Quantum Mechanics : : Homework 0X

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Problem 01

(a)

I always feel weird to use $\hat{P}$ operator

$$
\begin{aligned}
\langle \vec{r} | \hat{P}_x | E, m \rangle &= -i\hbar \frac{\mathrm{d}\Psi_{E,m}(r)}{\mathrm{d}x} \\
\langle \vec{r} | \hat{X}^2 | E, m \rangle &= \langle \vec{r} | \hat{X}^2 | \vec{r} \rangle \langle \vec{r} | E, m \rangle \\
&= x^2 \Psi_{E,m}(r)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle \vec{r} | \hat{H} | E, m \rangle = E\Psi_{E,m}(r) &= \frac{-\hbar^2}{2\mu} \left(\nabla^2 \Psi_{E,m}(r) \right) + \frac{\mu\omega^2}{2} \left(x^2 + y^2 \right) \Psi_{E,m}(r) \\
&= \frac{-\hbar^2}{2\mu} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{m^2}{r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} \right) \Psi_{E,m}(r) + \frac{\mu\omega^2}{2} r^2 \Psi_{E,m}(r) \\
&= \frac{-\hbar^2}{2\mu} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{m^2}{r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2\mu}{\hbar}\frac{\mu\omega^2}{2} r^2 \right) \Psi_{E,m}(r) \\
&\implies \frac{-\hbar^2}{2\mu} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{m^2}{r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mu^2\omega^2}{\hbar} r^2 + \frac{2\mu}{\hbar^2} E \right) \Psi_{E,m}(r) = 0 \\
&= -\omega\hbar b^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{m^2}{r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mu^2\omega^2}{\hbar} r^2 + \frac{2\mu}{\hbar^2} E \right) \Psi_{E,m}(r) = 0 \\
&= \left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{m^2}{y^2} + \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} - y^2 + \frac{2\mu}{\hbar^2}\frac{\hbar}{\mu\omega} E \right) \Psi_{E,m}(r) = 0 \\
&= \left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{m^2}{y^2} + \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} - y^2 + \frac{2}{\hbar\omega} E \right) \Psi_{E,m}(r) = 0
\end{aligned}
$$

$$
\boxed{\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} - \frac{m^2}{y^2} + 2\varepsilon \right] \psi_{\varepsilon,m}(y) = 0}
$$

(b)

$$
\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} - \frac{m^2}{y^2} + 2\varepsilon \right] y^\alpha = 0
$$

$$\begin{aligned}\alpha(\alpha-1)y^{\alpha-2}+\alpha y^{\alpha-2}-m^2y^{\alpha-2}+2\varepsilon y^{\alpha}&=0\\ \alpha(\alpha-1)+\alpha-m^2+2\varepsilon y^{\alpha-\alpha+2}&=0\\ \alpha^2-m^2+2\varepsilon y^2=0 &\implies \lim_{y\to 0}\alpha^2-m^2+2\varepsilon y^2=0\\ &\implies \alpha=|m|\end{aligned}$$

$$\boxed{\lim_{y\to 0}\psi_{\varepsilon,m}(y)=y^{|m|}}$$

(c)

