Honors Linear Algebra : : Class 15

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Duality

if $T \in \mathcal{L}(V,W)$ and the transformation over the dual space$T' \in \mathcal{L}(W',V')$

$$V \to^T W$$

$$V' \leftarrow^{T'} W'$$

then

$$\dim(\text{range } T) = \dim(\text{range } T')$$

Matrix of the dual of a linear map. We are going to write the matrix for both cases and compare them. Let's use $\{v_m\}$ for V basis and $\{w_m\}$ is basis for W. Then we have a matrix for T,

$$\mathcal{A} = \mathcal{M}(T)$$

Definition of this matrix is this,

$$T(\vec{v}_k) = \sum_{r=1}^{m} \mathcal{A}_{r,k} w_r$$

For the inverse map T' we are going to use the Dual Basis. There are n of these basis, so $\{\phi_n\}$ for going back to V and for W we have $\{\psi_m\}$.

$$\mathcal{C} = \mathcal{M}(T')$$

hence,

$$T'\psi_j = \sum_{r=1}^{n} \mathcal{C}_{r,j}\phi_r$$

Now compute the dual basis,

$$T'(\psi_j)(v_k)$$

$$= \sum_{r=1}^{n} \mathcal{C}_{r,j}\phi_r(v_k) = C_{k,j}$$

Another computation, where we have a composition of two linear maps.

$$(\psi_j T)(v_k) = \psi_j(Tv_k) = \psi_j\left(\sum_r \mathcal{A}_{r,k} w_r\right)$$

$$= \sum_r \mathcal{A}_{r,k}\psi_j(w_r) = A_{j,k}$$

We can see

$$\mathcal{C}_{k,j} = \mathcal{A}_{j,k}$$

$\mathcal{C}$ happens to be transpose of $\mathcal{A}$.

A good consequence is, for a matrix $\mathcal{A}$, the column rank is equal to the row rank. We saw the proof before. Here's a second proof.

Problem 3.133

Column Rank is equal to row rank. So suppose A is a matrix $A \in \mathbb{F}^{m,n}$. We want to use the fact we just had done in previous section. We need a T, and obvious way to get a linear mapping is to define the mapping from $\mathbb{F}^{n,1} \to^T \mathbb{F}^{m,1}$ by $T(\vec{x}) = A\vec{x}$, matrix multiplication.

Vertically m and n horizontally of A.

Column rank of A well we've seen that $T(x)$ is a linear combination of the columns of A, so the column rank of A is the dimension of the range of A

$$\begin{aligned} \text{column rank of } A &= \dim(\text{range } T) = \dim(\text{range } T') \\ &= \text{column rank of } A^T = \text{row rank of } A \end{aligned}$$

Exercise 09

The vector space is $\mathcal{P}_m(\mathbb{R})$ and the standard basis,

$$1, x, x^2, x^3, \ldots, x^m$$

The exercise is to find the Dual Basis. Let's call them,

$$\phi_0, \phi_1, \ldots, \phi_m$$

So,

$$\phi_k(1) = 0, \ldots, \phi_k(x^{k-1}) = 0, \phi_k(x^k) = 1, \ldots, \phi_k(x^m) = 0$$
$$p(x) = c_0 + c_1 x + \ldots + c_m x^m$$
$$\phi_k(p) = \phi_k(c_k x^k) = c_k$$

Separately to find c_k we can try doing a differentiation,

$$p^{(k)} = c_k k! + c_n x^n + \ldots$$

Set $x = 0$, then

$$c_k = \frac{p^{(k)}(0)}{k!} = \phi_k(p)$$

Exercise 32

The double dual space of V. Let have V vector space and V' the dual space that has all the functionals. But V' itself is a vector space, so it must have a dual of it's own, so V'' is dual of V'. Say $\Lambda \in \mathcal{L}(V, V'')$.

$$\Lambda(v) \text{ is a linear functional on } V'$$

$$\Lambda(v)(\phi) = \phi(v)$$

I need a linear functional on V'.

Now show that if V is finite dimensional then, Λ is a bijection.

Exercise 30

Polynomials