Classical Mechanics : : Homework 04

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Problem 01

$$\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = Ce^{i\omega t}} \rightarrow x(t) = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-\omega_0^2}} + Be^{-t\sqrt{\gamma^2-\omega_0^2}}\right) + \left(\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)e^{i\omega t}$$
$$= e^{-\gamma t}Ae^{t\sqrt{\gamma^2-\omega_0^2}} + e^{-\gamma t}Be^{-t\sqrt{\gamma^2-\omega_0^2}} + \left(\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)e^{i\omega t}$$
$$= Ae^{t[\sqrt{\gamma^2-\omega_0^2}-\gamma]} + Be^{-t[\sqrt{\gamma^2-\omega_0^2}+\gamma]} + \left(\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)e^{i\omega t}$$

$$\sqrt{\gamma^2 - \omega_0^2} - \gamma < 0 \implies \text{for } t \rightarrow \infty \implies Ae^{t[\sqrt{\gamma^2-\omega_0^2}-\gamma]} + Be^{-t[\sqrt{\gamma^2-\omega_0^2}+\gamma]} \rightarrow 0$$
$$\therefore \lim_{t\rightarrow\infty} x(t) = \left(\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)e^{i\omega t} \qquad \text{(steady state solution)}$$

a

External driving force

$$F = Ce^{i\omega t}$$

Work done with proper change of coordinates from $\mathrm{d}x$ to $\mathrm{d}t$, we seek the time interval to be a full cycle $t_i = 2n\pi/\omega$ to $t_f = 2(n+1)\pi/\omega$

$$\begin{aligned}
-W = \int_{x_i}^{x_f} F\,\mathrm{d}x &= \int_{t_i}^{t_f} F(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \int_{t_i}^{t_f} (Ce^{i\omega t})\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}e^{i\omega t}\right]\right)\mathrm{d}t \\
&= \int_{t_i}^{t_f} (Ce^{i\omega t})\left(\frac{i\omega C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}e^{i\omega t}\right)\mathrm{d}t \\
&= \int_{t_i}^{t_f} \frac{i\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}e^{2i\omega t}\,\mathrm{d}t \\
&= \frac{i\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\left(\frac{1}{2i\omega}e^{2i\omega t}\right)\Bigg|_{t_i}^{t_f} \\
&= \frac{C^2}{2(-\omega^2 + 2i\gamma\omega + \omega_0^2)}\left(e^{2i\omega t_f} - e^{2i\omega t_i}\right) \\
&= \frac{C^2}{2(-\omega^2 + 2i\gamma\omega + \omega_0^2)}\left(e^{2i\omega[2(n+1)\pi/\omega} - e^{2i\omega[2n\pi/\omega]}\right) \qquad \text{where } n \in \mathbb{Z}^+, n \gg 1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
-W = \int_{x_i}^{x_f} F\,\mathrm{d}x &= \int_{t_i}^{t_f} F(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \int_{t_i}^{t_f} (C\cos(\omega t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\cos(\omega t)\right]\right)\mathrm{d}t \\
&= \int_{t_i}^{t_f} (C\cos(\omega t))\left(\frac{\omega C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}(-\sin(\omega t))\right)\mathrm{d}t \\
&= \frac{-\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\int_{t_i}^{t_f} \sin(\omega t)\cos(\omega t)\mathrm{d}t \\
&= \frac{1}{2}\frac{-\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\int_{t_i}^{t_f} \sin(2\omega t)\mathrm{d}t \\
&= \frac{1}{2}\frac{-\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\left[-\cos(2\omega t)\cdot\frac{1}{2}\right]_{t_i}^{t_f} \\
&= \frac{1}{4}\frac{-\omega C^2}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\left[-\cos\left(2\omega\left(\frac{2(n+1)\pi}{\omega}\right)\right) + \cos\left(2\omega\left(\frac{2n\pi}{\omega}\right)\right)\right] \\
&= 0
\end{aligned}$$

