Quantum Mechanics Homework 01

September 7, 2024

Ahmed Saad Sabit, Rice University

Problem 1 (a)

Studying the problem

The initial configuration of the string

$$q(x,0) = f(x) = \begin{cases} \frac{2h}{L}x & 0 \leq x \leq \frac{L}{2} \\ 2h - \frac{2h}{L}x & \frac{L}{2} \leq x \leq L \end{cases}$$

$$\frac{\partial q(x,t)}{\partial t}\bigg|_{t=0} = \sum_{n=1}^{\infty} d_n \Omega_n \phi_n(x) = g(x) = 0 \implies \boxed{d_n = 0}$$

The general solution to the string equation (assumed solution is separable between time and position)

$$q(x,t) = \sum_{n=1}^{\infty} [c_n \cos(\Omega_n t) + d_n \sin(\Omega_n t)] \phi_n(x)$$

For $t = 0$ we get,

$$q(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

We are interested on finding the general solution of $q(x,t)$ that will hold for the future given this initial condition. The variables of our equation are obviously x, t and what we need to find out is c_n, d_n. The next sub-section will find out a solution for c_n (d_n is trivially zero given zero initial velocity).

Solving for c_n

Let us do the following computation now. Let us multiply both sides of the above equation with $\phi_p(x)$ where p represents the p-th term while we take a summation over the index of n.

$$f(x)\phi_p(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \phi_p(x)$$

Just so that we can invoke the inner product between orthonormal bases, we can take an integral with the following way

$$\begin{aligned} \int_0^L \mathrm{d}x\, f(x)\phi_p(x) &= \int_0^L \mathrm{d}x \left(\sum_{n=1}^{\infty} c_n \phi_n(x) \phi_p(x) \right) \\ &= \sum_{n=1}^{\infty} c_n \int_0^L \mathrm{d}x\, \phi_n(x)\phi_p(x) \\ &= \sum_{n=1}^{\infty} c_n \delta_{np} \frac{L}{2} \\ &= c_p \frac{L}{2} \end{aligned}$$

This above gives us the p-th term

$$c_p = \frac{2}{L}\int_0^L \mathrm{d}x\, f(x)\phi_p(x)$$

Using the explicit equation for the bases and also looking at the piecewise function, we can write,

$$\begin{aligned}
c_p &= \frac{2}{L}\int_0^L \mathrm{d}x\, f(x)\sin\left(\frac{p\pi x}{L}\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{p\pi x}{L}\right) + \int_{\frac{L}{2}}^{L} f(x)\sin\left(\frac{p\pi x}{L}\right)\right) \\
&= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2h}{L}x\sin\left(\frac{p\pi x}{L}\right) + \int_{\frac{L}{2}}^{L}\left(2h - \frac{2h}{L}x\right)\sin\left(\frac{p\pi x}{L}\right)\right) \\
&= \frac{2}{L}\left(\frac{hL}{\pi^2p^2}\left[2\sin\left(p\frac{\pi}{2}\right) - \pi p\cos\left(p\frac{\pi}{2}\right)\right] - \frac{hL}{\pi^2p^2}\left[2\sin(\pi p) - 2\sin\left(p\frac{\pi}{2}\right) - \pi p\cos\left(p\frac{\pi}{2}\right)\right]\right) \\
&= \frac{8h}{\pi^2p^2}\sin\left(p\frac{\pi}{2}\right)\left[1 - \cos\left(p\frac{\pi}{2}\right)\right]
\end{aligned}$$

Hence if I write this huge mess properly

$$\boxed{c_p = \frac{8h}{\pi^2p^2}\sin\left(\frac{p\pi}{2}\right)\left[1 - \cos\left(\frac{p\pi}{2}\right)\right]}$$