$$\begin{aligned}\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}-\frac{m^2}{y^2}+2\varepsilon\right]y^{|m|}f(y)&=0\\ \frac{\mathrm{d}}{\mathrm{d}y}\left(|m|y^{|m|-1}f(y)+y^{|m|}f'(y)\right)+\frac{1}{y}\left(|m|y^{|m|-1}f(y)+y^{|m|}f'(y)\right)-\frac{m^2}{y^2}y^{|m|}f(y)+2\varepsilon y^{|m|}f(y)&=0\\ \frac{\mathrm{d}}{\mathrm{d}y}\left(|m|y^{|m|-1}f(y)+y^{|m|}f'(y)\right)+\left(|m|y^{|m|-2}f(y)+y^{|m|-1}f'(y)\right)-m^2y^{|m|-2}f(y)+2\varepsilon y^{|m|}f(y)&=0\\ \left(|m|(|m|-1)y^{|m|-2}f(y)+|m|y^{|m|-1}f'(y)+|m|y^{|m|-1}f'(y)+y^{|m|}f''(y)\right)&+\\ +\left(|m|y^{|m|-2}f(y)+y^{|m|-1}f'(y)\right)-m^2y^{|m|-2}f(y)+2\varepsilon y^{|m|}f(y)&=0\\ y^{|m|-2}f(y)\left[|m|(|m|-1)+|m|-m^2\right]+y^{|m|-1}f'(y)\left[2|m|+1\right]+y^{|m|}\left[f''(y)+2\varepsilon f(y)\right]&=0\\ y^{|m|-1}f'(y)\left[2|m|+1\right]+y^{|m|}\left[f''(y)+2\varepsilon f(y)\right]&=0\\ f'(y)\left[2|m|+1\right]+y\left[f''(y)+2\varepsilon f(y)\right]&=0\\ f'(y)\frac{\left[2|m|+1\right]}{y}+\left[f''(y)+2\varepsilon f(y)\right]&=0\\ \frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y^2/2}u(y)\right)\frac{\left[2|m|+1\right]}{y}+\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(e^{-y^2/2}u(y)\right)+2\varepsilon\left(e^{-y^2/2}u(y)\right)\right]&=0\\ \left(-yu(y)+u'(y)\right)e^{-y^2/2}\frac{\left[2|m|+1\right]}{y}&+\\ +\left[\left(-u(y)-yu'(y)+u''(y)+y^2u(y)-yu'(y)\right)e^{-y^2/2}+2\varepsilon\left(e^{-y^2/2}u(y)\right)\right]&=0\\ \left(-yu(y)+u'(y)\right)\frac{\left[2|m|+1\right]}{y}&+\\ +\left[\left(-u(y)-2yu'(y)+u''(y)+y^2u(y)\right)+2\varepsilon u(y)\right]&=0\\ u''(y)+\left[\frac{2|m|+1}{y}-2y\right]u'+(2\varepsilon-1-2|m|-1)u&=0\end{aligned}$$

$$\boxed{u''(y)+\left[\frac{2|m|+1}{y}-2y\right]u'+(2\varepsilon-2|m|-2)u=0}$$

(c)

$$
\begin{aligned}
u(y) &= \sum_{p=0}^{\infty} c_p y^p \\
u'(y) &= \sum_{p=1}^{\infty} p c_p y^{p-1} = \sum_{p=0}^{\infty} (p+1) c_{p+1} y^p \\
u''(y) &= \sum_{p=2}^{\infty} p(p-1) c_p y^{p-2} = \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} y^p
\end{aligned}
$$

$$
u''(y) + \left[\frac{2|m|+1}{y} - 2y\right] u' + (2\varepsilon - 2|m| - 2)u = 0
$$

$$
\begin{aligned}
0 &= \sum_{p=0}^{\infty}(p+2)(p+1)c_{p+2}y^p + \left[\frac{2|m|+1}{y} - 2y\right]\sum_{p=0}^{\infty}(p+1)c_{p+1}y^p + (2\varepsilon - 2|m| - 2)\sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty}(p+2)(p+1)c_{p+2}y^p + \underbrace{\left[(2|m|+1)\sum_{p=-1}^{\infty}(p+2)c_{p+2}y^p - 2\sum_{p=0}^{\infty}(p)c_p y^p\right]}_{\text{term of interest where } c_1 = 0} + (2\varepsilon - 2|m| - 2)\sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty}(p+2)(p+1)c_{p+2}y^p + (2|m|+1)\sum_{p=0}^{\infty}(p+2)c_{p+2}y^p - 2\sum_{p=0}^{\infty}(p)c_p y^p + (2\varepsilon - 2|m| - 2)\sum_{p=0}^{\infty} c_p y^p \\
&= \sum_{p=0}^{\infty} y^p \left[(p+2)(p+1)c_{p+2} + (2|m|+1)(p+2)c_{p+2} - 2pc_p + (2\varepsilon - 2|m| - 2)c_p\right] \\
&= \sum_{p=0}^{\infty} y^p \left[\underbrace{\left[(p+2)(p+1) + (2|m|+1)(p+2)\right]}_{\text{two terms above}} c_{p+2} + \left[-2p + (2\varepsilon - 2|m| - 2)\right] c_p\right]
\end{aligned}
$$

