Solid State Physics : : Homework 03

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Problem 01

(a)

$$B = \frac{\hbar}{el_B^2} = 5048.51\,T$$

(b)

$$\omega_c = \frac{eB}{m} \to \hbar\omega_c \sim k_B T \implies B \sim \frac{k_B T m}{\hbar e}$$
$$B \sim 15.7\,T$$

(c)

Small Effective mass and Large Magnetic field means

$$\frac{eB}{m} \gg 1$$

So now our resistivity tensor can approximately behave like

$$\rho \sim \rho_0 \begin{bmatrix} 0 & \omega_c\tau \\ -\omega_c\tau & 0 \end{bmatrix}$$

It's quite obvious to see that a current $\begin{bmatrix} j \\ 0 \end{bmatrix}$ would be caused by electric field along

$$E_{1,0} = \rho\vec{j} \sim \rho_0 \begin{bmatrix} 0 & \omega_c\tau \\ -\omega_c\tau & 0 \end{bmatrix} \begin{bmatrix} j \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho_0 j\omega_c\tau \end{bmatrix}$$

Similarly for a current along $\begin{bmatrix} 0 \\ j \end{bmatrix}$ we have

$$E_{0,1} \sim \begin{bmatrix} \rho_0 j\omega_c\tau \\ 0 \end{bmatrix}$$

The current is "mostly" **perpendicular** to the direction of Electric field.

(d)

- Turn on the Magnetic field: now the carriers have a cyclotron frequency ω_c were m is the effective mass in $\omega_c = eB/m$.

- Now let's try to measure the resistance as shown in the figure of the problem. That requires us to apply a voltage difference (as in the figure). For this, the electric field also happens to point in the direction along the voltage difference.
- The current is also being measured **parallel** to the voltage difference as it can be seen in the figure. But from our analysis in (c) we just saw for *high magnetic field* and *low effective* mass where we can safely project the approximation $B/m \gg 1$ then *current flow is perpendicular to supplied electric field.*
- So the amount of current flow **along** the electric field in this given system is extremely low (given how well our B/m ratio is). This invokes a "Magnetoresistance" (Tanner would be happy to hear this) which explains why the resistance goes up.

Documented Calculator Assist

$$\frac{\left(1.054 \cdot 10^{-34}\right)}{\left(1.602 \cdot 10^{-19}\right)\left(0.361 \cdot 10^{-9}\right)^2} = 5048.515516$$

$$\frac{\left(1.38 \cdot 10^{-23}\right)(300)\left(0.07 \cdot 9.11 \cdot 10^{-31}\right)}{\left(1.054 \cdot 10^{-34}\right)\left(1.6 \cdot 10^{-19}\right)} = 15.65511148$$

Figure 1: ./fig/3/1.png

Problem 02

a

`https://proofwiki.org/wiki/Curl_of_Vector_Cross_Product`

$$\begin{aligned}
\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left(\frac{1}{2}\mathbf{B} \times \mathbf{r}\right) \\
&= (\mathbf{r} \cdot \nabla)\,\frac{1}{2}\mathbf{B} - \mathbf{r}\left(\nabla \cdot \frac{1}{2}\mathbf{B}\right) - (\frac{\mathbf{B}}{2} \cdot \nabla)\mathbf{r} + \frac{1}{2}\mathbf{B}\,(\nabla \cdot \mathbf{r}) \\
&= 0 - 0 - \sum_{j=1}^{3}\left(\sum_{n=1}^{3} \frac{B_n}{2}\frac{\partial r_j}{\partial x_n}\widehat{x_j}\right) + \sum_{n=1}^{3} \frac{1}{2}\,(3B_n\widehat{x_n}) \\
&= -\sum_{n=1}^{3} \frac{B_n}{2}\widehat{x_n} + \sum_{n=1}^{3} \frac{1}{2}\,(3B_n\widehat{x_n}) \\
&= \frac{1}{2}\sum_{n=1}^{3} -B_n\widehat{x_n} + 3B_n\widehat{x_n} \\
&= \sum_{n=1}^{3} B_n\hat{x}_n = \mathbf{B}
\end{aligned}$$

