Quantum Mechanics : : Homework 03

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Problem 01

(a)

I did the first matrix multiplication computation by hand and it aligns with what I got from Matlab. There's way too many multiplication to do so I am instead opting for matlab solution.

$$\sigma_1\kappa_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\kappa_1\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

I have thought enough how to convince the grader that I know how to do Matrix Multiplication, the most tedious approach being I show each and every single individual dot product on paper. That's too much work.

We just wanna prove that the commutator is zero

$$[\sigma_1\kappa_1] = \sigma_1\kappa_1 - \kappa_1\sigma_1 = 0$$

Very similarly we can keep doing it, one by one in MATLAB

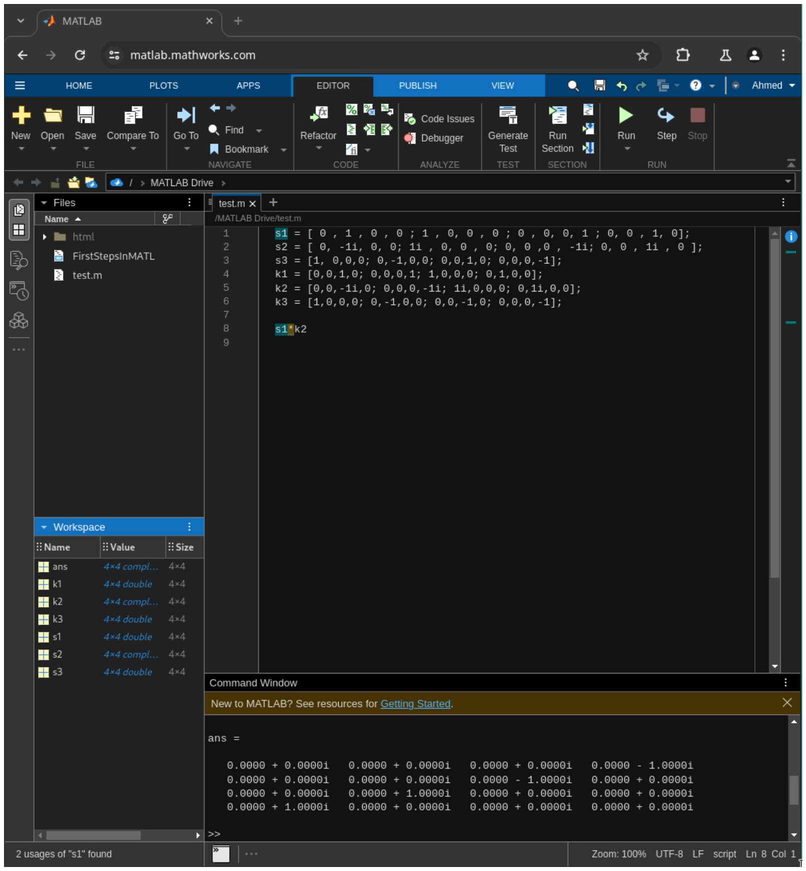

Figure 1: Computing $\sigma_1\kappa_2$

We find

$$\sigma_1\kappa_2 = \kappa_2\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad [\sigma_1, \kappa_2] = 0$$

$$\sigma_1\kappa_3 = \kappa_3\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \rightarrow \quad [\sigma_1, \kappa_3] = 0$$

$$\sigma_2\kappa_2 = \kappa_2\sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad [\sigma_2, \kappa_2] = 0$$

$$\sigma_2\kappa_3 = \kappa_3\sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \rightarrow \quad [\sigma_2, \kappa_3] = 0$$

b

Take σ_1 and do pen and paper calculation to find out the eigenvalues. I used mathematica to find out the determinant safely and we get the eigenvalues from the characteristic polynomial

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

Which gives,

$$\lambda = -1, -1, 1, 1$$

Set of eigenvectors for σ_1 being

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Now let's do this for the κ_2 and we get the characteristic equation

$$\lambda^4 - 2\lambda^2 + 1$$

And we get

$$\lambda = -1, -1, 1, 1$$

With Eigenvectors

$$\begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

By theory if there are common eigenvectors,

$$A|\phi\rangle = \phi_A|\phi\rangle$$

$$B|\phi\rangle = \phi_B|\phi\rangle$$

Then

$$AB|\phi\rangle = \phi_A\phi_B|\phi\rangle$$

Taking $\sigma_1\kappa_2$,

$$\sigma_1\kappa_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

The eigenvectors of these are

$$\begin{pmatrix} i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}$$

Looking at the eigenvectors we know there are NO common eigenvectors.