Requirement to keep everything 0 (also note $p = 2k$)

$$
\begin{aligned}
c_{p+2} &= \frac{2p - (2\varepsilon - 2|m| - 2)}{(p+2)(p+1) + (2|m|+1)(p+2)} \\
&= \frac{2p - (2\varepsilon - 2|m| - 2)}{p^2 + p + 2p + 2 + 2|m|p + 4|m| + p + 2} \\
&= \frac{2p - (2\varepsilon - 2|m| - 2)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&= 2\frac{p - (\varepsilon - |m| - 1)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&\text{for } \varepsilon = 2n + |m| + 1 \implies 2\frac{p - (2n + |m| + 1 - |m| - 1)}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&= 2\frac{p - 2n}{p^2 + (4 + 2|m|)p + 4 + 4|m|} \\
&\implies c_{2n+2} = 0 \impliedby p = 2n
\end{aligned}
$$

$$
c_{p+2} = 2\frac{p - (\varepsilon - |m| - 1)}{p^2 + (4 + 2|m|)p + 4 + 4|m|}
$$

Problem 02

(a)

$$\hat{x} = \frac{1}{b}\hat{X} \qquad\qquad \hat{p} = \frac{b}{\hbar}\hat{P}$$

$$\begin{aligned}
\hat{a}\left|n=0\right\rangle &= 0 \\
\frac{1}{\sqrt{2}}\left(\hat{x}+i\hat{p}\right)\left|n=0\right\rangle &= 0 \\
\frac{1}{\sqrt{2}}\left(\hat{x}+i\hat{p}\right)\hat{I}\left|n=0\right\rangle &= 0 \\
\frac{1}{\sqrt{2}}\left(\hat{x}+i\hat{p}\right)\left|p\right\rangle\left\langle p|n=0\right\rangle &= 0 \\
\implies \frac{1}{\sqrt{2}}\left\langle p'\right|\left(\hat{x}+i\hat{p}\right)\left|p\right\rangle\left\langle p|n=0\right\rangle &= 0 \\
\frac{1}{\sqrt{2}}\Big[\left\langle p'\right|\hat{x}\left|p\right\rangle + i\left\langle p'\right|\hat{p}\left|p\right\rangle\Big]\left\langle p|n=0\right\rangle &= 0 \\
\frac{1}{\sqrt{2}}\Big[\left\langle p'\right|\hat{x}\left|p\right\rangle\psi_0(p) + i\left\langle p'\right|\hat{p}\left|p\right\rangle\psi_0(p)\Big] &= 0 \\
i\hbar\frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}p}\psi_0(p) + \frac{b}{\hbar}ip\psi_0(p) &= 0 \\
\implies \frac{\hbar}{b}\frac{\mathrm{d}}{\mathrm{d}p}\psi_0(p) + \frac{b}{\hbar}p\psi_0(p) &= 0 \\
\frac{\hbar^2}{b^2}\frac{\mathrm{d}}{\mathrm{d}p}\psi_0(p) + p\psi_0(p) &= 0 \\
\frac{\hbar^2}{b^2}\frac{\mathrm{d}\psi_0(p)}{\psi_0(p)} &= -p\,\mathrm{d}p \\
\frac{\hbar^2}{b^2}\ln\left(\frac{\psi_0(p)}{\psi_0(0)}\right) &= -\frac{p^2}{2} \\
\ln\left(\frac{\psi_0(p)}{\psi_0(0)}\right) &= -\frac{b^2}{2\hbar^2}p^2 \\
\ln\left(\frac{\psi_0(p)}{\psi_0(0)}\right) &= -\frac{\hbar}{2m\omega\hbar^2}p^2 \\
\ln\left(\frac{\psi_0(p)}{\psi_0(0)}\right) &= -\frac{p^2}{2\omega m\hbar} \\
\psi_0(p) &= \psi_0(0)\,\mathrm{e}^{-p^2/2m\omega\hbar}
\end{aligned}$$

$$\begin{aligned}
|\psi_0(p)|^2 = \psi_0^2(0)\int_{-\infty}^{\infty}\mathrm{d}p\,e^{-p^2/m\omega\hbar} = \psi_0^2(0)\frac{\sqrt{2\pi}}{\sqrt{2/m\omega\hbar}} &= \psi_0^2(0)\sqrt{m\omega\pi\hbar} \\
\implies |\psi_0(p)|^2 &= \psi_0^2(0)\sqrt{m\omega\pi\hbar} \\
\implies \Psi_0(p) &= \frac{e^{-p^2/2m\omega\hbar}}{(m\omega\pi\hbar)^{1/4}}
\end{aligned}$$

