Homework 05 : : Classical Mechanics

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Answer Sheet

Problem 01

- Fourier representation:

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi(1-4n^2)} \right) \cos(2nx)$$

- Plots of the Representation: Plotted in solution.

Problem 02

- Work by impulse case

$$W = \frac{K^2}{2m}$$

Work by Damped Spring Computation:

$$W = \frac{K^2}{\sqrt{\omega_0^2 - \gamma^2}} \qquad \text{or, with mass } \frac{K^2}{m\sqrt{\omega_0^2 - \gamma^2}}$$

- Mistake of statement: (summary) impulse given by force is non-zero hence particle does achieve some speed hence $v(0) = 0$ cannot be assumed.

Problem 03

- Response solution

$$x(t) = F_0 \sum_{n=0}^{\infty} A^n G\left(t, \frac{n}{2}\tau\right) = F_0 \sum_{n=0}^{\infty} A^n \Theta(t - n\tau/2) \frac{e^{-\gamma(t-n\tau/2)}}{\omega} \sin\left[\omega(t - n\tau/2)\right]$$

- The amplitude of oscillation can be written as $\Lambda^n e^{-\gamma t} \frac{1}{\omega}$ where $\Lambda = Ae^{\gamma\tau/2}$ setting the bound

$$e^{-\gamma\tau/2} > A$$

- Exponential Increase.

Problem 01

First of all note that $y(x) = |\sin(x)|$ is an even function as

$$f(-x) = f(x)$$

This function representation is a periodic function where $\tau = \pi$

$$y(x) = \sin(x) \quad (0 \leq x \leq \pi = \tau)$$

The function repeats itself after every $x = \tau$. Basic common sense. For this our Fourier Series representation of this problem would be with period $\tau = \frac{2\pi}{\omega} = \pi$ as $\omega = 2$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega n x)$$

Solve for a_0:

$$\int_0^\tau f(x)\,\mathrm{d}x = \int_0^\tau a_0 \mathrm{d}x + \int_0^\tau \sum_{n=1}^{\infty} a_n \cos(\omega n x)\,\mathrm{d}x$$
$$\implies a_0 = \frac{1}{\tau}\int_0^\tau f(x)\,\mathrm{d}x$$

Solve for a_n:

$$\int_0^\tau f(x)\cos(\omega p x) = \int_0^\tau a_0 \cos(\omega p x)\,\mathrm{d}x + \sum_{n=1}^{\infty}\int_0^\tau a_n \cos(\omega n x)\cos(\omega p x)\,\mathrm{d}x$$
$$\int_0^\tau f(x)\cos(\omega p x) = a_p \frac{\pi}{\omega} \qquad \text{(2nd term is zero for } n \neq p\text{)}$$
$$a_p = \frac{\omega}{\pi}\int_0^\tau f(x)\cos(\omega p x)\,\mathrm{d}x$$

Putting together:

$$f(x) = \frac{1}{\pi}\int_0^\tau f(x)\,\mathrm{d}x + \frac{2}{\pi}\int_0^\tau f(x)\cos(2px)\,\mathrm{d}x$$

Computation of a_0:

$$a_0 = \frac{1}{\pi}\int_0^\tau \sin(x)\,\mathrm{d}x = \frac{2}{\pi}$$

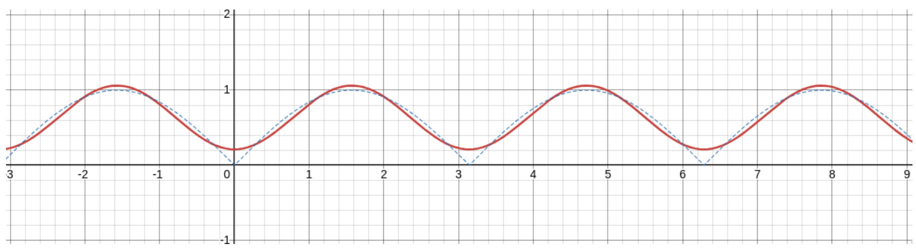
Computation of a_p :

$$\begin{aligned} a_p &= \frac{2}{\pi}\int_0^\tau \sin(x)\cos(2px)\,\mathrm{d}x \\ &= \frac{2}{\pi}\frac{2}{1-4p^2} \\ &= \frac{4}{\pi(1-4p^2)} \end{aligned}$$

