Mechanics : : Homework 08

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Problem 01

In the orbital frame the total force

$$F_{\text{tot}} = m\frac{v^2}{r} - \frac{k}{r^2}\exp(-r/a) = \frac{L^2}{mr^3} - \frac{k}{r^2}\exp(-r/a)$$

$$F_0 = 0 \tag{Equilibrium}$$

(a)

Let's find the first order perturbation from the equilibrium position $r = r_0$ for which $F(r_0) = F_0$. Instead of ρ, I use r.

$$\begin{aligned}
F_{\text{centrifugal}} &= \frac{L^2}{mr^3} \\
\frac{\mathrm{d}F_\omega}{\mathrm{d}r} &= -3\frac{L^2}{mr^4}
\end{aligned}$$

$$\begin{aligned}
F_{\text{central}} &= -\frac{k}{r^2}\exp(-r/a) \\
\frac{\mathrm{d}F_c}{\mathrm{d}r} &= \frac{2k}{r^3}\exp(-r/a) + \left(-\frac{k}{r^2}\right)\left(-\frac{1}{a}\right)\exp(-r/a) = \left(\frac{2k}{r^3} + \frac{k}{ar^2}\right)\exp(-r/a)
\end{aligned}$$

$$\mathrm{d}F_{\text{tot}} = (\mathrm{d}F_{\text{tot}} + F_0) - F_0 \approx m\ddot{r} = \left[\left(\frac{2k}{r^3} + \frac{k}{ar^2}\right)\exp(-r/a) - 3\frac{L^2}{mr^4}\right]\Delta r$$

We can invoke the condition of nudging from the equilibrium to solve for L using the following condition $F_0 = 0$

$$\frac{L^2}{mr^3} = \frac{k}{r^2}\exp(-r/a) \implies L^2 = mkr\exp(-r/a)$$

Using this on the equation we received above,

$$
\begin{aligned}
m\ddot{r} &= \left[\left(\frac{2k}{r^3} + \frac{k}{ar^2}\right)\exp(-r/a) - 3\frac{L^2}{mr^4}\right]\Delta r \\
&= \left[\left(\frac{2k}{r^3} + \frac{k}{ar^2}\right)\exp(-r/a) - 3\frac{mkr\exp(-r/a)}{mr^4}\right]\Delta r \\
&= \left[2\frac{k}{r^3} + \frac{k}{ar^2} - 3\frac{k}{r^3}\right]\exp(-r/a)\Delta r \\
&= \left[\frac{k}{ar^2} - \frac{k}{r^3}\right]\exp(-r/a)\Delta r
\end{aligned}
$$

$$
\implies \ddot{r} + \frac{k}{m}\exp(-r/a)\left[\frac{1}{r^3} - \frac{1}{ar^2}\right]\Delta r = 0
$$

Required condition for Simple Harmonic Oscillations

$$
\begin{aligned}
\frac{1}{r^3} - \frac{1}{ar^2} &> 0 \\
\frac{1}{r^3} &> \frac{1}{ar^2} \\
r^3 &< ar^2 \\
r &< a
\end{aligned}
$$

Hence as long as our radius of motion is within a we have small oscillations.

The angular frequency, please note that r here is such that $F(r) = F_0$ because this is a small nudge from equilibrium.

$$
\omega^2 = \frac{k}{m}\exp(-r/a)\left(\frac{1}{r^3} - \frac{1}{ar^2}\right)
$$

