Quantum Mechanics : : Homework 11

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Problem 01

General matrix representation of an operator X if basis vectors are given by $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$ is (*Sakurai eq 1.73*)

$$\begin{bmatrix} \langle 1 \mid X \mid 1\rangle & \langle 1 \mid X \mid 2\rangle & \langle 1 \mid X \mid 3\rangle & \langle 1 \mid X \mid 4\rangle & \langle 1 \mid X \mid 5\rangle \\ \langle 2 \mid X \mid 1\rangle & \langle 2 \mid X \mid 2\rangle & \langle 2 \mid X \mid 3\rangle & \langle 2 \mid X \mid 4\rangle & \langle 2 \mid X \mid 5\rangle \\ \langle 3 \mid X \mid 1\rangle & \langle 3 \mid X \mid 2\rangle & \langle 3 \mid X \mid 3\rangle & \langle 3 \mid X \mid 4\rangle & \langle 3 \mid X \mid 5\rangle \\ \langle 4 \mid X \mid 1\rangle & \langle 4 \mid X \mid 2\rangle & \langle 4 \mid X \mid 3\rangle & \langle 4 \mid X \mid 4\rangle & \langle 4 \mid X \mid 5\rangle \\ \langle 5 \mid X \mid 1\rangle & \langle 5 \mid X \mid 2\rangle & \langle 5 \mid X \mid 3\rangle & \langle 5 \mid X \mid 4\rangle & \langle 5 \mid X \mid 5\rangle \end{bmatrix}$$

(a)

The basis we are going to use,

$$\{|2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle, |2,-2\rangle\}$$

The first matrix is for

$$\hat{\vec{J}}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$\begin{aligned} \hat{J}^2 |2,2\rangle &= \hbar^2 6 |2,2\rangle, \\ \hat{J}^2 |2,1\rangle &= \hbar^2 6 |2,1\rangle, \\ \hat{J}^2 |2,0\rangle &= \hbar^2 6 |2,0\rangle, \\ \hat{J}^2 |2,-1\rangle &= \hbar^2 6 |2,-1\rangle, \\ \hat{J}^2 |2,-2\rangle &= \hbar^2 6 |2,-2\rangle \\ \implies \hat{J}^2 &= \hbar^2 \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

The second matrix is for

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

$$
\begin{aligned}
\hat{J}_z |2,2\rangle &= 2\hbar |2,2\rangle , \\
\hat{J}_z |2,1\rangle &= \hbar |2,1\rangle , \\
\hat{J}_z |2,0\rangle &= 0, \\
\hat{J}_z |2,-1\rangle &= -\hbar |2,-1\rangle , \\
\hat{J}_z |2,-2\rangle &= -2\hbar |2,-2\rangle \\
\implies \hat{J}_z &= \hbar \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}
\end{aligned}
$$

The third matrix is the (+) ladder matrix

$$
\hat{J}_\pm |j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle
$$

$$
\begin{aligned}
\hat{J}_+ |2,2\rangle &= 0, \\
\hat{J}_+ |2,1\rangle &= \hbar\sqrt{4}\,|2,2\rangle , \\
\hat{J}_+ |2,0\rangle &= \hbar\sqrt{6}\,|2,1\rangle , \\
\hat{J}_+ |2,-1\rangle &= \hbar\sqrt{6}\,|2,0\rangle , \\
\hat{J}_+ |2,-2\rangle &= \hbar\sqrt{4}\,|2,-1\rangle , \\
\implies \hat{J}_+ &= \hbar \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

The other third matrix for the (−) ladder matrix

$$
\begin{aligned}
\hat{J}_- |2,2\rangle &= \hbar\sqrt{4}\,|2,1\rangle , \\
\hat{J}_- |2,1\rangle &= \hbar\sqrt{6}\,|2,0\rangle , \\
\hat{J}_- |2,0\rangle &= \hbar\sqrt{6}\,|2,-1\rangle , \\
\hat{J}_- |2,-1\rangle &= \hbar\sqrt{4}\,|2,-2\rangle , \\
\hat{J}_- |2,-2\rangle &= 0. \\
\implies \hat{J}_- &= \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix}
\end{aligned}
$$

(b)

I will include the $\hbar^2$ factor at the end, for now consider $\hbar = 1$.

$$
\begin{aligned}
[J_z, J_+] &= J_zJ_+ - J_+J_z \\
&= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}
\end{aligned}
$$

