Problem 07 : : PHYS 201 Ahmed Sabit (ass15)

Problem 1

The amplitude is given by the equation

$$E(r,t) = \frac{E_m}{r}\sin(kr \pm \omega t)$$

The wave equation in three dimensions is given by

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi$$

In spherical coordinates we have the re-written equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

Solving for the given equation of electric field **Left Side**

$$\frac{\partial^2 E(r,t)}{\partial t^2} = -\omega^2 \frac{E_m}{r}\sin(kr \pm \omega t)$$

Solving for the **Right Side**

$$\begin{aligned}
c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial E(r,t)}{\partial r}\right) &= c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \left[-\frac{E_m}{r^2}\sin(kr \pm \omega t) + k\frac{E_m}{r}\cos(kr \pm \omega t)\right]\right) \\
&= c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(-E_m \sin(kr \pm \omega t) + krE_m \cos(kr \pm \omega t)\right) \\
&= c^2 \frac{1}{r^2}\left(-kE_m \cos(kr \pm \omega t) + kE_m \cos(kr \pm \omega t) - k^2 r E_m \sin(kr \pm \omega t)\right) \\
&= -c^2 \frac{k^2 E_m}{r}\sin(kr \pm \omega t) \\
&= -\omega^2 \frac{E_m}{r}\sin(kr \pm \omega t) && (c = \omega/k)
\end{aligned}$$

Woohoo we have both the left and right side equal.

$$c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial E(r,t)}{\partial r}\right) = \frac{\partial^2 E(r,t)}{\partial t^2} = -\omega^2 \frac{E_m}{r}\sin(kr \pm \omega t)$$

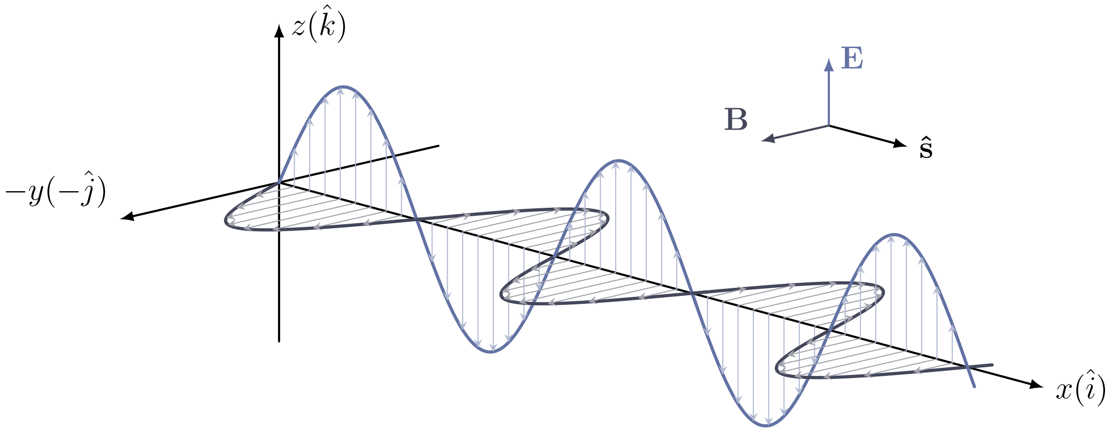
This validates that the given equation $E(r,t)$ is an isotropic spherical wave solution.

Problem 2

The electromagnetic wave in vacuum is given by (usual units)

$$\vec{E} = (2)\cos\left(\omega\left(t - \frac{x}{c}\right)\right)\hat{k}$$

This is a great place to copy paste TiKZ code from the internet! We are going to draw a generic wave for this purpose.



Verification of the directions, we require the Poynting vector to point $+x$ direction.

$$\vec{S} \propto \vec{E} \times \vec{B} \propto -\hat{z} \times \hat{y} = \hat{x}$$

Validating $-y$ to be the direction where $\vec{B}$ points.

$$\vec{B} = \left(\frac{2}{c}\right)\cos\left(\omega\left(t - \frac{x}{c}\right)\right)(-\hat{j})$$

(a)

The form is given by

$$\vec{B} = \left(\frac{2}{c}\right)\cos\left(\omega\left(t - \frac{x}{c}\right)\right)(-\hat{j})$$

Where the amplitude is $B_0 = 6.67 \times 10^{-9}\,T$

(b)

Average intensity (as shown in EMWaves3.pdf lecture file)

$$S_{\text{av}} = \frac{E_m^2}{2c\mu_0} = \frac{2}{c\mu_0} = 0.00530883746\,W/m^2$$

