Quantum Mechanics : : Homework 01

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Ahmed Saad Sabit, Rice University

Problem 1

The first statement in the problem in mathematical sense is

$$\langle x|\psi\rangle = \begin{cases} \psi(x) = c & -a < x < a \\ \psi(x) = 0 & \text{elsewhere} \end{cases}$$

The corresponding wave function in momentum representation is

$$\psi(p) = \langle p|\psi\rangle = \langle p|\hat{I}|\psi\rangle = \langle p| \int_{-\infty}^{\infty} \mathrm{d}x \; |x\rangle \langle x|\psi\rangle$$

Which gives us

$$\psi(p) = \int_{-\infty}^{\infty} \mathrm{d}x \; \langle p|x\rangle \, \psi(x) = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\psi(x)}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

This can be computed using what we have

$$\begin{aligned}
\psi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{a} c e^{-ipx/\hbar} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} c \left[\frac{1}{-ip/\hbar} (e^{-ipa/\hbar} - e^{ipa/\hbar}) \right] \\
&= \frac{c}{\sqrt{2\pi\hbar}} \left[\frac{\hbar}{ip} (e^{ipa/\hbar} - e^{-ipa/\hbar}) \right] \\
&= \frac{2c\hbar}{p\sqrt{2\pi\hbar}} \left[\frac{1}{2i} (e^{ipa/\hbar} - e^{-ipa/\hbar}) \right] \\
&= \frac{2c\hbar}{p\sqrt{2\pi\hbar}} \sin(ap/\hbar) \\
&= \sqrt{\frac{2\hbar c^2}{\pi p^2}} \sin(ap/\hbar)
\end{aligned}$$

