which now validates the $\pi/2$ angle difference.

$$\vec{s} = s\exp\left(-i\alpha/2\right)\begin{bmatrix}\exp(-i\phi/2)\cos(\theta/2)\\ \exp(i\phi/2)\sin(\theta/2)\end{bmatrix}$$

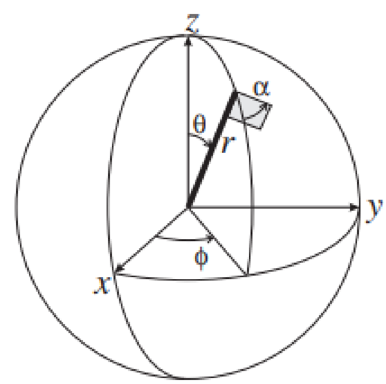


FIG. 1: A spinor. The spinor has a direction in space ('flag-pole'), an orientation about this axis ('flag'), and an overall sign (not shown). A suitable set of parameters to describe the spinor state, up to a sign, is $(r, \theta, \phi, \alpha)$, as shown. The first three fix the length and direction of the flagpole by using standard spherical coordinates, the last gives the orientation of the flag.

Problem 4

(a)

The states as time progress is well understood which is

$$
\begin{aligned}
|\psi_0\rangle &:= |m=1\rangle_z && (t=0)\\
|\psi_1\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}_1 t\right)|\psi_0\rangle && (0 \le t \le t_y)\\
|\psi_2\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}_2 (t-t_y)\right)\exp\left(-\frac{i}{\hbar}\hat{H}_1 t_y\right)|\psi_0\rangle && (t > t_y)
\end{aligned}
$$

The Hamiltonian is given by

$$
\hat{H}(t) = \begin{cases} \hat{H}_1 = |\gamma| B_y \hat{S}_y & 0 < t < t_y \\ \hat{H}_2 = |\gamma| B_z \hat{S}_z & t > t_y \end{cases}
$$

Crunch the computation of $|\psi_1\rangle$

$$
\begin{aligned}
|\psi_1(t_y)\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}_1 t_y\right)|\psi_0\rangle\\
&= \exp\left(-\frac{i}{\hbar}|\gamma| B_y \hat{S}_y \left(\frac{\pi}{2|\gamma| B_y}\right)\right)|\psi_0\rangle\\
&= \exp\left(-\frac{i}{\hbar}\hat{S}_y \frac{\pi}{2}\right)|\psi_0\rangle\\
&= \hat{R}_y\left(\frac{\pi}{2}\right)|\psi_0\rangle = \begin{pmatrix} \frac{1+\cos(\pi/2)}{2} \\ \frac{\sin(\pi/2)}{\sqrt{2}} \\ \frac{1-\cos(\pi/2)}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} && \text{(from previous problem solution)}
\end{aligned}
$$

Use this to finally find the required $|\psi_2\rangle$ and calling $|\psi_1(t_y)\rangle \equiv |\psi_1\rangle$

$$
\begin{aligned}
|\psi_2\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}_2 t + \frac{i}{\hbar}\hat{H}_2 \frac{\pi}{2|\gamma| B_y}\right)|\psi_1\rangle\\
&= \exp\left(-\frac{i}{\hbar}|\gamma| B_z \hat{S}_z + \frac{i}{\hbar}|\gamma| B_z \hat{S}_z \frac{\pi}{2|\gamma| B_y}\right)|\psi_1\rangle\\
&= \exp\left(-\frac{i}{\hbar}|\gamma| B_z \hat{S}_z + \frac{i}{\hbar}\frac{B_z}{B_y}\frac{\pi}{2}\hat{S}_z\right)|\psi_1\rangle\\
&= \exp\left(-\frac{i}{\hbar}|\gamma| B_z \hat{S}_z\right)\exp\left(\frac{i}{\hbar}\frac{B_z}{B_y}\frac{\pi}{2}\hat{S}_z\right)|\psi_1\rangle
\end{aligned}
$$

The matrix $\hat{S}_z$ with some scalar $\Lambda' = \Lambda/\hbar$ so that it also get's rid of $\hbar$ (for computational ease), and I don't really make much distinction from Λ, Λ', so you can imagine that $\hbar\Lambda \implies \Lambda$ where planks constant is sucked into the Λ

and we will take care of it at the end by pulling it out.

$$
\begin{aligned}
\hat{S}_z &= \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
\Lambda \hat{S}_z &= \begin{bmatrix} \hbar\Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar\Lambda \end{bmatrix} \underbrace{\Longrightarrow}_{\text{suck } \hbar \text{ into } \Lambda} \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Lambda \end{bmatrix} \\
\mathrm{e}^{i\Lambda\hat{S}_z} = \exp\left(i\Lambda\hat{S}_z\right) &= \hat{I} + i\frac{\Lambda\hat{S}_z}{1!} + i^2\frac{\Lambda^2\hat{S}_z^2}{2!} + i^3\frac{\Lambda^3\hat{S}_z^3}{3!} + i^4\frac{\Lambda^4\hat{S}_z^4}{4!} + \cdots \\
&= \hat{I} + \Lambda \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} - \frac{\Lambda^2}{2!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\Lambda^3}{3!} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} + \frac{\Lambda^4}{4!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cdots \\
&= \hat{I} - \frac{\Lambda^2}{2!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\Lambda^4}{4!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cdots + i \left(\Lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{\Lambda^3}{3!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \cdots \right) \\
&= \hat{I} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\Lambda^2}{2!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\Lambda^4}{4!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cdots \\
&\quad + \cdots + i \left(\Lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{\Lambda^3}{3!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \cdots \right) \\
&= \left(\hat{I} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) + \left(1 - \frac{\Lambda^2}{2!} + \frac{\Lambda^4}{4!} + \cdots \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + i \left(\Lambda - \frac{\Lambda^3}{3!} + \cdots \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) + \begin{bmatrix} \cos\Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos\Lambda \end{bmatrix} + \begin{bmatrix} i\sin\Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i\sin\Lambda \end{bmatrix} \\
&= \begin{bmatrix} \cos\Lambda + i\sin\Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\Lambda - i\sin\Lambda \end{bmatrix} \\
&= \begin{bmatrix} \exp(i\Lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i\Lambda) \end{bmatrix} \underbrace{\Longrightarrow}_{\text{push } \hbar \text{ out}} \begin{bmatrix} \exp(i\hbar\Lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i\hbar\Lambda) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(-\frac{i}{\hbar}|\gamma|B_z\hat{S}_z t\right) \exp\left(\frac{i}{\hbar}\frac{B_z}{B_y}\frac{\pi}{2}\hat{S}_z\right)|\psi_1\rangle \\
&= \begin{bmatrix} \exp(-i|\gamma|B_z t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i|\gamma|B_z t) \end{bmatrix} \begin{bmatrix} \exp\left(i\frac{B_z}{B_y}\frac{\pi}{2}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp\left(-i\frac{B_z}{B_y}\frac{\pi}{2}\right) \end{bmatrix} \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \\
&= \begin{bmatrix} \exp(-i|\gamma|B_z t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i|\gamma|B_z t) \end{bmatrix} \begin{bmatrix} (1/2)\exp\left(i\frac{B_z}{B_y}\frac{\pi}{2}\right) \\ 1/\sqrt{2} \\ (1/2)\exp\left(-i\frac{B_z}{B_y}\frac{\pi}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} (1/2)\exp\left(i\frac{B_z}{B_y}\frac{\pi}{2} - i|\gamma|B_z t\right) \\ 1/\sqrt{2} \\ (1/2)\exp\left(-i\frac{B_z}{B_y}\frac{\pi}{2} + i|\gamma|B_z t\right) \end{bmatrix}
\end{aligned}
$$

