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Section A

Explicit covariance matrix for particle measurement precision

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Abstract

We derive explicit and precise formulae for 3×3 error matrix of particle transverse momentum, direction and impact parameter. The error matrix elements are expressed as functions of up to fourth order statistical moments of the measured coordinates. The formulae are valid for any curvature and track length in case of negligible multiple scattering. The calculation is compared with formulae derived by Gluckstern for curvature and direction. We show that Gluckstern formulation is valid at the limit of small L/R , ratio between the track length and radius of curvature. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Search for massive new particles such as Higgs boson in high energy experiments involves measurement of high transverse momentum particles. Therefore an important feature of the detector is its capability to measure high p_t trajectory parameters with a good precision. In designing new detectors one normally has to use sophisticated program codes in order to understand the resolution requirements of a heterogeneous detector system often consisting of several detector types with variable resolutions. Explicit precision formulae easy to use are therefore of great interest. In this paper we derive new formulae for this purpose.

Earlier works exist on explicit calculation of particle measurement precision. Amongst the most

frequently cited is Gluckstern's evaluation of uncertainties in particle momentum and direction due to multiple scattering and measurement errors [1]. Gluckstern uses parabola description of the projected trajectory. In this early work no impact parameter uncertainties were considered. Since the discovery of higher flavour particles the impact parameter analysis has become an important method to enhance the experimental event samples with short living flavour particles. In a more recent work Innes [2] uses the Gluckstern formulation and derives analytic error estimates also for the impact error for homogeneous detector system of many detector layers. The emphasis there was to optimize errors in the low p_t regime where multiple scattering is important.

In uniform magnetic field and with negligible multiple scattering the particle trajectory can be modeled as a helix whose projection on the plane normal to the magnetic field is a circular arc. In this work we derive explicit covariance matrix by using

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the circle description instead of parabola approximation.

This article is organized in the following manner: In Section 2 we define the notations and give an introduction to the formalism of the problem. In Section 3 we derive the general solution for the covariance matrix independent of the trajectory length or curvature. In Section 4 we consider the validity range of the ‘straightening hypothesis’ which Gluckstern uses in his paper and in Section 5 we present simple error formulae applicable in the regime of high p_t tracks. Finally in Section 6 we consider a few examples and discuss also briefly the treatment of multiple scattering.

2. Problem formulation

The coordinate system (u, v) is defined such that the u - v -plane is normal to the magnetic field direction. It can be shown that in uniform magnetic field and in the absence of multiple scattering the offset ε_i of a measured point (u_i, v_i) from the particle trajectory can be written as [3]:

$$\varepsilon_i = \frac{1}{2} \rho q_i^2 - (1 + \rho d)(u_i \sin \phi - v_i \cos \phi) + \frac{1}{2} \rho d^2 + d, \quad (1)$$

where $q_i^2 = u_i^2 + v_i^2$. The offset ε_i (or residual) is the distance of closest approach (d.c.a.) of a point (u_i, v_i) from the trajectory. The three parameters ρ , ϕ and d are defined as follows:

- ρ = curvature ($= \pm 1/R$ where R is the radius of curvature),
- ϕ = direction of propagation at the point of closest approach,
- d = impact parameter = distance of closest approach to the origin.

The expression (1) is valid even over a full trajectory loop unlike the parabola approximation used in Refs. [1,2]. It is also to be noted that the expression (1) is valid at the straight line limit $\rho \rightarrow 0$. In the absence of point to point correlations the χ^2 func-

tion for the best fit parameters of the trajectory with N measured points (u_i, v_i) reads as

$$\chi^2 = \sum_{i=1}^N w_i \varepsilon_i^2, \quad (2)$$

where w_i are weights defined as $w_i = \sigma_i^{-2}$ and σ_i is the precision of the point (u_i, v_i) in the direction normal to the trajectory.

