Homework 09

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1 Problem 01

Consider V is finite dimensional and U and W are subspaces of W. By definition we have,

$$(U+W)^0 \text{ contains all } \phi \in V'$$

where for every $v \in U+W$, and $\phi(v) = 0$. This means that ϕ must be 0 for every $u \in U$ and every $w \in W$, otherwise it won't be 0 in $U+W$, which means $(U+W)^0 \subset U^0 \cap W^0$. For every other direction, let us suppose $\phi \in U^0$ and $\phi \in W^0$ hence,

$$\phi \in U^0 \cap W^0$$

From this we can write,

$$\phi(u) = 0$$

$$\phi(w) = 0$$

Then for all $u \in U$ and $w \in W$ we have,

$$\implies \phi(u+w) = 0 + 0 = 0 \quad \forall (u+w) \in U+W$$

Therefore $U^0 \cap W^0 \subset (U+W)^0$ since both directions of subsets hold we can say,

$$(U+W)^0 = U^0 \cap W^0$$

as required.

Problem 02

If U nad W are subspaces of V then $U \cap W$ are subspace too. Then we can write,

$$\dim(U \cap W)^0 = \dim V - \dim(U \cap W)$$

$$\dim U^0 = \dim V - \dim U$$

$$\dim W^0 = \dim V - \dim W$$

This all implies that,

$$\implies \dim U^0 + \dim W^0 = 2\dim V - \dim U - \dim W$$

Which then implies that,

$$\implies \dim(U^0 + W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) = 2\dim V - \dim U - \dim W - \dim(U^0 \cap W^0)$$

We had found out in the last problem that,

$$U^0 \cap W^0 = (U+W)^0$$

so from here,

$$\dim(U^0 \cap W^0) = \dim(U+W)^0 = \dim V - \dim(U+W)$$

$$\implies \dim(U^0 + W^0) = 2\dim V - \dim U - \dim W - (\dim V - \dim(U+W))$$

Now using what we had seen in,

$$\implies \dim(U^0 + W^0) = \dim V - \dim(U \cap W) = \dim(U \cap W)^0 \quad \text{LADR 2.43}$$

Then supposing that $\phi \in U^0$ and $\phi \in W^0$ so that,

$$\phi(u) = 0 = \phi(w) = 0$$

$$\implies \forall u \in U, \quad w \in W$$

then any vector $v \in U \cap W$ has $\phi(v) = \phi(u) = \phi(w)$ is equal to 0.

Hence,

$$\phi(v) = \phi(u) + \phi(w) = 0$$

and

$$(U \cap W)^0 \subset (U^0 + W^0)$$

Since one is a subset of the other and the dimensions are proved to be same hence,

$$U^0 + W^0 = (U \cap W)^0$$

Problem 3

Considering $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$ then,

$$0 = T^4 - I = (T^2 + I)\left(T^2 - I\right) = (T^2 + I)(T + I)(T - I)$$

Then at least one of the T^2 is equal to,

$$T^2 = -I$$

$$T = -I$$

$$T = I$$

However since $Iv = v$ then $\forall v \in V$ then I has at least eigenvalue $\lambda = 1$. Similarly $-I$ has eigenvalue -1.

Then $T = -I$ or $T = I$ contradict the original statement that T has no eigenvalues. There for the only remaining valid case is that

$$T^2 = -I$$

Problem 04

Let us write the matrix form of the following,

$$T^2 - (a+d)T + (ad - bc)I$$

$$\begin{pmatrix} a & c \\ b & c \end{pmatrix}\begin{pmatrix} a & c \\ b & c \end{pmatrix} - (a+d)\begin{pmatrix} a & c \\ b & c \end{pmatrix} + (ad - bc)I$$

I did the calculation on paper and we get from that,

$$\begin{pmatrix} bc - ad & 0 \\ 0 & bc - ad \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Problem 05

Since $\dim V = 2$ the minimal polynomial of T has at most 2 degree of coefficients satisfying,

$$c_0 I + c_1 T = -c_2 T^2 \implies c_0 I + c_1 \begin{pmatrix} a & c \\ b & d \end{pmatrix} = -c_2 \begin{pmatrix} a^2 + bc & c(a+d) \\ b(a+d) & bc + d^2 \end{pmatrix}$$

From setting the bound $b = c = 0$ and $a = d$ then,

$$\implies c_0 I + c_1 \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = -c_2 \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix}$$

$$\implies c_0 + c_1 a + c_2 a^2 = 0$$

At most one from c_0, c_1, c_2 can be 0 otherwise all three are 0 and the polynomial isn't anymore monic. In the case none are 0 then this forms a quadratic that has solutions for particular values of a, c_0, c_1, c_2 rather than any value of a therefore the only solutions to this occurs when exactly one of c_0, c_1, c_2 is 0. Moreover in this case with $b = c = 0$ and $a = d$ we have,

$$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI$$

and since I is invertible, T is also invertible.