Problem 3

(a)

The magnetic field inside a long solenoid is given by

$$B_z = \mu_0 n i$$

So the magnetic field energy density is given by

$$u_B = \frac{\mu_0 n^2 i^2}{2}$$

(b)

Total energy inside the solenoid is given by

$$E = u_M V = \frac{\mu_0 n^2 i^2}{2} \pi r^2 l$$

The derivative is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mu_0 \pi r^2 n^2 l \left[i \frac{\mathrm{d}i}{\mathrm{d}t} \right] < 0 \impliedby \frac{\mathrm{d}i}{\mathrm{d}t} < 0$$

The overall power is negative. The energy is decreasing. Makes a lot of sense because $\frac{\mathrm{d}i}{\mathrm{d}t}$ is negative.

(c)

We have linear B field change because of changing current. Only taking account of magnitude and running down a computation of one of the Maxwell's equation

$$E(2\pi r) = \frac{\partial B}{\partial t}(\pi r^2) \implies E = \frac{r}{2} \frac{\partial B}{\partial t}$$

B and i have linear dependence and because the above quantity is a derivative of B only with respect to time, we have linear decrease of current that implies that Electric Field is going to be a constant in time.

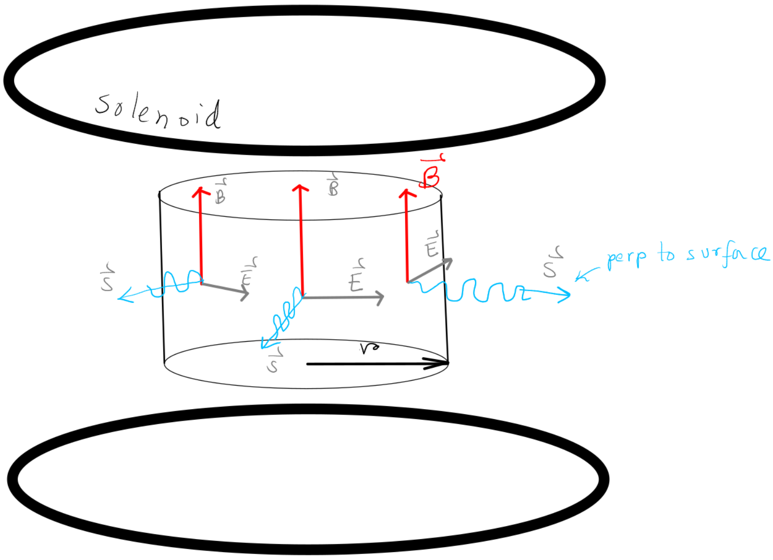
(d) + (e)

Power flowing through a theoretical cylinderical tube surface of radius r inside the solenoid (of same length l) is computed. Total electromagnetic power is computed from the poynting vector

$$S = \frac{EB}{\mu_0} = \frac{B}{\mu_0} \left(\frac{r}{2} \frac{\partial B}{\partial t} \right) = \frac{r}{2\mu_0} \left(B \frac{\partial B}{\partial t} \right)$$

$$P = SA = S(2\pi r l) = \left(\frac{r}{2\mu_0} B \frac{\partial B}{\partial t} \right) 2\pi r l = (\pi r^2 l) \frac{1}{\mu_0} B \frac{\partial B}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t} \left((\text{volume}) \frac{1}{\mu_0} \frac{B^2}{2} \right)$$

Above its absolutely the same answer we got in the previous sections, just the derivative of energy lose.



Figuur 1: Answer to (d) in a sense, the Poynting vector points towards $\vec{E} \times \vec{B}$

Hence total energy density inside the solenoid in terms of the magnetic field is

$$\frac{1}{2\mu_0} B^2$$

The electromagnetic radiation is also shown.

Problem 4

(a)

$$\begin{aligned} F &= \frac{IA}{c} = \frac{PA}{Ac} = \frac{P}{c} \\ &= \frac{1500}{c} = 5 \times 10^6 \end{aligned}$$

$$a = \frac{F}{m} = \frac{5 \times 10^6}{10 + 60} = 7.14 \times 10^{-8}$$

$$20 = \frac{1}{2} a t^2 \implies t = \sqrt{\frac{40}{a}} = 23656\, s = 6.5 \text{ hours}$$

cannot make it in time

(b)

Recompute the acceleration

$$a = \frac{F}{m} = 8.33 \times 10^{-8}$$

$$t = \sqrt{\frac{40}{a}} = 6.08 \text{ hours}$$

still does not make it. Sad. RIP.

