Honors Linear Algebra : : Homework 04

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Ahmed Saad Sabit, Rice University

Problem 01

A possible linear combination of the polynomials where the coefficients belong in $\mathbb{F}$,

$$a_1(1-x) + a_2x(1-x) + a_3x^2(1-x) + a_4x^3$$

This is equivalently

$$a_1 - a_1x + a_2x - a_2x^2 + a_3x^2 - a_3x^3 + a_4x^3$$

This is exactly same as the linear combination

$$b_1 + b_2x + b_3x^2 + b_4x^3$$

Here this linear combination equals 0 if and only if $b_1 = b_2 = b_3 = b_4 = 0$ because we know $1, x, x^2, x^3, \ldots$ are linearly independent. Hence they also form a basis for $4 - 1 = 3$ hence $\mathcal{P}_3(\mathbb{F})$

Problem 02: 2.27(f) example

The list is

$$(1, -1, 0), (1, 0, -1)$$

The condition is

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$$

We can see that $x + y + z = 0$ is satisfied for both of the vectors. Now, the linear combination is,

$$a(1, -1, 0) + b(1, 0, -1) = (a + b, -a, -b)$$

If it's linearly independent then

$$a(1, -1, 0) + b(1, 0, -1) = (a + b, -a, -b) = \vec{0}$$

if and only if $a = b = 0$. We can see that

$$\begin{aligned} a + b &= 0 \\ -a &= 0 \\ -b &= 0 \end{aligned}$$

Hence proven $a = b = 0$ hence they are linearly independent. As it spans the set wholly it's a perfect basis.

Problem 03: 2B3(a)

The subspace of $\mathbb{R}^5$ is defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$$

A vector in this space is

$$\left(z_1, \frac{z_1}{3}, z_2, \frac{z_2}{7}, z_3\right)$$

The map $f : \mathbb{R}^3 \to U$ help us simply find the basis for U. The basis for $\mathbb{R}^3$ is,

$$(1,0,0),(0,1,0),(0,0,1)$$

Mapped to U, we get the basis,

$$(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)$$

Problem 04: 2B3(b)

The existing basis is

$$(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)$$

Note that this basis cannot extend wholly to $\mathbb{R}^5$ because of the constraints over the set U. If we lift the constraint conditions and let $x_2 \neq f(x_1)$ and $x_4 \neq g(x_3)$ where $f, g : \mathbb{R} \to \mathbb{R}$ then we can easily extend to the additional vectors to form the basis for $\mathbb{R}^5$

$$(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)$$

These are obviously linearly independent from inspection.

Problem 05: 2B10

Definition of Direct Sum is for $V_1 \oplus V_2 \oplus \cdots V_l$ then it's elements can be written in only one way where $v_1 + v_2 + \cdots + v_l$ where $v_k \in V_k$. So for $V_1 \oplus V_2$, if $a \in V_1 \cap V_2$ and $a \neq \{0\}$ then there should be only one way to write this vector. But it's a contradiction because $a \in V_1$ so linear combination of V_1 basis can give a, so as the linear combination of the basis of V_2 (as $V_1 \cap V_2$ means both sets has the shared members).

From the problem definition we know the set $\{u_m\}$ and $\{w_n\}$ are each linearly independent. And because $V = U \oplus W$ hence $U \cap W = \{0\}$. This means that $\{w_n\}$ cannot be written as a linear combination of $\{v_m\}$ or vice versa. This hence means $\{v_m\}$ and $\{w_n\}$ are linearly independent.

If they are linearly independent, and having $V = U \oplus W$, $\{v_m\}, \{w_n\} \in V$. Thus every vector in V can be written uniquely by the basis of U and V, and hence $u_1, \ldots, u_m, w_1, \ldots, w_n$ forms the basis of V.

Problem 06

Note: Proof of the sum of spaces and their dimensions is added in the Appendix section.

$\dim U = \dim V = m$ means the number of basis required to span U and V is equal. Let $\vec{u_1}, \vec{u}_2, \ldots, \vec{u}_m$ basis spans U. Similarly , let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be basis of V. By definition the set $\{\vec{u}_m\}$ and $\{\vec{v}_m\}$ are linearly independent.

Now if U is a subspace of V then $U \subset V$. If so then all the members of U can be written as linearly combinations of V basis.

$$\{\vec{v}_m\} \to \{\vec{u}_m\}$$

All the vectors $\vec{v}_m$ can be linearly combined into $\vec{u}_m$ because $\vec{u}_m$ belong in V and hence $\vec{v}_m$ basis should be able to turn into that. But because the length of the list is same, then we can do a reverse transformation and write each vector of $\vec{v}_m$ in terms of $\vec{u}_m$. If we can write the basis of V in terms of U basis then that basically means both vector spaces "can" share the same basis.

If they share the same basis then they must be equal. Hence $U = V$.