(b)

$$
\begin{aligned}
\psi_0(x) = \langle x|n=0\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle \langle p|n=0\rangle \,\mathrm{d}p &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\Psi_0(p)\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \frac{e^{-p^2/2m\omega\hbar}}{(m\omega\pi\hbar)^{1/4}}\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \int_{-\infty}^{\infty} e^{ipx/\hbar - p^2/2m\omega\hbar}\,\mathrm{d}p \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \int_{-\infty}^{\infty} e^{\beta p - \frac{\alpha}{2}p^2}\,\mathrm{d}p \impliedby \begin{cases} \alpha &= \frac{1}{m\omega\hbar} \\ \beta &= \frac{ix}{\hbar} \end{cases} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left(\sqrt{\frac{2\pi}{\alpha}} e^{\beta^2/2\alpha}\right) \impliedby \begin{cases} \alpha &= \frac{1}{m\omega\hbar} \\ \beta &= \frac{ix}{\hbar} \end{cases} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left(\sqrt{2\pi m\omega\hbar}\, e^{-x^2(m\omega\hbar)/2\hbar^2}\right) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left(\sqrt{2\pi m\omega\hbar}\, e^{-x^2(m\omega)/2\hbar}\right) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(m\omega\pi\hbar)^{1/4}} \left(\sqrt{2\pi m\omega\hbar}\, e^{-x^2/2b}\right) \\
&= \frac{\sqrt{m\omega}}{(m\omega\pi\hbar)^{1/4}} \left(e^{-x^2/2b}\right) \\
&= \frac{(m\omega)^{1/4}}{(\pi\hbar)^{1/4}} \left(e^{-x^2/2b}\right) \\
&= \frac{1}{\pi^{1/4} b^{1/2}} e^{-x^2/2b}
\end{aligned}
$$

Problem 03

(a)

For $n = 1$ **:**

$$
\begin{aligned}
\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right) e^{-y^2/2} &= ye^{-y^2/2} - \frac{\mathrm{d}}{\mathrm{d}y} e^{-y^2/2} \\
&= ye^{-y^2/2} + ye^{-y^2/2} \\
&= 2ye^{-y^2/2}
\end{aligned}
$$

For $n = 2$ **:**

$$\begin{aligned}
\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)^2 e^{-y^2/2} &= \left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right)\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right) e^{-y^2/2} \\
&= \left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right) 2ye^{-y^2/2} \\
&= 2y^2e^{-y^2/2} - \frac{\mathrm{d}}{\mathrm{d}y}\left(2ye^{-y^2/2}\right) \\
&= 2y^2e^{-y^2/2} - \left(2e^{-y^2/2} - 2y^2e^{-y^2/2}\right) \\
&= 4y^2e^{-y^2/2} - 2e^{-y^2/2} \\
&= (4y^2 - 2)e^{-y^2/2}
\end{aligned}$$

$$\begin{aligned}
\psi_1(x) &= \frac{2ye^{-y^2/2}}{\sqrt{2b\sqrt{\pi}}} && (y = x/b) \\
\psi_2(x) &= \frac{(4y^2 - 2)e^{-y^2/2}}{2\sqrt{2b\sqrt{\pi}}}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha x^2/2} &= \sqrt{\frac{2\pi}{a}} \\
\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha x^2/2} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\sqrt{\frac{2\pi}{a}} \\
\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}\alpha}e^{-\alpha x^2/2} &= \sqrt{2\pi}\left(-\frac{1}{2}a^{-\frac{3}{2}}\right) \\
-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, x^2e^{-\alpha x^2/2} &= \frac{\sqrt{2\pi}}{a^{\frac{3}{2}}}\left(-\frac{1}{2}\right) \\
\int_{-\infty}^{\infty} \mathrm{d}x\, x^2e^{-\alpha x^2/2} &= \frac{\sqrt{2\pi}}{a^{3/2}}
\end{aligned}$$

