Computational Complex Analysis : : Class 35

April 5, 2024

Ahmed Saad Sabit, Rice University

The Schwarz-Pick lemma: we have a holomorphic function f defined on D. We are going to derive some inequalities similar to this: if $f(0) = 0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Suppose $f(z_1) = w_1$ and $f(z_2) = w_2$. Use Mobius functions,

$$\phi_a(z) = \frac{z-a}{1-\overline{a}z} \quad (a \in D)$$

$$\phi_a(a) = 0$$

$$\phi_a(0) = -a$$

$$\phi_a^{-1} = \phi_a$$

$$\begin{pmatrix} 1 & -a \\ -\overline{a} & 1 \end{pmatrix} \to \text{inv} \to \begin{pmatrix} 1 & a \\ \overline{a} & 1 \end{pmatrix}$$

$$\phi_{w_1} \circ f \circ \phi_{z_1}^{-1}$$

This maps 0 to 0. Schwarz says,

$$|\phi_{w_1} \circ f \circ \phi_{z_1}^{-1}(z)| \leq |z|$$

$$|\phi_{w_1} \circ f(z_2)| \leq |\phi_{z_1}(z_2)|$$

From this,

$$\left| \frac{f(z_2) - w_1}{1 - \overline{w_1} f(z_2)} \right| \leq \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|$$

$$\left| \frac{w_2 - w_1}{1 - \overline{w}_1 w_2} \right| \leq \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|$$

Detective method is to start with conformal equivalence for upper half. Then we realize we get Mobius transformation. Lemma: if $f(z) = \frac{az+b}{cz+d}$ is real for all real z, then a, b, c, d are essentially real.