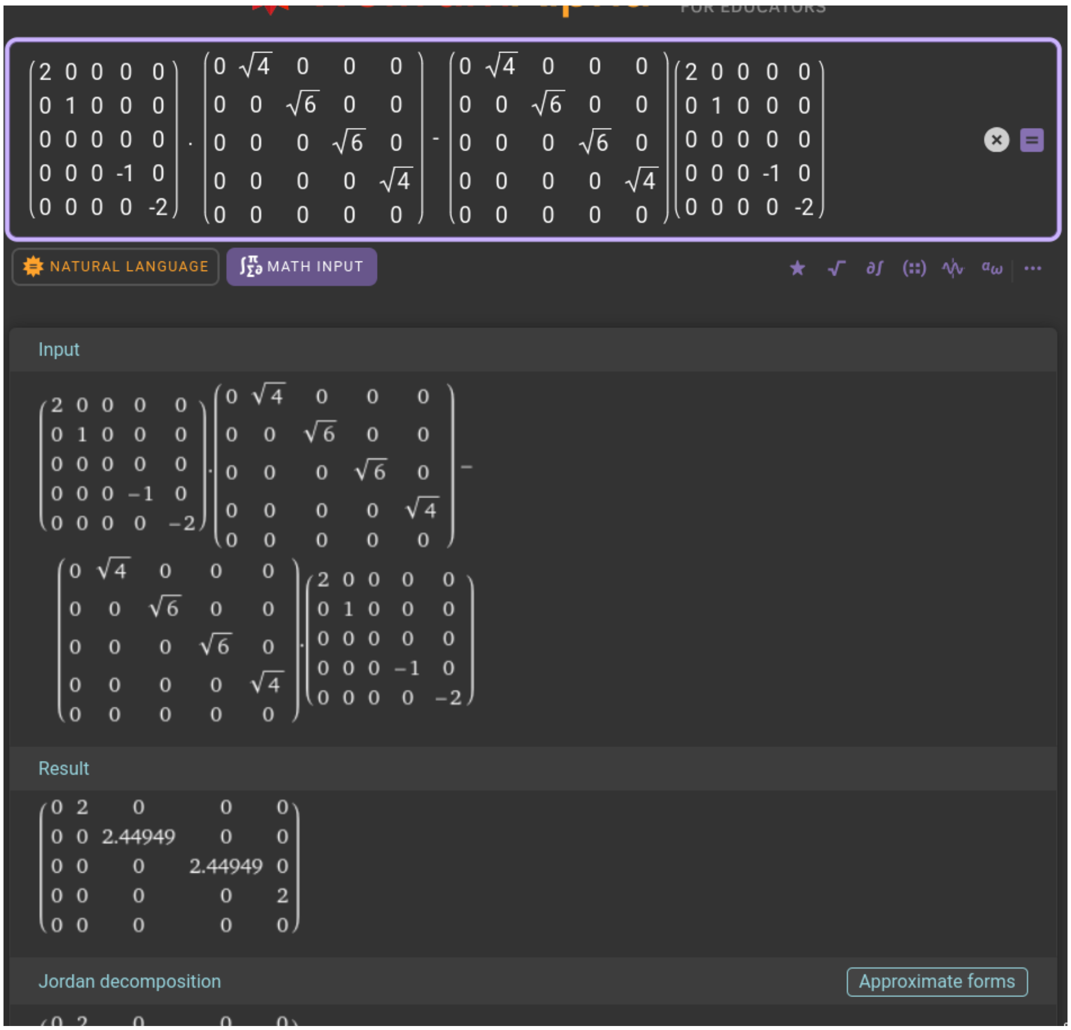

Figure 1: ./ss/11/1.png

$$
= \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \hbar^2 \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \hbar J_+
$$

$$
\implies [J_z, J_+] = \hbar J_+
$$

$$[J_z, J_-] = J_zJ_- - J_-J_z$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