Discussion on odd and even modes

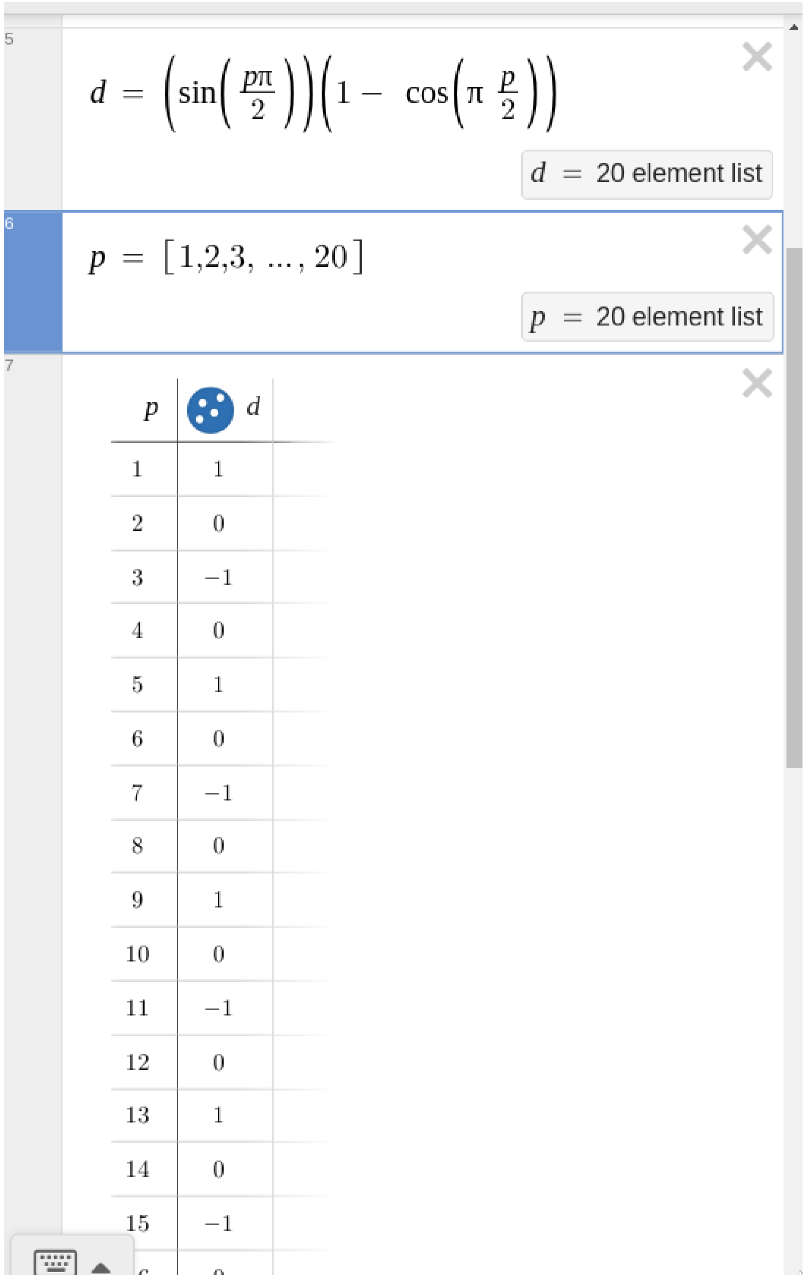

Figure 1: ss/graph-cn.png

We can see that every *even* numbers is giving us a 0 for c_p (d term in graph). The *odd* terms alter between -1 and 1. The contribution of even and odd modes is obvious from here.

Plotting the displacement with time

Let's set $L = 1$ and we will look at $0 \leq x \leq 1$.

As given $\Omega_n = k_n = \frac{n\pi}{L} = n\pi$.

For h let's pick $h = 0.4$.

Using hte c_p we derived above, the displacement function $q(x,t)$ is then,

$$q(x,t)_{(10)} = \sum_{n=1}^{10} c_n \cos(n\pi t)\sin(n\pi x)$$

Expanding c_n,

$$q(x,t)_{(10)} = \frac{8(0.4)}{\pi^2 n^2} \sum_{n=1}^{10} \sin\left(\frac{n\pi}{2}\right)\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] \cos\left(n\pi t\right)\sin(n\pi x)$$

And $\tau = \frac{2\pi}{\Omega_1} = \frac{2\pi}{\pi} = 2$

Plots of $q(x,t)$ where $\tau = 2$

$$q(x,t)_{(10)} = \frac{8(0.4)}{\pi^2}\sum_{n=1}^{10}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(n\pi t\right)\sin(n\pi x) \quad \text{and } x \in [0,1]$$

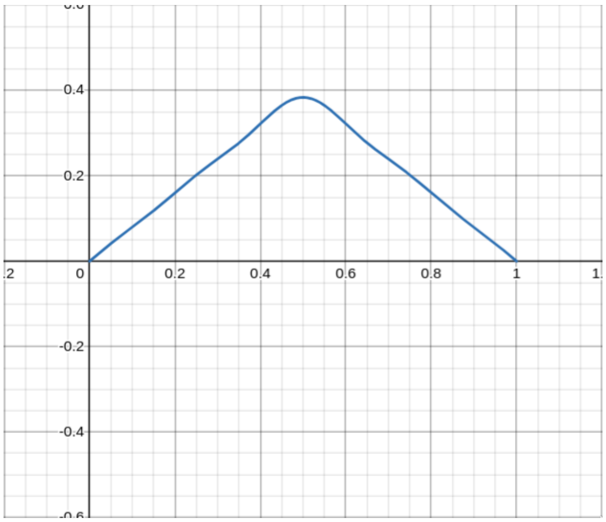

Figure 2: $t = 0$ and also $t = \tau$

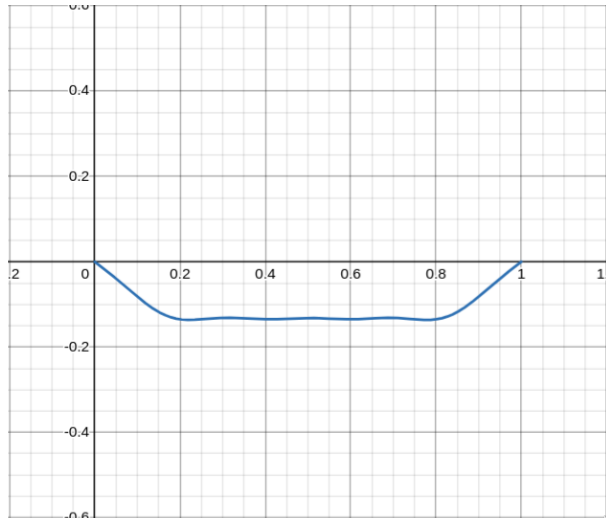

Figure 4: $t = \frac{\tau}{3} = \frac{2}{3}$

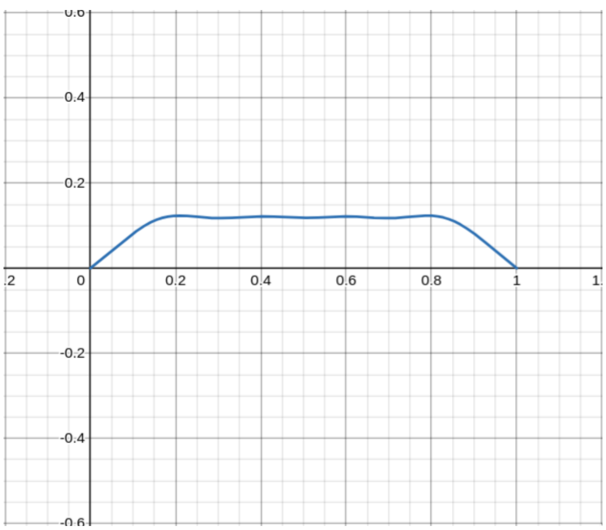

Figure 3: $t = 0.35$

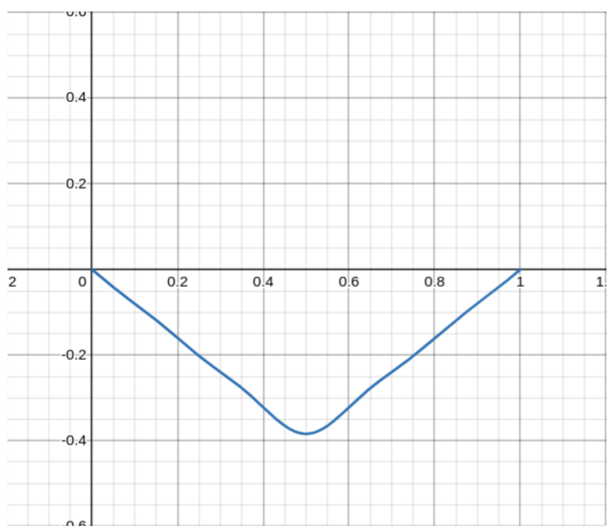

Figure 5: $t = \frac{\tau}{2} = 1$

Problem 1 (b)