The supremum of r is

$$
\mathrm{Sup}(r) = a
$$

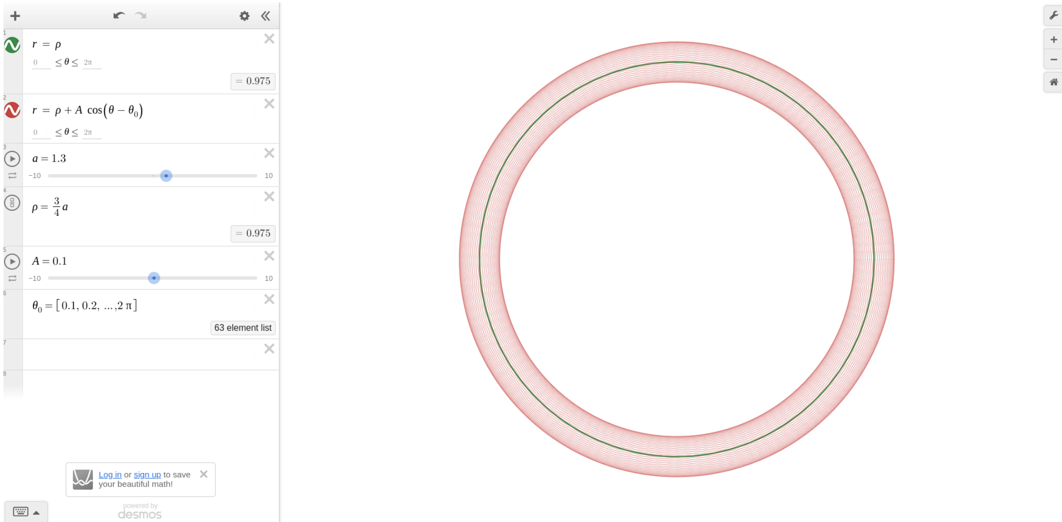
(b)

$$
\Delta\theta = \frac{L}{mr^2}\frac{2\pi}{\omega} = \sqrt{mkre^{-r/a}}\frac{1}{mr^2}\frac{2\pi}{\sqrt{\frac{k}{m}e^{-r/a}\left(\frac{1}{r^3} - \frac{1}{ar^2}\right)}} = 2\pi\sqrt{\frac{a}{a-r}}
$$

(c)

$$
r = \frac{3}{4}a \implies \Delta\Theta = 4\pi
$$

All possible orbit superimposed on one another gives the following diagram



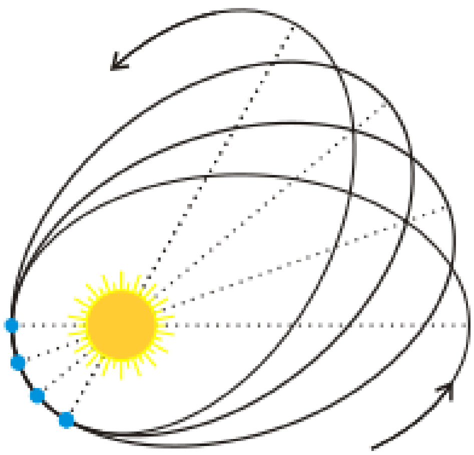


Figure 1: ./ss/8/1.png

Problem 02

The position of the particle in terms of (r, θ, ϕ) where $r = r(t)$ is a given function of time, the generalized coordinates can be $\theta = \theta(t) = q_\theta(t)$ and $\phi = \phi(t) = q_\phi(t)$.

(a)

$$\mathcal{L} = \frac{m}{2}\left(r(t)^2\dot{\theta}^2 + r(t)^2 \sin^2\theta \dot{\phi}^2\right) - mgr(t)\cos\theta$$

$$
\begin{aligned}
\mathcal{H} &= \sum_{n=1}^{2} \dot{q}_n \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \mathcal{L} \\
&= \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \\
&= \frac{m}{2}\left[\dot{\theta}\left(2\dot{\theta}r^2(t)\right) + \dot{\phi}\left(2\dot{\phi}r^2(t)\sin^2\theta\right)\right] - \frac{m}{2}\left(r(t)^2\dot{\theta}^2 + r(t)^2\sin^2\theta\dot{\phi}^2\right) + mgr(t)\cos\theta \\
&= \frac{m}{2}\left[\left(2\dot{\theta}^2r^2(t)\right) + \left(2\dot{\phi}^2r^2(t)\sin^2\theta\right)\right] - \frac{m}{2}\left(r(t)^2\dot{\theta}^2 + r(t)^2\sin^2\theta\dot{\phi}^2\right) + mgr(t)\cos\theta \\
&= \frac{m}{2}\left(r(t)^2\dot{\theta}^2 + r(t)^2\sin^2\theta\dot{\phi}^2\right) + mgr(t)\cos\theta = E(t)
\end{aligned}
$$

(b)

Solving for the generalized momentum

$$
p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}
$$

$$
\begin{aligned}
p_\theta &= mr^2(t)\dot{\theta} \\
p_\phi &= m\dot{\phi}^2r^2(t)\sin^2\theta
\end{aligned}
$$

