Quantum Mechanics : : Homework 02

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Problem 01

$$\begin{aligned}
[x_a, p_b] &= i\hbar\delta_{ab} \\
x_a p_b - p_b x_a &= i\hbar\delta_{ab} \\
x_a p_b - p_b x_a - i\hbar\delta_{ab} &= 0 \\
x_a p_b - i\hbar\delta_{ab} &= p_b x_a
\end{aligned}$$

We will be plugging in $p_j x_i = x_i p_j - i\hbar\delta_{ij}$ in the forthcoming solutions.

(a)

$$\begin{aligned}
[x_j, L_i] &= x_j \varepsilon_{imn} x_m p_n - \varepsilon_{imn} x_m p_n x_j \\
&= \varepsilon_{imn} x_j x_m p_n - \varepsilon_{imn} x_m p_n x_j \\
&= \varepsilon_{imn} x_j x_m p_n - \varepsilon_{imn} x_m (x_j p_n - i\hbar\delta_{jn}) \\
&= \varepsilon_{imn} x_j x_m p_n - \varepsilon_{imn} x_m x_j p_n + i\hbar\varepsilon_{imn}\delta_{jn} x_m \\
&= i\hbar\varepsilon_{imj} x_m \\
&= i\hbar\varepsilon_{mji} x_m
\end{aligned}$$

Let's try for sake of understanding in terms of real index

$$\begin{aligned}
[x_2, L_3] = x_2 L_3 - L_3 x_2 &= x_2(x_1 p_2 - x_2 p_1) - (x_1 p_2 - x_2 p_1) x_2 \\
&= x_2 x_1 p_2 - x_2 x_2 p_1 - x_1 p_2 x_2 + x_2 p_1 x_2 \\
&= x_2 x_1 p_2 - x_2 x_2 p_1 - x_1 (x_2 p_2 - i\hbar) + x_2 x_2 p_1 \\
&= x_2 x_1 p_2 - x_2 x_2 p_1 - x_1 x_2 p_2 + i\hbar x_1 + x_2 x_2 p_1 \\
&= i\hbar x_1 \\
&= i\hbar\varepsilon_{123} x_1
\end{aligned}$$

(b)

$$\begin{aligned}
[p_j, L_i] &= p_j \varepsilon_{imn} x_m p_n - \varepsilon_{imn} x_m p_n p_j \\
&= \varepsilon_{imn} p_j x_m p_n - \varepsilon_{imn} x_m p_n p_j \\
&= \varepsilon_{imn} (x_m p_j - i\hbar \delta_{mj}) p_n - \varepsilon_{imn} x_m p_n p_j \\
&= \varepsilon_{imn} x_m p_j p_n - i\hbar \varepsilon_{imn} \delta_{mj} p_n - \varepsilon_{imn} x_m p_n p_j \\
&= \varepsilon_{imn} x_m p_j p_n - i\hbar \varepsilon_{imn} \delta_{mj} p_n - \varepsilon_{imn} x_m p_j p_n \\
&= -i\hbar \varepsilon_{imn} \delta_{mj} p_n \\
&= -i\hbar \varepsilon_{ijn} p_n \\
&= -i\hbar \varepsilon_{nij} p_n
\end{aligned}$$

(c)

$$\begin{aligned}
\vec{r} \cdot \left(\vec{L} \times \vec{r} \right) &= \vec{r} \cdot ((\vec{r} \times \vec{p}) \times \vec{r}) \\
&= \vec{r} \cdot (\vec{p}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{p} \cdot \vec{r})) \\
&= \vec{r} \cdot \vec{p}(\vec{r} \cdot \vec{r}) - \vec{r} \cdot \vec{r}(\vec{p} \cdot \vec{r}) \\
&= (\vec{r} \cdot \vec{p})(\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{r})(\vec{p} \cdot \vec{r}) \\
&= 0
\end{aligned}$$