Problem 02

a

Total force on m_1 and m_2

$$m_1\ddot{x}_1 = -k_A x_1 + k_B(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k_B(x_2 - x_1) + k_C(L - x_2 - x_1)$$

The equilibriums are for which net force is zeroed

$$0 = -k_A x_1 + k_B(x_2 - x_1)$$

$$0 = -k_B(x_2 - x_1) + k_C(L - x_2 - x_1)$$

Trying to solve this by hand I get,

$$x_1 = \frac{k_B}{k_A + k_B} x_2$$

Individual solution

$$x_2^0 = \frac{k_A k_C + k_B k_C}{(k_B + k_C)(k_A + k_B) - (k_B - k_C)k_B} L$$

$$x_1^0 = \frac{k_B k_C}{(k_B + k_C)(k_A + k_B) - (k_B - k_C)k_B} L$$

$$-ax_1 + b\left(x_2 - x_1\right) = 0, \ -b\left(x_2 - x_1\right) + c\left(L(x) - x_2 - x_1\right) = 0$$

Solution

$$x_1 = \frac{bcL}{ab + ac + 2bc}, x_2 = -\frac{-acL - bcL}{ab + ac + 2bc}, \quad a \neq -\frac{2bc}{b + c}, a \neq -b, a + b \neq 0$$

Figure 2: Verifying that I haven't really messed up.

b

Apparently $\ddot{x}_i = \delta\ddot{x}_i$. And

$$x_2 - x_1 = x_2^0 - x_1^0 + \delta x_2 - \delta x_1$$

As we know the denominator of x_1^0, x_2^0 are the same, calling them as D,

$$x_2 - x_1 = \frac{k_A k_C L}{D} + \delta x_2 - \delta x_1$$

$$x_2 + x_1 = \frac{k_A k_C + 2k_B k_C}{D} L + \delta x_1 + \delta x_2$$

Thus through substituting our new formulas for $x_2 - x_1$ and $x_1 + x_2$ with equilibrium position in the newton's equations we can get,

$$m_1\delta\ddot{x}_1 = -\frac{k_A k_B k_C L}{D} - k_A\delta x_1 + \frac{k_A k_B k_C L}{D} + k_B(\delta x_2 - \delta x_1) = \boxed{-k_A\delta x_1 + k_B(\delta x_2 - \delta x_1)}$$

For the second equation,

$$m_2\delta\ddot{x}_2 = -\frac{k_A k_B k_C L}{D} - k_B(\delta x_2 - \delta x_1) + k_C L - \frac{k_A k_C^2 L + 2k_B k_C^2 L}{D} - k_C(\delta dx_2 + \delta dx_1)$$

$$m_2\delta\ddot{x}_2 = -\frac{k_Ak_Bk_CL}{D} - k_B(\delta x_2 - \delta x_1) + \frac{k_CDL - k_Ak_C^2L - 2k_Bk_C^2L}{D} + k_CL - k_C(\delta dx_2 + \delta dx_1)$$

Note that $D = (k_B + k_C)(k_A + k_B) - (k_B - k_C)k_B = k_Ak_B + k_Ak_C + 2k_Bk_C$, hence

$$\frac{k_CDL - k_Ak_C^2L - 2k_Bk_C^2L}{D} = \frac{k_Ak_Bk_C + k_Ak_C^2 + 2k_Bk_C^2 - k_Ak_C^2 - 2k_Bk_C^2}{D}L = \frac{k_Ak_Bk_C}{D}L$$

$$m_2\delta\ddot{x}_2 = -\frac{k_Ak_Bk_CL}{D} - k_B(\delta x_2 - \delta x_1) + \frac{k_Ak_Bk_C}{D}L + k_CL - k_C(\delta dx_2 + \delta dx_1)$$

