Solid State Physics : : Homework 0X

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Problem 1

(a)

Inverse of a matrix (from memory)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The original resistivity tensor is

$$\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}$$

So

$$\tilde{\sigma} = \tilde{\rho}^{-1} = \frac{1}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \begin{bmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{bmatrix}$$

This immediately gives us the solution for conductivity matrix

$$\begin{aligned} \sigma_{xx} &= \frac{\rho_{yy}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \\ \sigma_{xy} &= \frac{-\rho_{xy}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \\ \sigma_{yx} &= \frac{-\rho_{yx}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \\ \sigma_{yy} &= \frac{\rho_{xx}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \end{aligned}$$

(b)

The system at insulating for longitudinal conduction

$$\begin{aligned} \sigma_{xx} &= 0 \\ \sigma_{yy} &= 0 \end{aligned}$$

This implies that

$$\begin{aligned} \sigma_{xx} &= 0 \\ \sigma_{xy} &= \frac{-\rho_{xy}}{-\rho_{xy}\rho_{yx}} = \frac{1}{\rho_{yx}} \\ \sigma_{yx} &= \frac{-\rho_{yx}}{-\rho_{xy}\rho_{yx}} = \frac{1}{\rho_{xy}} \\ \sigma_{yy} &= 0 \end{aligned}$$

With the added effect

$$\rho_{xx} = \rho_{yy} = 0$$

Analysis: This is surprisingly absurd, we apparently have superconductivity (no resistance) along xx and yy.

Problem 2

a

The heuristic of this problem is to convert mass density into a number density

$$\rho = 2.32\frac{g}{cm^3} = 2.32\frac{g}{cm^3}\frac{mol}{mol} = \frac{2.32}{28.1}\frac{g}{cm^3}\frac{mol}{g} = 0.083\frac{mol}{cm^3} = 0.083(N_a)\frac{1}{10^{-6}}\frac{\text{particles}}{m^3} \approx 5\times 10^{28}\frac{\text{particles}}{m^3}$$

$$\boxed{5\times 10^{28} \text{ atoms per unit meter cubed}}$$

For doping, every millionth Si is replaced with P. $1M = 1\times 10^6$. Hence in total per unit meter cube we add

$$\frac{5\times 10^{28}}{10^6} = 5\times 10^{22}$$

P atoms. The carrier density of electrons hence is

$$\boxed{5\times 10^{22} \text{ electrons per unit meter cubed}}$$

(b)

I change to mks units and use the electron charge to mass ratio to solve

$$\mu = \frac{e\tau}{m} \to \tau = \frac{m\mu}{e} = 0.3\times\frac{1}{1.758\times 10^{11}}\times 1000(10^{-4})(s) = 1.706\times 10^{-13}\, s$$

$$\boxed{\tau = 1.706\times 10^{-13}\, s}$$

(c)

$$\sigma = \frac{e^2\tau n}{m} \to \tau = \frac{m}{\rho e^2 n} = \frac{9.1\times 10^{-31}}{2\times 10^{-5}\times(1.602\times 10^{-19})^2(5\times 10^{28})}s = 3.546\times 10^{-17}\, s$$

This is about 1000 times faster than (b) in terms of order of magnitude. Which makes sense because we seem to have a higher number density.

Estimate of a mean free path would be

$$l = \langle v\rangle\,\tau = 10^6\times 3.546\times 10^{-17}m = 3.546\times 10^{-11}m = 0.35A^\circ$$

Analysis: Google says me that an atom is roughly 1 to 2.5 angstroms. The electron gets scattered before it traverses around 0.35 angstroms, around $1/3$ of the size of the atom around. I can see why materials like these are called bad metals.

Drude model hence implies a bad metallicity for this metal. Drude model is successful in making a rough estimate of the badness of the metal, although it's already understood that it's not quantum mechanically accurate.

Problem 3

(a)

I am going to do something that's a mathematicians worse nightmare.

$$\begin{aligned} v_t = -v_x \tau \frac{\mathrm{d}v_x}{\mathrm{d}x} &= -\tau \left(\mathrm{d} \int \right) v_x \frac{\mathrm{d}v_x}{\mathrm{d}x} \\ &= -\tau \left(\frac{\mathrm{d}}{\mathrm{d}x} \int \right) v_x \mathrm{d}v_x \\ &= -\frac{\tau}{2} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) v_x^2 \\ &= -\frac{\tau}{2} \frac{\mathrm{d}(v_x^2)}{\mathrm{d}x} \\ &= -\frac{\tau}{2} \frac{\mathrm{d}(v^2/3)}{\mathrm{d}x} \\ &= -\frac{\tau}{6} \frac{\mathrm{d}(v^2)}{\mathrm{d}x} \\ &= -\frac{\tau}{6} \frac{\mathrm{d}(v^2)}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}x} \end{aligned}$$

For physical sense I like to write this as

$$v_t = \frac{\tau}{6} \frac{\mathrm{d}v^2}{\mathrm{d}T} \left(-\frac{\mathrm{d}T}{\mathrm{d}x} \right)$$

This makes a lot of sense. This says that v is positive along the direction of temperature decrease. I imagine this as flowing along the downward going (hence negative) ramp of temperature, as electrons from higher temperature region flows towards lower temperature region by the assist of a velocity difference.

