Integrals Exercise from Edwards

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Definition 1. *Box* in $\mathbb{R}^n$ is defined by

$$B = [a_1, b_1] \times \cdots [a_n, b_n]$$

We may also call them *Rectangles* in cases.

Definition 2. *Volume of a Box* is defined by

$$v(B) = |a_1 - b_1| \times \cdots \times |a_n - b_n|$$

Definition 3. *Area α of a Bounded Set S in $\mathbb{R}^n$* is given by the following two conditions for $\epsilon > 0$

- Finite collection of $p_1, p_2, \ldots, p_k$ non-overlapping rectangles contained in S, with

$$\sum_i^k a(p_i) > \alpha - \epsilon$$

- Finite collection of $P_1, P_2, \ldots, P_l$ rectangles that together contain S with

$$\sum_i^l a(P_i) < \alpha + \epsilon$$

Problem 1. Find the area under the curve (graph of -) $y = x^2$ in a closed interval $[0, 1]$ using the above definition of area.

Solution. TODO: please do it. □

Problem 2. Do the same above exercise with fundamental theorem.

Problem 3. If S and T have area, and $S \subset T$, then $a(S) \leq a(T)$.
Prove this statement.

Solution. Consider set of small rectangles inside T α_i such that their set $\{\alpha_i\} \subset T$. Similarly, consider the set of small rectangles inside S as $\{s_i\}$. These follow (obviously) $\{s_i\} \subset S$.

$$\{\alpha_i\} \subset T$$

$$\{s_i\} \subset S$$

Now let's invoke $S \subset T$. And define $\{s_i\}$ to be

$$\{s_i\} \subset S \cap \{\alpha_i\}$$

Note that I define $\{s_i\} \subset S \cap \{\alpha_i\}$ instead of $\{s_i\} = S \cap \{\alpha_i\}$ because the equality can cause boxes to be cut off weirdly. I only want full rectangles to be in $\{s_i\}$. Because of this, several rectangles will get cut off, so apparently it's obvious that

$$\sum a(s_i) \leq \sum a(\alpha_i)$$

$\square$