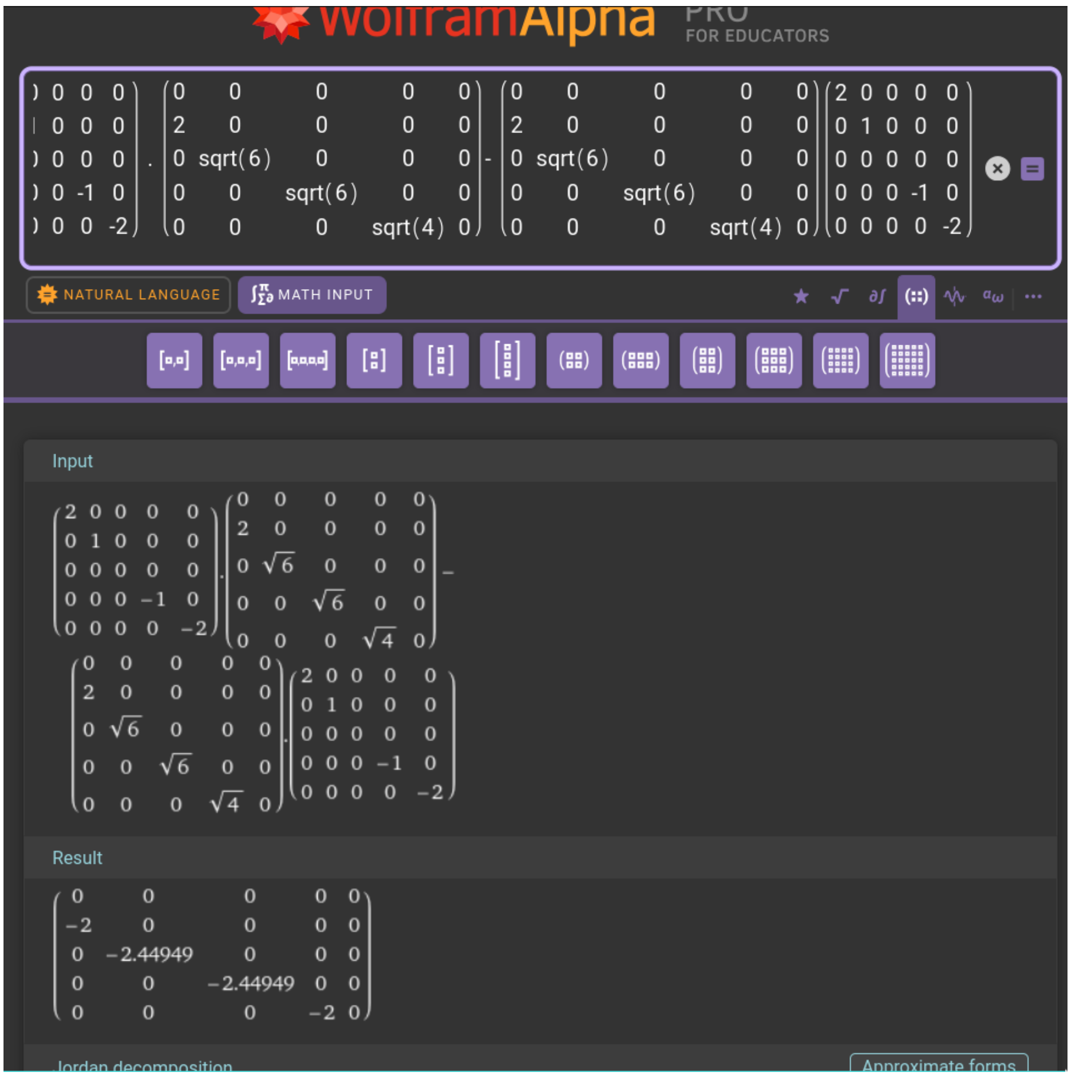

Figure 2:

$$= - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} \implies -\hbar^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} = -\hbar J_-$$