(c)

Using momentum conservation directly

$$0 = 60v + 10$$

Solving this gives with t

$$t = 20/(1/6) = 120\,\mathrm{s}$$

this time yes, the time is enough

Problem 5

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

(a)

The superposition would be

$$\begin{aligned}
\vec{E}_R + \vec{E}_L &= \hat{\imath}\left(E_m\cos(kz-\omega t) + E_m\cos(kz-\omega t+\phi_1)\right) \\
&+ \hat{\jmath}\left(E_m\sin(kz-\omega t+\phi_1) - E_m\sin(kz-\omega t)\right) \\
&= 2E_m\cos\left(\frac{2(kz-\omega t)+\phi_1}{2}\right)\cos\left(-\frac{\phi_1}{2}\right)\hat{\imath} \\
&+ 2E_m\cos\left(\frac{2(kz-\omega t)+\phi_1}{2}\right)\sin\left(\frac{\phi_1}{2}\right)\hat{\jmath} \\
&= 2E_m\cos\left((kz-\omega t)+\frac{\phi_1}{2}\right)\cos\left(\frac{\phi_1}{2}\right)\hat{\imath} \\
&+ 2E_m\cos\left((kz-\omega t)+\frac{\phi_1}{2}\right)\sin\left(\frac{\phi_1}{2}\right)\hat{\jmath} \\
&= \left(2E_m\cos\left(kz-\omega t+\frac{\pi}{6}\right)\right)\left[\cos(\pi/6)\hat{\imath} + \sin(\pi/6)\hat{\jmath}\right] \\
&= \left(2E_m\cos\left(kz-\omega t+\frac{\pi}{6}\right)\right)\left[\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}\right]
\end{aligned}$$

The wave can be thought of as going along the $\sqrt{3}\hat{\imath} + \hat{\jmath}$ direction with a wave form given by the bracketed thing in the left. This is proven to be linearly polarized

$$\psi(z,t) = 2E_m\cos(kz-\omega t+\pi/6)$$

This makes $\pi/6$ rad angle with the $\hat{x}$ axis.

(b)

$$S_{\mathrm{av}} = \frac{E^2}{2c\mu_0} = \frac{2E_m^2}{c\mu_0}$$

(c)

It's only the y component that passes through.

$$S_{\mathrm{y,\ av}}/S_{\mathrm{av}} = E_{\mathrm{y,m}}^2/E_m^2 = (1/2)^2 = \frac{1}{4}$$

(d)

$$E_R = \frac{(-E_m)^2}{E_m^2} = 1$$

$$E_L = \frac{E_m^2 \sin^2 \phi_1}{E_m^2} = \sin^2 \phi_1 = 0.75$$

Problem 06

$$\begin{aligned}
I_1 &= I_0 \cos^2(\theta_1 - \theta_0) \\
I_2 &= I_0 \cos^2(\theta_2 - \theta_1) \cos^2(\theta_1 - \theta_0) \\
&\vdots \\
I_n &= I_0 (\cos^2 \Delta\theta)^n && (\Delta\theta = \theta_2 - \theta_1 = \theta_1 - \theta_0) \\
I_3 &= I_0 \cos^6(\pi/6) = 0.42 I_0 \\
I_6 &= I_0 \cos^{12}(\pi/12) = 0.66 I_0 \\
I_9 &= I_0 \cos^{18}(\pi/18) = 0.76 I_0 \\
I_{90} &\approx 0.97 I_0 && \text{(using relevant approximations)}
\end{aligned}$$

We get the Highest for 9 filters.

Problem 7

(a)

The form could be written as

$$\vec{E} = \vec{E}_0 \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{p\pi z}{L}\right) \cos\left(t\frac{\pi c}{L}\sqrt{m^2 + n^2 + p^2}\right)$$

(b)

First harmonic is $m = p = n = 1$ which gives $\lambda = 2L = 50cm$

(c)

$$f = \frac{c}{\lambda}\sqrt{m^2 + n^2 + p^2}$$

(d)

$$\begin{aligned}
(m, n, p) = (1, 1, 1) &\implies f_1 = \frac{c}{\lambda}\sqrt{3} = 1.03 \times 10^4 \text{ Hz} \\
(2, 1, 1) &\implies f_2 = \frac{c}{\lambda}\sqrt{6} = 1.46 \times 10^4 \text{ Hz} \\
(2, 2, 1) &\implies f_3 = \frac{c}{\lambda}\sqrt{9} = 1.74 \times 10^4 \text{ Hz}
\end{aligned}$$