Solving for derivatives of generalized momentum

$$
\dot{p}_n = -\frac{\partial H}{\partial q_n}
$$

$$
\begin{aligned}
\dot{p}_\theta &= mgr(t)\sin\theta - mr^2(t)\dot{\phi}^2\sin\theta\cos\theta \\
\dot{p}_\phi &= 0
\end{aligned}
$$

Writing the Hamiltonian in terms of new variables

$$
\mathcal{H} = \frac{p_\theta^2}{2mr^2(t)} + \frac{p_\phi^2}{2mr^2(t)\sin^2\theta} - \frac{1}{2}m\dot{r}^2(t) + mgr(t)\cos\theta
$$

(c)

We can see that the Hamiltonian is equal to the total energy $T + V$. The constraints working here (constraint of sphere until in contact) is time independent, forces here are conservative - hence it makes sense.

There is a time dependence on one of the coordinate variables $r(t)$ which makes the system NOT to be conserving energy.

Problem 03

Position representation

$$
\vec{r} = \rho\hat{\rho} + z\hat{e}_z
$$

Velocity representation

$$
\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z
$$

$$
\begin{aligned}
|\vec{r}|^2 &= \rho^2 + z^2 \\
(\vec{r}\cdot\hat{e}_z) &= z \\
\vec{\omega}\times\vec{r} &= \rho\omega\hat{\phi}
\end{aligned}
$$

(a)

$$
\boxed{\mathcal{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{4}m\Omega^2(\rho^2 - 2z^2) - \frac{1}{2}m\rho^2\dot{\phi}\omega} \qquad (\omega = eB/mc)
$$

(b)

$$
\begin{aligned}
p_\rho &= \frac{\partial\mathcal{L}}{\partial\dot{q}_\rho} = m\dot{\rho} & \dot{\rho} &= \frac{P_\rho}{m} \\
p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{q}_\phi} = m\rho^2\dot{\phi} - \frac{1}{2}m\rho^2\omega & \dot{\phi} &= \frac{p_\phi + \frac{1}{2}m\rho^2\omega}{m\rho^2} \\
p_z &= \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z} & \dot{z} &= \frac{p_z}{m}
\end{aligned}
$$

Using the equation of Hamiltonian I solve

$$
\mathcal{H} = p_\rho\dot{\rho} + p_\phi\dot{\phi} + p_z\dot{z} - L
$$

$$
\boxed{\mathcal{H} = \frac{p_\rho^2}{m} + \frac{p_\phi^2}{2m\rho} + \frac{p_z^2}{2m} + \frac{\omega}{2}p_\phi + \frac{1}{8}m\left(\omega^2 - 2\Omega^2\right)\rho^2 + \frac{1}{2}m\Omega^2 z^2}
$$

(c)

$$
\begin{aligned}
\dot{\rho} &= \frac{\partial\mathcal{H}}{\partial p_\rho} = \frac{p_\rho}{m} & \dot{p_\rho} &= -\frac{\partial\mathcal{H}}{\partial\rho} = \frac{1}{4}m(\omega^2 - 2\Omega^2)\rho \\
\dot{\phi} &= \frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{m\rho} + \frac{\omega}{2} & \dot{p}_\phi &= -\frac{\partial\mathcal{H}}{\partial\phi} = 0 \\
\dot{z} &= \frac{\partial\mathcal{H}}{\partial p_z} = \frac{p_z}{m} & \dot{p}_z &= -\frac{\partial\mathcal{H}}{\partial z} = -m\Omega^2 z
\end{aligned}
$$

Note that for the z motion we have

$$
\ddot{z} + \Omega^2 z = 0
$$

The z motion is a simple harmonic oscillator with Ω frequency. Ω is important because it characterizes the effective entrapment of the particle in z axis.