$$\implies [J_z, J_-] = -\hbar J_-$$

$[J_+, J_-] = J_+J_- - J_-J_+$

$$= \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

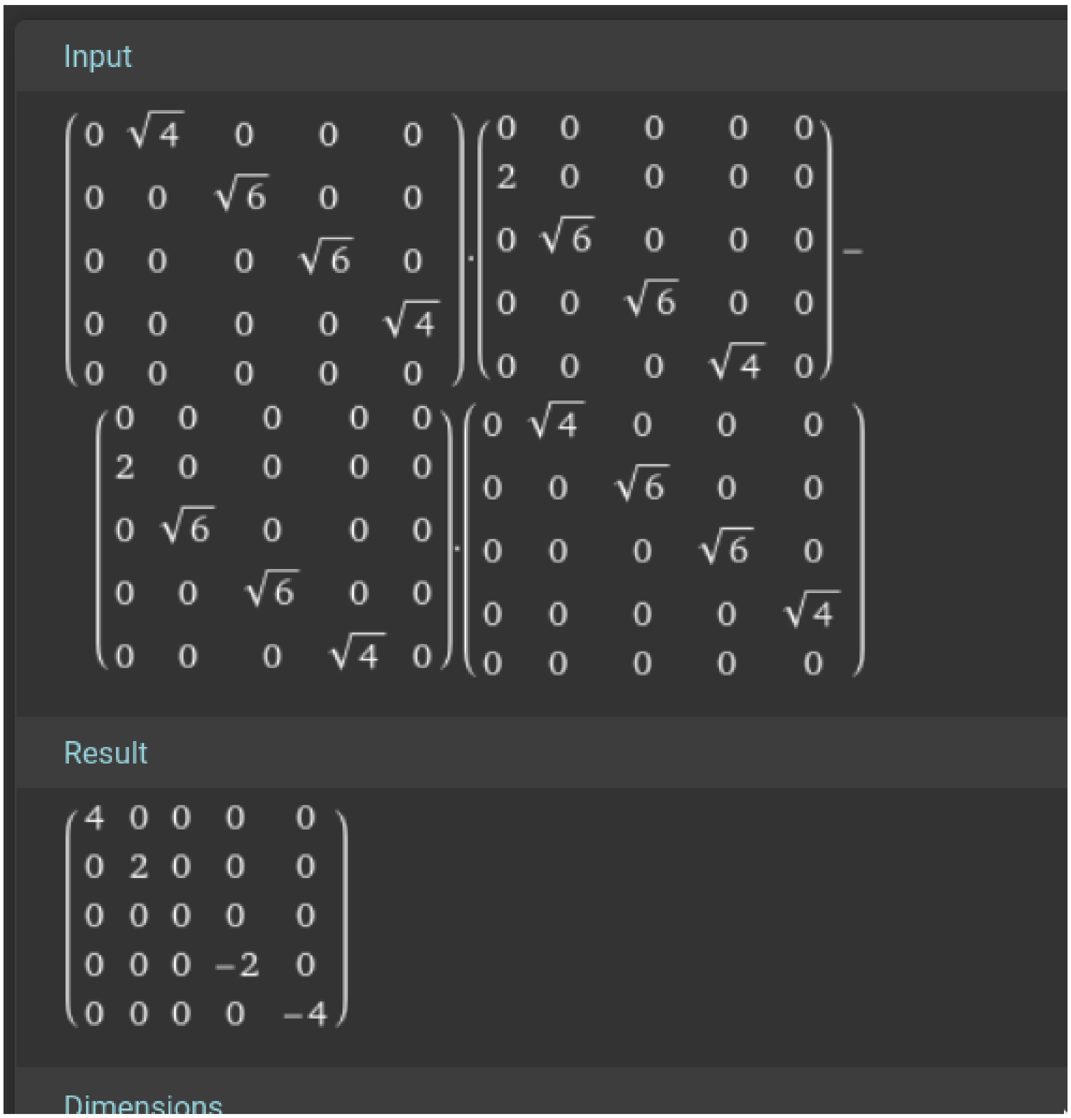

Figure 3: ./ss/11/3.png

$$= 2 \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \implies \hbar^2 \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} = 2\hbar J_z$$

$$[J_+, J_-] = 2\hbar J_z$$

We will compute

$$\frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

using matrix.

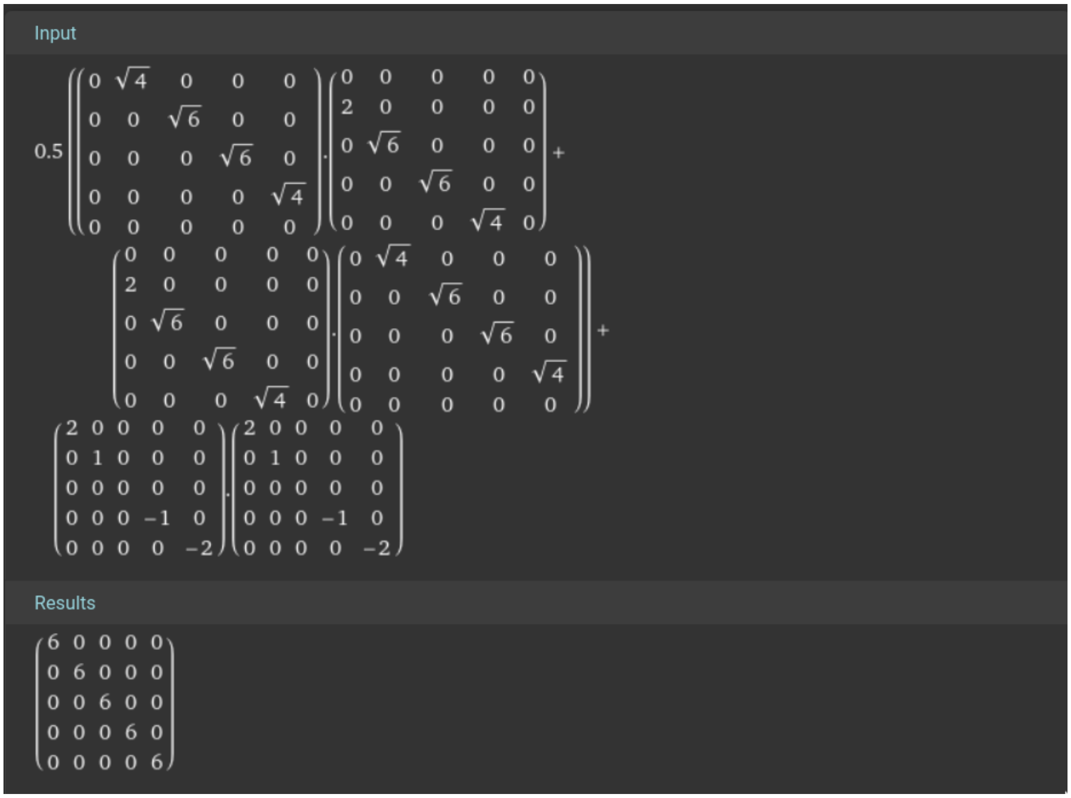

Figure 4: ./ss/11/4.png

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \implies \hbar^2 \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} = 6\hbar^2 \hat{I} = \vec{J}^2$$