Studying the Problem

Initially the string is tight and straight hence

$$q(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x) = f(x) = 0 \implies \boxed{c_n = 0}$$

$$\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} = g(x) = v_0 \theta \left(a - \left| x - \frac{L}{2} \right| \right)$$

Where $\theta(x)$ is a Heaviside step function (outputs 1 whenever input is 0 or positive). I am not going to waste my and graders time by re-writing everything I wrote above, the procedure we are going to follow is same as above.

Computation of d_n

$$\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} d_n \Omega_n \phi_n(x) \\
\int_0^L \mathrm{d}x\, g(x)\phi_p(x) &= \sum_{n=1}^{\infty} \int_0^L \mathrm{d}x\, d_n \Omega_n \phi_n(x)\phi_p(x) \\
\int_{\frac{L}{2}-a}^{\frac{L}{2}+a} v_0 \phi_p(x) &= d_p \Omega_p \frac{L}{2} \\
\int_{\frac{L}{2}-a}^{\frac{L}{2}+a} v_0 \sin(k_p x) &= d_p k_p \frac{L}{2} \\
\frac{v_0}{k_p} \left[-\cos(k_p x) \right]_{x=\frac{L}{2}-a}^{x=\frac{L}{2}+a} &= d_p k_p \frac{L}{2} \\
\cos\left(k_p \frac{L}{2} - k_p a \right) - \cos\left(k_p \frac{L}{2} + k_p a \right) &= \frac{d_p k_p^2 L}{2v_0} \\
\cos\left(\frac{n\pi}{2} - \frac{n\pi a}{L} \right) - \cos\left(\frac{n\pi}{2} + \frac{n\pi a}{L} \right) &= \frac{d_p n^2 \pi^2}{2 v_0 L}
\end{aligned}$$

This gives us

$$d_p = \frac{2v_0 L}{n^2\pi^2} \left[\cos\left(\frac{n\pi}{2} - \frac{n\pi a}{L} \right) - \cos\left(\frac{n\pi}{2} + \frac{n\pi a}{L} \right) \right] = \frac{4 v_0 L}{n^2 \pi^2} \sin\left(\frac{n\pi}{2} \right) \sin\left(\frac{n\pi a}{L} \right)$$

Discussion on Even and Odd modes

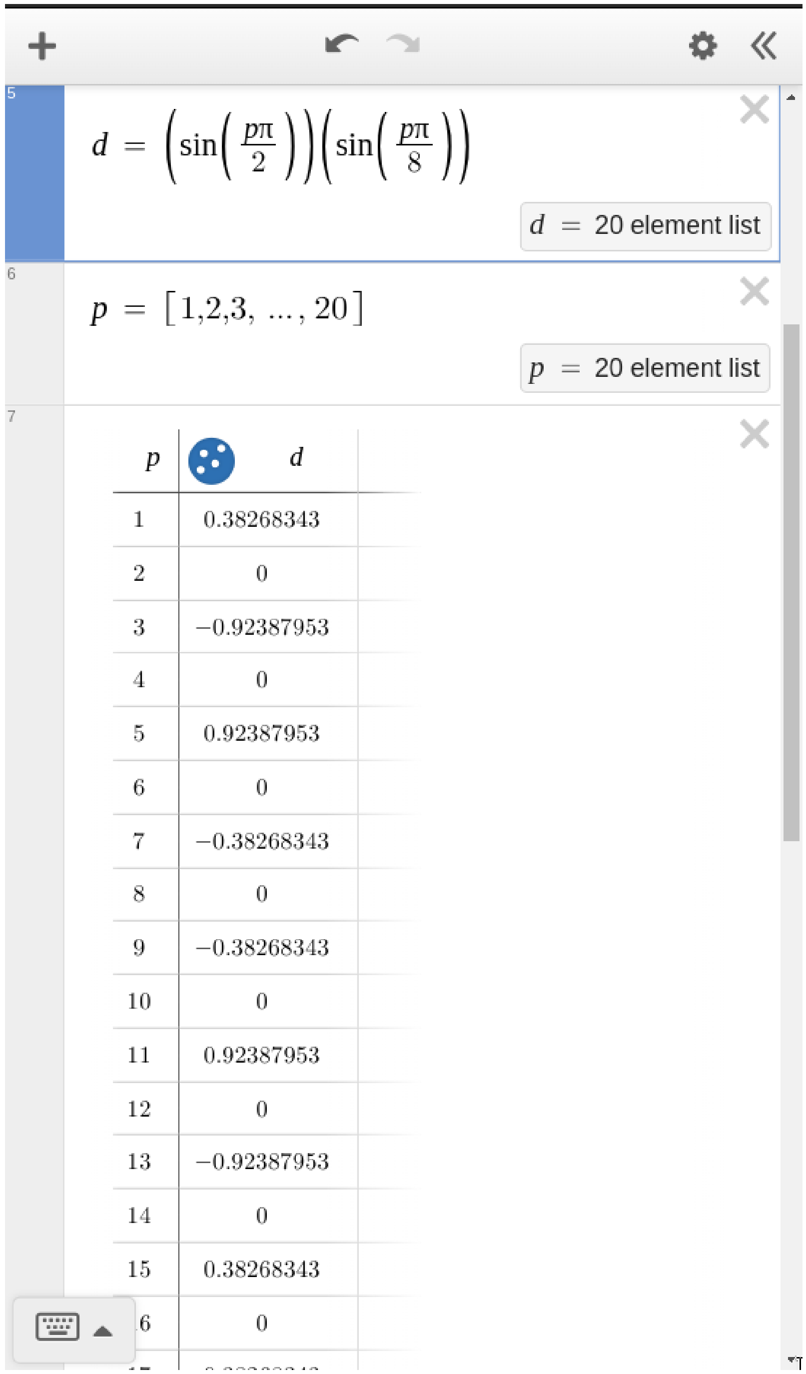

Figure 6: ss/graph-dn

Like before we can see every *even* modes are 0.