So after all these algebraic and matrix warfare

$$|\psi_2(t)\rangle = \begin{bmatrix} (1/2)\exp\left(i\frac{B_z}{B_y}\frac{\pi}{2} - i|\gamma|B_z t\right) \\ 1/\sqrt{2} \\ (1/2)\exp\left(-i\frac{B_z}{B_y}\frac{\pi}{2} + i|\gamma|B_z t\right) \end{bmatrix} \equiv \begin{bmatrix} (1/2)\exp(-i\theta) \\ 1/\sqrt{2} \\ (1/2)\exp(i\theta) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-i\theta} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2}e^{i\theta} \end{bmatrix}$$

(b)

For any desired state $|\Psi\rangle$ the probability of getting that

$$\langle\psi_2(t)|\Psi|\rangle^2 \implies \langle\Psi|\psi_2(t)\rangle\,\langle\psi_2(t)|\Psi\rangle$$

The desired state this case

$$\hat{S}_z\,|m=1\rangle_z = \hbar\,|m=1\rangle_z \implies |m=1\rangle_z = |\Psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle\psi_2(t)|\Psi\rangle^2 = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{-i\theta} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2}e^{i\theta} \end{bmatrix}\right)\left(\begin{bmatrix} \frac{1}{2}e^{i\theta} & \frac{1}{\sqrt{2}} & \frac{1}{2}e^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \frac{1}{4}$$

(c)

$$\hat{S}_z\,|m=0\rangle_z = |0\rangle \implies |\Psi\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\langle\psi_2(t)|\Psi\rangle^2 = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{-i\theta} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2}e^{i\theta} \end{bmatrix}\right)\left(\begin{bmatrix} \frac{1}{2}e^{i\theta} & \frac{1}{\sqrt{2}} & \frac{1}{2}e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{1}{2}$$

(d)

$$\hat{S}_x\,|m=0\rangle_x = |0\rangle \implies \begin{bmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_2 \\ m_1 + m_3 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies |m=0\rangle_x = |\Psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$
\begin{aligned}
\langle\psi_2(t)|\Psi\rangle^2 &= \frac{1}{2}\left(\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2}e^{-i\theta}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}e^{i\theta}\end{bmatrix}\right)\left(\begin{bmatrix}\frac{1}{2}e^{i\theta} & \frac{1}{\sqrt{2}} & \frac{1}{2}e^{-i\theta}\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\right)\\
&= \frac{1}{2}\left[\left(\frac{1}{2}e^{-i\theta}-\frac{1}{2}e^{i\theta}\right)\left(\frac{1}{2}e^{i\theta}-\frac{1}{2}e^{-i\theta}\right)\right]\\
&= -\frac{1}{2}\left[\left(\frac{1}{2}e^{-i\theta}-\frac{1}{2}e^{i\theta}\right)\left(\frac{1}{2}e^{-i\theta}-\frac{1}{2}e^{i\theta}\right)\right]\\
&= -\frac{1}{2}\left(\frac{1}{2}e^{-i\theta}-\frac{1}{2}e^{i\theta}\right)^2\\
&= -\frac{1}{8}\left(e^{-2i\theta}+e^{2i\theta}-2e^{-i\theta}e^{i\theta}\right)\\
&= \frac{1}{8}\left(2-e^{-2i\theta}-e^{2i\theta}\right)\\
&= \frac{1-\frac{e^{-2i\theta}+e^{2i\theta}}{2}}{4}\\
&= \frac{1-\cos(2\theta)}{4}\\
&= \frac{1}{2}\frac{1-\cos(2\theta)}{2}\\
&= \frac{1}{2}\sin^2\theta\\
&= \frac{1}{2}\sin^2\left(|\gamma|B_z t-\frac{\pi}{2}\frac{B_z}{B_y}\right)
\end{aligned}
$$

$$
\mathcal{P}_{(S_x=0)} = \frac{1}{2}\sin^2\left(|\gamma|B_z t-\frac{\pi}{2}\frac{B_z}{B_y}\right)
$$

Problem 5

(a)

To resist my sanity from leaving my head i am going to make some changes on the notation so

$$\varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\begin{aligned}
\vec{A} \times (\vec{B} \times \vec{C}) &= [\varepsilon_{ijk} a_j (\varepsilon_{kmn} b_m c_n)]_i \\
&= [\varepsilon_{kij} a_j (\varepsilon_{kmn} b_m c_n)]_i \\
&= [\varepsilon_{kij}\varepsilon_{kmn} a_j b_m c_n]_i \\
&= [(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n]_i \\
&= [\delta_{im}\delta_{jn} a_j b_m c_n - \delta_{in}\delta_{jm} a_j b_m c_n]_i \\
&= [\delta_{im}(\delta_{jn} a_j c_n) b_m - \delta_{in}(\delta_{jm} a_j b_m) c_n]_i \\
&= [(a_j c_j) b_i - (a_j b_j) c_i]_i \\
&= [(\vec{A}\cdot\vec{C}) b_i - (\vec{A}\cdot\vec{B}) c_i]_i \\
&= \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right) && \text{(taking sum over all component)}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mu}\cdot\vec{\mu}) = 2\vec{\mu}\cdot\frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} &= 2\vec{\mu}\cdot\left(\gamma\vec{\mu}\times\vec{B}(t)\right) \\
&= 0 && (\vec{\mu} \perp \gamma\vec{\mu}\times\vec{B}(t)) \\
&\implies \vec{\mu}(t)\cdot\vec{\mu}(t) = \text{const}
\end{aligned}$$

(c)

Solve for the magnetic field first in terms of $\vec{n}$

$$\begin{aligned}
\vec{B}(t) &= \vec{B}_1(t) + \vec{B}_2(t) \\
&= B_1(t)\vec{n}(t) + \left(-\frac{1}{\gamma}\vec{n}(t)\times\frac{\mathrm{d}}{\mathrm{d}t}\vec{n}(t)\right)
\end{aligned}$$

