Domácí úkol č. 4

B0B01MA1

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Upravená verze pro účely výuky

I. Zadání:

1. Spočtěte

$$\int_0^{\ln \sqrt{2}} \frac{6e^{6x} - 2e^{4x}}{e^{4x} - 3e^{2x} + 4} \, \mathrm{d}x$$

2. Spočtěte

$$\int_{\frac{e}{2}}^{\infty} \frac{3}{x(\ln^2 2x - 6\ln 2x + 13)} \, \mathrm{d}x$$

3. Spočtěte

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x - 4\cos x}{1 - 2\sin x - \cos^2 x} \, \mathrm{d}x$$

4. Spočtěte

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{5\sin x - \sin x \cdot \cos x}{\underbrace{\sin^2 x}_{1-\cos^2 x} + \cos x + 1} \, \mathrm{d}x$$

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II. Řešení:

1. Spočtěte

$$\int_0^{\ln\sqrt{2}} \frac{6e^{6x} - 2e^{4x}}{e^{4x} - 3e^{2x} + 4}\, dx = \left| \begin{array}{l} u = e^{2x} \\ du = 2e^{2x}\, dx \end{array} \right| = \int_1^2 \frac{2e^{2x}(3u^2 - u)}{u^2 - 3u + 4} \cdot \frac{du}{2e^{2x}} = \int_1^2 \frac{3u^2 - u}{u^2 - 3u + 4}\, du =$$

$$= \left| \begin{array}{l} \text{dělení} \\ \text{polynomů} \end{array} \right| = \int_1^2 \left(3 + \frac{8u - 12}{u^2 - 3u + 4} \right) du = [3u]_1^2 + \int_1^2 \frac{8u - 12}{u^2 - 3u + 4}\, du = \left| \begin{array}{l} t = u^2 - 3u + 4 \\ dt = (2u - 3)\, du \end{array} \right| =$$

$$= [3u]_1^2 + \int_2^2 \frac{4 \cdot (2u - 3)}{t} \cdot \frac{dt}{2u - 3} = [3u]_1^2 + \underbrace{4\int_2^2 \frac{dt}{t}}_{=0} = [3u]_1^2 = 6 - 3 = \underline{\underline{3}}$$

2. Spočtěte

$$\int_{\frac{e}{2}}^{\infty} \frac{3}{x(\ln^2 2x - 6\ln 2x + 13)}\, dx = \left| \begin{array}{l} u = \ln 2x \\ du = \frac{1}{x}\, dx \end{array} \right| = 3\int_1^{\infty} \frac{1}{x(u^2 - 6u + 13)} \cdot x\, du = 3\int_1^{\infty} \frac{1}{(u^2 - 6u + 9) + 4}\, du =$$

$$= 3\int_1^{\infty} \frac{1}{(u-3)^2 + 4}\, du = \left| \begin{array}{l} t = u - 3 \\ dt = 1\, du \end{array} \right| = 3\int_{-2}^{\infty} \frac{1}{t^2 + 4}\, dt = \frac{3}{4}\int_{-2}^{\infty} \frac{1}{\frac{t^2}{4} + 1}\, dt = \left| \begin{array}{l} v = \frac{t}{2} \\ dv = \frac{1}{2}\, dt \end{array} \right| = \frac{3}{2}\int_{-1}^{\infty} \frac{1}{v^2 + 1}\, dv =$$

$$= \frac{3}{2}[\arctan v]_{-1}^{\infty} = \frac{3}{2}\left(\lim_{v\to\infty} (\arctan v) - \arctan(-1) \right) = \frac{3}{2} \cdot \frac{\pi}{2} - \frac{3}{2}\left(-\frac{\pi}{4} \right) = \underline{\underline{\frac{9\pi}{8}}}$$

3. Spočtěte

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x - 4\cos x}{1 - 2\sin x - \cos^2 x}\, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array} \right| = \int_0^1 \frac{\cos x(u - 4)}{u^2 - 2u} \cdot \frac{du}{\cos x} = \int_0^1 \frac{u - 2}{u(u - 2)}\, du - 2\int_0^1 \underbrace{\frac{1}{u(u-2)}}_{g(u)}\, du =$$

$$= \underbrace{[\ln|u|]_0^1}_{S_f} + \underbrace{\int_0^1 -2g(u)\, du}_{S_g} = S_f + S_g = S$$