(d)

Because

$$
\dot{p}_\phi = \frac{\partial\mathcal{H}}{\partial\phi} = 0 \implies p_\phi = \text{const}
$$

ϕ is a cyclic coordinate.

$$\dot{\phi} = \frac{p_\phi}{m\rho} + \frac{\omega}{2}$$

(e)

$$\ddot{\rho} = \frac{\dot{p}_\rho}{m}$$

$$\ddot{\rho} = \frac{1}{m}\left(\frac{p_\phi^2}{m\rho^3} - \frac{1}{4}m\left(\omega^2 - 2\Omega^2\right)\rho\right)$$
$$\ddot{\rho} = \frac{p_\phi^2}{m^2\rho^3} - \frac{1}{4}\left(\omega^2 - 2\Omega^2\right)\rho$$

$$m\ddot{\rho} = -\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}\rho}$$
$$U_{\text{eff}}(\rho) = \int \mathrm{d}\rho\left(\frac{p_\phi^2}{m^2\rho^3} - \frac{1}{4}\left(\omega^2 - 2\Omega^2\right)\rho\right)$$
$$U_{\text{eff}}(\rho) = \frac{p_\phi^2}{2m\rho^2} + \frac{1}{8}m\left(\omega^2 - 2\Omega^2\right)\rho^2$$

Plotting, $U_{\text{eff}}(x) = \frac{a}{x^2} + bx^2$ a is always greater than 0. $b < 0$ when $\omega^2 - 2\Omega^2 < 0 \implies \omega < \sqrt{2}\Omega$ we have the following plot.

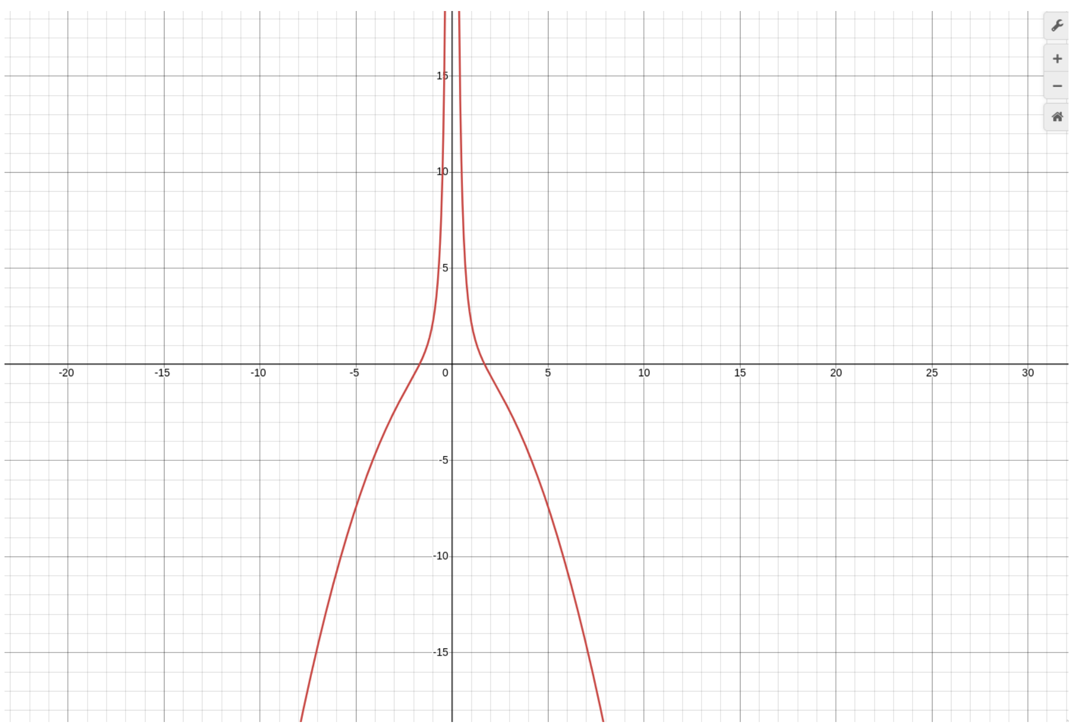


Figure 2: ./ss/8/3.png

When $\omega^2 - 2\Omega^2 > 0 \implies \omega > \sqrt{2}\Omega$ that means $b > 0$ we have