(c)

Consider the functions without the normalization constants in front

$$\begin{aligned}
\Psi_0(x) &= e^{-y^2/2} \\
\Psi_1(x) &= ye^{-y^2/2} \\
\Psi_2(x) &= (4y^2 - 2)e^{-y^2/2}
\end{aligned}$$

$$
\begin{aligned}
\langle 0|2\rangle \implies & \int_{-\infty}^{\infty} \mathrm{d}y\,(4y^2-2)e^{-y^2} \\
&= \int_{-\infty}^{\infty} \mathrm{d}y\,4y^2e^{-y^2} - \int_{-\infty}^{\infty} \mathrm{d}y\,2e^{-y^2} \\
&= \sqrt{2\pi}\left(\frac{4}{2^{3/2}} - \frac{2}{2^{1/2}}\right) \\
&= \sqrt{2\pi}\left(\frac{2}{2^{1/2}} - \frac{2}{2^{1/2}}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle 1|1\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\,y^2e^{-y^2} \\
&= b\int_{-\infty}^{\infty} \mathrm{d}y\,y^2e^{-y^2} \\
&= b\frac{\sqrt{2\pi}}{2^{3/2}} \\
&= b\frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2}^3} = b\frac{\sqrt{\pi}}{2}
\end{aligned}
$$

Coefficients of $\Psi_1(x)$ are

$$
C = \frac{2}{\sqrt{2b\sqrt{\pi}}} \implies C^2 = \frac{2}{b\sqrt{\pi}}
$$

Putting that in place

$$
\langle 1|1\rangle = C^2\int_{-\infty}^{\infty} \mathrm{d}x\,y^2e^{-y^2} = \frac{2}{b\sqrt{\pi}}b\frac{\sqrt{\pi}}{2} = 1
$$

Problem 04

(a)

$$
\begin{aligned}
\hat{\overline{X}} &= \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^T) \\
\hat{\overline{X}}\hat{\overline{X}} &= \frac{1}{2}\left(\hat{a}+\hat{a}^T\right)\left(\hat{a}+\hat{a}^T\right) \\
&= \frac{1}{2}\left(\hat{a}\hat{a}+\hat{a}^T\hat{a}^T+\hat{a}^T\hat{a}+\hat{a}\hat{a}^T\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\overline{P}} &= \frac{i}{\sqrt{2}}(\hat{a}^T-\hat{a}) \\
\hat{\overline{P}}\hat{\overline{P}} &= -\frac{1}{2}\left(\hat{a}^T-\hat{a}\right)\left(\hat{a}^T-\hat{a}\right) \\
&= -\frac{1}{2}\left(\hat{a}\hat{a}+\hat{a}^T\hat{a}^T-\hat{a}^T\hat{a}-\hat{a}\hat{a}^T\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle n|\,\hat{\overline{X}}\,|n\rangle &= \frac{1}{\sqrt{2}}\,\langle n|\,\hat{a}+\hat{a}^T\,|n\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^T|n\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\right) = 0 \\
\langle n|\hat{\overline{P}}|n\rangle &= \frac{i}{\sqrt{2}}\,\langle n|\hat{a}^T-\hat{a}|n\rangle \\
&= \frac{i}{\sqrt{2}}\left(\langle n|\hat{a}^T|n\rangle - \langle n|\hat{a}|n\rangle\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle n|\,\hat{\hat{X}}^2\,|n\rangle &= \frac{1}{2}\left(\langle n|\hat{a}\hat{a}|n\rangle + \langle n|\hat{a}^T\hat{a}^T|n\rangle + \langle n|\hat{a}\hat{a}^T|n\rangle + \langle n|\hat{a}^T\hat{a}|n\rangle\right)\\
&= \frac{1}{2}\left(0 + 0 + (n+1) + n\right)\\
&= \frac{1}{2}\left(2n+1\right)\\
&= n + \frac{1}{2}\\
\langle n|\,\hat{\hat{P}}^2\,|n\rangle &= -\frac{1}{2}\left(\langle n|\hat{a}\hat{a}|n\rangle + \langle n|\hat{a}^T\hat{a}^T|n\rangle - \langle n|\hat{a}\hat{a}^T|n\rangle - \langle n|\hat{a}^T\hat{a}|n\rangle\right)\\
&= -\frac{1}{2}\left(0 + 0 - ((n+1) + n)\right)\\
&= n + \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\Delta X &= \sqrt{\frac{\hbar}{m\omega}}\sqrt{\langle n|\hat{\hat{X}}^2|n\rangle - \langle n|\hat{\hat{X}}|n\rangle} = \sqrt{\frac{\hbar}{m\omega}}\sqrt{n+\frac{1}{2}}\\
\Delta P &= \hbar\sqrt{\frac{m\omega}{\hbar}}\sqrt{\langle n|\hat{\hat{P}}^2|n\rangle - \langle n|\hat{\hat{P}}|n\rangle} = \hbar\sqrt{\frac{m\omega}{\hbar}}\sqrt{n+\frac{1}{2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\implies \Delta X \Delta P &= \hbar\left(n+\frac{1}{2}\right)\\
\Delta X \Delta P &\geq \frac{\hbar}{2}\\
\Delta X \Delta P &= \frac{\hbar}{2} \iff n = 0
\end{aligned}
$$