We compute the rate of change of $\vec{\mu}$ and then dot product $\vec{n}$ with it to get the required solution

$$\begin{aligned}
\frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = \gamma\vec{\mu}\times\vec{B} &= \gamma B_1(t)(\vec{\mu}\times\vec{n}) + \vec{\mu}\times\left(-\vec{n}\times\frac{\mathrm{d}}{\mathrm{d}t}\vec{n}\right) \\
&= \gamma B_1(t)(\vec{\mu}\times\vec{n}) - \vec{\mu}\times\left(\vec{n}\times\frac{\mathrm{d}}{\mathrm{d}t}\vec{n}\right) \\
&= \gamma B_1(t)(\vec{\mu}\times\vec{n}) - \left(\vec{n}\left(\vec{\mu}\cdot\frac{\mathrm{d}\vec{n}}{\mathrm{d}t}\right) - \frac{\mathrm{d}\vec{n}}{\mathrm{d}t}(\vec{\mu}\cdot\vec{n})\right)
\end{aligned}$$

Now dotting $\vec{n}$ with both sides

$$\vec{n} \cdot \frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = 0 - \left(\vec{n} \cdot \vec{n} \left(\vec{\mu} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t} \right) - \vec{n} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t} \left(\vec{\mu} \cdot \vec{n} \right) \right) \qquad (\vec{n} \perp \vec{\mu} \times \vec{n})$$

$$\vec{n} \cdot \frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = - \left(\vec{\mu} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t} \right) + \vec{n} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t} \left(\vec{\mu} \cdot \vec{n} \right) \qquad (\vec{n} \cdot \vec{n} = 1)$$

$$\vec{n} \cdot \frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = - \left(\vec{\mu} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t} \right) + 0 \qquad (\vec{n} \cdot \tfrac{\mathrm{d}\vec{n}}{\mathrm{d}t} = 0 \text{ as } \vec{n} \cdot \vec{n} = 0)$$

$$\vec{n} \cdot \frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = -\vec{\mu} \cdot \frac{\mathrm{d}\vec{n}}{\mathrm{d}t}$$

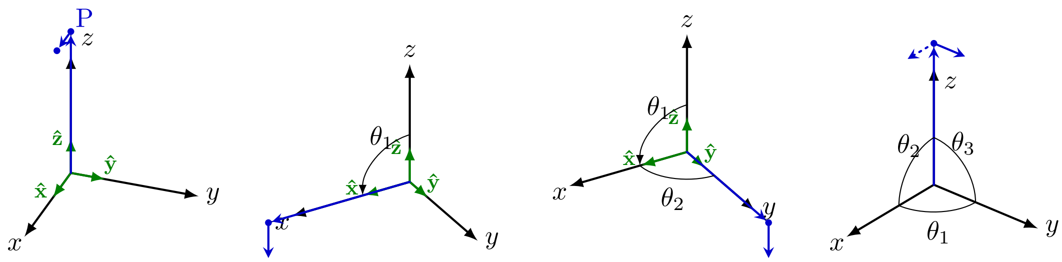
Now solving for the $\vec{n} \cdot \vec{\mu}$ derivative

$$\implies \frac{\mathrm{d}}{\mathrm{d}t} \left(\vec{n}(t) \cdot \vec{\mu}(t) \right) = \vec{\mu}(t) \frac{\mathrm{d}\vec{n}(t)}{\mathrm{d}t} + \vec{n}(t) \frac{\mathrm{d}\vec{\mu}(t)}{\mathrm{d}t} = 0$$

Hence approving $\vec{n} \cdot \vec{\mu}$ is a constant.

