Honors Multivariable Calculus : : Class 15

February 12, 2024

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1-variable 2nd order Taylor Polynomial $x = a$ is

$$f(a) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$$

This matches the 0th and 1-st derivative of f at a. If $x \approx a$

$$f(a+h) \approx f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2$$

When $h \approx 0$.

Now we are going to do this for multivariable function,

There are taylor functions that converge but not the function it's expanding.

Two Variable function

$$f(0,0) + (\text{something})x + (\text{something})y$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y$$

But we should have x^2, y^2, xy terms

$$\text{incoorect!} = f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + x^2 + y^2 + xy$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}f(\vec{0})x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(\vec{0})y^2 + \frac{\partial^2 f}{\partial x \partial y}(\vec{0})xy$$

$$f(0,0) + \frac{\partial f}{\partial x}(\vec{0})x + \frac{\partial f}{\partial y}(\vec{0})y + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}f(\vec{0})x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(\vec{0})y^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial y}(\vec{0})xy + \frac{1}{2!}\frac{\partial^2 f}{\partial y \partial x}(\vec{0})yx$$

What about even more terms? We will have additional

$$\frac{1}{3!}\frac{\partial^3 f}{\partial x^3}x^3 + \frac{1}{3!}\frac{\partial^3 f}{\partial y^3}y^3 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}x^2 y + \frac{1}{2}\frac{\partial^3 f}{\partial y^2 \partial x}y^2 x$$

Think about this

$$\frac{1}{3!}(xxy + xyx + yxx)$$

So the k'th order terms for the taylor polynomials for some function $f : \mathbb{R}^n \to \mathbb{R}^m$ are (an outline first, exact one later)

$$\sum_{i_k \in \{1 \ldots k\}} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \left(x_{i_1} x_{i_2} \cdots x_{i_k}\right)$$

If it is centered at $\vec{a}$ then $\vec{a} = (a_1, a_2, \ldots, a_n)$ then replace $\vec{x}_i$ with $\vec{x}_i - \vec{a}_i$

How correct is our taylor series?

Single variable

$$f(a+h) = f(a) + f'(a)h + \ldots + \frac{1}{k!}f^{(k)}(a)h^k + R_k(a,h)$$

Here $R_k(a,h)$ is the remainder of k order.

Facts about R_k: If f is C^k near a then

$$\lim_{h\to 0}\frac{R_k(a,h)}{h^k} = 0$$

If f is C^{k+1} then

$$\begin{aligned} R_k(a,h) &= \int_a^{a+h}\frac{(a+h-x)^k}{k!}f^{(k+1)}(x)\mathrm{d}x \\ &= \frac{1}{(k+1)!}f^{k+1}(c)h^{k+1} \end{aligned}$$