Honors Multivariable Calculus : : Class 18

February 19, 2024

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Example

Classify the critical points of $f(x, y) = x^2y + xy^2 + x^2 - x$. We are looking for places where the derivative is identically zero. This being a nice function, derivative exists everywhere.

$$df = 0$$

$$df = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

Hence,

$$0 = f_x = 2xy + y^2 + 2x - 1$$
$$0 = f_y = x^2 + 2xy$$

From the second equation

$$x^2 + 2xy = x(x + 2y) = 0$$

Hence, for $x = 0$,

$$y^2 - 1 = 0 \quad \text{where } y = \pm 1$$

For $x = -2y$,

$$-4y^2 + y^2 - 4y - 1 = 0 \quad y = -1 \text{ or } y = -\frac{1}{3}$$

The critical points are

$$(0, \pm 1)$$

$$(2, -1), (\frac{2}{3}, -\frac{1}{3})$$

Now the partials of second kind are

$$\frac{\partial^2 f}{\partial x^2} = 2y + 2$$
$$\frac{\partial^2 f}{\partial y^2} = 2x$$
$$\frac{\partial^2 f}{\partial x \partial y} = 2x + 2y$$

Hessian at $(2/3, -1/3)$

$$\begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}$$

Calculate the Eigenvalue from characteristic polynomial,

$$\lambda^2 - \text{trace}\,\lambda + (?) = 0$$

$$\lambda^2 - \frac{8}{3}\lambda + \frac{4}{3} = 0$$

This being positive gives us a local minimum.

Let's try another one $(0, -1)$, we just plug in again

$$\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

$$\lambda = 2, -2$$

Eigenvectors we get are

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Spectral theorem says that for the symmetric matrices the eigenspaces are going to be orthogonal to each other.