Plotting the displacement with time

Let's set $L = 1$, then $a = 1/8$. Let's set $v_0 = 1.5$. As before $\tau = 2$.

Plots of $q(x,t)$ where $\tau = 2$

$$q(x,t)_{(10)} = \frac{4(1.5)}{\pi^2} \sum_{n=1}^{10} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(n\pi(1/8)\right) \sin\left(n\pi t\right) \sin\left(n\pi x\right) \quad \text{and } x \in [0,1]$$

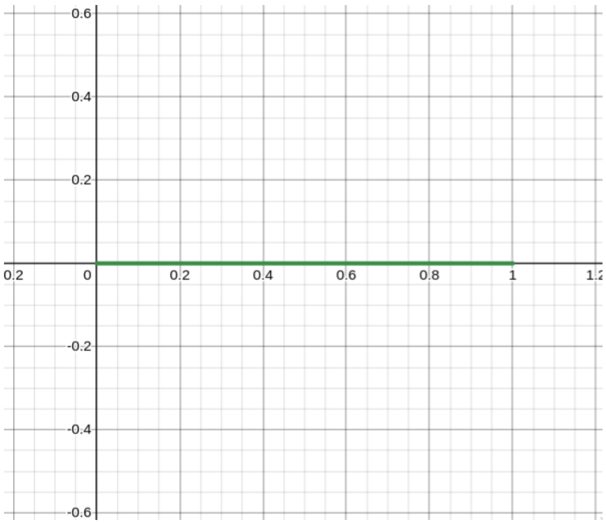

Figure 7: $t = 0$ and also $t = \tau$

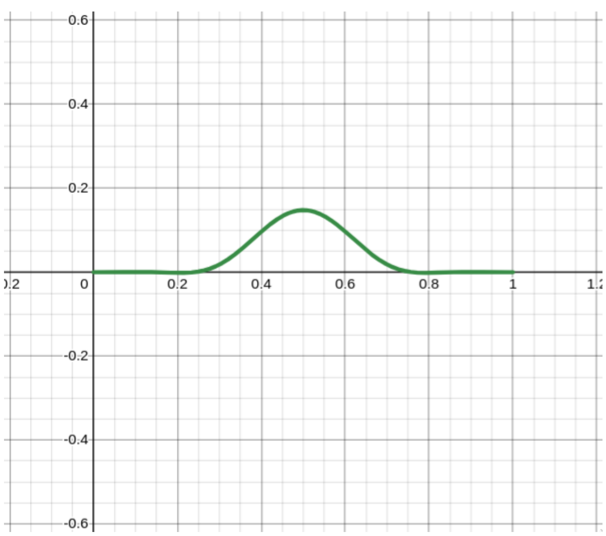

Figure 8: $t = \frac{\tau}{20} = 0.1$

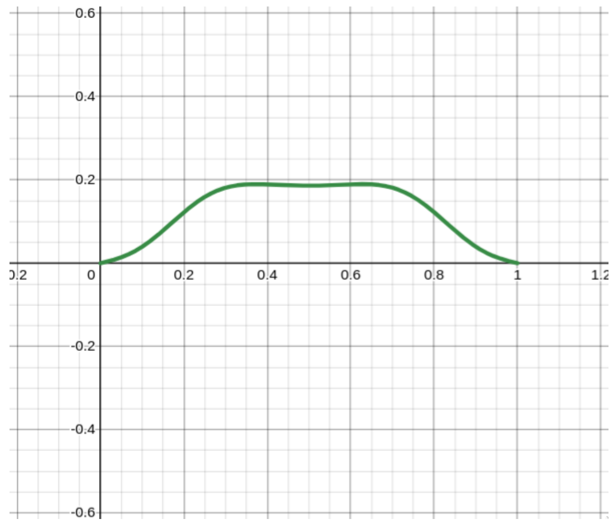