(b)

$$\mathcal{H} = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} + g\mu_B\mathbf{B} \cdot \sigma + V(r)$$

Expanding $(\mathbf{p}+e\mathbf{A})^2$

$$(\mathbf{p}+e\mathbf{A})^2 = \mathbf{p}^2 + e(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}) + e^2\mathbf{A}^2$$

$\mathbf{A}$ is a function of position, so we can ignore order of dot product,

$$\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p} = 2\mathbf{p}\cdot\mathbf{A}$$

Thus,

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{e}{2m}(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}) + \frac{e^2}{2m}\mathbf{A}^2 + g\mu_B\mathbf{B}\cdot\sigma + V(r)$$

$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$

$$\mathbf{p}\cdot\mathbf{A} = \mathbf{p}\cdot\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \frac{1}{2}\mathbf{B}\cdot(\mathbf{p}\times\mathbf{r})$$

which gives:

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{e}{2m}\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) + \frac{e^2}{2m}\left(\frac{1}{4}|\mathbf{B}\times\mathbf{r}|^2\right) + g\mu_B\mathbf{B}\cdot\sigma + V(r)$$

(c)

First of all we refer to a formula table and also use the vector identity from previous homework,

$$\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) = (\mathbf{p}\times\mathbf{r})\cdot\mathbf{B} = \mathbf{L}\cdot\mathbf{B}$$

This lets us rewrite

$$\frac{e}{2m}\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) = \frac{e}{2m}\mathbf{B}\cdot\mathbf{L}$$

Using $\mu_B = e\hbar/2m$ we solve

$$\frac{e}{2m}\mathbf{B}\cdot\mathbf{L} = \mu_B\frac{\mathbf{B}\cdot\mathbf{L}}{\hbar}$$

Substitute in what we had gotten,

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{e}{2m}\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) + \frac{e^2}{2m}\left(\frac{1}{4}|\mathbf{B}\times\mathbf{r}|^2\right) + g\mu_B\mathbf{B}\cdot\sigma + V(r)$$

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \mu_B\frac{\mathbf{B}\cdot\mathbf{L}}{\hbar} + \frac{e^2}{2m}\left(\frac{1}{4}|\mathbf{B}\times\mathbf{r}|^2\right) + g\mu_B\mathbf{B}\cdot\sigma + V(r)$$

$$\boxed{\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \mu_B\mathbf{B}\cdot\left(\frac{\mathbf{L}}{\hbar} + g\sigma\right) + \frac{e^2}{2m}\left(\frac{1}{4}|\mathbf{B}\times\mathbf{r}|^2\right) + V(r)}$$

(d)

First of all

$$\begin{aligned}\mathbf{B} \times \mathbf{r} &= B\,(\hat{z}) \times (x\hat{x} + y\hat{y}) \\ &= -Bx\hat{y} - By\hat{x} \\ |\mathbf{B} \times \mathbf{r}|^2 &= B^2(x^2 + y^2)\end{aligned}$$

For quantum mechanics, x, y are position operator.

Larmor Diamagnetism Term Hamiltonian (perturbation)

$$\langle\phi|\,\mathcal{H}_D\,|\phi\rangle = \frac{e^2B^2}{8m}\,\langle\phi|\,x^2 + y^2\,|\phi\rangle$$

So now our concern is how to compute $\langle\phi|x^2|\phi\rangle$ and $\langle\phi|y^2|\phi\rangle$

We know that

$$\sum_{i=1}^{3}\langle\phi|r_i^2|\phi\rangle = 1 \tag{normalization}$$

And through symmetry arguments (somewhat statistical mechanical sense) we can write

$$\langle\phi|x^2|\phi\rangle = \langle\phi|y^2|\phi\rangle = \langle\phi|z^2|\phi\rangle = \frac{1}{3}$$

So

$$\langle\phi|\,\mathcal{H}_D\,|\phi\rangle = \frac{e^2B^2}{8m}\,\langle\phi|\,x^2 + y^2\,|\phi\rangle = \langle\phi|\,\mathcal{H}_D\,|\phi\rangle = \frac{e^2B^2}{8m}\left(\frac{2}{3}\,\langle\phi|r^2|\phi\rangle\right)$$