A highly precise explicit solution of the minimization problem (2) can be written as [3]

$$\begin{aligned} \phi &= \frac{1}{2} \arctan(2Q/P), \\ \rho &= 2(\sin \phi \sigma_{uq^2} - \cos \phi \sigma_{vq^2}) / \sigma_{q^2 q^2}, \\ d &= -\frac{1}{2} \rho \langle q^2 \rangle + \langle u \rangle \sin \phi - \langle v \rangle \cos \phi, \end{aligned} \quad (3)$$

where $Q = \sigma_{q^2 q^2} \sigma_{uv} - \sigma_{uq^2} \sigma_{vq^2}$ and $P = \sigma_{q^2 q^2} (\sigma_{uu} - \sigma_{vv}) - \sigma_{uq^2}^2 + \sigma_{vq^2}^2$. The coefficients $\sigma_{uu}, \sigma_{uv}, \dots, \sigma_{q^2 q^2}$ are the six statistical covariances of the measurements u_i, v_i and q_i^2 . The covariance between any two quantities, say a and b , is defined in the standard manner as $\sigma_{ab} = \langle ab \rangle - \langle a \rangle \langle b \rangle$ where $\langle a \rangle$ denotes a weighted average, e.g., $\langle u \rangle = \sum w_i u_i / \sum w_i$.

The emphasis of this paper is to consider the precision of the three parameters ρ , ϕ and d in explicit terms. The covariance matrix of these parameters is calculated as the inverse of the symmetric weight matrix W whose elements are

$$W_{kl} = \sum_i w_i \frac{\partial \varepsilon_i}{\partial \alpha_k} \frac{\partial \varepsilon_i}{\partial \alpha_l}, \quad (4)$$

where we have denoted $\rho = \alpha_1$, $\phi = \alpha_2$ and $d = \alpha_3$. In an arbitrary frame of reference the explicit calculation of Eq. (4) and its inverse is infeasible. Using the covariance matrix invariance properties, however, one can invert the W matrix in a straightforward manner as shown in the next section.

3. Explicit covariance matrix

When one rotates the coordinate system around the origin by an angle ϕ_0 and shifts the origin by d_0 in the direction normal to the trajectory, the parameters ρ , ϕ and d transform as: $\rho \rightarrow \rho$, $\phi \rightarrow \phi - \phi_0$ and $d \rightarrow d - d_0$. It follows that the

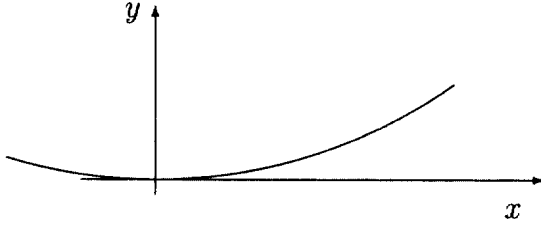


Fig. 1.

covariance matrix is invariant under this kind of transformation. The trick to make the analytic inversion of the weight matrix (4) feasible is based on this invariance. Before the inversion the coordinate system is rotated and translated such that

$$d = \phi = 0, \quad (5)$$

i.e. the new x -axis is tangential to the trajectory at the origin (see Fig. 1). We denote the new coordinate axes as x and y .

The transformation from (u_i, v_i) to (x_i, y_i) goes as

$$\begin{aligned} x_i &= +u_i \cos \phi + v_i \sin \phi, \\ y_i &= -u_i \sin \phi + v_i \cos \phi - d, \end{aligned} \quad (6)$$

where ϕ and d are computed from Eq. (3).

3.1. General case

In the coordinate system of Fig. 1 the three derivatives needed in Eq. (4) are simply

$$\frac{\partial \varepsilon_i}{\partial \rho} = \frac{1}{2} r_i^2, \quad \frac{\partial \varepsilon_i}{\partial \phi} = -x_i, \quad \frac{\partial \varepsilon_i}{\partial d} = 1 + \rho y_i \quad (7)$$

with $r_i^2 = x_i^2 + y_i^2$. The algebra to calculate the weight matrix and its inverse can be found in Appendix A. As the result we get the covariance matrix elements as functions of statistical moments of the measured coordinates:

$$\begin{aligned} \sigma_{\rho\rho} &= C[4\sigma_{xx} - 4\rho^2(\langle x^2 \rangle^2 - \langle x \rangle \langle x r^2 \rangle \\ &\quad + \tfrac{1}{4}\rho^2 \langle x r^2 \rangle^2)], \\ \sigma_{\rho\phi} &= C[2\sigma_{xr^2} - \rho^2(2\langle x^2 \rangle \langle x r^2 \rangle \\ &\quad - \langle r^2 \rangle \langle x r^2 \rangle - \langle x \rangle \langle r^4 \rangle + \tfrac{1}{2}\rho^2 \langle r^4 \rangle \langle x r^2 \rangle)], \end{aligned}$$