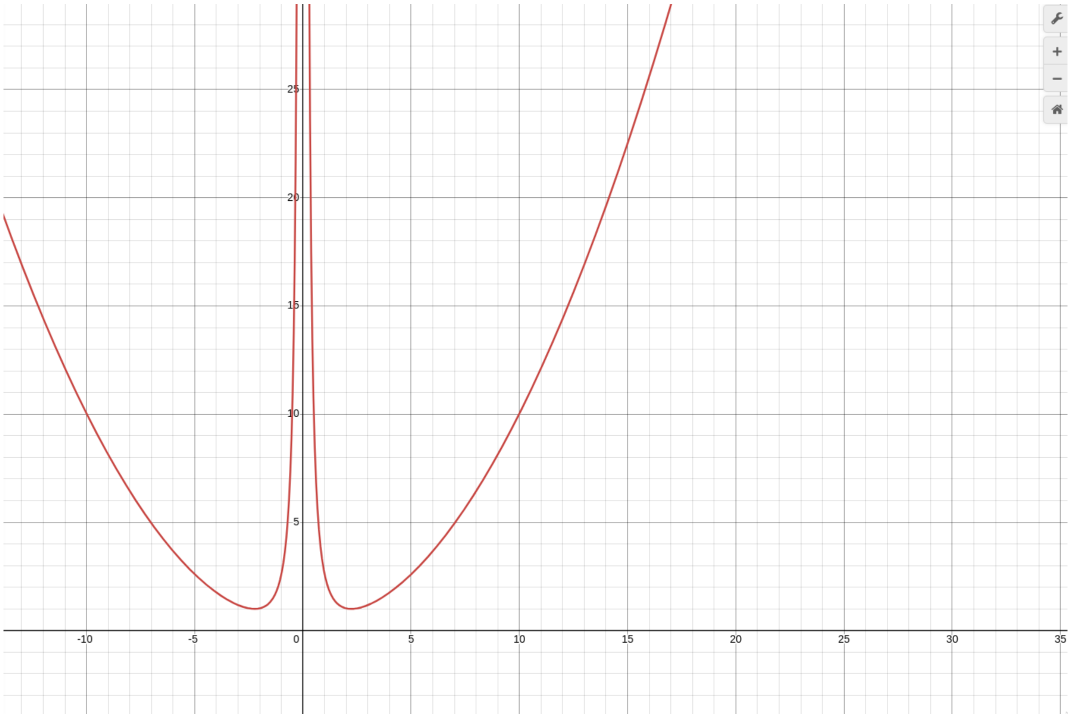


Figure 3: ./ss/8/4.png

Problem 04

The speed gained from the impulse adds to the radial component of the overall velocity. That speed v_r is given by

$$mv_r = I \implies v_r = \frac{I}{m}$$

The tangential speed already is

$$v_t^2 = \frac{GM}{r_0} = \frac{k}{mr_0}$$

(a)

The orbit initially before impulse is a circular orbit with distance from the center being r_0. The velocity is v_t.

When the impulse has been applied, we now have a velocity $\vec{v} = v_t\hat{\theta} - v_r\hat{r}$, at a distance r_0 from the "center" (which is now a focus of the orbit). We know there must exist an elliptical orbit that keeps the old center in the focus and satisfies the following speed - position relation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

where a is the semi-major axis.

Solving for a would give us helpful information about the geometry of the orbit. Right after the impulse,

$$
\begin{aligned}
v^2 &= \frac{k}{m}\left(\frac{2}{r_0} - \frac{1}{a}\right) \\
\frac{k}{mr_0} + \frac{I^2}{m^2} &= \frac{k}{m}\left(\frac{2}{r_0} - \frac{1}{a}\right) \\
\frac{1}{r_0} + \frac{I^2}{km} &= \left(\frac{2}{r_0} - \frac{1}{a}\right) \\
\frac{I^2}{mk} &= \frac{1}{r_0} - \frac{1}{a} \\
\frac{1}{a} &= \frac{1}{r_0} - \frac{I^2}{mk} \\
a &= \frac{1}{\frac{1}{r_0} - \frac{I^2}{mk}}
\end{aligned}
$$

We have solved for a being

$$
a = \frac{1}{\dfrac{1}{r_0} - \dfrac{I^2}{mk}}
$$

The eccentricity is given by

$$
\varepsilon = \sqrt{1 + 2\frac{EL^2}{mk^2}}
$$

The energy here is

$$
E = -\frac{1}{2}\frac{GMm}{r_0} + \frac{1}{2}mv_r^2 = -\frac{1}{2}\frac{GMm}{r_0} + \frac{1}{2}m\frac{I^2}{m^2} = \frac{I^2}{2m} - \frac{k}{2r_0} \implies \frac{1}{2}\left(\frac{I^2}{m} - \frac{k}{r_0}\right)
$$