(b)

Resulting drift velocity only in terms of e, m, τ, E_x can be solved through the probabilistic differentia lequation

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -e\vec{E} - \vec{p}/\tau$$

In steady state for drift velocity we end up with

$$m\vec{v}_d = -e\tau\vec{E} \implies v_d = -\frac{e\tau}{m} E_x$$

(c)

$$
\begin{aligned}
v_t + v_d &= 0 \\
-\frac{\tau}{6}\frac{\mathrm{d}v^2}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}x} - \frac{e\tau}{m}E_x &= 0 \\
\frac{\tau}{6}\frac{2}{m}c_v\frac{\mathrm{d}T}{\mathrm{d}x} &= -\frac{e\tau}{m}E_x \\
\frac{\tau}{6}\frac{2}{m}c_v\frac{\mathrm{d}T}{\mathrm{d}x} &= \frac{e\tau}{m}\frac{\mathrm{d}V}{\mathrm{d}x} \\
\frac{1}{3}c_v\frac{\mathrm{d}T}{\mathrm{d}x} &= e\frac{\mathrm{d}V}{\mathrm{d}x} \\
\frac{c_v}{3e}\frac{\mathrm{d}T}{\mathrm{d}x} &= \frac{\mathrm{d}V}{\mathrm{d}x} \\
\frac{3k_B/2}{3e}\frac{\mathrm{d}T}{\mathrm{d}x} &= \frac{\mathrm{d}V}{\mathrm{d}x} \\
\frac{k_B}{2e}\frac{\mathrm{d}T}{\mathrm{d}x} &= \frac{\mathrm{d}V}{\mathrm{d}x}
\end{aligned}
$$

Comparing we see that apparently

$$\frac{k_B}{2e} = -S \implies S = -\frac{k_B}{2e}$$

Problem 4

(a)

Let's find the kinetic energy of **one** single particle with $k = k_i$

$$
\begin{aligned}
E_k &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(p/m)^2 \\
&= \frac{p^2}{2m} \\
&= \frac{\hbar^2 k^2}{2m} \\
\implies E_k(k_i) &= \frac{\hbar^2 k_i^2}{2m}
\end{aligned}
$$

Philosophy: I think we are always very careless about notations. I don't like it. k is the general wavevector k, where k_i is a specific value we are interested in. Now let's find the number of particles that are moving around in $k = k_i$ state. Mathematically being specific $k = \sqrt{\vec{k}\cdot\vec{k}} = \sqrt{k_x^2 + k_y^2 + k_z^2}$ where k behaves like the scalar radius of fermi ball in k space.

We know that in 3D k-space (is it really mathematically a space? I know that $\vec{k} \in \mathbb{R}^3$ but what's the mathematical rule that makes it a space? I prefer the physics version though, slapping the word space and just going with the feeling).

The separation between each state is given by $2\pi/L$ (lecture recording, I enjoyed that lecture like a movie (it was so interesting)). There is one particle per every box of length $2\pi/L$ hence we could possibly write

$$\rho = \frac{1}{(2\pi/L)^3} = \frac{V}{8\pi^3}$$

The number of particles in a state k is (drum roll) : 0. Every point particle in k space only intersects the manifold, but not contained. Being mathematically specific, we count k points of states that are inside the "volume". The manifold of $r = k$ has no containing volume so no state is contained.

