Zermelo-Fraenkel Axioms

1. **Axiom of extensionality:** Two sets are equal iff they have the same members.
$$\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \leftrightarrow a = b)$$

2. **Empty set axiom:** There is a set $\varnothing$ with no elements.
$$\exists a \forall x (\neg (x \in a))$$

3. **Pairing axiom:** If a and b are sets, there exists a set, $\{a, b\}$, whose members are exactly a and b
$$\forall a \forall b \exists c (\forall x (x \in c \leftrightarrow ((x = a) \vee (x = b))))$$
special case: $\{a, a\}$ denoted $\{a\}$ - **singleton set**

4. **Union Axiom:** If a is a set, there exists a set $\bigcup a$ whose members are the members of members of a
$$\forall a \exists b \forall x (x \in b \leftrightarrow \exists y ((x \in y) \wedge (y \in a)))$$
notation: $\bigcup a, b$ is denoted $a \cup b$

5. **Powerset Axiom:** If a is a set, there is a set, $\mathcal{P}(a)$, whose members are the subsets of a.
$$\forall a \exists b \forall x (x \in b \leftrightarrow x \subseteq a)$$
shorthand: $a \subseteq b$ stands for $\forall x (x \in a \rightarrow x \in b)$

6. **Separation Axiom (Scheme):** (alias Subset, alias Selection) For any admissible formula $\varphi(x)$ and for any set a there is a set
$$\{x \in a : \varphi(x)\}$$
whose members are those members of a which satisfy $\varphi(x)$

7. **Axiom of Infinity:** There exists an inductive set, that is:
$$\exists a (\varnothing \in a \wedge \forall x (x \in a \rightarrow x \cup \{x\} \in a))$$

8. **Replacement Axiom (Scheme):** The range of a partial function whose domain is a set is itself a set. Let $\varphi(x, y)$ be an admissible formula such that $\forall s \exists t ((\varphi(s, t) \wedge \forall u (\varphi(s, u) \rightarrow u = t)))$ (that is, φ is a class function). Then:
$$\forall a \exists b \forall y (y \in b \leftrightarrow \exists x (x \in a \wedge \varphi(x, y)))$$

9. **Axiom of Foundation:** (alias Regularity) Every non-empty set is well-founded (that is, contains an element minimal w.r.t. $\in$)
$$\forall a (a \neq \varnothing \rightarrow (\exists x (x \in a \wedge x \cap a = \varnothing)))$$