Problem 02

$$
\begin{aligned}
J_+ &= J_x + iJ_y \\
J_- &= J_x - iJ_y \\
J_+ + J_- &= 2J_x \\
\implies J_x &= \frac{1}{2}\left(J_+ + J_-\right) \\
\implies J_x^2 &= \frac{1}{4}\left(J_+ + J_-\right)\left(J_+ + J_-\right) = \frac{1}{4}\left(J_+^2 + J_+J_- + J_-J + +J_-^2\right) = \frac{1}{4}\left(J_+^2 + J_+J_- + J_-J_+ + J_-^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{now, } J_x\left|j,m\right\rangle &= \frac{1}{2}J_+\left|j,m\right\rangle + \frac{1}{2}J_-\left|j,m\right\rangle = C_1\left|j,m+1\right\rangle + C_2\left|j,m-1\right\rangle \\
\implies \langle j,m|J_x|j,m\rangle &= 0 \\
J_x^2\left|j,m\right\rangle &= \frac{1}{4}(D_1\left|j,m+2\right\rangle + D_2\left|j,m-2\right\rangle) + \frac{1}{4}\left(\hbar^2\sqrt{j(j+1)-(m-1)m}\sqrt{j(j+1)-m(m-1)}\right)\left|j,m\right\rangle \\
&+\frac{1}{4}\left(\hbar^2\sqrt{j(j+1)-(m+1)m}\sqrt{j(j+1)-(m+1)m}\right)\left|j,m\right\rangle \\
\implies \langle j,m|J_x^2|j,m\rangle &= \frac{1}{4}\left(\hbar^2 j(j+1) - \hbar^2 m(m-1) + \hbar^2 j(j+1) - \hbar^2 m(m+1)\right) \\
&= \frac{\hbar^2}{4}\left(j(j+1) - m(m-1) + j(j+1) - m(m+1)\right) \\
&= \frac{\hbar^2}{4}\left(j(j+1) - m^2 + m + j(j+1) - m^2 - m\right) \\
&= \frac{\hbar^2}{2}\left(j(j+1) - m^2\right) \\
&= \frac{\hbar^2}{2}\left(j + j^2 - m^2\right) \\
\Delta J_x = \Delta J_y = \Delta J_\perp &= \sqrt{\langle j,m|J_x^2|j,m\rangle - \left(\langle j,m|J_x|j,m\rangle\right)^2} \\
&= \sqrt{\frac{\hbar^2}{2}\left(j + j^2 - m^2\right)}
\end{aligned}
$$

$$
\boxed{\Delta J_\perp = \sqrt{\frac{\hbar^2}{2}(j + j^2 - m^2)}}
$$

- $m = 0$ sets $\Delta J_\perp$ to be maximized through $\Delta J_\perp = \sqrt{\frac{\hbar^2}{2} j(j+1)}$
- $m = \pm j$ the possible value of m sets $\Delta J_\perp = \sqrt{\frac{\hbar^2}{2} j}$

Problem 03

(a)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

1. Partial Derivative $\frac{\partial}{\partial x}$:

For $r = \sqrt{x^2 + y^2 + z^2}$:

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

For $\theta = \cos^{-1}\left(\frac{z}{r}\right)$:

$$\frac{\partial \theta}{\partial x} = -\frac{z}{r^3 \sin\theta}$$

For $\phi = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2}$$

The chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{x}{r} + \frac{\partial}{\partial \theta}\left(-\frac{z}{r^3 \sin\theta}\right) + \frac{\partial}{\partial \phi}\left(-\frac{y}{x^2 + y^2}\right)$$

Now, in terms of spherical coordinates:

$$\frac{x}{r} = \sin\theta\cos\phi$$

$$\frac{y}{r} = \sin\theta\sin\phi$$

$$\frac{z}{r} = \cos\theta$$

Thus:

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\boxed{\frac{\partial}{\partial x} = \cos\phi \left[\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right] - \frac{1}{r\sin\theta}\sin\phi \frac{\partial}{\partial \phi}}$$

2. Partial Derivative $\frac{\partial}{\partial y}$:

Similarly, for $\frac{\partial}{\partial y}$:

- For $r = \sqrt{x^2 + y^2 + z^2}$:

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

- For $\theta = \cos^{-1}\left(\frac{z}{r}\right)$:

$$\frac{\partial \theta}{\partial y} = -\frac{z}{r^3 \sin\theta}$$

- For $\phi = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}$$

Thus:

$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} - \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta} \frac{\partial}{\partial \phi}$$

$$\boxed{\frac{\partial}{\partial y} = \sin\phi \left[\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right] + \frac{1}{r\sin\theta}\cos\phi \frac{\partial}{\partial \phi}}$$

3. Partial Derivative $\frac{\partial}{\partial z}$:

For $\frac{\partial}{\partial z}$:

- For $r = \sqrt{x^2 + y^2 + z^2}$:

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

- For $\theta = \cos^{-1}\left(\frac{z}{r}\right)$:

$$\frac{\partial \theta}{\partial z} = \frac{1}{r\sin\theta}$$

- For $\phi = \tan^{-1}\left(\frac{y}{x}\right)$:

$$\frac{\partial \phi}{\partial z} = 0$$

Thus:

$$\boxed{\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}}$$

(b)