appendix: unnecessary things only for aesthetic purposes

The TiKZ figure




```
\begin{center}
% 3D AXIS with spherical coordinates
\tdplotsetmaincoords{60}{110}
\begin{tikzpicture}[scale=1.8,tdplot_main_coords]

  % VARIABLES
  \def\l{0.3} % length scale dark unit vector
  \def\rvec{1.2}
  \def\thetavec{0}
  \def\phivec{50}

  % AXES
  \coordinate (O) at (0,0,0);
  \tdplotsetcoord{P}{\rvec}{\thetavec}{\phivec}
  \draw[dashed,mydarkblue] (O)  -- (Pxy);
  \draw[thick,->] (0,0,0) -- (1,0,0) node[below left=-3]{$x$};
  \draw[thick,->] (0,0,0) -- (0,1,0) node[right=-1]{$y$};
  \draw[thick,->] (0,0,0) -- (0,0,1) node[anchor = south west]{$z$};
  \draw[unit vector] (0,0,0) -- (1.3*\l,0,0) node[above=3,left=-1,scale=0.8]{$\vu{x}$};
  \draw[unit vector] (0,0,0) -- (0,.9*\l,0) node[right=2,above=-1,scale=0.8]{$\vu{y}$};
  \draw[unit vector] (0,0,0) -- (0,0,\l) node[left,scale=0.8]{$\vu{z}$};

  % VECTORS
  \draw[dashed,mydarkblue] (P)  -- (Pxy);
  \draw[dashed,mydarkblue] (P)  -- (Pz);
  \draw[dashed,mydarkblue] (Py) -- (Pxy) -- (Px);
  \node[circle,inner sep=0.9,fill=myblue]
(P’) at ({\rvec*sin(\thetavec)*cos(\phivec)},{\rvec*sin(\thetavec)*sin(\phivec)},
{\rvec*cos(\thetavec)}) {};
\node[circle,inner sep=0.9,fill=myblue]
(P’’) at ({\rvec*sin(\thetavec)*cos(\phivec) + 0.3 },{\rvec*sin(\thetavec)*sin(\phivec)},
{\rvec*cos(\thetavec)}) {};
  \draw[vector] (O)  -- (P’) node[above right=-2] {P};
  \draw[vector] (P’) -- (P’’);


\end{tikzpicture}
\tdplotsetmaincoords{60}{150}
\begin{tikzpicture}[scale=1.8,tdplot_main_coords]

  % VARIABLES
  \def\l{0.3} % length scale dark unit vector
  \def\rvec{1.2}
  \def\thetavec{90}
  \def\phivec{0}

  % AXES
  \coordinate (O) at (0,0,0);
  \tdplotsetcoord{P}{\rvec}{\thetavec}{\phivec}
  \draw[dashed,mydarkblue] (O)  -- (Pxy);
  \draw[thick,->] (0,0,0) -- (1,0,0) node[below left=-3]{$x$};
  \draw[thick,->] (0,0,0) -- (0,1,0) node[right=-1]{$y$};
  \draw[thick,->] (0,0,0) -- (0,0,1) node[above=-1]{$z$};
  \draw[unit vector] (0,0,0) -- (1.3*\l,0,0) node[above=3,left=-1,scale=0.8]{$\vu{x}$};
  \draw[unit vector] (0,0,0) -- (0,.9*\l,0) node[right=2,above=-1,scale=0.8]{$\vu{y}$};
  \draw[unit vector] (0,0,0) -- (0,0,\l) node[left,scale=0.8]{$\vu{z}$};

  % VECTORS
  \draw[dashed,mydarkblue] (P)  -- (Pxy);
  \draw[dashed,mydarkblue] (P)  -- (Pz);
  \draw[dashed,mydarkblue] (Py) -- (Pxy) -- (Px);
  \node[circle,inner sep=0.9,fill=myblue]
(P’) at ({\rvec*sin(\thetavec)*cos(\phivec)},{\rvec*sin(\thetavec)*sin(\phivec)},
{\rvec*cos(\thetavec)}) {};
  \draw[vector] (O)  -- (P’) node[above right=-2] {};

  % ARCS
  \tdplotsetthetaplanecoords{\phivec}
  \tdplotdrawarc[->,tdplot_rotated_coords]{(0,0,0)}{0.4}{0}{\thetavec}
{right=2,above}{$\theta_1$}
\draw[vector](P) -- (1.2,0,-0.3);

\end{tikzpicture}
\tdplotsetmaincoords{60}{150}
\begin{tikzpicture}[scale=1.8,tdplot_main_coords]

  % VARIABLES
  \def\l{0.3} % length scale dark unit vector
  \def\rvec{1.2}
  \def\thetavec{90}
  \def\phivec{90}

  % AXES
  \coordinate (O) at (0,0,0);
  \tdplotsetcoord{P}{\rvec}{\thetavec}{\phivec}
  \draw[dashed,mydarkblue] (O)  -- (Pxy);
  \draw[thick,->] (0,0,0) -- (1,0,0) node[below left=-3]{$x$};
  \draw[thick,->] (0,0,0) -- (0,1,0) node[right=-1]{$y$};
  \draw[thick,->] (0,0,0) -- (0,0,1) node[above=-1]{$z$};
  \draw[unit vector] (0,0,0) -- (1.3*\l,0,0) node[above=3,left=-1,scale=0.8]{$\vu{x}$};
  \draw[unit vector] (0,0,0) -- (0,.9*\l,0) node[right=2,above=-1,scale=0.8]{$\vu{y}$};
  \draw[unit vector] (0,0,0) -- (0,0,\l) node[left,scale=0.8]{$\vu{z}$};

  % VECTORS
  \draw[dashed,mydarkblue] (P)  -- (Pxy);
  \draw[dashed,mydarkblue] (P)  -- (Pz);
  \draw[dashed,mydarkblue] (Py) -- (Pxy) -- (Px);
  \node[circle,inner sep=0.9,fill=myblue]
(P’) at ({\rvec*sin(\thetavec)*cos(\phivec)},{\rvec*sin(\thetavec)*sin(\phivec)},
{\rvec*cos(\thetavec)}) {};
  \draw[vector] (O)  -- (P’) node[above right=-2] {};
  \draw[vector](P) -- (0,1.2,-0.3);
  % ARCS


\tdplotsetthetaplanecoords{0}

\tdplotdrawarc[->,tdplot_rotated_coords]{(0,0,0)}{0.4}{0}{\thetavec}
{right=2,above}{$\theta_1$}

\tdplotdrawarc{(0,0,0)}{0.4}{0}{\phivec}{anchor=north}{$\theta_2$}


\end{tikzpicture}
\tdplotsetmaincoords{60}{130}
\begin{tikzpicture}[scale=1.8,tdplot_main_coords]

  % VARIABLES
  \def\l{0.3} % length scale dark unit vector
  \def\rvec{1.2}
  \def\thetavec{0}
  \def\phivec{50}

  % AXES
  \coordinate (O) at (0,0,0);
  \tdplotsetcoord{P}{\rvec}{\thetavec}{\phivec}
  \draw[dashed,mydarkblue] (O)  -- (Pxy);
  \draw[thick,->] (0,0,0) -- (1,0,0) node[below left=-3]{$x$};
  \draw[thick,->] (0,0,0) -- (0,1,0) node[right=-1]{$y$};
  \draw[thick,->] (0,0,0) -- (0,0,1) node[anchor = north west]{$z$};


  % VECTORS
  \draw[dashed,mydarkblue] (P)  -- (Pxy);
  \draw[dashed,mydarkblue] (P)  -- (Pz);
  \draw[dashed,mydarkblue] (Py) -- (Pxy) -- (Px);
  \node[circle,inner sep=0.9,fill=myblue]
(P’) at ({\rvec*sin(\thetavec)*cos(\phivec)},{\rvec*sin(\thetavec)*sin(\phivec)},
{\rvec*cos(\thetavec)}) {};
  \draw[vector] (O)  -- (P’) node[above right=-2] {};
  \draw[dotted, vector](P) -- (0.3,0,1.2);
  \draw[vector](P) -- (0,0.3,1.2);

  % ARCS


  \tdplotdrawarc{(0,0,0)}{0.4}{90}{0}{anchor=north}{$\theta_1$}
  \tdplotsetrotatedcoords{-90}{-90}{0}
  \tdplotdrawarc[tdplot_rotated_coords]{(0,0,0)}{0.4}{90}{0}{anchor=south}{$\theta_2$}
  \tdplotsetrotatedcoords{0}{-90}{0}
  \tdplotdrawarc[tdplot_rotated_coords]{(0,0,0)}{0.4}{90}{0}{anchor=south}{$\theta_3$}


\end{tikzpicture}
\end{center}
```

general solution to time independent schrodinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

is nontrivial if $\hat{H}$ depend on time but trivial if independent of time

$$\hat{H}(t) = |\gamma|\vec{\hat{S}}\,\vec{B}(t) =$$
$$\left(\hat{S}_x B_x(t) + \hat{S}_y B_y(t) + \hat{S}_z B_z(t)\right)|\gamma|$$

solving for magnetic field,

$$\vec{B}(t) = \begin{cases} B_y \vec{n}_y & 0 \le t \le t_0 \\ B_z \vec{n}_z & t_0 < t < \infty \end{cases}$$

so,

$$\vec{H}(t) = \begin{cases} |\gamma|\hat{S}_y B_y & \text{before } t_0 \\ \hat{S}_z B_z |\gamma| & \text{after} \end{cases}$$

$\hat{H}_1(t)$ and then it becomes $\hat{H}_2(t)$

writing small like this is quite esxy.

The initial state is $t=0$

$|\psi_0\rangle = |m=1\rangle$ inter connected

example $\hat{S}_z = \pm\hbar = \hbar$, we are required to find the final state for spin $t > t_y$.