Let us consider the case c_2 is 0. Then $c_0 + c_1 a = 0$ that implies $c_0 = -c_1 a$ and the polynomial is $c_1 z - c_1 a \to z - a$.

Considering $c_1 = 0$ then $c_0 + c_2 a^2 = 0$ that means, $c_0 = -c_2 a^2$. The polynomial we get from this is $c_2 z^2 - c_2 a^2 \to z^2 - a^2$.

Note that LADR 5.32 says that $c_0 \neq 0$.

The lowest degree polynomial is found if $c_2 = 0$ hence, when $b =; c = 0$ and $a = d$ the minimal polynomial is $z - a$.

2 Problem 05

Let V be finite dimensional and $T \in \mathcal{L}(V)$. Defining

$$A \in \mathcal{L}(\mathcal{L}(V)) \quad A(S) = TS$$

where $S \in \mathcal{L}(V)$. The set of A eigenvalues are those λ such that

$$A(S) = \lambda_A S$$

while T satisfy $Tv = \lambda_T v$. Then taking,

$$A(S) = \lambda_A S = TS = T(Sv) = \lambda_T S$$

As long as $S \neq 0$,

$$\lambda_A S = \Lambda_T S \implies \lambda_A = \lambda_T$$

This means that the both eigenvalues are the same.

Problem 06

We can consider the matrix and break it down in the following way using the definition of determinants

$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & \\ b & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ c & 0 & 1 \end{pmatrix} = 1 - ab$$

Problem 07

$$\det\begin{bmatrix}\vec{u}+\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}$$

We can break this down in the following way,

$$\det\begin{bmatrix}\vec{u} & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}+\det\begin{bmatrix}\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}$$

We can do this for each next terms,

$$\begin{aligned}\det\begin{bmatrix}\vec{u} & \vec{u} & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}&+\det\begin{bmatrix}\vec{u} & \lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}\\&+\det\begin{bmatrix}\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}\end{aligned}$$

The first term contains the same instance of $\vec{u}$ in two columns so it's determinant is 0. This gives,

$$\det\begin{bmatrix}\vec{u} & \lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}+\det\begin{bmatrix}\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}$$

Expand the third term like this, and repeated process will yield us,

$$\begin{aligned}&=\det\begin{bmatrix}\vec{u} & \lambda\vec{e}_2 & \cdots & \lambda\vec{e}_n\end{bmatrix}+\det\begin{bmatrix}\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}\\&=\det\begin{bmatrix}\lambda\vec{e}_2 & \lambda\vec{e}_3 & \cdots & \lambda\vec{e}_n\end{bmatrix}+\det\begin{bmatrix}\lambda\vec{e}_1 & \vec{u}+\lambda\vec{e}_2 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}\\&=\lambda^{n-1}+\lambda\left(\lambda^{n-2}+\lambda\det\begin{bmatrix}\vec{u}+\lambda\vec{e}_3 & \cdots & \lambda\vec{e}_{n-1}\end{bmatrix}\right)\end{aligned}$$

The second matrix can be written as the exact same as previous one with the substituation of $n\to n-1$ and $\vec{u}$ with length $n-1$,

$$=\lambda^{n-1}+\lambda\left(\lambda^{n-2}+\lambda\det\begin{bmatrix}\vec{u}+\lambda\vec{e}_3 & \cdots & \vec{u}+\lambda\vec{e}_n\end{bmatrix}\right)$$

So this we can write,

$$\begin{aligned}&\lambda^{n-1}+\lambda\left(\lambda^{n-2}+\lambda\left(\lambda^{n-3}+\cdots+\det\begin{bmatrix}\vec{u}+\lambda\vec{e}_n\end{bmatrix}\right)\right)\\&=\lambda^{n-1}+\lambda\left(\lambda^{n-2}+\lambda\left(\lambda^{n-3}+\cdots+\lambda+1\right)\right)\end{aligned}$$

Looking at the pattern of multiplication, apparently,

$$\boxed{n\lambda^{n-1}}$$