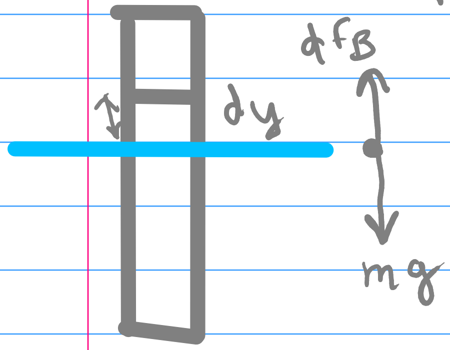
Homework 04

95515

problem 01



mass of water displaced

$$dm = \rho dV = \rho A dy$$

boyant force

$$dF_B = -g dm$$
$$= -\rho g A dy$$

dF_B is the additional force that works while displaced from equilibrium position, so,

$$m\ddot{dy} = -\rho g A dy$$

$$\rightarrow \quad \ddot{dy} + \frac{\rho g A}{m} dy = 0$$

so, $$\omega = \sqrt{\frac{\rho g A}{m}}$$

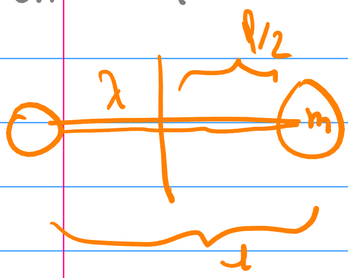
$$\frac{2\pi}{T} = \sqrt{\frac{g\rho A}{m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{g\rho A}}}$$

- This is a simple harmonic oscillator.

problem 02

moment of inertia of rod : $I_{rod} = \frac{1}{12} m_r l^2$



$$= \frac{1}{12} (\lambda l) l^2$$

$$= \frac{1}{12} \lambda l^3$$

moment of inertia of two balls $= 2\left(\left(\frac{l}{2}\right)^2 m\right)$

$$= 2 \frac{l^2}{4} m$$

$$= \frac{ml^2}{2}$$

total moment of inertia

$$I = \frac{1}{12} \lambda l^3 + \frac{m l^2}{2} = \left(\frac{1}{12} \lambda l + \frac{m}{2}\right) l^2$$

Keep this in memory.

For small displacement $d\theta = \theta$ torque applied can cause torsion

$$\tau = -k\theta$$

or, $$I\ddot{\theta} + k\theta = 0$$

or, $$\underbrace{\ddot{\theta} + \frac{k}{I}\theta = 0}$$

Simple Harmonic Motion (small disp)

we need to solve for k, so look at initial case,

starting conditions

$$\tau_0 = k\theta_0$$

$$\Rightarrow \frac{l}{2} F = k\,\theta_F$$

or,
$$\frac{lF}{2\theta_F} = k \Rightarrow \text{constant}$$

hence, looking at SHM

$$\ddot{\theta} + \frac{k}{I}\theta = 0 \Rightarrow \ddot{\theta} + \omega^2\theta = 0$$

so,
$$\omega = \sqrt{\frac{k}{I}}$$

$$= \sqrt{\frac{lF}{2\theta_F\left(\frac{1}{12}\lambda l + \frac{m}{2}\right)l^2}}$$

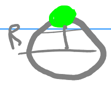
$$= \sqrt{\frac{12F}{2\theta_F(\lambda l + 6m)\,l}}$$

period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2\theta_F\,(7l+6m)\,l}{12F}}$$

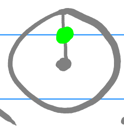
problem 03

basically a moment of inertia computation

$$I\left(\right.$$  $$\left.\right) = \underbrace{I_0 + MR^2}_{\text{parallel axis theorem}}$$

$$= \frac{1}{2}MR^2 + MR^2$$

$$= \frac{3}{2}MR^2$$

$$I\left(\right.$$  $$\left.\right) = \frac{1}{2}MR^2 + M\left(\frac{R}{2}\right)^2$$

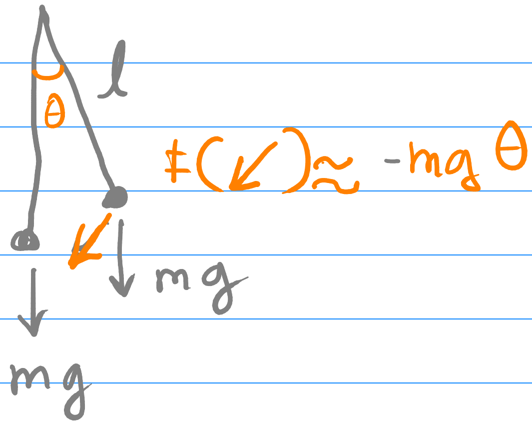
$$= \frac{1}{2}MR^2 + M\frac{R^2}{4}$$

$$= \frac{4}{4}\left(\frac{1}{2}MR^2 + M\frac{R^2}{4}\right)$$

$$= \frac{1}{4}\left(2MR^2 + MR^2\right)$$

$$= \frac{3}{4} MR^2$$

Now, for small displacement θ, torque measured from pivot to center of mass



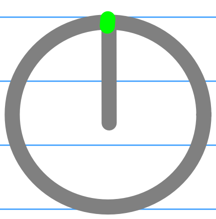
$$\tau = l F_\perp$$
$$= -l m g \theta$$