(a) evolution,

$$|\psi_1\rangle = e^{-\frac{i}{\hbar}\hat{H}_1 t}|\psi_0\rangle$$

after this

$$|\psi_2\rangle = e^{-\frac{i}{\hbar}\vec{H}_2(t-t_0)}\,e^{-\frac{i}{\hbar}\hat{H}_1 t_0}|\psi_0\rangle$$

is there a cleaner way to write this?

$$|\psi_1\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}_1 t\right)|\psi_0\rangle \quad t < t_0$$
$$|\psi_2\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}_2(t-t_0)\right)\exp\left(-\frac{i}{\hbar}\hat{H}_1(t_0)\right)|\psi_0\rangle,\ t > t_0$$

and then,

$$|\psi_2\rangle = \exp\left(-\frac{i}{\hbar}|\gamma|\hat{S}_z B_z\left(t - \frac{\pi}{2|\gamma|B_y}\right)\right)\exp\left(-\frac{i}{\hbar}\hat{S}_y B_y \frac{|\gamma|\pi}{2|\gamma|B_y}\right)|\psi_0\rangle$$
$$= \exp\left(-\frac{i}{\hbar}|\gamma|\hat{S}_z B_z\left(t - \frac{\pi}{2|\gamma|B_y}\right)\right)\exp\left(-\frac{i}{\hbar}\frac{\hat{S}_y}{2}\pi\right)|\psi_0\rangle$$
$$\sim \exp\left(-\frac{i}{\hbar}\hat{S}_y\left(\frac{\pi}{2}\right)\right)|\psi_0\rangle$$
$$\sim \hat{R}_y(\pi/2)|\psi_0\rangle$$

and now, (b), probability of getting $S_z = \hbar$ means $\langle\psi_2|S_z|\psi_s\rangle^2 = \hbar$?

the state with $S_z = \hbar$ is $|\psi_0\rangle$ and hence,

$\langle\psi_0|\psi_2\rangle^2 =$ probability

what we get is, that thing is pretty long lol. I don't quite know what to really do here lol.

$$\hat{J}_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \hat{J}_z|m\rangle = m|m\rangle$$

$$\det(\hat{J}_z - mI) = 0$$

we get $m = -1, 0, 1$

$m=0$ means, state with 0 eigenvalue

$$e^{-i\theta\hat{J}_z}|m=0\rangle = e^{0}|m=0\rangle = \underset{\text{eigenvector}}{|m=0\rangle}$$

$m = \pm1$,

$$e^{-i\theta\hat{J}_z}|m=\pm1\rangle = \underbrace{e^{\pm i\theta}}_{\text{basically a complex phase}}|m=\pm1\rangle$$

what is a complex phase for state?

now, $|n_z\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$ state (our basis)

$|n_x\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $|n_y\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$|n_z\rangle, |n_x\rangle, |n_y\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

eigenstate $|m=0\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$

eigenstate $|m=+1\rangle = \begin{bmatrix}1\\i\\0\end{bmatrix}$ $\quad |0\rangle, |+1\rangle, |-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\i\\0\end{bmatrix}, \begin{bmatrix}1\\-i\\0\end{bmatrix}$

eigenstate $|m=-1\rangle = \begin{bmatrix}1\\-i\\0\end{bmatrix}$

$\hbar$

? -1 $\quad 0 \quad$?

$$R_y(0) \rightarrow \begin{bmatrix} \frac{1+\cos(\pi/2)}{2} \\ \sin(\pi/2) \\ \frac{1-\cos(\pi/2)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

problems 1

1 The basis vectors are $|\uparrow\rangle_z , |\downarrow\rangle_z$

ein $\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\hat{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

find the eigenvector $\hat{S}_x$ and $\hat{S}_y$. Express the eigenvectors

$|\uparrow\rangle_x , |\downarrow\rangle_x \quad |\uparrow\rangle_y \quad |\downarrow\rangle_y$

$(\hat{S}_x - \hat{I}m)|\uparrow\rangle_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda\hat{I} = \cancel{\frac{\hbar}{2}\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}} \rightarrow$

$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} - \lambda I = \begin{bmatrix} -\lambda & a \\ a & -\lambda \end{bmatrix} \rightarrow \lambda^2 - a^2 = 0$

$(\lambda+a)(\lambda-a) = 0$

$\therefore \lambda = a, -a$

$= \frac{\hbar}{2}, -\frac{\hbar}{2}$

eigenvalue $\frac{\hbar}{2}, -\frac{\hbar}{2}$ now from there

$\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\hbar}{2}(1)\hat{I} \rightarrow \frac{\hbar}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 = x_2 = 1$

$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

✓ $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\quad x_1 = -x_2 \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1$

$$\boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

2nd $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\begin{bmatrix} -\lambda x_1 - i x_2 \\ i x_1 - \lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \rightarrow$

$\lambda^2 - [-i \cdot i] = \lambda^2 - [-i^2] = \lambda^2 - [1] = (\lambda+1)(\lambda-1) \quad \lambda = 1, -1$

$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 i \\ x_1 i - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ $x_1 = -x_2 i$, $x_1 i = x_2$ $\quad x_1 = 1, x_2 = i$ $\quad \boxed{\begin{bmatrix} 1 \\ i \end{bmatrix}}$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{matrix} 0+1 \\ i+0 \end{matrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ ✓

$$\begin{bmatrix} 1 & -i \\ i & +1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 i \\ x_1 i + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x_1 = x_2 i \\ x_1 i = -x_2 \end{matrix} \qquad \begin{matrix} x_1 = i \\ x_2 = 1 \end{matrix}$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ -1 \end{bmatrix} = -1\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\boxed{\underset{1}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}}, \underset{-1}{\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}}}$$

(b) $|\psi\rangle = \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\Delta S_x^2 = \langle\psi|\hat{S}_x\hat{S}_x|\psi\rangle + \left(\langle\psi|\hat{S}_x|\psi\rangle\right)^2$

compute $\langle\psi|\hat{S}_x|\psi\rangle = \sum\langle\psi|\hat{S}_x|\lambda_i\rangle\langle\lambda_i|\psi\rangle$

$= \langle\psi|\hat{S}_x|\uparrow\rangle_z\,\alpha + \langle\psi|\hat{S}_x|\downarrow\rangle_z\,\beta$

$\uparrow\uparrow \quad \uparrow\downarrow$

$\downarrow\uparrow \quad \downarrow\downarrow$

$= {}_z\langle\uparrow|\hat{S}_x|\uparrow\rangle_z\,\alpha\alpha^* + {}_z\langle\downarrow|\hat{S}_y|\uparrow\rangle_z\,\alpha\beta^* +$

${}_z\langle\uparrow|\hat{S}_x|\downarrow\rangle_z\,\beta\alpha^* + {}_z\langle\downarrow|\hat{S}_x|\downarrow\rangle\,\beta\beta^*$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad = (\alpha\beta^* + \alpha^*\beta)^2$

$[1 \; 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \qquad \hat{S}_x\hat{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0+1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{I}$

$[0 \; 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \langle\psi|\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}|\psi\rangle = \alpha\alpha^* + \beta\beta^*$

$[1, 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Rightarrow \sqrt{\alpha\alpha^* + \beta\beta^* - (\alpha\beta^* + \alpha^*\beta)^2}$

ψ eigen state $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \sqrt{1 + 1 - (1+1)^2}$

$\sqrt{2 - 4} =$

$\langle \hat{S}_x^2 \rangle - \langle S_x \rangle^2$

$2 - 4$

$1 - \left(\frac{1}{2} + \frac{1}{2}\right)^1 = 0$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $|\psi\rangle = \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z$

$(\vec{n})_x = \langle\psi|\sigma^1|\psi\rangle$

$= \langle\psi|\lambda_i\rangle\langle\lambda_i|\sigma^1|\lambda_j\rangle\langle\lambda_j|\psi\rangle$