$$\sigma_{\phi\phi} = C[\sigma_{r^2 r^2} + \rho^2 \langle r^4 \rangle (\langle y^2 \rangle - \tfrac{1}{4}\rho^2 \langle r^4 \rangle)],$$

$$\begin{aligned} \sigma_{\rho d} &= C[2\langle x \rangle \langle x r^2 \rangle - 2\langle x^2 \rangle \langle r^2 \rangle \\ &\quad - \rho^2(\langle x r^2 \rangle^2 - \langle x^2 \rangle \langle r^4 \rangle)], \end{aligned}$$

$$\sigma_{\phi d} = C[\langle x \rangle \langle r^4 \rangle - \langle r^2 \rangle \langle x r^2 \rangle],$$

$$\sigma_{dd} = C[\langle x^2 \rangle \langle r^4 \rangle - \langle x r^2 \rangle^2], \quad (8)$$

where $C = \{S_w[\sigma_{xx}\sigma_{r^2 r^2} - (\sigma_{xr^2})^2]\}^{-1}$ and S_w is the sum of weights. The above covariance formulae hold for any curvature and track length even for a closed trajectory loop in the absence of multiple scattering. We have verified these analytic formulae by comparison with a code [4] which performs numerically the calculation and inversion of the weight matrix (4) and found no significant difference.

The error matrix (8) depends on the choice of origin. We call the origin as the *reference point* of the error matrix. The *propagation* of errors to a new reference point on the trajectory is simply performed by transforming the coordinate system such that the new x -axis is tangential to the trajectory at the new reference point and using the transformed coordinates in Eq. (8). Notice, however, that the curvature variance $\sigma_{\rho\rho}$ is independent of the reference point.

3.2. Fit with vertex constraint

In case the emission point (vertex) of the particle is known to a certain precision, one simply adds it as an extra measured point with appropriate weight and uses the formulation derived above.

In the special case that the trajectory is forced to go through the vertex the formulation becomes somewhat simpler: the impact parameter d vanishes at the vertex and the covariance matrix becomes a 2×2 matrix. Its derivation is straightforward and we give only the result in the following. We transform the origin of the coordinate system to the vertex point and perform a rotation such that the track direction coincides with the x -axis (as in Fig. 1). In this coordinate system the covariance

matrix of ρ and ϕ becomes:

$$\begin{aligned}\sigma_{\rho\rho} &= C' 4\langle x^2 \rangle, \\ \sigma_{\rho\phi} &= C' 2\langle xr^2 \rangle, \\ \sigma_{\phi\phi} &= C' \langle r^4 \rangle\end{aligned}\quad (9)$$

with $C' = \{S_w[\langle x^2 \rangle \langle r^4 \rangle - \langle xr^2 \rangle^2]\}^{-1}$. The covariance matrix (9) is defined with the vertex as the reference point.

3.3. Tracks in zero magnetic field

In case of zero magnetic field one fits a straight line and the equation of residuals is:

$$\varepsilon_i = -u_i \sin \phi + v_i \cos \phi + d. \quad (10)$$

Noticing again that the covariance matrix is invariant in rotation and translation normal to the trajectory, we compute the covariance matrix in the system where $\phi = d = 0$. A brief calculation gives

$$\begin{aligned}\sigma_{\phi\phi} &= (S_w \sigma_{xx})^{-1}, \\ \sigma_{\phi d} &= (S_w \sigma_{xx})^{-1} \langle x \rangle, \\ \sigma_{dd} &= (S_w \sigma_{xx})^{-1} \langle x^2 \rangle,\end{aligned}\quad (11)$$

where x is the coordinate measured along the track. One can see that the ϕ variance $\sigma_{\phi\phi}$ is invariant in translation of the origin along the trajectory whereas the impact parameter variance σ_{dd} is a function of the position. The error propagation of σ_{dd} is again made simply by choosing the new reference point as the origin in the formulae (11). One concludes also that the correlation between ϕ and d vanishes at the weighted centre of the trajectory.

From Eq. (11) one can derive an illustrative formula for the impact parameter error $\Delta d = \sqrt{\sigma_{dd}}$ in the straight line case:

$$\Delta d = S_w^{-1/2} \oplus \Delta\phi x_r, \quad (12)$$

where $\Delta\phi = \sqrt{\sigma_{\phi\phi}}$ and $x_r = x - \langle x \rangle$ is the distance of the reference point from the weighted centre of track. This means that the impact parameter error is a quadratic sum of the impact error at the centre of track ($= 1/\sqrt{S_w}$) and the product of direction error times the distance from the centre.