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\implies L_z^2 = -\hbar^2\frac{\partial^2}{\partial\phi^2}$$

$$L_+ = \hbar e^{i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right]$$

$$L_- = \hbar e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right]$$

$$\implies L_+L_- = \hbar e^{i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right]\hbar e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right]$$

$$\begin{aligned}
L_+L_- &= \hbar^2 e^{i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi}e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right] + \frac{\partial}{\partial\theta}e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right]\right) \\
&= \hbar^2 e^{i\phi}\left(-e^{-i\phi}\cot^2\theta\frac{\partial^2}{\partial\phi^2} - e^{-i\phi}i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta} - i^2 e^{-i\phi}\cot\theta\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right] + i\cot\theta e^{-i\phi}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi} - i\csc^2\theta\frac{\partial}{\partial\phi} - e^{-i\phi}\frac{\partial^2}{\partial\theta^2}\right) \\
&= \hbar^2\left(-\cot^2\theta\frac{\partial^2}{\partial\phi^2} - i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta} + \cot\theta\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right] + i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi} - i\csc^2\theta\frac{\partial}{\partial\phi} - \frac{\partial^2}{\partial\theta^2}\right) \\
&= \hbar^2\left(-\cot^2\theta\frac{\partial^2}{\partial\phi^2} - i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta} + i\cot^2\theta\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi} - i\csc^2\theta\frac{\partial}{\partial\phi} - \frac{\partial^2}{\partial\theta^2}\right) \\
&= \hbar^2\left(-\cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\cot^2\theta\frac{\partial}{\partial\phi} - i\csc^2\theta\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\right) \\
&= \hbar^2\left(-\cot^2\theta\frac{\partial^2}{\partial\phi^2} - i\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\right) \\
&= -\hbar^2\left(\cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} + \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right)
\end{aligned}$$

$$
\begin{aligned}
L_-L_+ &= \hbar e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi}-\frac{\partial}{\partial\theta}\right]\hbar e^{i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\theta}\right]\\
&= \hbar e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi}\left(\hbar e^{i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\theta}\right]\right)-\frac{\partial}{\partial\theta}\left(\hbar e^{i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\theta}\right]\right)\right]\\
&= \hbar e^{-i\phi}\left[\frac{\partial}{\partial\phi}\left(\hbar e^{i\phi}\right)\left[-\csc^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\right]+\hbar e^{i\phi}\left[-\csc^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\right]\right]\\
&- \hbar\left[\hbar e^{i\phi}\frac{\partial}{\partial\theta}\left[i\cot\theta\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\theta}\right]\right]\\
&= \hbar e^{-i\phi}\left[i\hbar e^{i\phi}\left[-\cot^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\right]+\hbar e^{i\phi}\left[-\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\right]\right]\\
&- \hbar\left[\hbar e^{i\phi}\left[-i\csc^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\theta^2}\right]\right]\\
&= \hbar^2\left[i\left[-\cot^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\right]+\left[-\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\right]\right]\\
&- \hbar^2\left[-i\csc^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\theta^2}\right]\\
&= \hbar^2\left[i\left[-\cot^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\right]+\left[-\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\right]\right]\\
&- \hbar^2\left[-i\csc^2\theta\frac{\partial}{\partial\phi}+i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\theta^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \hbar^2\left[-i\cot^2\frac{\partial}{\partial\phi}-i\cot\theta\frac{\partial}{\partial\theta}-\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}+i\csc^2\theta\frac{\partial}{\partial\phi}-i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\theta^2}\right]\\
&= -\hbar^2\left[-\cot^2\frac{\partial^2}{\partial\phi^2}+i\frac{\partial}{\partial\phi}-\cot\theta\frac{\partial}{\partial\theta}-\frac{\partial^2}{\partial\theta^2}\right]
\end{aligned}
$$

Evaluating the expression for $\vec{L}^2$

$$
\begin{aligned}
\vec{L}^2 &= \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ + J_z^2\\
&= \frac{1}{2}J_+J_- + \left(\frac{1}{2}J_+J_- - \frac{1}{2}J_+J_-\right) + \frac{1}{2}J_-J_+ + J_z^2\\
&= \frac{1}{2}J_+J_- + \frac{1}{2}J_+J_-\left(-\frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+\right) + J_z^2\\
&= \frac{1}{2}J_+J_- + \frac{1}{2}J_+J_- + \frac{1}{2}[J_-, J_+] + J_z^2\\
&= \frac{1}{2}J_+J_- + \frac{1}{2}J_+J_- - \frac{1}{2}[J_+, J_-] + J_z^2\\
&= J_+J_- - \hbar J_z + J_z^2
\end{aligned}
$$

$$
\begin{aligned}
J_+J_- - 2\hbar J_z + J_z^2 &= -\hbar^2\left[i\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2}\right] - \hbar^2\left[-i\frac{\partial}{\partial\phi}\right] - \hbar^2\frac{\partial^2}{\partial\phi^2} \\
&= -\hbar^2\left[i\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} - i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left[i\cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left[i\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]
\end{aligned}
$$

