Honors Linear Algebra : : Homework 02

January 25, 2024

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1 Problem 01

Consider the base case. If $\vec{u}_1$ is a member of U, then $\lambda_1 \vec{u}_1$ is a member either. So this is for $\mu = 1$. Let's consider the case μ. Then $\vec{u}_\mu$ is a vector member, and hence is $\lambda_\mu \vec{u}_\mu$ Consider the case $\lambda_{\mu+1}\vec{u}_{\mu+1}$. Let's add the two vectors, which is a member of the subspace,

$$\lambda_\mu \vec{u}_\mu + \lambda_{\mu+1}\vec{u}_{\mu+1} = \vec{A}_\mu \in U$$

This being a member of the subspace let's try adding the next term too,

$$\vec{A}_\mu + \lambda_{\mu+2}\vec{u}_{\mu+2} \in U$$

Two vectors being a member, well adding the third one works too. Hence the induction is true for the whole series.

$$\lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2 + \cdots + \lambda_m \vec{u}_m \in U$$

2 Problem

(a) Let's assume $(x_1, x_2, x_3) \in \mathbb{F}^3$ where $\mathbb{F}^3$ is either $\mathbb{R}^3$ or $\mathbb{C}^3$. $0 \in U$ for this case $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. We can take two solutions $\vec{x}$ and $\vec{x}'$ which still satisfy additive closure.

$$(x_1 + x_1') + 2(x_2 + x_2') + 3(x_3 + x_3') = 0$$

This is also closed multiplication closure.

$$\lambda x_1 + 2\lambda x_2 + 3\lambda x_3 = 0$$

3 Problem

Define the "zero function" in $\mathbb{R}^{[0,1]}$ as $z : [0,1] \to \mathbb{R}$ such that $z(x) = 0$ for all $x \in [0,1]$. For $z \in U$, z being continuous, which it is, and $\int_0^1 z = b$. Since $\int_0^1 z = \int_0^1 0 = 0$, the latter is true if and only if $b = 0$. Thus, $0 \in U$ if and only if $b = 0$.

Let's consider the two functions f and g, it's closed under addition

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0$$

It's weird to take integral without a dx.

Multiplicative closure is trivial

$$c\int_0^1 f = 0 = \int_0^1 cf$$

4 Problem

$0 \in U$ with $(0, 0, 0)$. This satisfies the conditions.

Additive closure $(a, b, c) + (a', b', c') = (a + a', b + b', c + c')$ with conditions $a^3 = b^3$ and $a'^3 = b'^3$. So, this is $(a+a', a+a', c+c')$. But then $(a+a') = (a+a')$ so this is still a member of U. Thus additive closure is achieved.

Multiplicative closure well $(a, b, c) \to (ka, kb, kc)$, then $(ka)^3 = (kb)^3$ with $k^3(a^3) = k^3(b^3)$. This is still a member of the subspace.

5 Problem

$0 \in U$ because $(0, 0, 0)$ is valid solution.

Additive closure can't perfectly work because for $\mathbb{C}$ $a^3 = b^3; a \neq b$. This is not closed so this is not a subspace.

6 Problem

Both subset must have $\{0\}$.

Additive closure, well two vectors $u, v \in V_1 \cap V_2$, so the vectors exist in both subspaces. Now, because it's a subspace, and both have u, v, then both should also have $u + v$ in common. Similarly all linear combination must be common in a way. The summation can be extended to a sum.

Multiplicative closure is also true because if they contain v, then they also contain all λ such that λv.

So this intersection is a subspace too.

7 Problem

Let's consider two set V and W. Let each subspaces contain vectors that are not member of the other. The can have vector v and w, and their combinations do NOT exist in the union because they are different separate vectors. So we can't have it as a subspace.

So we can't have pairs of vectors that are not common in both sets, hence either the two subsets should be exactly similar or they have to contain the other.

8 Problem

$0 \in U$ so $0 \in U + U$. Consider two vectors in U such that $\vec{u}_1$ and $\vec{u}_2$. The summation must be a member of the U itself, and also $ku_1, ku_2 \in U$ hence $U + U$ is also a subspace.

9 Problem

Let V_1 and V_2 such that there are some vectors in V_1 that don't belong in V_2. Then $V_1 + U$ is going to have vectors that don't belong in $V_2 + U$ so they can't be equal. So $V_1 = V_2$.

10 Problem

Let's consider a subspace $(x, x, 0, 0)$, this is a subspace because it contains $(0, 0, 0, 0)$ and $(x, x, 0, 0) + (y, y, 0, 0) = (x + y, x + y, 0, 0)$ which is a member of the subspace. $\lambda(x, x, 0, 0)$ is $(\lambda x, \lambda x, 0, 0)$ which is a subspace member too. Similarly $(0, 0, m, m)$ is a subspace too.

The direct sum would be with $x, m \in \mathbb{F}$ is

$$(x, x, 0, 0) \oplus (0, 0, m, m) = (x, x, m, m)$$

This is a subspace direct sum because $(x, x, 0, 0) \cap (0, 0, m, m) = \{0\}$