With $m_1 = m_2 = m$, we finalize

$$\boxed{m\delta\ddot{x}_2 = -k_B(\delta x_2 - \delta x_1) - k_C\delta x_2}$$

$$\boxed{m\delta\ddot{x}_1 = -k_A\delta x_1 + k_B(\delta x_2 - \delta x_1)}$$

c

We can divide both sides of equations with m and name $k_I/m = \omega_I^2$

$$\delta\ddot{x}_1 = -\omega_A^2\delta x_1 + \omega_B^2(\delta x_2 - \delta x_1)$$

$$\delta\ddot{x}_2 = -\omega_B^2(\delta x_2 - \delta x_1) - \omega_C^2\delta x_2$$

The homogenous set of equation,

$$\delta\ddot{x}_1 = (-\omega_A^2 - \omega_B^2)\delta x_1 + \omega_B^2\delta x_2$$
$$\delta\ddot{x}_2 = \omega_B^2\delta x_1 + (-\omega_B^2 - \omega_C^2)\delta x_2$$

This is very equivalently

$$\begin{pmatrix}\delta\ddot{x}_1\\ \delta\ddot{x}_2\end{pmatrix} = \begin{pmatrix}(-\omega_A^2 - \omega_B^2) & \omega_B^2\\ \omega_B^2 & (-\omega_B^2 - \omega_C^2)\end{pmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}$$

Simplify notation to avoid myself to getting hospitalized.

$$\begin{pmatrix}\delta\ddot{x}_1\\ \delta\ddot{x}_2\end{pmatrix} = \begin{pmatrix}-A - B & B\\ B & -B - C\end{pmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}$$

The characteristic equation is

$$\frac{1}{m^2}(k_Ak_B + k_Ak_C + k_Bk_C) + \frac{1}{m}(k_A + 2k_B + k_C)\lambda + \lambda^2 = 0$$

$$\lambda = \frac{1}{2m}\left(\pm\sqrt{4k_B^2 - (k_A - k_C)^2} - k_A - 2k_B - k_C\right)$$

$$\boxed{\lambda_1 = -\omega_1^2 = \frac{1}{2m}\left(k_D - k_A - 2k_B - k_C\right)}$$

$$\boxed{\lambda_2 = -\omega_2^2 = \frac{1}{2m}\left(-k_D - k_A - 2k_B - k_C\right)}$$

$$\omega_1 = \sqrt{\frac{(k_A + k_C + 2k_B) - k_D}{2m}}$$
$$\omega_2 = \sqrt{\frac{(k_A + k_C + 2k_B) + k_D}{2m}}$$

The reason why we can equate the eigenfrequency with eigenvalue in such a way is given in the following box

Showing eigenfrequency of the matrix here itself is also the eigenvalue

$$\begin{pmatrix} \delta\ddot{x}_1 \\ \delta\ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -A - B & B \\ B & -B - C \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}$$

$$\delta\ddot{x}_1 = (-\omega_A^2 - \omega_B^2)\delta x_1 + \omega_B^2 \delta x_2$$
$$\delta\ddot{x}_2 = \omega_B^2 \delta x_1 + (-\omega_B^2 - \omega_C^2)\delta x_2$$

Let's say $\delta x_i = A_i e^{-i\lambda t}$ (using λ instead of ω to reduce eyesore).

$$(A_1 e^{-i\lambda t})(-1)\lambda^2 = (-\omega_A^2 - \omega_B^2)\delta x_1 + \omega_B^2 \delta x_2$$
$$(A_2 e^{-i\lambda t})(-1)\lambda^2 = \omega_B^2 \delta x_1 + (-\omega_B^2 - \omega_C^2)\delta x_2$$

$$-(\delta x_1)\lambda^2 = (-\omega_A^2 - \omega_B^2)\delta x_1 + \omega_B^2 \delta x_2$$
$$-(\delta x_2)\lambda^2 = \omega_B^2 \delta x_1 + (-\omega_B^2 - \omega_C^2)\delta x_2$$

$$0 = (-\omega_A^2 - \omega_B^2 + \lambda^2)\delta x_1 + \omega_B^2 \delta x_2$$
$$0 = \omega_B^2 \delta x_1 + (-\omega_B^2 - \omega_C^2 + \lambda^2)\delta x_2$$