Normalization gives us a clue what could be $\psi(x)$ which is

$$\int \psi(x)^2 \mathrm{d}x = c^2 2a = 1 \rightarrow c = \sqrt{\frac{1}{2a}}$$

Hence

$$\psi(p) = \sqrt{\frac{\hbar}{\pi a p^2}} \sin(ap/\hbar)$$

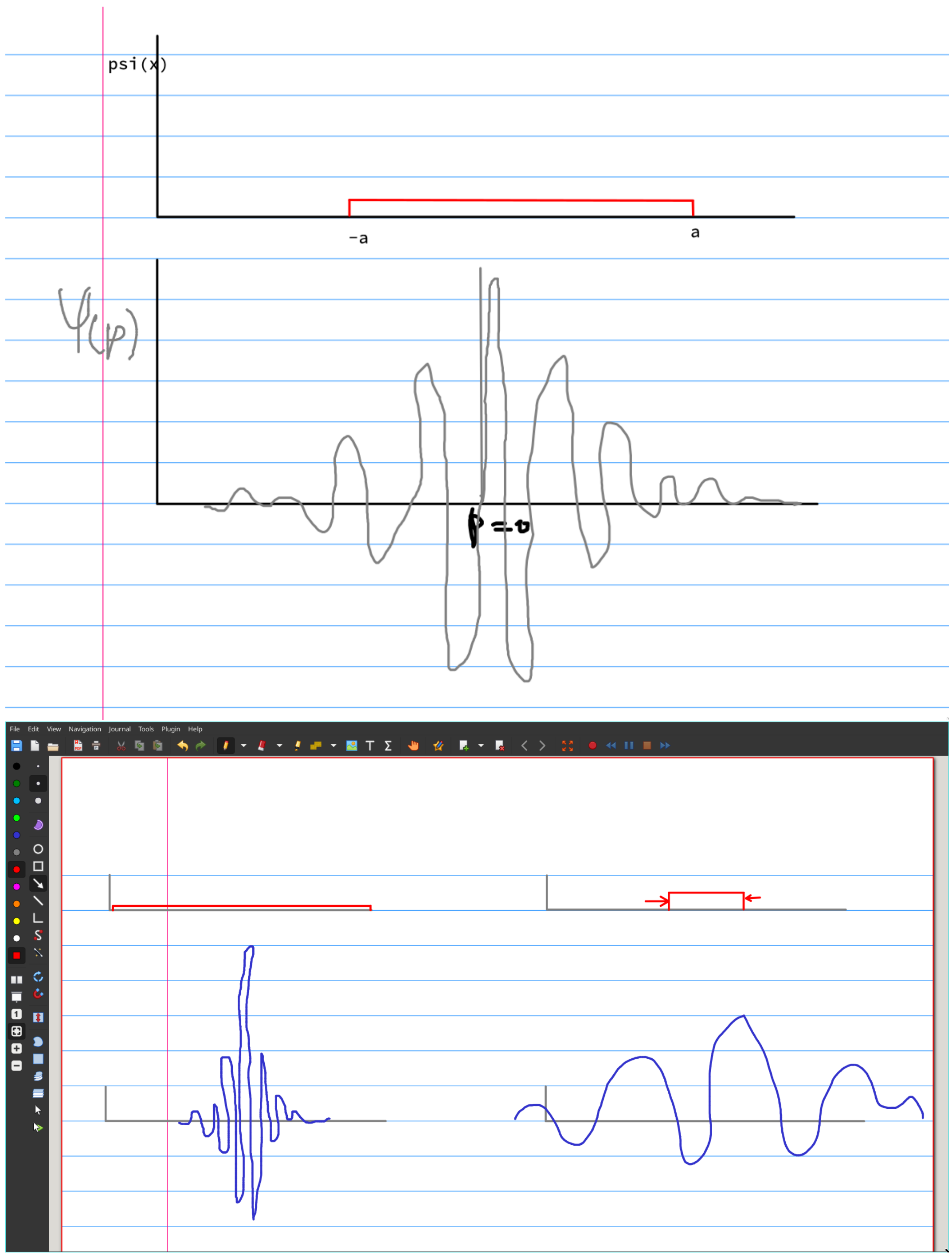

Figure 1: For large and small a

For $a \to 0$**:** In this case we have an extremely high amount of localization. This will make p very large because of uncertainty. Evidently from what we say from normalization of $\psi(x)$ the wavefunction will apparently look like a spike centered at origin.

For $a \to \infty$ This will localize p in a sense by making the error small, where we do have a dirac delta-like spike centered in origin.

Problem 2

(a)

Projection operator will satisfy the relation $\Lambda^2 = \Lambda$, hence computing Λ^2

$$\begin{aligned}\Lambda^2 &= \frac{1}{2}(I+P)\frac{1}{2}(I+P)\\ &= \frac{1}{4}(I^2 + IP + PI + P^2)\\ &= \frac{1}{4}(I + 2P + I)\\ &= \frac{1}{4}(2I + 2P)\\ &= \frac{1}{2}(I+P)\\ &= \Lambda\end{aligned}$$

We find that $\Lambda^2 = \Lambda$ hence Λ is a projection operator (proved).

(b)

Similar to the previous analysis we did

$$\begin{aligned}\overline{\Lambda}^2 &= (I-\Lambda)(I-\Lambda)\\ &= I - I\Lambda - \Lambda I + \Lambda^2\\ &= I - 2\Lambda + \Lambda\\ &= I - \Lambda\\ &= \overline{\Lambda}\end{aligned}$$

Hence proving $\overline{\Lambda}$ is also a projection operator. Now, if $\overline{\Lambda}$ and Λ are orthogonal, they will satisfy the relation $\hat{A}\hat{A}_\perp = \hat{A}_\perp\hat{A} = 0$

$$\begin{aligned}\overline{\Lambda}\Lambda &= (I-\Lambda)\Lambda\\ &= I\Lambda - \Lambda^2\\ &= \Lambda - \Lambda\\ &= 0\end{aligned}$$

$$\begin{aligned}\Lambda\overline{\Lambda} &= \Lambda(I-\Lambda)\\ &= \Lambda I - \Lambda^2\\ &= \Lambda - \Lambda\\ &= 0\end{aligned}$$

Hence both Λ and $\overline{\Lambda}$ are orthogonal projection operators (proved).

(c)

Let T be a non trivial projection operator. This means this T will satisfy $T^2 = T$. Then

$$T^{-1}T = I$$

Introduce a T on the right

$$(T^{-1}T)T = T$$

This yields

$$T^{-1}T^2 = T$$

Now, by using the rule of a non trivial projection operator

$$T^{-1}T = T$$

This shows (after comparing with the first equation $T^{-1}T = I$) that $T = I$. This is a trivial projection operator, which leaves the vector as it was. So T cannot have an inverse otherwise it's just a trivial projection (the identity operator).