$$\underline{g(u)} = \frac{1}{u(u-2)} = \frac{A}{u} + \frac{B}{u-2}$$

$$A + B = 0 \Rightarrow A = -B$$

$$-2A = 1 \Rightarrow A = -\frac{1}{2} \Rightarrow B = \frac{1}{2}$$

$$g(u) = -\frac{1}{2} \cdot \frac{1}{u} + \frac{1}{2} \cdot \frac{1}{u-2}$$

$$S_g = -2\left(\int_0^1 -\frac{1}{2} \cdot \frac{1}{u}\, du + \int_0^1 \frac{1}{2} \cdot \frac{1}{u-2}\, du \right) = [\ln|u|]_0^1 - [\ln|u-2|]_0^1$$

$$S = S_f + S_g = [\ln|u|]_0^1 + [\ln|u|]_0^1 - [\ln|u-2|]_0^1 = 0 - \lim_{u\to 0} \ln|u| + 0 - \lim_{u\to 0} \ln|u| - 0 + \ln 2 = +\infty + \infty + \ln 2 = \underline{\underline{+\infty}}$$

$$\text{Diverguje} \Rightarrow \text{existuje.}$$

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4. Spočtěte

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{5\sin x - \sin x \cdot \cos x}{\underbrace{\sin^2 x}_{1-\cos^2 x} + \cos x + 1}\, dx = \Big|\substack{u = \cos x \\ du = -\sin x\, dx}\Big| = \int_{\frac{1}{2}}^{0} = \frac{\sin x(5-u)}{1-u^2+u+1}\cdot\frac{du}{-\sin x} = \int_{\frac{1}{2}}^{0} \frac{5-u}{(u-2)(u+1)}\, du =$$

$$= 5\int_{\frac{1}{2}}^{0} \underbrace{\frac{1}{(u-2)(u+1)}}_{f(u)}\, du + \int_{\frac{1}{2}}^{0} \underbrace{\frac{-u}{(u-2)(u+1)}}_{g(u)}\, du = \underbrace{\int_{\frac{1}{2}}^{0} f(u)\, du}_{S_f} + \underbrace{\int_{\frac{1}{2}}^{0} -g(u)\, du}_{S_g} = S_f + S_g = S$$

$$\underline{f(u)} = \frac{1}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1}$$

$$A + B = 0 \Rightarrow A = -B$$

$$A = -B \wedge A - 2B = 1 \Rightarrow -3B = 1 \Rightarrow B = -\frac{1}{3} \wedge A = \frac{1}{3}$$

$$f(u) = \frac{1}{3}\cdot\frac{1}{u-2} - \frac{1}{3}\cdot\frac{1}{u+1}$$

$$S_f = 5\int_{\frac{1}{2}}^{0} f(u) = \frac{5}{3}\int_{\frac{1}{2}}^{0} \frac{1}{u-2}\, du - \frac{5}{3}\int_{\frac{1}{2}}^{0} \frac{1}{u+1}\, du = \frac{5}{3}\Big([\ln|u-2|]_{\frac{1}{2}}^{0}\Big) - \frac{5}{3}\Big([\ln|u+1|]_{\frac{1}{2}}^{0}\Big) =$$

$$= \frac{5}{3}(\ln 2 - (\ln 3 - \ln 2)) - \frac{5}{3}(\ln 1 - (\ln 3 - \ln 2)) = \frac{5}{3}(\ln 2 - 0) = S_f$$

$$\underline{g(u)} = \frac{u}{(u-2)(u+1)} = \frac{(u-2)}{(u-2)(u+1)} + \frac{2}{(u-2)(u+1)} = \frac{1}{u+1} + \frac{2}{3}\left(\frac{1}{u-2} - \frac{1}{u+1}\right)$$

$$S_g = -\int_{\frac{1}{2}}^{0} g(u) = -\int_{\frac{1}{2}}^{0} \frac{1}{u+1}\, du - \frac{2}{3}\int_{\frac{1}{2}}^{0} \frac{1}{u-2}\, du + \frac{2}{3}\int_{\frac{1}{2}}^{0} \frac{1}{u+1}\, du =$$

$$= -[\ln|u+1|]_{\frac{1}{2}}^{0} - \frac{2}{3}[\ln|u-2|]_{\frac{1}{2}}^{0} + \frac{2}{3}[\ln|u+1|]_{\frac{1}{2}}^{0} = -(\ln 1 - (\ln 3 - \ln 2)) - \frac{2}{3}(\ln 2 - (\ln 3 - \ln 2)) + \frac{2}{3}(\ln 1 - (\ln 3 - \ln 2)) =$$

$$= \ln 3 - 0 - \ln 2 - \frac{2}{3}(\ln 2 - 0) = S_g$$

$$S = S_f + S_g = \frac{5}{3}\ln 2 + \ln 3 - \ln 2 - \frac{2}{3}\ln 2 = \frac{3}{3}\ln 2 - \ln 2 + \ln 3 = \underline{\underline{\ln 3}}$$

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