Comment: This makes sense because the graph of $x(t)$ doesn't go up or down. It stays steady which gives us the sanity check of $W = 0$ for $t \to \infty$. The driving force does positive work against the drag overshooting the mass, but then does negative work again to balance the overshoot and repeat. This is my intuition for the steady state.

b

$$\begin{aligned}
W &= \int_{x_I}^{x_f} (-F)\,\mathrm{d}x \\
&= \int_{x_i}^{x_f} \left(2\gamma \frac{\mathrm{d}x}{\mathrm{d}t}\right)\,\mathrm{d}x \\
&= \int_{t_i}^{t_f} 2\gamma \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{t_i}^{t_f} 2\gamma \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \mathrm{d}t \\
&= \int_{t_i}^{t_f} 2\gamma \left(\frac{-\omega C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\sin(\omega t)\right)^2 \mathrm{d}t \\
&= 2\gamma \left(\frac{-\omega C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)^2 \int_{t_i}^{t_f} \sin^2(\omega t)\,\mathrm{d}t \\
&= 2\gamma \left(\frac{-\omega C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)^2 \left(\frac{\pi}{\omega}\right) \\
&= \frac{2\pi\gamma\omega C^2}{\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right)^2}
\end{aligned}$$

$$\frac{1}{2}mv^2(t) = K(t) \implies \left\langle \frac{2K(t)}{m} \right\rangle = \langle v^2(t) \rangle$$

$$x(t) = \frac{C}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\cos(\omega t)$$

$$\begin{aligned}
x(t) &= A\cos(\omega t) \\
v(t) &= -\omega A \sin(\omega t) \\
v^2(t) &= \omega^2 A^2 \sin^2(\omega t) \\
\langle v^2(t) \rangle &= \frac{1}{2}\omega^2 A^2 \\
\langle K \rangle &= \frac{m}{4}\omega^2 A^2 \\
&= \frac{m}{4}\omega^2 C^2 \left(\frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2}\right)\left(\frac{1}{-\omega^2 - 2i\gamma\omega + \omega_0^2}\right) \\
&= \boxed{\frac{m}{4}\left(\frac{\omega^2 C^2}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}\right)} \\
&= \frac{m}{4}\left(\frac{\omega^2 C^2}{4\gamma^2\omega^2}\right) && \text{(at resonance } \omega = \omega_0\text{)} \\
&= \frac{mC^2}{16\gamma^2}
\end{aligned}$$

For $\omega = n\omega_0$

$$
\begin{aligned}
\langle K \rangle &= \frac{m}{4} \frac{n^2\omega_0^2 C^2}{\omega_0^4 \left(1-n^2\right)^2 + 4\gamma^2 n^2 \omega_0^2} \\
&= \frac{m}{4} \frac{C^2}{\omega_0^2 \left(\frac{1-n^2}{n}\right)^2 + 4\gamma^2}
\end{aligned}
$$

For one octave higher,

$$\frac{1-n^2}{n} = \frac{1-2^2}{2} = \frac{1-4}{2} = -\frac{3}{2}$$

For one octave lower,

$$\frac{1-n^2}{n} = \frac{1-(1/2)^2}{(1/2)} = \frac{1-\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{2}$$

Let any higher octave be called $2^k = N$ and a lower octave by same distance be $2^{-k} = n$. Then it's obvious,

$$\frac{1-N^2}{N} = \frac{1-2^{2k}}{2^k} = 2^k(2^{-k}-2^k)\frac{1}{2^k} = \frac{1}{2^{-k}}(2^{-2k}-1) = -\frac{1-(2^{-k})^2}{2^{-k}} = -\frac{1-n^2}{n}$$

$$\implies \left(\frac{1-N^2}{N}\right)^2 = \left(-\frac{1-n^2}{n}\right)^2 \implies \langle K(\omega = 2^k\omega_0)\rangle = \langle K(\omega = 2^{-k}\omega_0)\rangle$$

Hence we proved for two equally distant octaves the kinetic energy satisfies

$$\boxed{\langle K(2^k\omega_0)\rangle = \langle K(2^{-k}\omega_0)\rangle}$$