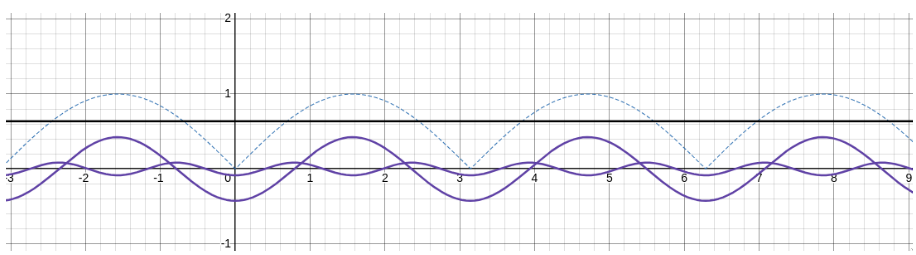
Hence the series repsentation is

$$\boxed{f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left(\frac{4}{\pi(1-4n^2)}\right)\cos(2nx)}$$

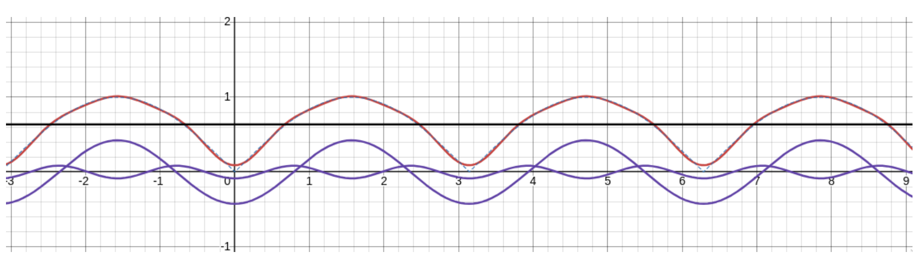
Visual Analysis:



Figuur 1: $n = 1$ for summation, dotted line for actual $|\sin(x)|$ plot.



Figuur 2: First three terms of the series are plotted.



Figuur 3: Red plot shows the summation of the three lines (can be compared with blue dotted exact plot).

Problem 02

Impulsive force on free particle:

$$\int F(t)\,\mathrm{d}t = \int K\delta(t)\,\mathrm{d}t = K = \Delta P = mv - 0 = mv \implies v = \frac{K}{m}$$

NOTE: As instructed in office hours, if we consider the particle to behave like a free particle, then work done here is $W = (1/2)mv^2 - 0 = K^2/2m$

$$W = \frac{K^2}{2m}$$

Damped Spring System: To keep things simple consider $m = 1$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t) = K\delta(t)$$

We had defined Green's Function for this specific use case

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)x(t) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)G(t,0)\cdot K = K\delta(t)$$

$$x(t) = KG(t,0) = \frac{\Theta(t)}{\omega}\sin(\omega t)\,e^{-\gamma t} \qquad (\omega^2 = \omega_0^2 - \gamma^2)$$

$$\begin{aligned} v(0) = \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0} &= \lim_{\epsilon\to 0}\frac{x(\epsilon) - x(0)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\frac{K}{\omega}\sin(\omega\epsilon)e^{-\gamma\epsilon}\right) \\ &\approx \frac{K}{\omega} \end{aligned}$$

Work done by the impulsive force:

$$\begin{aligned} W = \int_{-\infty}^{\infty} F(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\,\mathrm{d}t &= \int_{-\infty}^{\infty} F(t)v(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} K\delta(t)v(t)\,\mathrm{d}t \\ &= K\int_{-\infty}^{\infty} v(t)\delta(t)\,\mathrm{d}t \\ &= Kv(0) \end{aligned}$$