Problem 02

(a)

$$
\begin{aligned}
[r^2, L] &= \left[\sum_{n=1}^{3} x_n x_n, \varepsilon_{ijk} x_j p_k\right] \\
&= \sum_{n=1}^{3} \varepsilon_{ijk} x_n x_n x_j p_k - \sum_{n=1}^{3} \varepsilon_{ijk} x_j p_k x_n x_n \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j (x_n p_k - i\hbar \delta_{kn}) x_n\right) \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j (x_n p_k - i\hbar \delta_{kn}) x_n\right) \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j x_n p_k x_n + \varepsilon_{ijk} i\hbar \delta_{kn} x_j x_n\right) \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j x_n (x_n p_k - i\hbar \delta_{nk}) + \varepsilon_{ijk} i\hbar \delta_{kn} x_j x_n\right) \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_j x_n x_n p_k + i\hbar \varepsilon_{ijk} \delta_{nk} x_j x_n + \varepsilon_{ijk} i\hbar \delta_{kn} x_j x_n\right) \\
&= \sum_{n=1}^{3} \left(\varepsilon_{ijk} x_n x_n x_j p_k - \varepsilon_{ijk} x_n x_n x_j p_k + i\hbar \varepsilon_{ijk} \delta_{nk} x_j x_n + \varepsilon_{ijk} i\hbar \delta_{kn} x_j x_n\right) \\
&= \sum_{n=1}^{3} \left(i\hbar \varepsilon_{ijk} \delta_{nk} x_j x_n + \varepsilon_{ijk} i\hbar \delta_{kn} x_j x_n\right) \\
&= \sum_{n=1}^{3} \left(2\varepsilon_{ijn} i\hbar x_j x_n\right)
\end{aligned}
$$

Looking at the term

$$\varepsilon_{ijn} x_j x_n = (x_j x_n - x_n x_j)_i = 0$$

This proves

$$[r^2, L] = 0$$

(b)

$$
\begin{aligned}
\vec{L}\cdot\vec{r} = L_1x_1 + L_2x_2 + L_3x_3 = \sum_{n=1}^{3}(L_n r_n) &= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}(\varepsilon_{nij}x_i p_j)x_n \\
&= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}x_i p_j x_n \\
&= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}x_i (x_n p_j - i\hbar\delta_{nj}) \\
&= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}x_i x_n p_j - i\hbar\sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}\delta_{nj}x_i \\
&= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}x_n x_i p_j \underbrace{-i\hbar\sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}\delta_{nj}x_i}_{\varepsilon_{nij}\delta_{nj}=\varepsilon_{nin}=0} \\
\text{also, } \vec{r}\cdot\vec{L} &= \sum_{n=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{nij}x_n x_i p_j \\
&= \varepsilon_{123}x_1x_2p_3 + \varepsilon_{132}x_1x_3p_2 + \\
&+ \varepsilon_{231}x_2x_3p_1 + \varepsilon_{213}x_2x_1p_3 + \\
&+ \varepsilon_{312}x_3x_1p_2 + \varepsilon_{321}x_3x_2p_1 \\
&= x_1x_2p_3 - \mathbf{x_1x_3p_2} + \\
&+ x_2x_3p_1 - x_2x_1p_3 + \\
&+ \underbrace{\mathbf{x_3x_1p_2}}_{\text{pairs cross out}} - x_3x_2p_1 = 0
\end{aligned}
$$

This expanded proof basically solves $\vec{r}\cdot\vec{L} = \vec{L}\cdot\vec{r} = 0$.

(c)

$$
\vec{W} = \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{e^2}{r}\vec{r}
$$

$$
\begin{aligned}
\vec{L}\cdot\vec{W} &= \vec{L}\cdot\left(\frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{e^2}{r}\vec{r}\right) \\
&= \vec{L}\cdot\left(\frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p})\right) - \left(\frac{e^2}{r}\vec{L}\cdot\vec{r}\right)
\end{aligned}
$$

From $(\vec{L}\cdot\vec{r}) = 0$ from part b. Using scalar triple product that says $\vec{r}\cdot(\vec{r}\times\vec{p}) = 0$.