or, $\tau = -l m g \theta$

or, $I \ddot{\theta} + l m g \theta$

or, $\ddot{\theta} + \underbrace{\frac{l m g}{I}}_{\omega^2} \theta = 0$

<u>a</u>



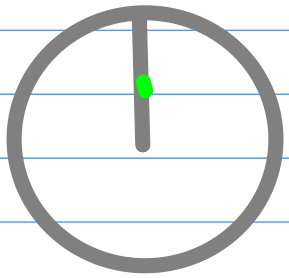
$l = R$

$I = \frac{3}{2} m R^2$

so, $\omega^2 = \frac{R m g}{\frac{3}{2} m R^2} = \frac{2g}{3R}$

so, $\boxed{\omega(0) = \sqrt{2g/3R}}$

b



$$w^2 = \frac{Rmg}{2\left(\frac{3}{4} mR^2\right)}$$

$$= \frac{2\,Rmg}{3\,mR^2} = \frac{2g}{3R}$$

$$\boxed{\omega = \sqrt{2g/3R}}$$

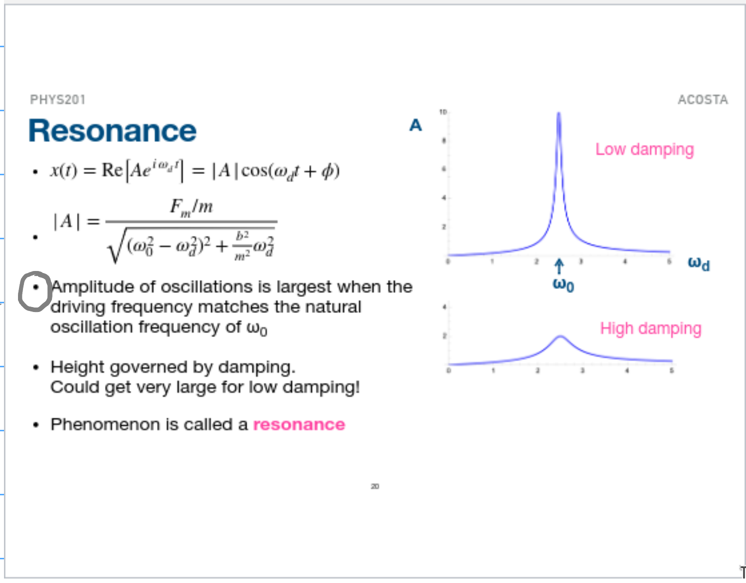
problem 04

(a)

steady state $x(t) = |A| \cos(\omega_d t + \phi)$

transient state $x(t) = |A| e^{-(b/2m)t} \cos(\omega t + \phi)$

now notice that this system is at resonance.



Resonance as per lecnote (OSC 4) is $\omega = \omega_d$

So both terms have same frequency.

But over time, transient term will die out because of e^{-kt} $(k>0)$ term dependence.

(b)

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{(b/m)}$$

two oscillators, similar,

$$Q_1 = 1000 \Rightarrow \frac{\omega_0}{\gamma_1} = 1000 \Rightarrow \gamma_1 = \frac{\omega_0}{1000}$$

$$Q_2 = 200 \Rightarrow \frac{\omega_0}{\gamma_2} = 200 \Rightarrow \gamma_2 = \frac{\omega_0}{200}$$

So, $\gamma_2 > \gamma_1$

now, transient terms have

$$e^{-\frac{\gamma_1}{2}t} \quad \text{and} \quad e^{-\frac{\gamma_2}{2}t}$$

because $\gamma_2 > \gamma_1$, $e^{-\frac{\gamma_2}{2}t}$ is going to die out faster than $e^{-\frac{\gamma_1}{2}t}$

$$\boxed{\text{So, } Q=200 \text{ will go to steady solution first.}}$$

problem 05

$$\omega^2 = \frac{k}{m} = \frac{400\ N/m}{1\ kg} = 400\ [\omega]$$

$$\frac{b^2}{4m^2} = \left(\frac{b}{2m}\right)^2 = \left(\frac{0.02}{2(1)}\right)^2 = (0.01)^2 < \omega^2$$