Figure 9: $t = \frac{\tau}{3} = \frac{2}{3}$

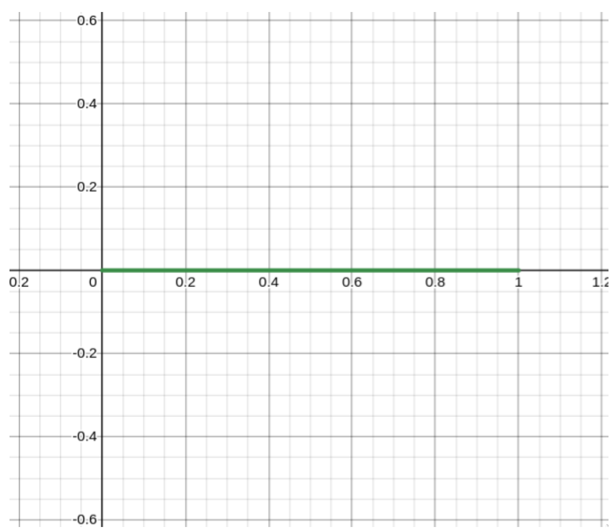

Figure 10: $t = \frac{\tau}{2} = 1$

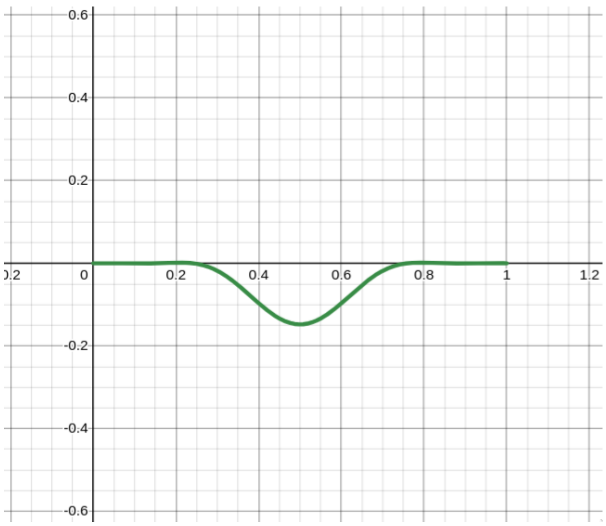

Figure 11: $t = 1.1$

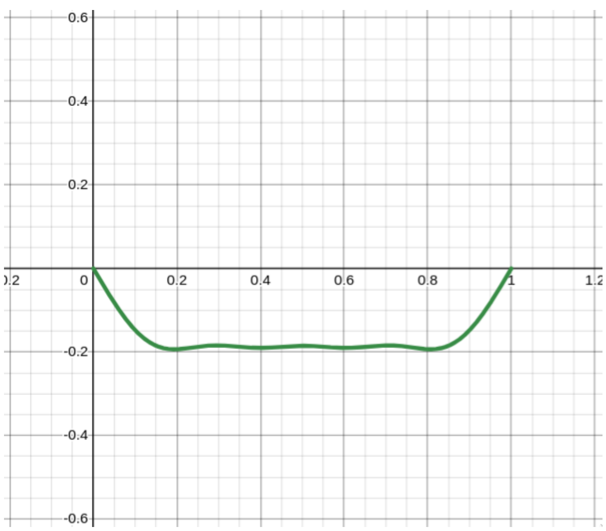

Figure 12: $t = 1.5$

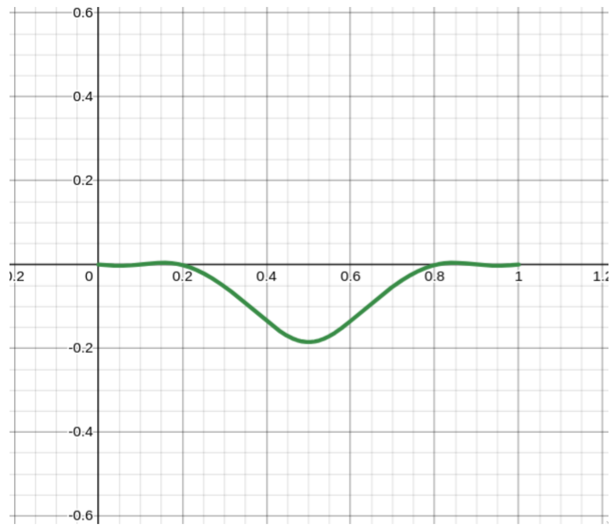

Figure 13: $t = 1.85$

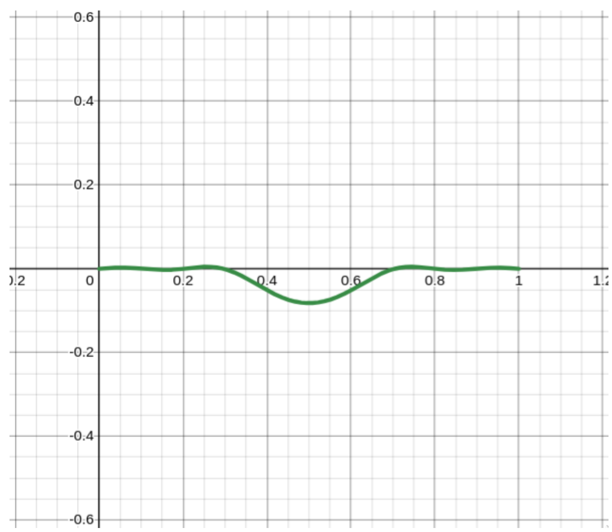

Figure 14: $t = 1.95$

Plotting the Initial Velocity

We can simply take the time derivative and make a plot. The more iterations we make, we should get close to a heaviside function as stated in the initial velocity profile.