The angular momentum here is

$$
L^2 = m^2 v_t^2 r_0^2 = m^2 \frac{k}{mr_0} r_0^2 = mkr_0
$$

Plug them in and solve for ε

$$
\begin{aligned}
\varepsilon &= \sqrt{1 + 2\frac{EL^2}{mk^2}} \\
&= \sqrt{1 + \frac{2}{mk^2}\frac{1}{2}\left(\frac{I^2}{m} - \frac{k}{r_0}\right)(mkr_0)} \\
&= \sqrt{1 + \frac{1}{mk^2}\left(I^2 k r_0 - mk^2\right)} \\
&= \sqrt{1 + \left(\frac{I^2 r_0}{mk} - 1\right)} \\
&= \sqrt{\frac{I^2 r_0}{mk}} = I\sqrt{\frac{r_0}{mk}}
\end{aligned}
$$

The relation between Semi-Minor axis b, Semi-Major axis a, and Semi-Latus Rectum ℓ

$$
\begin{aligned}
b &= a\sqrt{1 - \varepsilon^2} \\
\ell &= a(1 - \varepsilon^2)
\end{aligned}
$$

Computing the Semi-Latus Rectum ℓ

$$
\ell = \frac{r_0}{\frac{1}{r_0} - \frac{I^2}{mk}}\left(\frac{1}{r_0} - \frac{I^2}{mk}\right) = r_0
$$

$$\boxed{\varepsilon = I\sqrt{\frac{r_0}{mk}}}$$

$$\boxed{\ell = r_0}$$

(b)