Re-writing this whole mess is basically

$$0 = \begin{pmatrix} (-\omega_A^2 - \omega_B^2) - H & \omega_B^2 \\ \omega_B^2 & (-\omega_B^2 - \omega_C^2) - H \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}$$

Hence solving the eigenvalue $H = -\lambda^2$ for the matrix $\begin{pmatrix} -A - B & B \\ B & -B - C \end{pmatrix}$ basically yields us with the required λ^2 in $A_i e^{-i\lambda t}$

Corresponding eigenvectors are

$$|\omega_1\rangle = \begin{pmatrix} \dfrac{k_C - k_A - k_D}{2k_B} \\ 1 \end{pmatrix}$$

$$|\omega_2\rangle = \begin{pmatrix} \dfrac{k_C - k_A + k_D}{2k_B} \\ 1 \end{pmatrix}$$

d

For ridiculous k_C,

$$k_D = \sqrt{4k_B^2 + (k_C - k_A)^2} = \sqrt{4k_B^2 + k_C^2\left(1 - \frac{k_A}{k_C}\right)^2} \approx k_C\left(1 - \frac{k_A}{k_C}\right) = k_C - k_A$$

Then

$$\omega_1 = \sqrt{\frac{k_A + k_C + 2k_B - k_C + k_A}{2m}} = \sqrt{\frac{k_A + k_B}{m}}$$
$$\omega_2 = \sqrt{\frac{k_A + k_C + 2k_B + k_C - k_A}{2m}} = \sqrt{\frac{k_C + k_B}{m}}$$
$$|\omega_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$|\omega_2\rangle = \begin{pmatrix} \frac{k_C - k_A}{k_B} \\ 1 \end{pmatrix}$$

e

$k_A = k_C = 0$ hence,

$$\omega_1 = \omega_2 = \sqrt{\frac{k_B}{m}}$$

Problem 03

a

For unitary matrix we know

$$\det(I) = \det(U^t U) = \det(U^t)\det(U) = 1$$

Hence,

$$\det(U^t \Omega U) = \det(U^t)\det(\Omega U) = \det(U^t)\det(\Omega)\det(U) = \det(\Omega)$$

Proven.

b

Note that what we have here is a diagonal matrix. Determinant of a diagonal matrix is given by product of all diagonal elements of the matrix. Hence, for the given matrix,

$$\det U = e^{i\omega_1} e^{i\omega_2} \cdots e^{i\omega_n} = e^{i(\sum_{i=1}^{n} \omega_i)}$$

Now take a look at $\log U$, it is ,

$$\begin{pmatrix} i\omega_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & i\omega_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & i\omega_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & i\omega_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega_n \end{pmatrix}$$

Trace of this matrix is simply

$$\operatorname{Tr}\log U = i\left(\sum_{i=1}^{n}\omega_i\right)$$

For this,

$$\det U = e^{\operatorname{Tr}\log U}$$

Proven.

Problem 04

a

$$(\vec{n}\cdot\vec{\sigma})^2 = (\vec{n}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) = \sum_{i=1}^{3}\sum_{j=1}^{3} n_i n_j \sigma_i \sigma_j$$

We will break the sum between $i = j$ and $i \neq j$ cases, we get,

$$= \sum_{\mu\neq\nu} n_\mu n_\nu \sigma_\mu \sigma_\nu + \sum_{k=1}^{3} n_k^2 \sigma_k \sigma_k$$

Because of being real numbers, obviously $n_\nu n_\mu = n_\mu n_\nu$. The first sum goes over both (μ, ν) indices and also (ν, μ) indices. Hence because $\sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu = 2\delta_{\mu\nu} I$, and as $\mu \neq \nu$ we have the first term equal 0.