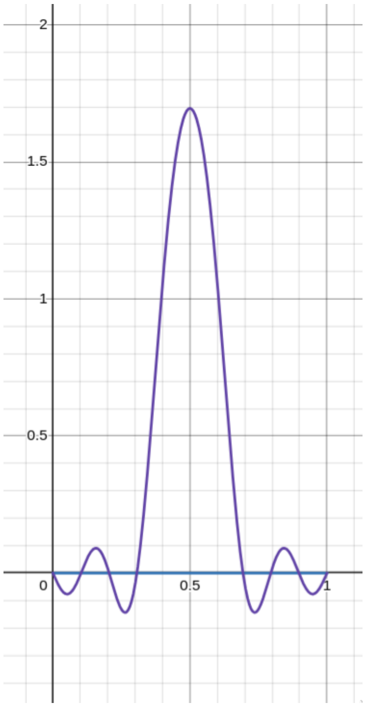

Figure 15: Plotted for $n = 10$

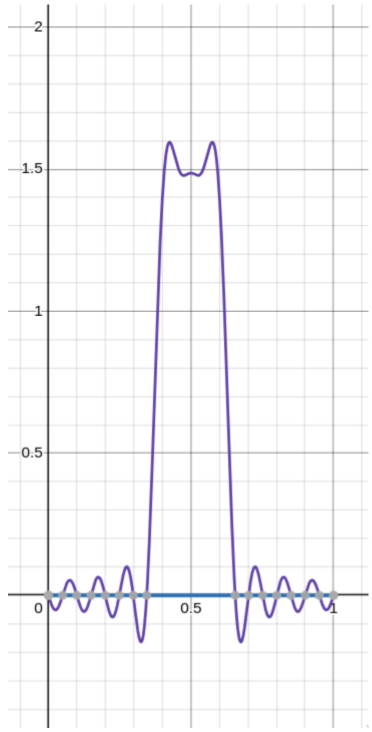

Figure 16: Plotted for $n = 20$

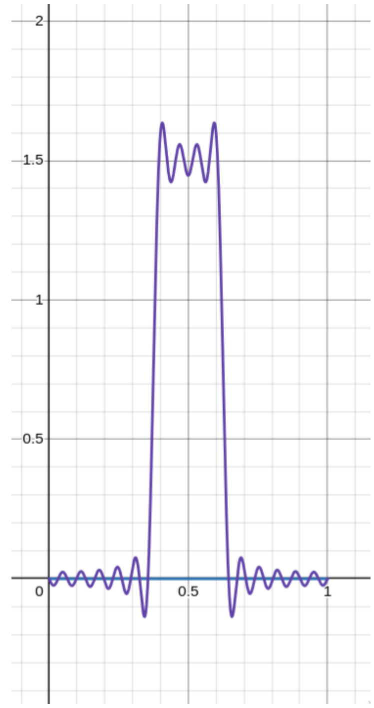

Figure 17: Plotted for $n = 30$

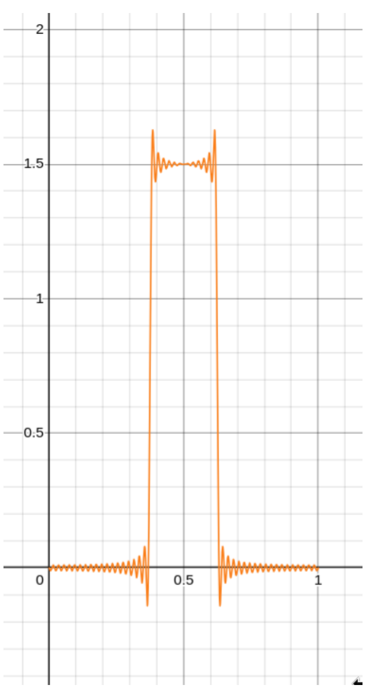

Figure 18: Plotted for $n = 100$

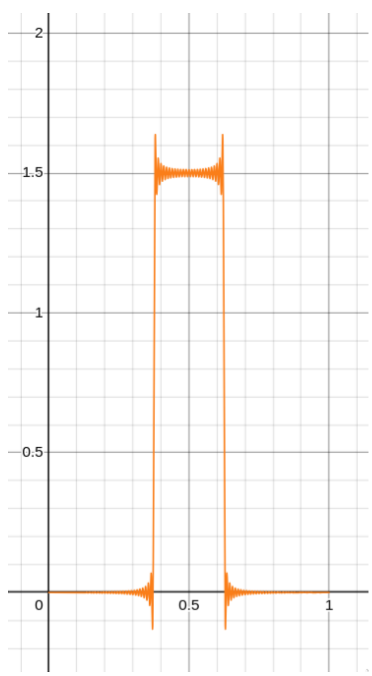

Figure 19: Plotted for $n = 200$

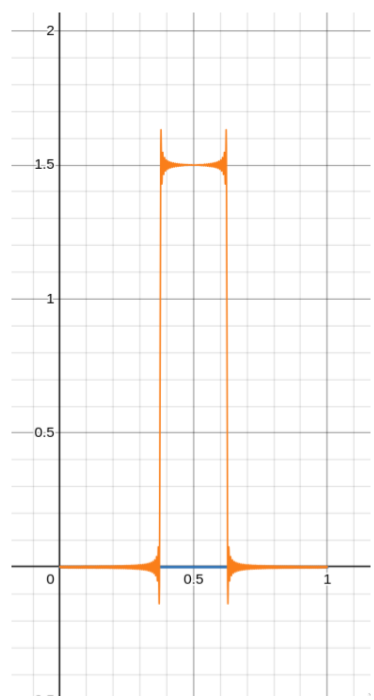

Figure 20: Plotted for $n = 300$

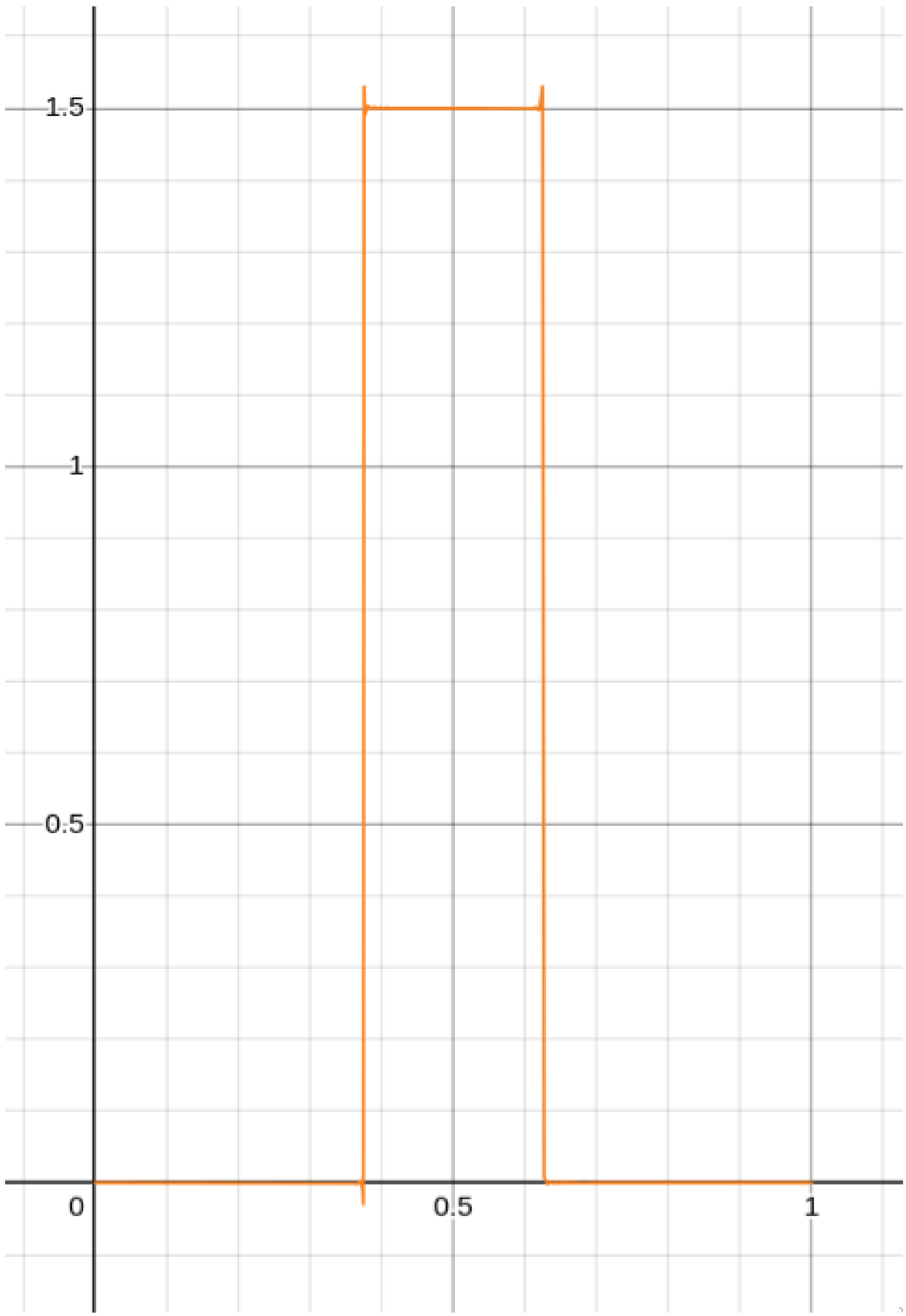

Figure 21: Waited 2 minutes to try $n = 10,000$. We have the exact initial conditions we wanted. This is absolutely beautiful to do this myself.

Problem 2

Linear Dependence equation

$$a\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}+b\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}+c\begin{pmatrix}-2 & -1\\0 & -2\end{pmatrix}=0$$

If the three above equations are linearly independent we are going to have the only possible solution of the equality being zero as $\{a,b,c\}=\{0,0,0\}$. We get three different equations

$$\begin{aligned}b-2c&=0\\a+b-c&=0\\b-2c&=0\\0+0&=0\end{aligned}$$

Where the I didn't count the last one as it's not really anything. If we solve the system of equation above we are left with (did the computation by hand on paper)

$$\begin{aligned}a&=-c\\b&=2c\\c&=\text{can be anything}\end{aligned}$$

a,b,c doesn't necessarily need to be 0 to yield above equation to be zero. This system is *NOT linearly independent.* For example, setting $c=1$

$$-\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}+2\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}+\begin{pmatrix}-2 & -1\\0 & -2\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

Problem 3

Deduction

$$\begin{aligned}|v+w|^2&=\langle v+w|v+w\rangle\\&=\langle v|v+w\rangle+\langle w|v+w\rangle\\&=\langle v|v\rangle+\langle v|w\rangle+\langle w|v\rangle+\langle w|w\rangle\\&=|v|^2+|w|^2+\langle v|w\rangle+\langle v|w\rangle^*\\&=|v|^2+|w|^2+2\text{Re}\left(\langle v|w\rangle\right)\\&\leq|v|^2+|w|^2+2|\langle v|w\rangle|\\&\leq|v|^2+|w|^2+2|v||w|\\&\leq\left(|v|+|w|\right)^2\end{aligned}$$

This shows that

$$|v+w|\leq|v|+|w|$$

Equality condition

Above we can see

$$\begin{aligned}|v+w|^2&=\langle v+w|v+w\rangle\\&=|v|^2+|w|^2+2\text{Re}\left(\langle v|w\rangle\right)\end{aligned}$$

For the equality to hold we strictly need

$$2\mathrm{Re}\left(\langle v|w\rangle\right) = 2|v||w|$$

Let's try $|v\rangle = a|w\rangle$.

$$2\mathrm{Re}\left(\langle aw|w\rangle\right) = 2a\mathrm{Re}\left(\langle w|w\rangle\right) = 2|aw||w| = 2a|w||w|$$

