# Computational Complex Analysis : : Homework 09

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## Problem 01

Consider the Mobius Transformation because it preserves circles to circles and lines to lines. A particular interest is,

$$\frac{1}{z-\alpha}$$

So if we imagine the point $\alpha$ to be moved rightwards to infinity we will end up creating the other and inner circle to become parallel lines. The circles between the lines are going to be symmetrically positioned in a way that the touching points between the "inner circles" is going to be a straight line. Call this line $L$.

If we bring the $\alpha$ point back to it's original position, because at infinity we had $L$, it's going to become a circle as $\alpha$ comes to finite distance.

## Problem 02

We know that,

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Morph the $z$ to $2z$ so that we can apply $\sin 2\theta = 2 \sin \theta \cos \theta$.

$$\frac{\sin 2\pi z}{2\pi z} = \frac{2 \sin \pi z \cos \pi z}{2\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)$$

We can divide $\sin \pi z / \pi z$ equation in both sides through the following and bring the $\cos \pi z$ in one side,

$$\cos \pi z = \frac{\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}$$

Every even terms of the numerator are factored off by the denominator whenever we have an even $n$. So $n$ is only odd and the product is taken over the odd $n$. From this straight forward calculation it's apparent that,

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{z^2}{\left(n + \frac{1}{2}\right)^2}\right)$$

## Problem 03

$z = e^{i\theta}$ is going to help us simplify this,

$$\int_0^{2\pi} e^{e^{i\theta}} \, \mathrm{d}\theta = e^z \mathrm{d}\theta$$

Now we need to shift variables, so,

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = ie^{i\theta} = iz$$

This with the forbidden multiplication of $\mathrm{d}\theta$ on both sides,

$$\mathrm{d}z = iz\,\mathrm{d}\theta$$

We have a more well functioning integral where it's taken over a circle of radius 1 in $\mathbb{C}$ plane

$$\int_{\mathrm{C}} \frac{e^z}{iz}\mathrm{d}z$$

The residue at the singularity at $z = 0$ that happens inside the path

$$\operatorname{Res}\left(\frac{e^z}{iz}, 0\right) = \frac{e^0}{i} = \frac{1}{i}$$

We know from the residue theorem

$$\frac{1}{2\pi i}\int_{\mathrm{C}} \frac{e^z}{iz}\mathrm{d}z = \frac{1}{i}$$

Hence what get is for the integral

$$\boxed{2\pi}$$

## Problem 04

The singularity exists at $z = -1$. As we are taking the integration over the $i$ axis, consider the semicircle of radius $R$ that has it's flat side on $i$ axis. Let's find the residue first

$$\operatorname{Res}\left(\frac{e^z}{(z+1)^4}, -1\right) = \frac{(e^{-1})^3}{3!} = \frac{1}{6e^3}$$

Now, the integral, denoting $C$ to be the curved path of $R$ radius

$$\oint \frac{e^z}{(z+1)^4}\mathrm{d}z = \int_C \frac{e^z}{(z+1)^4}\mathrm{d}z + \int_{-iR}^{iR} \frac{e^z}{(z+1)^4}\mathrm{d}z$$

Looking at the $\int_C$, the real part of $z$ in the $e^z$ in denominator is goes smaller with increase of $R$ because the $\mathrm{Re}(z) < 0$. $e^{-R}$ will go to zero faster than $(-R+1)^4$ so $\int_C$ will eventually converge to zero as we set $R \to \infty$.

From this, with the simple application of residue theorem on $\oint$

$$\boxed{\int_{-iR}^{iR} \frac{e^z}{(z+1)^4} = \frac{2\pi i}{6e^3} = \frac{\pi i}{3e^3}}$$

## Problem 05

Our very own $B(\alpha, \beta)$ is,

$$B(\alpha, \beta) = 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}\,\mathrm{d}\theta$$

Setting the conditions to get a $\cos^{2m} x$

$$2\beta - 1 = 2m \implies \beta = \frac{2m+1}{2}$$

And to get a $\sin x = 1$

$$2\alpha - 1 = 0 \implies \alpha = \frac{1}{2}$$

Putting this together,

$$B\left(\frac{1}{2}, \frac{2m+1}{2}\right) = 2\int_0^{\pi/2} \cos^{2m} x \,\mathrm{d}x$$

Now by drawing the simple graph of this function we know that the bounds of this integration

$$2\int_0^{\pi/2} = \int_0^{\pi}$$

Hence from our intuition regarding odd-even functions it's very easy to see that

$$\int_0^{2\pi} = 4\int_0^{\pi/2}$$

It gives us the following form,

$$2B\left(\frac{1}{2}, \frac{2m+1}{2}\right) = \int_0^{2\pi} \cos^{2m} x \,\mathrm{d}x$$

Now we know (found this in wikipedia)

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{m+n}{mn} \Bigg/ \binom{m+n}{m}$$

So,

$$B\left(\frac{1}{2}, \frac{2m+1}{2}\right) = \frac{1/2 + (2m+1)/2}{(2m+1)/4} \Bigg/ \binom{\frac{1}{2} + \frac{2m+1}{2}}{\frac{1}{2}} = 4\,\frac{m+1}{2m+1} \Bigg/ \binom{m+1}{1/2}$$

The answer we got is for the integral as $2B(\frac{1}{2}, \frac{2m+1}{2})$ shows

$$8\,\frac{m+1}{2m+1} \Bigg/ \binom{m+1}{1/2}$$