Note that we cannot make the statement $v(0) = 0$ as the impulsive force has already imparted a momentum on this spring system, hence $|v(0)| > 0$.

$$W = Kv(0) = K\frac{K}{\omega} = \frac{K^2}{\sqrt{\omega_0^2 - \gamma^2}}$$

If we have to consider mass, it's trivial (inspired from Natural Units)

$$W = \frac{K^2}{m\sqrt{\omega_0^2 - \gamma^2}}$$

Problem 03

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \sum_{n=0}^{\infty} A^n \delta\left(t - \frac{n}{2}t\right)$$

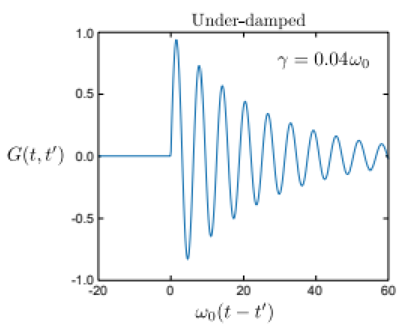
We know that for the simple case

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \delta(t - t') \implies x(t) = G(t, t')$$

We can manipulate this to take the form as given by the force

$$\begin{aligned}
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \delta\left(t - \frac{n}{2}\tau\right) &\implies x(t) = G\left(t, \frac{n}{2}\tau\right) \\
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = A^n\delta\left(t - \frac{n}{2}\tau\right) &\implies x(t) = A^n G\left(t, \frac{n}{2}\tau\right) \\
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \sum_{n=0}^{\infty} A^n\delta\left(t - \frac{n}{2}\tau\right) &\implies x(t) = F_0 \sum_{n=0}^{\infty} A^n G\left(t, \frac{n}{2}\tau\right)
\end{aligned}$$

Defining $\omega = \sqrt{\omega_0^2 - \gamma^2}$



$$G(t, t') = \Theta(t - t')\frac{e^{-\gamma(t-t')}}{\omega} \sin\left(\omega(t - t')\right)$$

Hence the response is given by

$$x(t) = F_0 \sum_{n=0}^{\infty} A^n \Theta(t - n\tau/2)\frac{e^{-\gamma(t-n\tau/2)}}{\omega} \sin\left[\omega(t - n\tau/2)\right]$$

Or if you, dear grader, would like a concise form

$$x(t) = F_0 \sum_{n=0}^{\infty} A^n G\left(t, \frac{n}{2}\tau\right)$$

Let's look at the peaks now. They are given by at any $n\tau/2 < t < (n+1)\tau/2$

$$A^n e^{-\gamma(t-n\tau/2)} \cdot \frac{1}{\omega} = A^n\left(e^{-\gamma t}e^{n\gamma\frac{\tau}{2}}\right)\frac{1}{\omega} = \left(Ae^{\gamma\tau/2}\right)^n e^{-\gamma t}\frac{1}{\omega} = \Lambda^n e^{-\gamma t}\frac{1}{\omega}$$

We require $\Lambda < 1$ otherwise Λ^n will blow up over time. The exponent term $e^{-\gamma t}$ ensures we have a decay.

$$Ae^{\gamma\tau/2} < 1 \implies A < e^{-\gamma\tau/2}$$

Now coming to the fight between Jay and Kay, we can write the amplitude as $e^{-\gamma\tau/2} < A < 1$. This obviously causes the terms to increase in magnitude every interval of $\tau/2$. Hence it is true that there will be an exponential explosion of amplitude.

appendix: trashcan

$$\begin{aligned} v(0) = \dot{x}(t) &= \frac{1}{\omega K}\left(\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\right) + \frac{\Theta(t)}{\omega K}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sin\left(\omega t\right) e^{-\gamma t}\right) \\ &= \frac{\delta(t)}{\omega K} + \frac{\Theta(t)}{K}\cos(\omega t)e^{-\gamma t} + \frac{\Theta(t)}{\omega K}\sin(\omega t)(-\gamma)e^{-\gamma t} \end{aligned}$$