For the second term with same indices we see $2\sigma_k\sigma_k = 2I$ hence $\sigma_k\sigma_k = I$. Hence the sum overall ends up being,

$$(\vec{n}\cdot\vec{\sigma})^2 = (n_1^2 + n_2^2 + n_3^2)I = I$$

As (n_1, n_2, n_3) is a unit vector. Proven.

b

I will do this first,

$$i = i \quad i^2 = -1 \quad i^3 = i^2 i = -i \quad i^4 = 1$$

$$U_1 = \exp(-i\vec{\phi}\cdot\vec{\sigma}) = 1 + (-i\phi\vec{n}_\phi\cdot\vec{\sigma}) + \frac{1}{2!}(-i\phi\vec{n}_\phi\cdot\vec{\sigma})^2 + \frac{1}{3!}(-i\phi\vec{n}_\phi\cdot\vec{\sigma})^3 + \frac{1}{4!}(-i\phi\vec{n}_\phi\cdot\vec{\sigma})^4 + \frac{1}{5!}(-i\phi\vec{n}_\phi\vec{\sigma})^5 + \cdots$$

Isolating the terms individually, firstly we note that even power on $\vec{n}_\phi\cdot\vec{\sigma}$ is going to be I (last problem).

Every even terms hence become the series,

$$1 - \frac{1}{2!}(\phi^2)I + \frac{1}{4!}(\phi^4)I + \cdots$$

For every odd terms of power n, the previous term $n-1$ is even so, $(\vec{n}_\phi\cdot\vec{\sigma})^{(n-1)+1} = (\vec{n}_\phi\cdot\vec{\sigma})$. Like this every odd terms become,

$$i\left(-\phi + \frac{1}{3!}\phi^3 - \frac{1}{5!}\phi^5\right)(\vec{n}_\phi\cdot\vec{\sigma})$$

So the two even and odd terms together combine to form,

$$U_1 = \cos(\phi)\hat{I} - i\sin(\phi)(\vec{n}_\phi\cdot\hat{\vec{\sigma}})$$

Proven.

c

$\vec{\phi} = \phi\vec{n}_\phi$ so using that

$$U_1 = \exp(-i\phi\vec{n}_\phi\cdot\vec{\sigma})$$

$$\log(U_1) = -i\phi\,(\vec{n}_\phi\cdot\vec{\sigma})$$

$$\log(U_1) = -i\phi(n_\phi^1\sigma^1 + n_\phi^2\sigma^2 + n_\phi^3\sigma^3)$$

$$= -i\phi\left(n_\phi^1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + n_\phi^2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + n_\phi^3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

$$\mathrm{Tr}(\log U_1) = -i\phi \left(n_\phi^1 \mathsf{Tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_\phi^2 \mathsf{Tr} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_\phi^3 \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = 0$$

As we know

$$\det U_1 = e^{\mathrm{Tr}\log(U_1)} = e^0 = 1$$

Proven.

Problem 05

The core idea (after a lot of thinking) is that the introduction of a λ factor and taking derivative with respect to that brings down the operator $\hat{A}$ from $e^{\lambda \hat{A}}$. It's like applying a trick to get the stubborn kid off the tree house.

Let's start with $g(\lambda) = e^{\lambda A} B e^{-\lambda A}$ as instructed. Taking derivatives at $\lambda = 0$

$$g(0) = B$$

$$g'(0) = (e^{\lambda A} A B e^{-\lambda A} + e^{\lambda A} B(-A) e^{-\lambda A})_{\lambda=0} = AB - BA = [A, B] = 0$$

$$g''(0) = \left(e^{\lambda A} A[A, B] e^{-\lambda A} + e^{\lambda A}[A, B](-A) e^{-\lambda A}\right)_{\lambda=0} = e^{\lambda A}[A, [A, B]] e^{-\lambda A}\big|_{\lambda=0} = [A, [A, B]]$$

$$\vdots \quad \text{(induction)}$$

$$g^{(k)}(0) = [A, [A, \cdots (k - \text{times}) \cdots, [A, B] \cdots]] = [A, \cdot]^k B$$

As given in the series,

$$g(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} g^k(0)$$

But we just solved for the particular form of $g^k(0)$, which gives us,

$$g(1) = e^A B e^{-A} = B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]] + \cdots$$

Proven.