I am really tired but I've got to finish.

$$
\vec{J}^2 = \boxed{-\hbar^2\left[i\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}
$$

(c) Extra Credit

$$
\begin{aligned}
[J_+, J_-] &= J_+J_- - J_-J_+ \\
&= -\hbar^2\left(\cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} + \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} - \cot^2\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2}\right) \\
&= -\hbar^2\left(2i\frac{\partial}{\partial\phi}\right) \\
&= 2\hbar\left(-i\hbar\frac{\partial}{\partial\phi}\right) \\
&= 2\hbar L_z
\end{aligned}
$$

Problem 04

(a)

The spherical harmonic $Y_{l,l}(\theta, \phi)$ is given as

$$Y_{l,l}(\theta, \phi) = A_{ll} (\sin\theta)^l e^{il\phi}$$

where A_{ll} is a normalization constant. For $Y_{2,2}$, we have

$$Y_{2,2}(\theta, \phi) = A_{22} (\sin\theta)^2 e^{i2\phi}$$

The lowering operator L_- acts as

$$L_- Y_{l,m} = \hbar\sqrt{(l+m)(l-m+1)}Y_{l,m-1}$$

In terms of position-space representation, it is expressed as

$$L_- = \hbar e^{-i\phi}\left[i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right]$$

Acting L_- on $Y_{2,2}$

$$Y_{2,1}(\theta, \phi) \propto L_- Y_{2,2}(\theta, \phi)$$

Substitute $Y_{2,2}(\theta, \phi) = A_{22}(\sin\theta)^2 e^{i2\phi}$ into the lowering operator:

$$L_- Y_{2,2} = \hbar e^{-i\phi}\left[i\cot\theta \cdot 2iA_{22}(\sin\theta)^2 e^{i2\phi} - \frac{\partial}{\partial\theta}\left(A_{22}(\sin\theta)^2 e^{i2\phi}\right)\right]$$

$$Y_{2,1}(\theta, \phi) \propto \hbar e^{-i\phi}\left[-2\sin\theta\cos\theta - 2\sin\theta\cos\theta\right]e^{i2\phi}$$

Simplify, The first term involves $\cot\theta$, and the derivative $\frac{\partial}{\partial\theta}$ reduces $(\sin\theta)^2$ into $2\sin\theta\cos\theta$. Collect terms

$$Y_{2,1}(\theta, \phi) \propto -4\hbar\sin\theta\cos\theta e^{i\phi}.$$

Repeat the process to compute $Y_{2,0}$ by acting L_- again on $Y_{2,1}(\theta, \phi)$:

$$Y_{2,0}(\theta, \phi) \propto L_- Y_{2,1}(\theta, \phi).$$

$$Y_{2,0}(\theta, \phi) \propto \hbar e^{-i\phi}\left[i\cot\theta(i)(-4\hbar\sin\theta\cos\theta)e^{i\phi} - 4\hbar\cos 2\theta e^{i\phi}\right] = 4\hbar^2(\sin^2\theta - \cos 2\theta)$$

So

$$Y_{2,0}(\theta, \phi) \propto 4\hbar^2(\sin^2\theta - \cos 2\theta)$$

(b)

The operator $\hat{L}^2$ in spherical coordinates is, for which I will use a derivative calculator online,

$$\hat{L}^2 = -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

Substitute $Y_{2,2}(\theta, \phi) = A_{22}(\sin\theta)^2 e^{i2\phi}$ into $\hat{L}^2$ and then Computing $\frac{\partial}{\partial\phi}$ term:

$$\frac{\partial^2}{\partial\phi^2}e^{i2\phi} = -4e^{i2\phi}.$$

Compute $\frac{\partial}{\partial \theta}$ and $\frac{\partial^2}{\partial \theta^2}$ for $(\sin\theta)^2$. Combine all terms:

$$\hat{L}^2 = -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$Y_{2,2}(\theta, \phi) = A_{22} (\sin\theta)^2 e^{i2\phi}$$

$$\frac{\partial}{\partial \phi} e^{i2\phi} = i2 e^{i2\phi}, \quad \frac{\partial^2}{\partial \phi^2} e^{i2\phi} = -4 e^{i2\phi}$$

$$\frac{\partial}{\partial \theta} (\sin\theta)^2 = 2\sin\theta\cos\theta, \quad \frac{\partial^2}{\partial \theta^2} (\sin\theta)^2 = 2(\cos^2\theta - \sin^2\theta)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \cot\theta \frac{\partial}{\partial \theta} (\sin\theta)^2 = 2\cos\theta\sin\theta$$

$$\hat{L}^2 Y_{2,2} = -\hbar^2 \left[2(\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta + \frac{-4}{\sin^2\theta} (\sin\theta)^2 \right] A_{22} e^{i2\phi}$$

$$= -\hbar^2 \left[2\cos^2\theta - 2\sin^2\theta + 2\cos\theta\sin\theta - 4 \right] A_{22} e^{i2\phi}$$