4. Straightening hypothesis

Gluckstern [1] makes the assumption that the errors are independent of curvature and derives the 2×2 error matrix elements $\sigma_{\rho\rho}$, $\sigma_{\rho\phi}$ and $\sigma_{\phi\phi}$ by fitting parabola to a sample of hits assumed to be measured in zero magnetic field. Gluckstern states that this ‘straightening hypothesis’ can be proved, but he bypasses the proof. In the following we show that this hypothesis is an approximation valid at the limit of small ρL where L is the track length in xy .

4.1. Counter examples

In order to show that Gluckstern’s hypothesis is inexact at large curvature we compute $\sigma_{\rho\rho}$ of Eq. (8) in special cases and compare the result with the one obtained with a straightened track. Let us take a half circle going through the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ for which $\rho = 1$. When straightened, the corresponding points become $(-\frac{1}{2}\pi, 0)$, $(0, 0)$ and $(\frac{1}{2}\pi, 0)$. Assuming equal precision ε (normal to the trajectory) for all points we get

$$\sigma_{\rho\rho}(\text{exact}) = \frac{\varepsilon^2}{2};$$

$$\sigma_{\rho\rho}(\text{Gluckstern}) = \frac{96\varepsilon^2}{\pi^4} \simeq 0.99\varepsilon^2$$

which implies about factor 2 difference. When adding $(0, 2)$ as a fourth point, the trajectory makes three quarters of a full circle and we get

$$\sigma_{\rho\rho}(\text{exact}) = \frac{\varepsilon^2}{4};$$

$$\sigma_{\rho\rho}(\text{Gluckstern}) = \frac{81\varepsilon^2}{(1.5\pi)^4} \simeq 0.16\varepsilon^2.$$

In the following section we take a closer look and find the validity range of the straightening hypothesis.

4.2. Validity range

In order to simplify the calculation we study the validity range of Gluckstern straightening hypothesis in the special case in which: (1) points are uniformly spaced, (2) equally precise and (3) the

number of points N is large. We calculate the curvature variance $\sigma_{\rho\rho}$ in Eq. (8) as a function of the track length and compare the result with the Gluckstern formula.

Since $\sigma_{\rho\rho}$ is invariant under translation of origin along the trajectory, we place the origin at the centre of track. It follows then from the assumptions (1) and (2) that the terms $\langle x \rangle$ and $\langle xr^2 \rangle$ vanish so that $\sigma_{\rho\rho}$ in Eq. (8) becomes:

$$\sigma_{\rho\rho} = \frac{4}{N} \varepsilon^2 \frac{1 - \rho^2 \langle x^2 \rangle}{\langle r^4 \rangle - \langle r^2 \rangle^2}. \quad (13)$$

To calculate the mean values in Eq. (13) we use the assumption (3) and replace necessary summations by integration. Using $x = 2R \sin \varphi \cos \varphi$ and $r = 2R \sin \varphi$ which are valid when the trajectory is tangential to the x -axis at the origin (see Fig. 1), we can perform the necessary integrations along the trajectory as a function of the azimuth angle φ . The φ variable translates to the path length variable s through $\varphi = -\frac{1}{2}\rho/s$ so that the result is a function of the track length L .

The calculation is fairly lengthy and the answer is a somewhat complicated trigonometric function for which we have derived a series expansion including terms up to the second order in ρL :

$$\sigma_{\rho\rho} = \frac{720}{N} \frac{\varepsilon^2}{L^4} \left[1 - \frac{1}{21} (\rho L)^2 + \dots \right]. \quad (14)$$

Similar expansions can be obtained for the other elements of the covariance matrix.

In the limit $|\rho L| \rightarrow 0$ the Eq. (14) approaches the Gluckstern [1] formula derived under the same conditions. One concludes that with the straightening hypothesis the curvature variance gets overestimated by a factor of about $|\rho L|^2/21$ which is less than 2% for $|\rho L| < 0.65$. This is equivalent to saying that the estimate of $\Delta p_t/p_t$ by straightening method is in error by less than 1% for tracks with total turning angle less than about 40 degrees.