$|v\rangle = a|w\rangle$ satisfies this. If $|v\rangle$ has component perpendicular to $|w\rangle$ (in a way that $|v\rangle$ can be broken down into constituent factors of basis vectors such that inner product with $|w\rangle$ gives zero), then some of the value of 2Re($\langle v|w\rangle$) is getting lost. Having the two vectors parallel gives equal to their norm. This is more of a physical vector-like intuition.

Problem 4(a)

I will write $\hat{\sigma}^n$ as simply σ^n for this problem. I did the multiplication by hand.

$$(\sigma^1)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^2)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^3)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We had been already defined

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All of these result matrix above is the identity matrix that validates

$$(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \sigma^0$$

Problem 4(b)

We are required to solve for $A^{\mu,\nu}$ where

$$A^{\mu,\nu} = \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu$$

Note that it is obvious

$$A^{\mu,\nu} = A^{\nu,\mu}$$

Computing each of the matrix multiplications, and also referring to previous computations

$$A^{k,k} = A^{1,1} = A^{2,2} = A^{3,3} = 2(\sigma^k)^2 = 2\sigma^0 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} A^{1,2} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$A^{2,3} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^{1,3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Different indexes cause a zero-matrix, and similar causes a double of identity matrix. From here we can easily figure out that

$$A^{\mu,\nu} = 2\sigma^0 \delta_{\mu,\nu}$$

Problem 4(c)

I am going to borrow the computations I did last problem

$$\mathsf{Tr}[\sigma^1\sigma^1] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^2\sigma^2] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^3\sigma^3] = \mathsf{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\mathsf{Tr}[\sigma^1\sigma^2] = \mathsf{Tr} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^2\sigma^1]$$

$$\mathsf{Tr}[\sigma^2\sigma^3] = \mathsf{Tr} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^3\sigma^2]$$

$$\mathsf{Tr}[\sigma^1\sigma^3] = \mathsf{Tr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 = \mathsf{Tr}[\sigma^3\sigma^1]$$

From this we can see that same-index gives 2 and different gives 0. From this it's obvious

$$\mathsf{Tr}[\sigma^\mu\sigma^\nu] = 2\delta_{\mu\nu}$$

Problem 4(d)

Expanding the equation of the operator

$$\hat{V} = \sum_{i=1}^{3} V_i \sigma^i = V_1 \sigma^1 + V_2 \sigma^2 + V_3 \sigma^3$$