Problem 05

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega \right\rangle(t) &= \frac{i}{\hbar}\left\langle [H,\Omega]\right\rangle(t)\\
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a \right\rangle(t) &= \frac{i}{\hbar}\left\langle [H,a]\right\rangle(t)\\
&= \frac{i}{\hbar}\left\langle \left[\hbar\omega\left(a^T a + \frac{1}{2}I\right), a\right]\right\rangle\\
&= i\omega\left\langle \left[\left(a^T a\right), a\right]\right\rangle\\
&= -i\omega\left\langle a \right\rangle\\
\int \frac{1}{\langle a\rangle}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a\right\rangle\,\mathrm{d}t &= \int -i\omega\,\mathrm{d}t\\
\left\langle a\right\rangle(t) &= \left\langle a\right\rangle(0)e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega \right\rangle (t) &= \frac{i}{\hbar}\left\langle [H, \Omega] \right\rangle (t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a \right\rangle (t) &= \frac{i}{\hbar}\left\langle [H, a] \right\rangle (t) \\
&= \frac{i}{\hbar}\left\langle \psi \right| Ha - aH \left| \psi \right\rangle \\
&= \frac{i}{\hbar}\left[\left\langle \psi \right| Ha \left| \psi \right\rangle - \left\langle \psi \right| aH \left| \psi \right\rangle\right] \\
&= \frac{i(\hbar\omega)}{\hbar}\left[\left\langle \psi \right| \left(a^T a + \frac{1}{2}I\right) a \left| \psi \right\rangle - \left\langle \psi \right| a \left(a^T a + \frac{1}{2}I\right) \left| \psi \right\rangle\right] \\
&= i\omega\left[\left\langle \psi \right| a^T aa \left| \psi \right\rangle - \left\langle \psi \right| aa^T a \left| \psi \right\rangle\right] \\
&= i\omega\left[\left\langle \psi(n) \right| a^T aa \left| \psi(n) \right\rangle - \left\langle \psi(n) \right| aa^T a \left| \psi(n) \right\rangle\right] && \text{(rewrite with } n\text{-th energy eigenstate)} \\
&= i\omega\left[\sqrt{n-1}\left\langle \psi(n) \right| a^T a \left| \psi(n-1) \right\rangle - \sqrt{n-1}\left\langle \psi(n) \right| aa^T \left| \psi(n-1) \right\rangle\right] \\
&= i\omega\left[\sqrt{(n-2)(n-1)}\left\langle \psi(n) \right| a^T \left| \psi(n-2) \right\rangle - \sqrt{n(n-1)}\left\langle \psi(n) \right| a \left| \psi(n) \right\rangle\right] \\
&= i\omega\left[\sqrt{(n-2)(n-1)^2}\left\langle \psi(n) \right| \left| \psi(n-1) \right\rangle - \sqrt{n(n-1)^2}\left\langle \psi(n) \right| \left| \psi(n-1) \right\rangle\right] \\
&= i\omega\left[\sqrt{(n-2)(n-1)^2} - \sqrt{n(n-1)^2}\right]\left\langle \psi(n) | \psi(n-1) \right\rangle
\end{aligned}
$$