new frequency of oscillation

$$\omega^2 = \omega_0^2 - \frac{b^2}{4m^2}$$

(a)

$$x(t) = |A|\, e^{-\frac{b}{2m}t} \cos(\omega t + \phi)$$

to become half $\quad e^{-\frac{b}{2m}t} = \frac{1}{2}$

or, $\quad -\frac{b}{2m}t = \ln(1/2)$

or, $\quad t = -\frac{2m}{b}\ln(1/2)$

$$t = -\frac{2(1)}{0.02}\ln(1/2)\ [s]$$

$$\boxed{t = 69.31\ s}$$

(b) total energy is given by

$$E = \frac{1}{2} w_0^2 A^2 \text{ for ordinary case}$$

but amplitude reduces with energy dissipation

also here $\frac{\beta}{2m} << w_0$ so

$$E \approx \frac{1}{2} m A^2 w_0^2 e^{-\gamma t}$$

Note that if that beta/2m << w0 doesn't hold, then there will be a sin/cos term interfering with E. My logic is for this beta/2m being very small than w0, the motion almost behaves like simple harmonic oscillation with amplitude slowly decreasing.

i am avoiding a rigorous derivation of this intution but it is proven here (that I am right with this assumption): https://www.entropy.energy/scholar/node/damped-harmonic-oscillator-energy

$$\Delta E = E(t) - E(0)$$

$$= \frac{1}{2} m A^2 w^2 \left(1 - e^{-\frac{\beta}{2m} t}\right)$$

$$= \frac{1}{2} m A^2 \left[w_0^2 - \frac{b^2}{4m^2}\right] \left(1 - e^{-\frac{\beta}{2m} t}\right)$$

$$= \left(\frac{1}{2}\right)(1\,kg)(0{\cdot}1\,m)^2 \left[400 - (0{\cdot}01)^2\right]\left(1 - e^{-0{\cdot}01 t}\right)$$

$$\approx 2\left(1 - e^{-0{\cdot}01 t}\right)$$

now,

$$\Delta E = 1.6 \text{ J}$$

so,

$$2\left(1-e^{-0.01t}\right) = 1.6 \text{ J}$$

$$1-e^{-0.01t} = \frac{1.6}{2}$$

$$1-\frac{1.6}{2} = e^{-0.01t}$$

$$\ln(0.2) = -0.01t$$

$$t = \frac{\ln(0.2)}{-0.01}$$

$$t = 160.94 \text{ s}$$

$$N = \frac{t}{\tau} = \frac{t\omega}{2\pi} \approx \frac{(160.94)(\sqrt{400})}{2\pi}$$

$$\boxed{\approx 512 \text{ cycles}}$$

problem 06

(a) $\frac{\mathcal{E}}{L} = \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q$

$w_0^2 = \frac{1}{LC}$

$f = \frac{w_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

plug in values

$$\boxed{f = 201 \text{ Hz}}$$

(b) $Q = \frac{w_0}{R/L} = \frac{1264 \cdot 0.5}{200}$

$$\boxed{Q = 3.16}$$

(c) $\frac{|q_{res}|}{2} = |q|$

$$\frac{\varepsilon}{2L\omega_0\omega_{res}\sqrt{\left(\frac{\omega_0}{\omega_{res}}-\frac{\omega_{res}}{\omega_0}\right)^2+\frac{1}{Q^2}}}$$

$$=\frac{\varepsilon}{L\omega_0\omega_d\sqrt{\left(\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}\right)^2+\frac{1}{Q^2}}}$$

using computer I solve the numerical left side,

$$1.97\times10^{-4}=\frac{100}{(0.5)(1264)\omega_d\sqrt{\left(\frac{1264}{\omega_d}-\frac{\omega_d}{1264}\right)^2+\frac{1}{(3.16)^2}}}$$

computer helps

$$\omega_d = 796,\ 1549$$

$$\boxed{f_1 = \frac{796}{2\pi} = 126.7\ Hz}$$

$$\boxed{f_2 = \frac{1549}{2\pi} = 246.\ Hz}$$

$$\boxed{\begin{aligned} FWHM &= 246.5 - 126.7 \\ &= 119.8\ Hz \end{aligned}}$$

(d) $\phi_1 = \tan^{-1}\left[\frac{-1}{Q\left(\frac{1264}{\omega_1} - \frac{\omega_1}{1264}\right)}\right]$

$Q = 3.16, \quad \omega_1 = 796$

$$\boxed{\phi_1 = -0.32\ Rad}$$

$\phi_2 = \tan^{-1}\left[\frac{-1}{Q\left(\frac{1264}{\omega_2} - \frac{\omega_2}{1264}\right)}\right]$

$w_2 = 1549$

gives $\boxed{\phi_2 = 0.66 \text{ Rad}}$