Problem 7: 2C13

Note: Proof of the sum of spaces and their dimensions is added in the Appendix section.

Consider the dimension of sum of the two subspaces,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 10 - \dim(U \cap W)$$

If $\dim(U \cap W) = \{0\}$ then $\dim(U + W) = \dim U + \dim W$ (which intuitively means the two vectors spaces share no elements), but then $\dim(U + V) = 10$. This is contradiction because $U, W \subset \mathbb{V}$ where $\dim \mathbb{V} = 9$. Hence $\dim(U \cap W)$ must be 1, hence proven $U \cap W \neq \{0\}$

Problem 08: 2C17

Note: Proof of the sum of spaces and their dimensions is added in the Appendix section.

We know that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

That gives us the inequality,

$$\dim(V_1 + V_2) \leq \dim V_1 + \dim V_2$$

Similarly,

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3)$$

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3)$$

We can find the trivial inequality

$$\dim(V_1 + V_2 + V_3) \leq \dim V_1 + \dim V_2 + \dim V_3$$

Like so we can see that this can be generalized to any length of subspaces because the $\dim(U \cap V)$ terms are always subtracted from the sum $\dim U + \dim V$. Hence proven.

$$\dim(V_1 + \ldots + V_m) \leq \dim V_1 + \ldots + \dim V_m$$

Problem 09

Note: Proof of the sum of spaces and their dimensions is added in the Appendix section.

Definition of Direct Sum is for $V_1 \oplus V_2 \oplus \cdots V_l$ then it's elements can be written in only one way where $v_1 + v_2 + \cdots + v_l$ where $v_k \in V_k$. So for $V_1 \oplus V_2$, if $a \in V_1 \cap V_2$, then there should be only one way to write this vector. But it's a contradiction because $a \in V_1$ so linear combination of V_1 basis can give a, so as the linear combination of the basis of V_2 (as $V_1 \cap V_2$ means both sets has the shared members).

Hence there can be no shared vectors hence $V_1 \cap V_2 = \{0\}$.

Now we know

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

If it's a direct sum then $\dim(V_1 \cap V_2) = 0$,

$$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$$

We can try

$$\dim((V_1 \oplus V_2) \oplus V_3) = \dim(V_1 \oplus V_2) + \dim(V_3) = \dim V_1 + \dim V_2 + \dim V_3$$

Like so we can extend this sum for any number k, hence proven

$$\dim(V_1 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_m) = \dim V_1 + \dim V_2 + \dim V_3 + \cdots + \dim V_m$$

Problem 10

- Case $U = \{0\}$
  $\{0\} \in \mathbb{F}^3$ because of additive inverse condition on vector space. So if $U = \{0\}$ then $U \in \mathbb{F}$ and hence $U \subset \mathbb{F}^3$.
- Case $U = \text{span}(\vec{v})$ such that $\vec{v} \in \mathbb{F}^3$
  If $\vec{v} \in \mathbb{F}^3$ then $\{a\vec{v} \in \mathbb{F}^3 : a \in \mathbb{F}\}$ from vector space conditions. But this is also the exact same definition of a span $\text{span}(\vec{v})$ so $U \subset \mathbb{F}^3$.
- Case $U = \text{span}(\vec{v}, \vec{w})$ while $\vec{v}, \vec{w} \in \mathbb{F}^3$
  Given $\vec{v}, \vec{w} \in \mathbb{F}^3$ and if $\vec{v}$ and $\vec{w}$ are linearly independent then from the conditions of a vector space $\{a\vec{v} + b\vec{w} \in \mathbb{F}^3 : a, b \in \mathbb{F}\}$. But turns out this is exactly the same definition of a span because $\text{span}(\vec{v}, \vec{w}) = \{a\vec{v} + b\vec{w} : a, b \in \mathbb{F}\}$. So $U \subset \mathbb{F}^3$.
- Case $U = \mathbb{F}^3$
  Then U is exactly same as $\mathbb{F}^3$ hence it also must have all the members of $\mathbb{F}^3$ hence a subset (that happens to be equal.)

Proof of $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Consider $\{v_m\}$ to be a basis for $V_1 \cap V_2$ hence $\dim(V_1 \cap V_2) = m$. Being the basis this set $\{v_m\}$ is independent in both V_1 and V_2.

Fetching some other vectors $\{u_j\}$ with this already existing $\{v_m\}$ can give us the basis for V_1. The dimension of V_1 will be hence $m + j$. Similarly for V_2 considering a list of vectors $\{w_k\}$ its dimension is $m + k$.

$$\dim(V_1 + V_2) = m + j + k = (m + j) + (m + k) - m = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

$\{v_m)\}$ is a basis itself so it's linearly independent. Because it unites with $\{u_j\}$ to form basis for V_1 then the set $\{v_m\} \cup \{u_j\}$ should also be linearly independent. Similarly $\{w_k\}$ is linearly independent because it forms basis for V_2.