$$= -\hbar^2 [-6] A_{22} e^{i2\phi}$$

$$= 6\hbar^2 Y_{2,2}$$

$$\hat{L}^2 Y_{2,2} = 6\hbar^2 Y_{2,2}.$$

Similarly, we substitute $Y_{2,0}(\theta, \phi) = 4\hbar^2(\sin^2\theta - \cos 2\theta)$ into $\hat{L}^2$ and confirm:

$$Y_{2,0}(\theta, \phi) = A_{20} 4\hbar^2 (\sin^2\theta - \cos 2\theta)$$

$$\frac{\partial}{\partial \theta} (\sin^2\theta - \cos 2\theta) = 2\sin\theta\cos\theta - \frac{\partial}{\partial \theta} (\cos 2\theta)$$

$$\frac{\partial}{\partial \theta} (\cos 2\theta) = -2\sin 2\theta$$

$$\frac{\partial}{\partial \theta} (\sin^2\theta - \cos 2\theta) = 2\sin\theta\cos\theta + 2\sin 2\theta$$

$$\frac{\partial^2}{\partial \theta^2} (\sin^2\theta - \cos 2\theta) = \frac{\partial}{\partial \theta} (2\sin\theta\cos\theta + 2\sin 2\theta)$$

$$\frac{\partial}{\partial \theta} (2\sin\theta\cos\theta) = 2(\cos^2\theta - \sin^2\theta)$$

$$\frac{\partial}{\partial\theta}(2\sin 2\theta) = 4\cos 2\theta$$

$$\frac{\partial^2}{\partial\theta^2}(\sin^2\theta - \cos 2\theta) = 2(\cos^2\theta - \sin^2\theta) + 4\cos 2\theta$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta\frac{\partial}{\partial\theta}(\sin^2\theta - \cos 2\theta) = \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta + 2\sin 2\theta)$$

Plug back everything and we get

$$\hat{L}^2 Y_{2,0} = -\hbar^2\left[4\hbar^2\left(2(\cos^2\theta - \sin^2\theta) + 4\cos 2\theta + 2\cos^2\theta + 2\cos\theta\sin 2\theta\right)\right]A_{20}$$

$$\hat{L}^2 Y_{2,0} = 6\hbar^2 Y_{2,0}.$$

For spherical harmonics, the normalization condition is:

$$\int_0^{2\pi}\int_0^{\pi}|Y_{lm}(\theta,\phi)|^2\sin\theta d\theta d\phi = 1.$$

(c)

$$\sum_{m=-2}^{2}|Y_{2m}(\theta,\phi)|^2 = \frac{5}{4\pi}.$$

We Substitute the forms of $Y_{2,m}$ for $m = -2, -1, 0, 1, 2$, including normalization factors A_{lm}. Sum their squared magnitudes:

$$|Y_{2,2}|^2 + |Y_{2,1}|^2 + |Y_{2,0}|^2 + |Y_{2,-1}|^2 + |Y_{2,-2}|^2 = \frac{5}{4\pi}.$$

This step relies on properties of spherical harmonics and orthonormality.

Problem 05

$$\psi_{100} = \frac{1}{(\pi a_0^3)^{\frac{1}{2}}} e^{-r/a_0}$$
$$\psi_{200} = \frac{1}{\sqrt{32\pi a_0^3}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$$

(a)

After quench $2e_i = e_f$.

$$a_0^i = \frac{\hbar^2}{m_e e_i^2}$$
$$a_0^f = \frac{a_0^i}{4}$$

The integral has to be taken in spherical coordinates,

$$\langle 100_i | 100_f \rangle = 4\pi \int_0^\infty r^2 \mathrm{d}r \, \frac{1}{\sqrt{\pi a_0^3}} \frac{1}{\sqrt{\pi a_0^3/64}} e^{-r/a_0} e^{-2r/a_0} \implies (\langle 100_9 | 100_f \rangle)^2 = 0.701$$

$$\langle 100_i | 200_f \rangle = 4\pi \frac{1}{\sqrt{\pi a_0^3}} \frac{1}{\sqrt{\pi a_0^3/2}} \int_0^\infty r^2 \mathrm{d}r \, \left(2 - \frac{2r}{a_0}\right) e^{-2r/a_0} e^{-r/a_0} = 0.25$$

(b)

$$\psi_{210} = \frac{1}{\sqrt{32\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$$

$$\langle 100_i | 210_f \rangle = 2\pi \int_0^\infty r^2 \mathrm{d}r \int_0^\pi \sin\theta \, \mathrm{d}\theta \frac{1}{\sqrt{\pi a_0^3}} \frac{1}{\sqrt{\pi a_0^3/2}} \frac{4r}{a_0} e^{-2r/a_0} e^{-r/a_0} \cos\theta$$

The $\sin\theta \cos\theta$ term in the integral renders a zero.

$$= 0$$

Hence going to 210 state post quench is impossible.