Which sums up to (with shorter notation)

$$\boxed{\delta\varepsilon = \frac{e^2B^2}{12m}\,\langle r^2\rangle}$$

(e)

One can relate the magnetic moment of a system to the free energy of that system. In a uniform magnetic field $\mathbf{B}$, the free energy $\mathbf{F}$ can be related to the magnetic moment $\mathbf{M}$ of the system as

$$\mathrm{d}F = -S\mathrm{d}T - \mathbf{M}\cdot\mathrm{d}\mathbf{B}$$

Through constant temperature we can hence relate that to what we had found above to solve for one atom

$$m = -\frac{\mathrm{d}\,\langle\phi|\mathcal{H}_D|\phi\rangle}{\mathrm{d}B} = -\frac{e^2}{6m}\,\langle r^2\rangle\,B$$

$$\boxed{m = -\frac{e^2}{6m}\,\langle r^2\rangle\,B}$$

From here the magnetization for a macroscopic piece

$$M = -\frac{N}{V}\frac{e^2}{6m}\,\langle r^2\rangle\,B$$

Now from common sense

$$\mathbf{M} = \chi\mathbf{H} = \frac{\chi}{\mu_0}\mathbf{B}$$

Combine

$$\boxed{M = -\mu_0\frac{N}{V}\frac{e^2}{6m}\,\langle r^2\rangle}$$

(f)

$$\begin{aligned}\langle r^2\rangle = \langle\psi|r^2|\psi\rangle &= \int_0^\infty r^2\Psi^*(r)\Psi(r)\,\mathrm{d}V\\ &= 4\pi\int_0^\infty r^2\Psi^*(r)\Psi(r)\,r^2\mathrm{d}r\end{aligned}$$

Boot up mathematica. For the integral we get

$$\boxed{\langle r^2\rangle = 3a_0^2}$$

From the equation for moment per atom we can see $\chi = \frac{m\mu_0}{B}$

$$\frac{\chi}{\mu_0} = -\frac{e^2}{6m}3a_0^2 = -\frac{e^2a_0^2}{2m}$$

We got in Mathematica

$$\chi = -4.95\times 10^{-33}$$

That's for one atom. For a mol of atom we get $\chi_{\mathrm{mol}} = -2.98\times 10^{-9}$

Appendix:
Solid State Physics , Homework 03

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Integral calculation to compute average $\langle r^2 \rangle$

Define the wavefunction

In[3]:=
```
f[r_] = (Pi * a0^3)^(-1/2) * Exp[-r / a0]
```

Out[3]=

$$\frac{e^{-\frac{r}{a_0}}}{\sqrt{\pi}\sqrt{a_0^3}}$$

Check the normalization

In[11]:=
```
4 *Pi *Integrate[r^2*f[r]^2, {r, 0, Infinity}]
```

Out[11]=

1 if $\mathrm{Re}[a_0] > 0$

Take the integral as required for the problem

In[13]:=
```
4* Pi*Integrate[r^4 *(f[r])^2, {r, 0, Infinity}]
```

Out[13]=

$3 a_0^2$ if $\mathrm{Re}[a_0] > 0$

Numerical Calculation of Larmor χ

Out[14]=

$$-\frac{h^2 a_0}{2 m^2}$$

In[15]:= `h = 1.054571817 × 10^-34`

Out[15]=

1.05457×10^{-34}

In[16]:= `a0 = 5.29 × 10^-10`

Out[16]=

5.29×10^{-10}

`m = 9.1093837 × 10^(-31)`

Out[17]=

9.10938×10^{-31}

In[20]:= `e = 1.602 * 10^-19`

Out[20]=

1.602×10^{-19}

In[28]:= `μ0 = 4 Pi * 10^-7`

Out[28]=

$$\frac{\pi}{2\,500\,000}$$

In[29]:= `A0 = 6.022 * 10^23`

Out[29]=

6.022×10^{23}

Computation of χ per atom

In[32]:=

$$-\frac{a_0\verb|^|2 * e\verb|^|2 * \mu_0}{2\,m}$$

Out[32]=

-4.95367×10^{-33}