$= \langle\psi|\uparrow\rangle\langle\uparrow|\sigma|\uparrow\rangle\langle\uparrow|\psi\rangle +$
$\langle\psi|\uparrow\rangle\langle\uparrow|\sigma|\downarrow\rangle\langle\downarrow|\psi\rangle +$
$\langle\psi|\downarrow\rangle\langle\downarrow|\sigma|\uparrow\rangle\langle\uparrow|\psi\rangle +$
$\langle\psi|\downarrow\rangle\langle\downarrow|\sigma|\downarrow\rangle\langle\downarrow|\psi\rangle$

$= \alpha^*\alpha(\sigma^1_{11}) + \alpha^*\beta(\sigma^1_{12}) + \beta^*\alpha(\sigma^1_{21}) + \beta^*\beta(\sigma_{22})$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_1 \rightarrow (\vec{n}_x) = \alpha^*\beta + \beta^*\alpha = \cancel{(\alpha^*\beta)^2} + \cancel{(\beta^*\alpha)^2} + 2(\alpha^*\alpha\beta\beta^*)$

$(\vec{n}_y) = -i\alpha^*\beta + i\beta^*\alpha = i^2[-\alpha^*\beta + \beta^*\alpha]^2 = \cancel{(-1)[(\alpha\beta^*)^2 + (\alpha^*\beta)^2}$
$- 2(\alpha^*\alpha\beta\beta^*)$

$(\vec{n}_z) = \alpha^*\alpha - \beta^*\beta = (\alpha^*\alpha)^2 + (\beta\beta^*)^2 - 2(\alpha^*\alpha \times \beta^*\beta)$

$\alpha^2 + \beta^2 = 1$ $\quad = 2(\alpha\alpha^*\beta\beta^*) = 2\left(\cancel{\frac{1}{2}\cdot\frac{1}{2}}\right) =$

$\alpha^2 - 2\alpha\beta + \beta^2 =$

$\alpha^2 + \beta^2 + 2(\alpha^*\alpha\,\beta^*\beta) = 1$ $\langle\psi|\psi\rangle$ ohie

$\vec{n}\cdot\hat{S}|\psi\rangle = [(\hat{n}_x)\hat{x} + (\hat{n}_y)\hat{y} + (\hat{n}_z)\hat{z}]\{S\hat{x} + S$

$\vec{n}\cdot\hat{S} = (n_x)S_1 + (n_2)S_2 + (n_3)S_3$

$= \frac{\hbar}{2}n_x\sigma_1 + \frac{\hbar_2}{2}n_x\sigma_2$

$\langle\psi|\vec{n}\cdot\hat{S}|\psi\rangle = \langle\psi|$

$\hat{\vec{S}} = \hat{S}_x \hat{x} + \cdots$

$\hat{\vec{n}} = (\vec{n})_x \hat{x} + \cdots$

$(\vec{n})_x \hat{S}_x + \cdots |\psi\rangle$

$\frac{\hbar}{2}(\vec{n})_x \hat{\sigma}' + \cdots |\psi\rangle$

$\frac{\hbar}{2}(\vec{n}_x)\hat{\sigma}'|\psi\rangle + \cdots$

$\langle\psi| \frac{\hbar}{2}(\vec{n}_x)\hat{\sigma}'|\psi\rangle$

$\frac{\hbar}{2}(\vec{n}_x)\langle\psi|\hat{\sigma}'|\psi\rangle$

$\frac{\hbar}{2}(\vec{n}_x)(\vec{n}_x) + \cdots$

$\frac{\hbar}{2} \cdots (1)$

$\frac{\hbar}{2}(\vec{n}_x)\begin{bmatrix} \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix} i$

$\frac{\hbar}{2}\left([\alpha' \ \beta'] \begin{bmatrix} \beta \\ \alpha \end{bmatrix}\right)\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

$\frac{\hbar}{2}\left([\alpha' \ \beta'] \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}\right)\begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$

$\frac{\hbar}{2}\left([\alpha' \ \beta'] \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}\right)\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$

$[n_x \ n_y \ n_z]\begin{bmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \ \boxed{\times}$

$\cancel{(\alpha^*\beta + \beta^*\alpha)\beta +}$

$\cancel{\alpha^*\beta^*} \ (\alpha^*\beta + \beta^*\alpha)\beta + (-\alpha^*\beta + \beta^*\alpha)(-\beta) + (\alpha^*\alpha - \beta^*\beta^*)\alpha$

$(\alpha^*\beta + \beta^*\alpha)\alpha + (-\alpha^*\beta + \beta^*\alpha)(\alpha) + (\alpha^*\alpha - \beta^*\beta^*)(-\beta)$

$\sigma|\psi\rangle\langle\psi|T\rangle = \sigma|T\rangle$?

$\alpha^*\beta\beta + \alpha\beta\beta^* + \alpha^*\beta\beta - \alpha\beta\beta^* + \alpha^*\alpha\alpha - \alpha\beta^*\beta^*$

$\alpha^*\cancel{\alpha}\beta + \alpha\alpha\beta^* - \alpha\alpha^*\beta + \alpha\alpha\beta^* - \cancel{\alpha^*\alpha}\beta + \beta^*\beta^*\beta$

$\alpha\alpha\beta^* - \alpha\alpha^*\beta + \beta^*\beta^*\beta$

$\langle\psi|\Pi\rangle = \langle\lambda|\Pi\rangle \Rightarrow |\psi\rangle = |\lambda\rangle$

$\langle\psi|\Pi\rangle - \langle\lambda|\Pi\rangle = 0$

or, $(\langle\psi| - \langle\lambda|)|\Pi\rangle = 0$

$|\Pi\rangle \neq 0$ then $\langle\psi| - \langle\lambda| = 0$

$\sigma|T\rangle$

$= \sigma I|T\rangle$

$= \sigma \sum|P_i\rangle\langle P_i|T\rangle$

3. products of rotations

$[\hat{\sigma}^a, \hat{\sigma}^b] = 2\delta^{a,b}\hat{I}$ (Clifford Algebra) $\quad [\hat{\sigma}^a, \hat{\sigma}^b] = 2i\epsilon^{abc}\hat{\sigma}^c$

$\hat{\sigma}^a\hat{\sigma}^b = \delta^{a,b}\hat{I} + i\epsilon^{abc}\hat{\sigma}^c$ $\quad$ {PHYS 311}

(a) $\hat{U} = e^{S_y\theta_3} \; S^z\theta_2 \; S_y\theta_1$

$$e^{-i\frac{\hat{S}_y}{\hbar}\theta_1} = e^{-i\frac{\sigma^2}{2}\theta_1} = \cos\left(\frac{\theta_1}{2}\right)\hat{I} - i\hat{\sigma}^2\sin\left(\frac{\theta_1}{2}\right)$$

$$\hat{U} = \cos\left(\frac{|\vec{\phi}|}{2}\right)\hat{I} - i\vec{n}_\phi\cdot\hat{\vec{\sigma}}\sin\left(\frac{|\vec{\phi}|}{2}\right)$$