Multiply σ^p where $p \in \{1, 2, 3\}$

$$\hat{V}\sigma^p = V_1 \sigma^1 \sigma^p + V_2 \sigma^2 \sigma^p + V_3 \sigma^3 \sigma^p$$

Taking the trace and using the property $\mathsf{Tr}(A + B) = \mathsf{Tr}(A) + \mathsf{Tr}(B)$

$$\mathsf{Tr}(\hat{V}\sigma^p) = V_1 \left(2\delta_{1p}\right) + V_2 \left(2\delta_{2p}\right) + V_3 \left(2\delta_{3p}\right)$$

From this using the definition of the $\delta_{\mu,p}$ we can simply write,

$$V_p = \frac{1}{2}\mathsf{Tr}(\hat{V}\sigma^p)$$

Using the form

$$\hat{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we can find the coefficients V_p

$$\hat{V}\sigma^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies V_0 = \frac{a+d}{2}$$

$$\hat{V}\sigma^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \implies V_1 = \frac{b+c}{2}$$

$$\hat{V}\sigma^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} ib & -ia \\ id & -ic \end{pmatrix} \implies V_2 = i\frac{b-c}{2}$$

$$\hat{V}\sigma^3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \implies V_3 = \frac{a-d}{2}$$

So our representation is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{ib-ic}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I've checked this in Wolfram Alpha and it seems to work.

Problem 5(a)

From the question, my understanding of the basis is

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Projection operator

$$P_1 = |1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_2 = |2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We've determined the elements of the matrices.

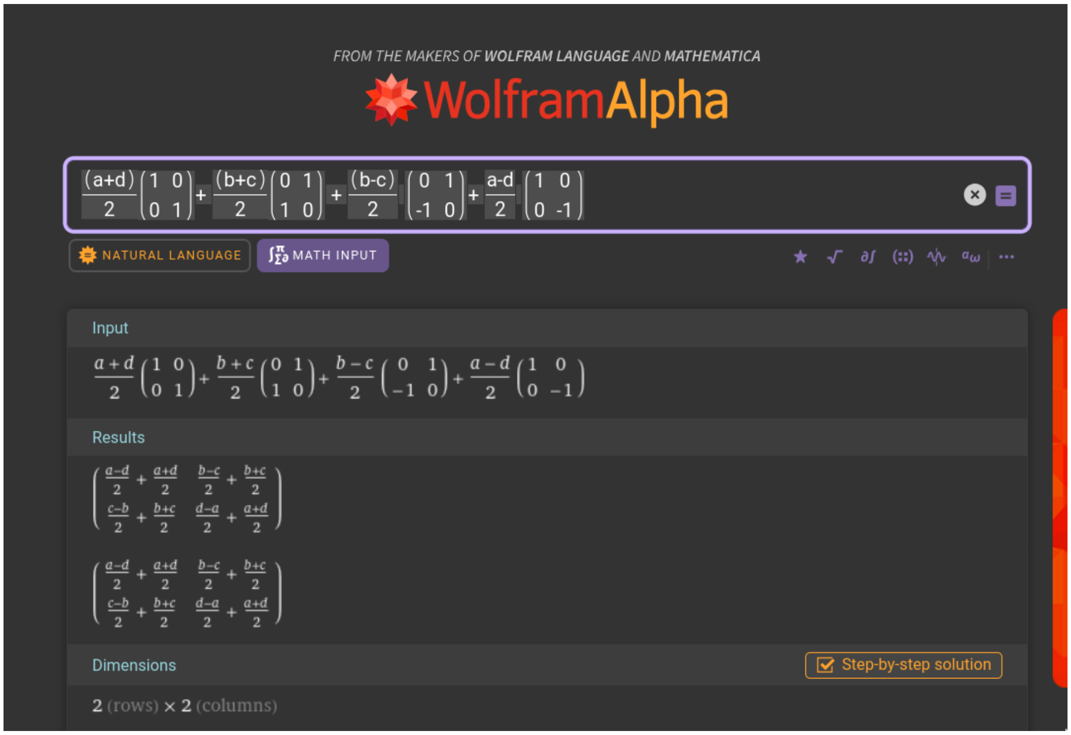

Figure 22: ss/pauli-basis.png

Problem 5(b)

Doing the matrix multiplication by hand

$$P_1P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_2P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_1P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_2P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

My intuition for the projection operator is basically

$$P_iP_j = (|i\rangle\langle i|)(|j\rangle\langle j|) = |i\rangle\langle i|j\rangle\langle j| = |i\rangle\delta_{ij}\langle j| = \delta_{ij}|i\rangle\langle j| = |k\rangle\langle k|$$

where $k = \{i, j\}$

Problem 5(c)

Computation

$$|V\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$P_V = |V\rangle\langle V| = \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) = \frac{1}{2}\left(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|\right)$$

$$P_V = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right] = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

Now computing

$$\begin{aligned}
P_1 P_V P_2 &= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right]\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}
\end{aligned}$$

We could have also done this directly like

$$\begin{aligned}
P_1 P_V P_2 &= \frac{1}{2}\left(|1\rangle\langle 1|\right)\left(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 1|\right)\left(|2\rangle\langle 2|\right)\\
&= \frac{1}{2}\left(|1\rangle\langle 1|\right)\left[|1\rangle\langle 1|2\rangle\langle 2| + |1\rangle\langle 2|2\rangle\langle 2| + |2\rangle\langle 1|2\rangle\langle 1| + |2\rangle\langle 2|2\rangle\langle 2|\right]\\
&= \frac{1}{2}\left(|1\rangle\langle 1|\right)\left(|1\rangle\langle 2| + |2\rangle\langle 2|\right)\\
&= \frac{1}{2}\left(|1\rangle\langle 1|1\rangle\langle 2| + |1\rangle\langle 1|2\rangle\langle 2|\right)\\
&= \frac{1}{2}|1\rangle\langle 2|\\
&= \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}
\end{aligned}$$

Analysis

We can keep three polarizers, each at a increasing angle, then components of lights get filtered as the pass through, remaining light falls on the screen behind.