Honors Multivariable Calculus : : Homework 1x

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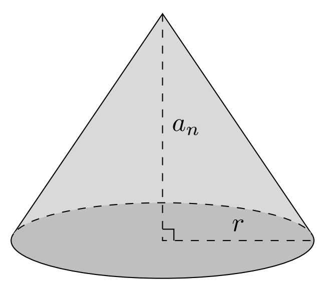
Problem 01

Aid for my brain: Let the compact region be $R \in \mathbb{R}^{n-1}$ where $n = 3$. Common sense tells us R is basically a disk in $\mathbb{R}^2$ or for now $x - y$ plane if we want what is in the figure. The tip of the cone is $\vec{a}$.
I think from the question our R doesn't really need to be necessarily a disk. Now the region is all the lines that join from $\vec{a}$ to R. If $\vec{x} \in R$ then considering a linear map $\gamma_{\vec{x}} : [0, 1] \to \mathbb{R}^n$ such that $\gamma_{\vec{x}}(0) = \vec{a}$ and $\gamma_{\vec{x}}(1) = \vec{x} \in R$



The line segment is set of all points such that,

$$\Gamma = \{s \in \mathbb{R}^n : s = \vec{a}t + (1-t)\vec{x} \text{ where } t \in [0,1], \vec{x} \in R, \vec{a} \in \mathbb{R}^n\}$$

Here $x \in R$. Every point $\vec{p} \in \Gamma$ is a memeber of the cone.

The volume of the region Γ (which is the defined cone) is going to be,

$$\int_\Gamma 1 = \text{Volume}$$

Now the burden is to find a region Γ.

The trick we are going to apply is to translate a relatively more simple shape into our cone shape by the following definition.

Definition 1. Define the D region to be $R \times [0, 1]$. Volume of this shape is

$$\int_D 1 = \text{vol}(R)$$

We need to find a transformation $T : D \to \Gamma$. From definition of cone we can write,

$$T(x_1, x_2, \ldots, x_{n-1}, x_n) = x_n(\vec{a} - (x_1, \ldots, x_{n-1}, 0)) + (x_1, \ldots, x_{n-1}, 0)$$

In vector form,

$$T(x_1, \dots, x_n) = \begin{pmatrix} x_n a_1 - x_n x_1 + x_1 \\ \vdots \\ x_n a_{n-1} - x_n x_{n-1} + x_{n-1} \\ x_n a_n \end{pmatrix}$$

T is injective (check appendix after this problem). So we can compute,

$$\mathrm{d}T = \begin{bmatrix} 1-x_n & 0 & \cdots & 0 & a_1 - x_1 \\ 0 & 1-x_n & \cdots & 0 & a_2 - x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-x_n & a_{n-1} - x_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

dT being an upper triangular matrix,

$$\det(\mathrm{d}T) = (1-x_n)^{n-1} a_n$$

Now through change of coordinates we can find the integral,

$$\int_D 1 = \int_\Gamma 1 \cdot T |\det(\mathrm{d}T)|$$

Because $1 \cdot T = 1$,

$$\begin{aligned} &= \int_\Gamma |\det(\mathrm{d}T)| \\ &= \int_\Gamma |(1-x_n)^{n-1} a_n| \\ &= \int_\Gamma (1-x_n)^{n-1} a_n \quad \text{(integration over } x_n > 0) \\ &= a_n \int_\Gamma (1-x_n)^{n-1} \\ &= a_n \int_R \int_0^1 (1-x_n)^{n-1} \mathrm{d}x_n \end{aligned}$$

We can carry out the integration using $u = 1 - x_n$ substitution method and we can get,

$$a_n \mathrm{vol}(R) \left[\frac{u^n}{n} \right]_0^1$$

Hence,

$$\boxed{\frac{a_n}{n} \mathrm{vol}(R)}$$

T is injective

For this transformation to be valid, T must be injective other than places that are content zero. We can see that T is injective everywhere on it's domain except for the point where the $x_n = 1$.

T is only valid if $x_n = 1$ "place" or $R \times \{1\}$ is content zero in $\mathbb{R}^n$.

Consider $f : R \subset R^{n-1} \to \mathbb{R}$ given by $f(\vec{x}) = 1$. Note that the graph of f is precise $R \times \{1\}$. Since our region R is compact we know that the graph of a continuous function is content zero. Therefore $R \times \{1\}$ is content zero.

Problem 02

Let D denote the region described by the n dimensional solid ellipsoid

$$\frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1$$

Here a_i are positive. The volume is,

$$\int_D 1$$

We need to do a transformation now, so define $T : D' \to D$ defined by

$$T(x_1, \ldots, x_n) = (a_1 x_1, \ldots, a_n x_n)$$

where D' is the unit ball in $\mathbb{R}^n$. Then note that T is surjective to D and injective on it's domain.

$$\mathrm{d}T = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}$$

So easily,

$$\det(\mathrm{d}T) = a_1 \cdots a_n$$

Now we perform the change of coordinates and compute,

$$\int_D 1 = \int_{D'} 1 \cdot T |\det(\mathrm{d}T)|$$

Carrying out the computation we simply have,

$$\int_{D'} |a_1 \cdots a_n|$$

So from here,

$$= (a_1 \cdots a_n)(\text{volume of the unit ball in } \mathbb{R}^n)$$

$$\boxed{(a_1 \cdots a_n) \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}}$$

Problem 03

Denote the region enclosed by the blue curve and red line by D. The average value of f on this region is,

$$\overline{f} = \frac{\int_D f}{\mathrm{vol}(D)}$$

We first need to compute $\int_D f$. So consider,

$$T : D' \to D \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

From here we can see,

$$|\det(\mathrm{d}T)| = r$$

We can compute the integral $\int_D f$ as,

$$\int_D f = \int_{D'} f \cdot T \cdot r = \int_{D'} (r^2 \cos^2\theta + r^2 \sin^2\theta) r$$