$c^{x+y} = cxcy - sxsy$

$cxcy =$ $c^{x-y} = cxcy + s$

$e^{-i\frac{S_y}{\hbar}\theta_1} =$

$$\left[\cos\left(\frac{\theta_3}{2}\right)\hat{I} - i\hat{\sigma}^2\sin\left(\frac{\theta_3}{2}\right)\right]\left[\cos\left(\frac{\theta_2}{2}\right)\hat{I} - i\hat{\sigma}^3\sin\left(\frac{\theta_2}{2}\right)\right]$$
$$\left[\cos\left(\frac{\theta_1}{2}\right)\hat{I} - i\hat{\sigma}^2\sin\left(\frac{\theta_1}{2}\right)\right]$$

$$\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\hat{I}\cdot\hat{I} - \left[i\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right)\right]\hat{I}\cdot\hat{\sigma}^3$$
$$- \left[i\sin\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\right]\hat{\sigma}^2\cdot\hat{I}$$
$$+ i^2\hat{\sigma}^2\hat{\sigma}^3\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\theta_2}{2}\right)$$

$\sigma^2\sigma^3 =$

2 3 1

$\cancel{\cos\left(\frac{\theta_3}{2}\right)\cos}$

$$\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\hat{I} - i\left[\hat{\sigma}^3\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right) + \hat{\sigma}^2\sin\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\right]$$
$$+ (-i)\left[\hat{\sigma}_1\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right]$$

$$\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\hat{I} - i\left[\hat{\sigma}_1\sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2} + \hat{\sigma}^2\frac{\sin\theta_3}{2}\cos\frac{\theta_2}{2} + \hat{\sigma}^3\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right.$$

$$\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{3}\hat{I} - i\left[\hat{\sigma}_1 \sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + \hat{\sigma}_2\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + \hat{\sigma}^3\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right]$$

$$\left[\cos\frac{\theta_1}{2}\hat{I} - i\hat{\sigma}^2\sin\frac{\theta_1}{2}\right]$$

$$\cos\frac{\theta_1}{2}\cos\frac{\theta_3}{3}\cos\frac{\theta_2}{2}\hat{I} - i\Big[\hat{\sigma}_1\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2} +$$

$$\hat{\sigma}_2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} +$$

$$\hat{\sigma}_3\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2_3}\Big] +$$

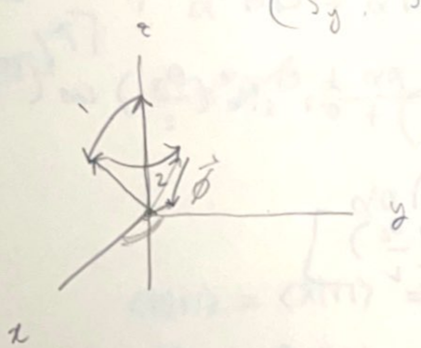
$$(-i\sigma^2)\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$

$$\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{3} = \frac{1}{2}\left(\cos\left(\frac{\theta_2}{2}+\frac{\theta_3}{3}\right) + \cos\left(\frac{\theta_2}{2}-\frac{\theta_3}{2}\right)\right)$$

$$e^{-i/\hbar\,(\hat{S}_y\theta_3 + \hat{S}_z\theta_2 + \hat{S}_y\theta_1)}$$

commutator does not hold for rotation

$$(\hat{S}_y,\ \hat{S}_z,\ \hat{S}_y)\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} \ ?$$



?

θ

$$\vec{\phi} = \theta_1\hat{S}_y\, n_y$$

$$|\vec{\phi}| = \phi_1$$

$$\hat{U} = \cos\left(\frac{|\vec{\phi}|}{2}\right)\hat{I} - i\,\vec{n}_\phi\cdot\vec{\sigma}\,\sin\left(\frac{|\vec{\phi}|}{2}\right)$$

$$e^{-i\frac{S_y}{\hbar}\theta_1} = e^{-i\frac{\sigma^2}{2}\theta_1} = e^{-i\sigma^2\left(\frac{\theta_1}{2}\right)}$$

$$= \cos\frac{\theta_1}{2}\hat{I} - i\sin\left(\frac{\theta_1}{2}\right)\underline{\hat{\sigma}^2}$$

$$|\vec{\phi}| = \theta_1 = \sqrt{\phi_1^2+\phi_2^2+\phi_3^2}$$

$$\vec{n}_\phi\cdot\hat{\sigma} = n_x\sigma^1 + \underbrace{n_2\sigma^2}_{\sigma^2} + \sigma^3 n_3$$

$$\vec{n}_\phi = \frac{\vec{\phi}}{|\vec{\phi}|} = \frac{a\hat{x}+b\hat{y}+c\hat{z}}{|\vec{\phi}|}$$

$$n_2 = 1 = \frac{\theta}{\theta}\sigma^2$$

$$\hat{U} = \cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}I - i\left[\hat{\sigma}_1\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + \hat{\sigma}_2\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + \hat{\sigma}_3\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right]$$

$$\hat{U} = e^{-i[\,]} = \sum \frac{-i\sigma\theta}{2}$$

$$e^x = \sum\frac{x^n}{n!}$$

$$= \sum\frac{\frac{i\theta}{2}}{n!}\sigma^{n!}$$

$$\cos\left(\frac{|\vec{\phi}|}{2}\right) = \cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right)$$

$$\cos\left(\frac{a+b}{2}\right) = c(A)c(B) - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$$

has to zero

$$\sum\frac{\left(-\frac{i\theta_1}{2}\right)^n}{n!}\sigma^n$$

$$\sum_j\sum_n \frac{\left(\frac{-i\theta_1}{2}\right)^n\left(\frac{-i\theta_2}{2}\right)^j}{n!\,j!}\sigma_y^n\sigma_z^j$$

$$c\frac{\theta_2}{2}c\frac{\theta_3}{2}I - i\sigma_1\frac{\sin\theta_2}{2}\sin\frac{\theta_3}{2}$$

$$c\frac{\theta_2}{2}c\frac{\theta_3}{2}I - s\frac{\theta_2}{2}s\frac{\theta_3}{2} + s\frac{\theta_2}{2}s\frac{\theta_3}{2} - i$$

$$\left(\sigma_1\sigma_1 - i\sigma_1\right)s\frac{\theta_2}{2}s\frac{\theta_3}{2}$$

$$\sigma_1\sigma_1 - \sigma_1\sigma_3$$

1 2 3 1 2 3

$\sigma^1\sigma^1 = I$

$\sigma^1\sigma^2 = i\sigma^3 \qquad \sigma^3\sigma^2 = -i\sigma^1$

$\sigma^2\sigma^3 = i\sigma^1 \qquad \sigma^2\sigma^1 = -i\sigma^3$

$\sigma^3\sigma^1 = i\sigma^2 \qquad \sigma^1\sigma^3 = -i\sigma^2$

$c\frac{\theta_2}{2}c\frac{\theta_3}{3}\hat{I} - \left[i\sigma_1 \sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + i\hat{\sigma}_2 \cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} + i\hat{\sigma}_3 \sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right.$

