Analysis HW 01

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Problem 01

a

Suppose we have $x^n = k$ where $k \in \mathbb{N}$ and we know that $x \notin \mathbb{N}$. Then assuming for the sake of contradiction that the root $x \in \mathbb{Q}$. This means

$$x = \frac{p}{q}$$

where p, q are natural numbers and coprime.

$$\frac{p^n}{q^n} = \implies \frac{p^n}{k} = q^n$$

q^n is a natural number. This implies that p^n/k is also a natural number. Writing

$$\frac{p}{k}(p^{n-1}) \in \mathbb{N}$$

and since obviously $p^{n-1} \in \mathbb{N}$ we also have $\frac{p}{k} \in \mathbb{N}$. Saying, p is divisible by k.

Now let's re-write

$$\frac{p^n}{k} = q^n \implies \frac{p^n}{k^2} = \frac{q^n}{k}$$

From which

$$\left(\frac{p^2}{k^2}\right) p^{n-2} = \left(\frac{q}{k}\right) q^{n-1}$$

Similarly like before, we know that the left hand side is a natural number. This is true for all n greater or equal to 2 but it's trivial to prove $n \neq 1$. Hence meaning

$$\frac{q}{k} q^{n-1} \in \mathbb{N}$$

As $q^{n-1} \in \mathbb{N}$, so $\frac{q}{k}$ must also be a natural number. Therefore q is divisible by k. This is a contradiction because p, q are supposedly co-prime. Hence the assumption that the roots are rational is false.

b

We have proved that the roots are in no way rational. We can possibly have natural numbers which are roots, and from the proof above, irrational in a sense that it is no way rational.

Problem 02

a

We have $x = A|B$, and $x > O^*$. Then let's define

$$C = \{p \in \mathbb{Q} : p < 0 \text{ or } pq < 1 | \forall q \in A\} \quad \text{and} \quad D = C^C$$

Looking at C we can tell that it forms a cut.

First let's show that if $x = A|B$ and $y = C|D$ then $xy = 1^*$.

$$xy = \{p \in \mathbb{Q} : p < ac, a > 0, c > 0, a \in A, c \in C\}|\{\text{rest of } \mathbb{Q}\}$$

From this construction it is apparent that $ac < 1$. Hence from the idea of cuts we know that $xy = 1^*$, because for any $p \in xy$ we have $p \in 1^*$. There does not exist an element $p \in 1^*$ such that $p \notin xy$ because if $p < 1$, then $p < ac$ for some $a \in A$ and $c \in C$, so $p \in xy$.

b

If $x < 0^*$ such that $x = A|B$ then there exists $-x$ such that $-x = A^*|B^*$ and $-x > 0^*$. Let's define $-y > 0^*$ where $-y = C^*|D^*$ and we define C^* the same way we defined C in the previous solution.

$$C^* = \{p \in \mathbb{Q} : pq < 1 \mid \forall q \in A\}$$

$D^* = C^{*C}$. We know that C is a valid cut. From our $-y$ we can take it's additive inverse in $y = C \mid D$. We know this value must exist. Finally because $-y > 0^*$ we know that $y < 0^*$ and from here we use the definition of a cut multiplication

$$xy = (-x)(-y), \quad x < 0^* \text{ and } y < 0^*$$

Because our new values of $-x$ and $-y$ are equivalent to previous solution, we can draw along the line of last solution.

c

Suppose x has two multiplicative inverses y_1 and y_2 such that $xy_1 = 1$ and $xy_2 = 1$. Suppose for sake of contradiction that they are not equal to each other. Because $\mathbb{R}$ is an ordered field, $y_1 > y_2$ (without loss of generality) we can define

$$y_1 = C_1|D_1$$

$$y_2 = C_2|D_2$$

then $C_2 \subset C_1$ but they are not equal. Implying existance of $r \in \mathbb{Q}$ such that $r \in C_1$ and $r \notin C_2$.

If $x > 0^*$, we have $rp < 1$ for all $p \in A$ based on the construction of C_1. However the construction of C_2 says that r should therefore also be in C_2 which is a contradiction. If $x < 0^*$ we use the exact same way of proving but with $rp > 1$ for all $p \in A$.

Problem 03

a

From definition b being lub(S), $\forall s \in S$

$$b \geq s$$

Given $\epsilon > 0$, $b - \epsilon$ is not an upper bound because

$$b - \epsilon < b$$

Thus there must exist $s \in S$ such that $s \geq b - \epsilon$ and hence

$$b - \epsilon \leq s \leq b$$

b

Counter-example. $S = \{1, 2\}$ where lub$(S) = 2$. If $\epsilon \leq 1$, there doesn't exist any s in S such that $b - \epsilon < s < b$. Hence the statement is not true.

c

By definition if $x = A|B$, then for any $a \in A$, $x > a$ as $x \in \mathbb{R}$.

Now let's say that x is NOT the least upper bound, then there must exist some y such that $y < x$ and is another upper bound. But from the definition of cuts, $y = C|D$, and then $C \subset A$ because $y < x$, and $C \neq A$.

y being upper bound, $\forall a \in A$, then there exist $c \in C$ such that $c > a$. But this would end up meaning $a \in C$ and hence $A \subset C$. This is a contradiction, hence x must be the least upper bound.

Problem 04

Take $x = A|B$ where $A = \{r \in \mathbb{Q} | r < 0$ or $r^2 < 2\}$ then by doing cut multiplication

$$x^2 = E|F$$

$$E = \{p \subset \mathbb{Q} | r_1 \in A, r_2 \in A, r_1 > 0, r_2 > 0, p < r_1 \cdot r_2\}$$

$$F = E^c$$

Proving $E|F = 2^*$ would prove $x = \sqrt{2}$.

As $r_1^2 < 2$ and $r_2^2 < 2$ then $(r_1^2)(r_2^2) < (2 \cdot 2)$ that gives us $(r_1 r_2)^2 < 4$ and $r_1 r_2 < 2$. That proves that $E|F$ is a cut of 2 and $x = \sqrt{2}$.

Problem 05

The greatest lower bound property of real numbers is that if $S \subset \mathbb{R}$, $s \neq \varnothing$, and S has a lower bound (a value of $x \in \mathbb{R}$ such that $x \leq s$ where $\forall s \in S$), then S has also a greatest lower bound. If x is the greatest lower bound of S, and $x' > x$ then x' is not a lower bound.

We can proof this in the following.

Let's have a set S that is a subset of real number. Let's say that S is bounded below that means there exists L such that any element $s \in S$ satisfies $L \leq s$. Let's define

$$B := \{b \in \mathbb{R} | b \text{ is a lower bound of S}\}$$

As we defined S has a bound in the bottom, B is non-empty.

Furthermore, this whole set B in and of itself is the set of all lower bound of S. Any member $b \in B$ satisfies $b \leq s$ where any element $s \in S$.

But conversely, every element of S is an upper bound of the set B. This apparently means the least upper bound lub(B) exists. Now lets say x is lub(B), then if $x' < x$ x' is not lub(B). But, this means that there eixsts $b \in B$ such that $b > x'$. Therefore x' cannot be glb(S) because $x' > x$ then $x' \notin B$ and x' is not a lower bound of S.