Problem 03

(a)

Let's create the equation in the problem first to motivate the solution. The central potential Hamiltonian is given by

$$H = \frac{\vec{p}^2}{2m} + V(r)$$

The eigenequation for this is

$$H|E,l,m\rangle = E|E,l,m\rangle$$

Now we know the radial equation (Sakurai 3.230) that

$$\frac{1}{2m}\langle\vec{r}|\,\vec{p}^2\,|\Psi\rangle = -\left(\frac{\hbar^2}{2m}\right)\nabla^2\langle\vec{r}|\Psi\rangle = -\left(\frac{\hbar^2}{2m}\right)\left(\frac{\partial^2}{\partial r^2}\langle\vec{r}|\Psi\rangle + \frac{2}{r}\frac{\partial}{\partial r}\langle\vec{r}|\Psi\rangle - \frac{1}{\hbar^2 r^2}\langle\vec{r}|\vec{L}^2|\Psi\rangle\right)$$

Using this above equation with $\Psi = R_{El}Y_{l,m}$ and $\vec{L}^2|E,l,m\rangle = l(l+1)\hbar^2|E,l,m\rangle$ we get

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right]R_{El}(r) = ER_{El}(r)$$

Let us make the following substitution as instructed in the problem $R_{El}(r) = u(r)/r$ then we get the following equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}u_{El} + \left(\frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right)u_{El}(r) = Eu_{El}(r)$$

With some basic algebraic manipulation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u_{El} - \frac{l(l+1)}{r^2}u_{El} + k^2u_{El} - \frac{2mV(r)}{\hbar^2}u_{El}(r) = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u_{El} + k^2u_{El} - \left(\frac{l(l+1)}{r^2} + \frac{2mV(r)}{\hbar^2}\right)u_{El}(r) = 0$$

What we get for effective potential is

$$\tilde{V}(r) = \frac{2m}{\hbar^2}V(r) + \frac{l(l+1)}{r^2}$$

(b)

According to the question at $r \to \infty$ we will end up getting

$$\tilde{V}(r) = \frac{2m}{\hbar^2}V(r) + \frac{l(l+1)}{r^2} \to 0$$

Which makes sense because if $V(r)$ goes to zero faster than $1/r$, then it's also obvious that $\frac{1}{r^2}l(l+1)$ will also go faster than $1/r$. Hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u + k^2 u = 0$$

Because $E = \frac{\hbar^2}{2m}k^2$ hence positive energy corresponds to positive k for which we get a solution of the form $u \propto e^{ikr}$ which qualitatively means that the particle is unbounded. And that is true because $E > 0$. For unbounded states $E < 0$ we get

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u = k^2 u$$

This means $u \propto e^{-kr}$ which is a bound state solution that says that as we go far away from the "nucleus" the more unlikely it is to find the particle. This is true given the bounded condition.

(c)

Now in our problem $V(r)$ goes to zero like $\frac{1}{r}$. And $1/r^2$ definitely goes to zero faster than $1/r$ so again we have the exact same case as (b).

Our answers are verified because every Hydrogenic wavefunctions mentioned in the Problem (referenced to Townsend) has an e^{-r} term that exactly agrees to our findings in $E < 0$ bound states, leaving no room of confusion that what we have is qualitatively valid.

Problem 04

(a)

$$\begin{aligned}
[p_i, r^2] &= \left[p_i, \sum_j^3 x_j x_j\right] \\
&= [p_i, x_j x_j] \\
&= p_i x_j x_j - x_j x_j p_i \\
&= p_i x_j x_j - x_j p_i x_j + x_j p_i x_j - x_j x_j p_i \\
&= [p_i, x_j]x_j + x_j[p_i, x_j] \\
&= -i\hbar\delta_{ij} \cdot x_j - x_j \cdot (i\hbar\delta_{ij}) \\
&= -2i\hbar x_i
\end{aligned}$$

(b)

We will reuse some mathematical trick steps from (a).

$$
\begin{aligned}
[r^2, p^2] &= \left[\sum_i^3 x_i x_i, \sum_j^3 p_j p_j\right] \\
&= [x_i x_i, p_j p_j] \\
&= [x_i x_i, p_j] p_j + p_j [x_i x_i, p_j] \\
&= -[p_j, x_i x_i] p_j - p_j [p_j, x_i x_i] \\
&= -[p_j, x_i] x_i p_j - x_i [p_j, x_i] p_j - p_j [p_j, x_i] x_i - p_j x_i [p_j, x_i] \\
&= (i\hbar\delta_{ij}) x_i p_j + x_i (i\hbar\delta_{ij}) p_j + p_j (i\hbar\delta_{ij}) x_i + p_j x_i (i\hbar\delta_{ij}) \\
&= 2i\hbar(x_i p_i + p_i x_i) \\
&= 2i\hbar(x_i p_i + x_i p_i - i\hbar) \\
&= 4i\hbar x_i p_i + 2\hbar^2
\end{aligned}
$$