But number of particles in a state $k < k_i < k + \mathrm{d}k$ is (which can be measured because this has a non-zero measure as a 3D open set)

$$\mathrm{d}N(k_i) = \rho\,\mathrm{d}\tau = \frac{V}{8\pi^3}(4\pi k_i^2 \mathrm{d}k) = \frac{V}{2\pi^2}k_i^2\mathrm{d}k$$

So the total energy of $\mathrm{d}N(k_i)$ (number of particles in k_i) state is given by

$$\mathrm{d}E(k_i) = \mathrm{d}N(k_i)\varepsilon(k_i) = \frac{V}{2\pi^2}k_i^2\mathrm{d}k \cdot \frac{\hbar^2 k_i^2}{2m}$$

We can integrate this from $k = 0$ to $k = k_F$ through

$$E_F = \frac{\hbar^2 V}{4\pi^2 m}\int_0^{k_F} k_i^4 \mathrm{d}k = \frac{\hbar^2 V}{4\pi^2 m}\frac{k_i^5}{5}\bigg|_{k_i=0}^{k_i=k_F} = \frac{\hbar^2 V k_F^5}{20\pi^2 m}$$

Now let's do a variable flip, we know that total number of particles in the k-ball from 0 to k_F are

$$N = \rho V = \frac{V}{8\pi^3}\frac{4}{3}\pi k_F^3 = \frac{V k_F^3}{6\pi^2}$$

And we already have the dispersion relation of energy that can give us the surface energy

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

Using this we can write flip the variables for E_F

$$E_F = \frac{\hbar^2 V k_F^3}{20\pi^2 m} = \frac{1}{20}\left(\frac{V k_F^3}{\pi^2}\right)\left(\frac{\hbar^2 k_F^2}{m}\right) = \frac{6}{10}\left(\frac{V k_F^3}{6\pi^2}\right)\left(\frac{\hbar^2 k_F^2}{2m}\right) = \frac{3}{5}N\varepsilon_F$$

$$\boxed{E_F = \frac{3}{5}N\varepsilon_F}$$

(b)

In 2-Dimensions, we are constrained to wavefunctions span along two axes. Hence our states are contained in $k = \sqrt{k_x^2 + k_y^2}$. We are still going to use the energy relation $\varepsilon(k_i) = \frac{\hbar^2 k_i^2}{2m}$.

We are required to derive ε_F which is the surface energy. That would be given by

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

Electron density (in k space)

$$n = \frac{1}{(2\pi/L)^2} = \frac{L^2}{4\pi^2} = \frac{A}{4\pi^2}$$

Number of electron in ε energy is given by

$$\mathrm{d}A = 2\pi k_i \mathrm{d}k$$

$$\mathrm{d}N = n\mathrm{d}A = \frac{A}{4\pi^2}2\pi k_i \mathrm{d}k = \frac{A}{2\pi}k_i\mathrm{d}k = \frac{A}{4\pi}\mathrm{d}(k_i^2) = \frac{A}{4\pi}\frac{2m}{\hbar^2}\mathrm{d}\varepsilon = A\frac{m}{\hbar^2}\mathrm{d}\varepsilon$$

$$\frac{\mathrm{d}N}{A} = g(\varepsilon)\,\mathrm{d}\varepsilon = \frac{m}{\hbar^2}\mathrm{d}\varepsilon$$

$$\frac{N}{A} = \int_0^{\varepsilon_F} g_{\text{2D}}(\varepsilon)\mathrm{d}\varepsilon = \frac{m\varepsilon_F}{\pi\hbar^2} \implies \pi\frac{\hbar^2 N}{mA} = \varepsilon_F$$

We find

$$\boxed{\varepsilon_F = \frac{\pi\hbar^2 N}{mA}}$$

(c)

Kinetic Energy of any particle with k_i was shown to be

$$E_k(k_i) = \frac{\hbar^2 k_i^2}{2m}$$

Number of particle per area in this state is

$$\frac{N_\varepsilon}{A} = g(\varepsilon)\mathrm{d}\varepsilon = \frac{m}{\hbar^2}\,\mathrm{d}\varepsilon \implies N_\varepsilon = A\frac{m}{\hbar^2}\mathrm{d}\varepsilon$$

Total energy (kinetic) of N_ε particles are

$$\mathrm{d}E(\varepsilon) = N_\varepsilon\varepsilon = A\frac{m}{\hbar^2}\varepsilon\mathrm{d}\varepsilon$$

So integrating this we get (also invoking what we had found $N/A = m\varepsilon_F/\pi\hbar^2$)

$$E_{\text{tot}} = \frac{mA}{2\hbar^2}\varepsilon_F^2 = \frac{m\varepsilon_F}{\hbar^2}\frac{A\varepsilon_F}{2} = \frac{N}{A}\pi\frac{A\varepsilon_F}{2} = \frac{\pi}{2}N\varepsilon_F$$

$$\boxed{E_{\text{tot}} = \frac{\pi}{2}N\varepsilon_F}$$

(d)

Maximum energy (surface energy)

$$\varepsilon_F = \frac{\pi\hbar^2 N}{mA} = \frac{1}{2}mv_F^2 \implies \boxed{v_F = \sqrt{\frac{2\pi\hbar^2 N}{m^2 A}}}$$