From here,

$$\int_{D'} r^3 = \int_0^{2\pi} \int_0^{\theta} r^3 \mathrm{d}r\mathrm{d}\theta$$

Computing the integral gives,

$$\frac{8}{5}\pi^5$$

Now we are required to evaluate the volume.

$$\int_D 1 = \int_{D'} (1 \cdot T) \cdot r = \int_0^{2\pi} \int_0^{\theta} r\mathrm{d}r\mathrm{d}\theta$$

Solving this gives us,

$$\frac{4}{3}\pi^3$$

So the division of the two results gives us the average value of f on D,

$$\frac{6}{5}\pi^2$$

Problem 04

Similar to the first problem let's start by a simple volume first, define the region $R \times [0, 2\pi]$. From here we define

$$T : R \times [0, 2\pi] \to D$$

with

$$T(x, y, z) = (x, y \cos z, y \sin z)$$

Note how T is a surjective map onto D and is injective at every points except the content zero portion that is $z = 0$ and $z = 2\pi$.

We will now compute,

$$\mathrm{d}T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & -y \sin z \\ 0 & \sin z & y \cos z \end{pmatrix}$$

From here,

$$\det(\mathrm{d}T) = 1 \cdot \det \begin{pmatrix} \cos z & -y \sin z \\ \sin z & y \cos z \end{pmatrix}$$

$$\det(\mathrm{d}T) = y$$

Now we can carry out this integration,

$$\int_D 1 = \int_{R \times [0,2\pi]} 1 \cdot T |\det(\mathrm{d}T)| = \int y = \int_R \int_0^{2\pi} y dz$$

Solving this get's us,

$$2\pi \int_R y$$

Now from here we compute,

$$2\pi \cdot \text{avg of y on R } \cdot \mathrm{vol}(R)$$

We get,

$$2\pi \int_R y$$

From this we showed that,

$$\mathrm{vol}(D) = \int_D 1 = 2\pi(\text{avg of y on R}) \cdot \mathrm{vol}(R)$$

1 Problem 05

The unit ball on $\mathbb{R}^4$ is defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$$

Let denote D on this region the unit ball, and to find the unit ball volume we find $\int_D 1$.

The coordinates can be changed by the following $T : D' \to D$ through,

$$T(r_1, \theta, r_2, \theta_2) = (r_1 \cos\theta, r_1 \sin\theta_1, r_2 \cos\theta_2, r_2 \sin\theta_2)$$

where $D' = S_1 \times [0, 2\pi], S_2 \times [0, 2\pi]$ such that, $\forall r_1 \in S$ and $\forall r_2 \in S_2$ we have $r_1^2 + r_2^2 \leq 1$ and $r_1, r_2 \geq 0$.

$$\mathrm{d}T = \begin{bmatrix} \cos\theta_1 & -r_1 \sin\theta_1 & 0 & 0 \\ \sin\theta_1 & r_1 \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -r_2 \sin\theta_2 \\ 0 & 0 & \sin\theta_2 & r_2 \cos\theta_2 \end{bmatrix}$$

Now that $\mathrm{d}T$ is a block matrix, therefore we calculate its determinant as,

$$\det(\mathrm{d}T) = \det\begin{pmatrix} \cos\theta_1 & -r_1 \sin\theta_1 \\ \sin\theta_1 & r_1 \cos\theta_1 \end{pmatrix} \cdot \det\begin{pmatrix} \cos\theta_2 & -r_2 \sin\theta_2 \\ \sin\theta_2 & r_2 \cos\theta_2 \end{pmatrix}$$
$$= (r_1 \cos^2\theta_1 + r_1 \sin^2\theta_1)\left(r_2 \cos^2\theta_2 + r_2 \sin^2\theta_2\right) = r_1 r_2$$

Through change of variables,

$$\int_D 1 = \int_{D'} 1 \cdot T|\det(\mathrm{d}T)|$$
$$= \int_{D'} r_1 r_2$$

This integral can be computed,

$$\int_{D'} r_1 r_2 = \int_{S_1 \times S_2} \int_0^{2\pi} \int_0^{2\pi} r_1 r_2 \mathrm{d}\theta_1 \mathrm{d}\theta_2$$
$$= \int_{S_1 \times S_2} r_1 r_2 (4\pi^2)$$

Inside this region we have $r_1^2 + r_2^2 \leq 1$. This is a unit circle and through change of ariables again,

$$G : S \to S_1 \times S_2$$
$$G(R, \alpha) = G(R\cos\theta, R\sin\alpha)$$

where $S = [0, 1] \times [0, \frac{\pi}{2}]$ we constrain the input $\alpha \in [0, \frac{\pi}{2}]$ upto content zero.

$$\mathrm{d}G = \begin{bmatrix} \cos\alpha & -R\sin\alpha \\ \sin\alpha & R\cos\alpha \end{bmatrix}$$
$$\det(\mathrm{d}G) = R\cos^2\alpha + R\sin^2\alpha = R$$

Then by the change of variables it follows,

$$\int_{S_1 \times S_2} r_1 r_2 = \int_S (R\cos\alpha)(R\sin\alpha) \cdot R$$
$$= \int_0^{\pi/2} \int_0^1 R^3 \cos\alpha \sin\alpha \mathrm{d}R \mathrm{d}\alpha$$

Completing this integral we find,

$$\boxed{\frac{1}{8}}$$

Using this in our previous integral,

$$\int_{D'} r_1 r_2 = 4\pi^2 \left(\frac{1}{8}\right) = \frac{\pi^2}{2}$$

Hence the volume is $\pi^2/2$.