$cx_2\, cx_3\left(\frac{\sigma^1\sigma^1 + \sigma^2\sigma^2 + \sigma^3\sigma^3}{2}\right)$

$e^{-i\frac{\sigma^2}{2}\theta_1}\, e^{i\frac{\sigma^2}{2}\theta_1} = \cos\left(\frac{\theta_1}{2}\right)$

$\underbrace{c\frac{\theta_2}{2}c\frac{\theta_3}{3}\hat{I} - s\frac{\theta_2}{2}s\frac{\theta_3}{2}I} + \underbrace{s\frac{\theta_2}{2}s\frac{\theta_3}{2}I} - i\sigma_1 \sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}$

$s\frac{\theta_2}{2}s\frac{\theta_3}{2}\sigma_1\sigma_1 - i\sigma_1 \sin \sin$

$\sigma_1\sigma_1 + \sigma_3\sigma_2$

$\sigma_1\sigma_1 - \sigma_2\sigma_3$

$\cos\left(\frac{\theta_2}{2} + \frac{\theta_3}{2}\right)\hat{I} + (\sigma_1\sigma_1 - \sigma_2\sigma_3)\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}$

$-\sigma^3\sigma^1 + \cancel{\sigma^1\sigma^3}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2}$

$-\sigma^1\sigma^2 + \cancel{\sigma^2\sigma^1}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$

$\sigma^3\sigma^1$

$$\left(\cos\left(\frac{\theta_2}{2}\right)\hat{I} - i\hat{\sigma}^3 \sin\left(\frac{\theta_2}{2}\right)\right)\left(\cos\left(\frac{\theta_1}{2}\right)\hat{I} - i\hat{\sigma}^2 \cancel{\ } \sin\frac{\theta_1}{2}\right)$$

$$\cos\left(\frac{\theta_2}{2}\right)\hat{I}\cos\left(\frac{\theta_1}{2}\right)\hat{I} - i\hat{\sigma}^2 \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}$$

$$- i\hat{\sigma}^3 \sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) + \underbrace{\cancel{\hat{\sigma}^3\hat{\sigma}^2}}_{-i\hat{\sigma}^1} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

$$c\frac{\theta_2}{2}c\frac{\theta_1}{2} - i\hat{\sigma}^2 \frac{\sin\theta_1}{2}\cos\frac{\theta_2}{2} - i\sin\theta_2 \cos$$

$$c\left(\frac{\theta_2}{2} + \frac{\theta_1}{2}\right) \quad \left(I - i\hat{\sigma}^1\right)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

$$\boxed{e^{ia(\hat{n}\cdot\vec{\sigma})} = I\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a}$$

$$-i\sigma^2 - i\sigma^3$$

$$-i\sigma^2 + i\sigma^2$$

$$-i\sigma^2 \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - i\sigma^3 \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}$$

$$\cancel{-i\sigma^2(\sin}$$

$$-i\sigma^2\left(\sin\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right) - \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\right)$$

$$-i\sigma^3\left(\sin\left(\frac{\theta_2}{2} + \frac{\theta_1}{2}\right) - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)$$

$$\left(-i\sigma^2 - i\sigma^3\right)\left(\sin\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right)\right)$$

$$-i\sigma^2\left(\frac{\sin\theta_2}{2}\cos\frac{\theta_1}{2}\right) +$$

$$i\sigma^3 ($$

4. $\hat{H}(t) = |\gamma| \hat{\vec{S}} \cdot \vec{B}(t)$

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$\hat{J}_z |m\rangle = m|m\rangle \qquad m \in \{1, 0, -1\}$

$|m = \pm 1\rangle = \frac{1}{\sqrt{2}} \left(|\vec{n}_x\rangle + i|\vec{n}_y\rangle \right) \qquad |m=0\rangle = |\vec{n}_z\rangle$

$\hat{R}_z(\theta)|m\rangle = e^{-i\theta \hat{J}_z} |m\rangle = e^{-im\theta} |m\rangle$

the states $|m = \pm 1\rangle$ acquires phases under rotation

i did this yesterday.

$\vec{B}(t) = \begin{cases} B_y \vec{y}_y & 0 \le t \le t_y & H_1 \\ B_z \vec{n}_z & t \ge t_y & H_2 \end{cases}$

---

init state $t = 0 \quad |\psi_0\rangle = |m = 1\rangle \qquad \hat{S}_z = +\hbar$

evolution until $t_y = \frac{\pi}{2|\gamma| B_y}$ compute final state $t > t_y$

---

so, propagator,

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \cancel{\#}\, e^{-\frac{i\hat{H}(t)t}{\hbar}} |\psi_0\rangle$$

$$0 \le t \le t_y \qquad = e^{-\frac{i\hat{H}_1 t}{\hbar}} |\psi_0\rangle$$

$$= e^{\left(-i\frac{H_1 t}{\hbar}\right)} |\psi_0\rangle$$

---

after,

$$|\psi(t)\rangle = U(t)|\psi(t_B)\rangle$$

$$= e^{-\frac{iH_2(t-t_B)}{\hbar}} e^{\frac{iH t_B}{\hbar}} |\psi_0\rangle$$

$\vec{A}\times(\vec{B}\times\vec{C}) = \hat{A}\times[B_b C_c \varepsilon_{abc}]$ I FORGOT HOW THEY COME INTO BEING

$\left(e^{-i\frac{\sigma^3}{2}\theta_2}\right)\left(e^{-i\frac{\sigma^2}{2}\theta_1}\right)$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$= \left(\hat{I} + \left(-i\frac{\sigma^3}{2}\theta_2\right) + \frac{1}{2}\left(-i\frac{\sigma^3}{2}\theta_2\right)^2 + \frac{1}{3!}\left(-i\frac{\sigma^3}{2}\theta_2\right)^4 + \cdots\right)$

$= ($

$= \cancel{\hat{I} + \left(-i\frac{\sigma^3\sigma^2}{2}\theta_2\right.} = \hat{I}(\text{series}) +$

$\left(\hat{I} + \left(-i\frac{\sigma^3}{2}\theta_2\right) + \frac{1}{2}\left(-i\frac{\sigma^3}{2}\theta_2\right)^2 + \cdots\right)\left(\hat{I} + \left(-i\frac{\sigma^2}{2}\theta_1\right) + \frac{1}{2}\left(-i\frac{\sigma^2}{2}\theta_1\right) + \cdots\right)$

$\hat{I} + \left(-i\frac{\sigma^2}{2}\theta_1\right) + \frac{1}{2}\left(-i\frac{\sigma^2}{2}\theta_1\right) +$

$-\sigma \quad = e^{x+y} \rightarrow e^{-\frac{i}{h}(\hat{S}_y\theta_3 + S_z\tilde{\theta}_2 = \hat{S}_y\theta_1)}$