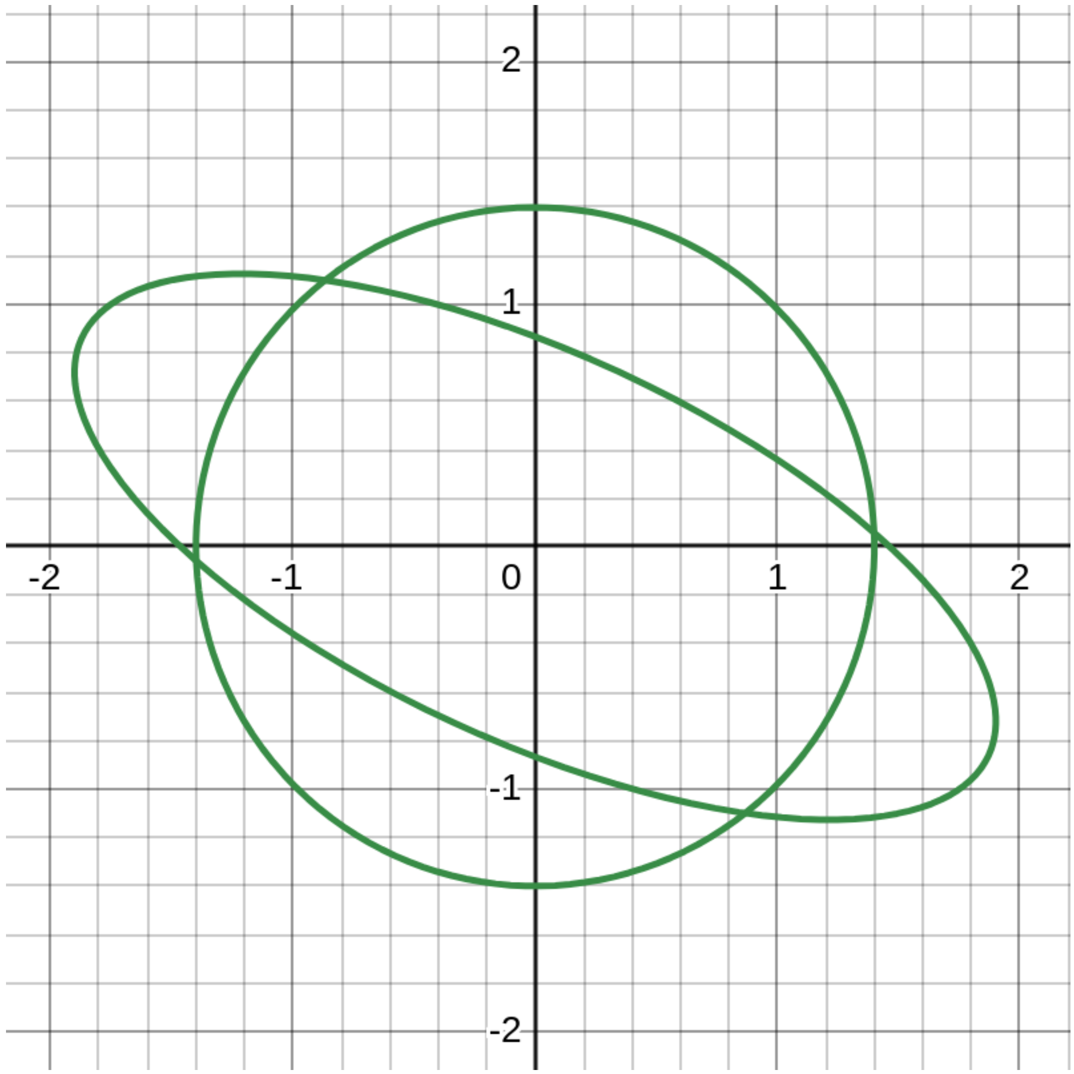


Figure 4: Inward impulse and outward impulse will have similar orbit shapes.

If there's an **inward impulse**, then assuming there's a **counter clockwise** moving orbit, the ellipse intersection with circle orbit is going to be (where transition happens) will be in 1st Quadrant, 3rd Quadrant (when ellipse is intersecting **into** the circular orbit).

Outward impulse is easy to understand, for this case it's the 2nd and 4th quadrant of coordinate system intersection. Ellipse is intersection **out** of the circular orbit.

(c)

$$\overline{F} = \frac{1}{T}\int_0^T \frac{1}{r^2}\,\mathrm{d}t$$
$$E = -\frac{GM}{2a}$$
$$\frac{I^2}{2m^2} - \frac{GM}{2r_0} = -\frac{GM}{2a}$$
$$\boxed{\frac{1}{a} = \frac{1}{r_0} - \frac{I^2}{6Mm^2}}$$

The manuever is counterproductive because a increases if it was going towards the star. The impulse should be applied away from the star to decrease a.

Last Homework: Problem 05

(a)

The cross product $\mu r\hat{\phi}$. Hence

$$\boxed{\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \frac{q\mu\rho^2\dot{\phi}}{(\rho^2+z^2)^{\frac{3}{2}}}}$$

(b)

$$p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho} \qquad \dot{\rho} = \frac{p_\rho}{m}$$
$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + \frac{q\mu\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}} \qquad \dot{\phi} = \frac{1}{m\rho^2}\left(p_\phi - \frac{q\mu\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}}\right)$$
$$p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z} \qquad \dot{z} = \frac{p_z}{m}$$

I did this following computation on paper. What I got.

$$\boxed{\mathcal{H} = \frac{p_\rho^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2m\rho^2}\left(p_\phi - \frac{q\mu\rho^2}{(\rho^2+z^2)^{\frac{3}{2}}}\right)^2}$$

(c)

$H = E$, yes the hamiltonian is equal to the total energy E. There is no explicit time dependence so energy is conserved.

(d)

Hamiltonian doesn't depend explicitly on ϕ so ϕ is a cyclic coordinate. Therefore

$$\ell = p_\phi = m\rho^2\dot{\phi} + \frac{q\mu\rho^2}{(\rho^2+z^2)^{3/2}}$$

(e)

$$
\begin{aligned}
\dot{\rho} &= \frac{p_\rho}{m} \\
\dot{p_\rho} &= \frac{1}{m\rho^3}\left(p_\phi - \frac{q\mu\rho^2}{r^3}\right)^2 - \frac{1}{m\rho^2}\left(2\left[p_\phi - \frac{q\mu\rho^2}{r^3}\right]\left[-2\frac{q\mu\rho}{r^3} + 3\frac{q\mu\rho^3}{r^5}\right]\right) \\
\dot{z} &= \frac{p_z}{m} \\
\dot{p_z} &= -\frac{2}{m\rho^2}\left(p_\phi - \frac{q\mu\rho^2}{r^3}\right)\left(3\frac{q\mu\rho^2 z}{r^5}\right)
\end{aligned}
$$

