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Problem 01

(a)

Directional derivative is taken along a direction, say $\vec{v}$ such that $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \in \mathbb{R}$. From the definition of a directional derivative

$$D_{\vec{v}} f(0,0) = \lim_{h \to 0} \frac{f(ah, bh) - f(0,0)}{h} = \frac{\frac{a^2bh^3}{(a^2+b^2)h^2}}{h} = \frac{a^2 b}{a^2+b^2}$$

(b)

Let's assume that f is differentiable at the origin. Then

$$D_{\vec{v}} f(\vec{a}) = \sum_{n=1}^{N} v_n D_n f(\vec{a})$$

Here $N = 2$. If $\vec{p} = \vec{0}$ and $\vec{v} = (a, b)$ then

$$D_{\vec{v}} f(\vec{p}) = \sum_{n=1}^{N} v_n D_n f(\vec{p}) = aD_1 f(0,0) + bD_2 f(0,0)$$

$$= a\left(\frac{a^2(0)}{a^2}\right) + b\left(\frac{(0)b}{b^2}\right) = 0$$

This is a contradiction because we had already solved $D_{\vec{v}} f(0,0) = \dfrac{a^2 b}{a^2+b^2}$ yet the partial differentiation addition rule doesn't sum up to the directional derivative, hence contradicting the initial assumption of f being differentiable.

(c)

If f is continuous at the origin then

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

If f has a limit then

$$|f(x,y) - f(0,0)| < \epsilon$$

would mean $|(x,y) - (0,0)| < \delta$, for some $\epsilon, \delta > 0$. Considering $f(0,0) = 0$ as stated in the function definition (and also considering the function to be continuous), let's check if the limit exists and if it's $f(0,0) = 0$.

$$\left| \frac{x^2 y}{x^2 + y^2} \right| < \epsilon$$

Say $|x| < \gamma$ and $|y| < \gamma$, then it implies

$$\frac{\gamma}{2} < \epsilon$$

Hence meaning that $|x| < 2\epsilon$ and $|y| < 2\epsilon$. This implies

$$\sqrt{x^2 + y^2} < 2\sqrt{2}\epsilon$$

Setting $2\sqrt{2}\epsilon = \delta$ gives us the following as $\sqrt{x^2 + y^2} = |(x, y) - (0, 0)|$

$$|(x, y) - (0, 0)| < \delta$$

Hence the limit exists, and it also happens to be equal to $f(0, 0) = 0$. So the function is continuous.

Problem 02

(a)

The simple single variable derivative where we are free to consider $x \neq 0$,

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

I did the simple derivative on paper.

(b)

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^2 \sin(1/h)$$

The $\sin(\frac{1}{h})$ is constrained within the range $[-1, 1]$ so it does not blow up to infinity. The coefficient h^2 does approach to 0, and as the $\sin 1/h$ factor is not growing, we can safely say that the limit is 0. So,

$$f'(0) = 0$$

(c)

If the function has a limit, the sequence $\{f(\vec{x}_k)\}$ will always converge to the limit L for any possible sequence $\{\vec{x}_k\}$ that also converges to a limit $\vec{a}$. If $\{f(\vec{a})\} = L$ then we can safely say this function is continuous.

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Let $x = \frac{1}{\theta}$, and $x \to 0$ for $\theta \to \infty$ (trivial). So, if f' is continuous,

$$\lim_{x \to a} f'(x) = f'(a)$$

$$\lim_{\theta \to \infty} f'(1/\theta) = \lim_{\theta \to \infty} \frac{2}{\theta} \sin(\theta) - \cos(\theta)$$

Using the previous similar reasoning we know that $\frac{2}{\theta} \sin\theta \to 0$ for $\theta \to 0$. But $\cos\theta$ can be anything in between -1 and 1 given θ. This limit can't exist because $\cos\theta$ is anything in the range $[-1, 1]$

Problem 03

Theorem 1. Folland Theorem 2.19:

Let f be a function defined on an open set in $\mathbb{R}^n$ that contains the point $\vec{a}$. Suppose that the partial derivatives $\frac{\partial f}{\partial x_j}$ for all j exist around the local neighborhood of $\vec{a}$ and they are continuous. Then f is differentiable at $\vec{a}$.

(a)

Let $(f_1(\vec{x}), f_2(\vec{x}))$ be a vector $\vec{v}$. Then differentiablity means

$$\frac{f(\vec{v}+\vec{h}) - f(\vec{v}) - \vec{c}\cdot\vec{h}}{|\vec{h}|} \to 0$$

as $\vec{h} \to 0$ where $\vec{c} = (\partial_1 f(\vec{v}), \partial_2 f(\vec{v}))$.