(f)

$$
\boxed{E = \frac{p_\rho^2}{2m} + \frac{1}{2m}\left(\frac{p_\phi}{\rho} - \frac{q\mu}{\rho^2}\right)}
$$

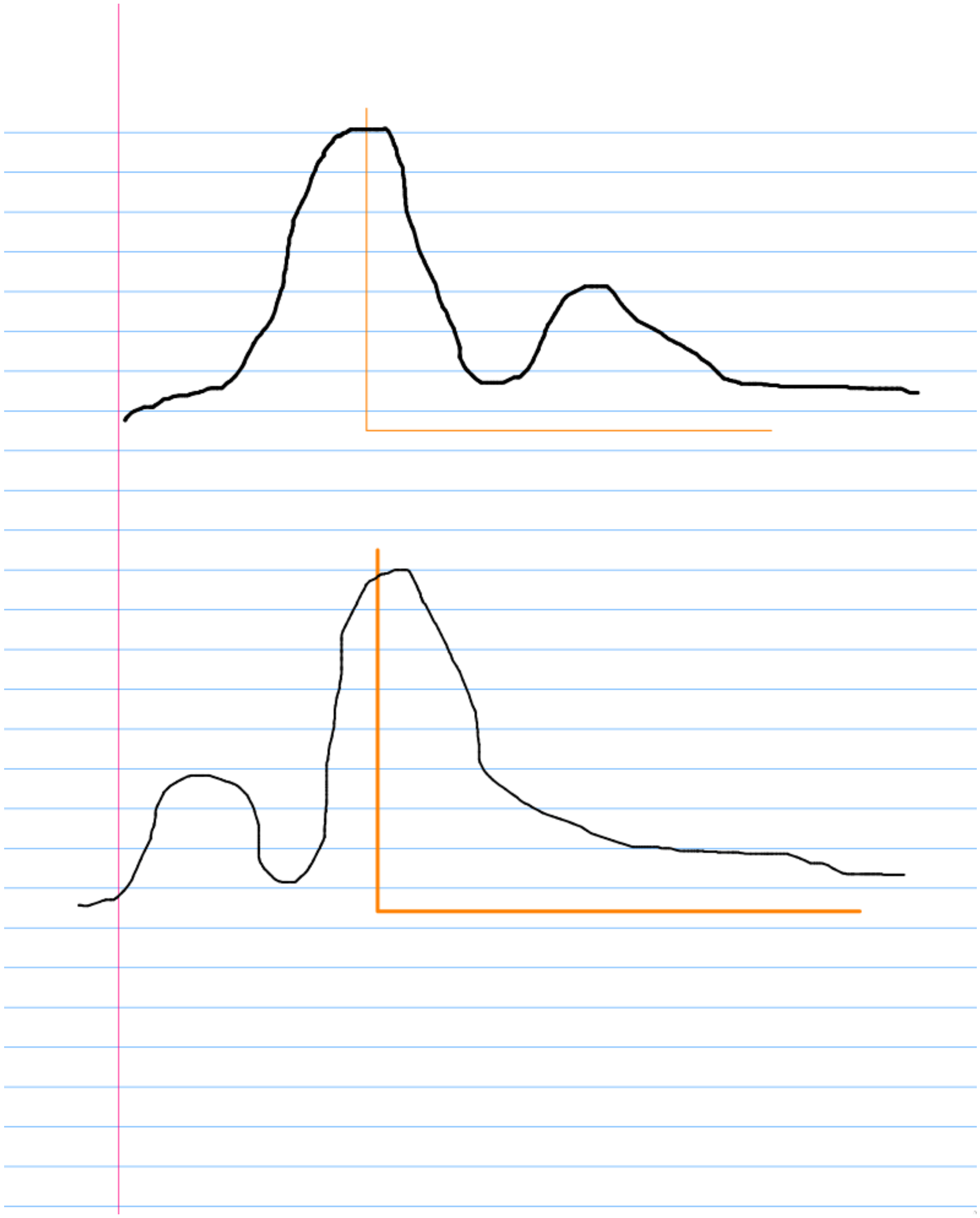


Figure 5: ./ss/8/6.png