Usually one is interested in the detector performance at the high p_t limit. The above condition implies $p_t [\text{GeV}] = 0.3BL/|\rho L| > 0.46B[\text{T}] L[\text{m}]$. For as high as a 4 Tesla field and 1 m projected track length the limit is as low as $p_t > 1.8 \text{ GeV}$.

5. Practical error formulae

In the previous section we showed that for practical purposes one gets precise error formulae by performing the calculation in the trajectory system, i.e. as a function of length along the track, and neglecting terms higher or equal to second order in ρL . The elements of the curvature, direction and impact parameter covariance matrix become then:

$$\begin{aligned} \sigma_{\rho\rho} &= 4D[\langle s^2 \rangle - \langle s \rangle^2] \quad (= 4D\sigma_{ss}), \\ \sigma_{\rho\phi} &= 2D[\langle s^3 \rangle - \langle s \rangle \langle s^2 \rangle] \quad (= 2D\sigma_{ss^2}), \\ \sigma_{\phi\phi} &= D[\langle s^4 \rangle - \langle s^2 \rangle^2] \quad (= D\sigma_{s^2s^2}), \\ \sigma_{\rho d} &= 2D[\langle s \rangle \langle s^3 \rangle - \langle s^2 \rangle^2], \\ \sigma_{\phi d} &= D[\langle s \rangle \langle s^4 \rangle - \langle s^2 \rangle \langle s^3 \rangle], \\ \sigma_{dd} &= D[\langle s^2 \rangle \langle s^4 \rangle - \langle s^3 \rangle^2], \end{aligned} \quad (15)$$

where $D = \{S_w[\sigma_{ss}\sigma_{s^2s^2} - (\sigma_{ss^2})^2]\}^{-1}$ and s denotes the path length coordinate. With these formulae the particle measurement precisions can be calculated with just a few lines of program code. They are valid for any spacing of points and variable detector resolutions. The formulae are in error by less than 1% for trajectory turning angle L/R up to about 0.65 radians.

For small curvatures one can further simplify Eq. (15) by replacing the path length s by the radial coordinate: $s_i \leftarrow r_i$. The precision suffers a little so that the 1% accuracy for $\Delta\rho = \sqrt{\sigma_{\rho\rho}}$ is valid for $L/R < 0.3$ i.e. for $p_t > 4 \text{ GeV}$, if $B = 4 \text{ Tesla}$ and $L = 1 \text{ m}$.

In the following we consider a few special cases and derive simple expressions for the covariance matrix elements.

5.1. Uniform spacing of points

With equal spacing $\Delta s = L/(N-1)$ and equal weights of points we can derive even simpler expressions for the covariances (15). We have $s_i = s_0 + (i-1)\Delta s$. Hence the mean values in Eq. (15) involve computation of the four series $S_N(k) = \sum_{i=1}^N i^k$ where k takes values from 1 to 4. The calculation is a fairly straightforward algebra

and we give only the result in the following:

$$\begin{aligned}
 \sigma_{\rho\rho} &= \frac{720}{N} \frac{\varepsilon^2}{L^4} a_N, \\
 \sigma_{\rho\phi} &= -\frac{720}{N} \frac{\varepsilon^2}{L^3} a_N q, \\
 \sigma_{\phi\phi} &= \frac{720}{N} \frac{\varepsilon^2}{L^2} a_N \left[q^2 + \frac{1}{60} b_N \right], \\
 \sigma_{\rho d} &= \frac{360}{N} \frac{\varepsilon^2}{L^2} a_N \left[q^2 - \frac{1}{12} c_N \right], \\
 \sigma_{\phi d} &= -\frac{360}{N} \frac{\varepsilon^2}{L} a_N \left[q^2 - \frac{1}{20} d_N \right] q, \\
 \sigma_{dd} &= \frac{180}{N} \varepsilon^2 a_N \left[q^4 - \frac{1}{10} q^2 d_N + \frac{1}{80} e_N \right], \quad (16)
 \end{aligned}$$

where ε is the detector resolution and the ‘propagation parameter’ q is defined as $q = (s - \langle s \rangle)/L$. The constants

$$\begin{aligned}
 a_N &= \frac{(N-1)^3}{(N^2-4)(N+1)}, & b_N &= \frac{N^2-4}{(N-1)^2}, \\
 c_N &= \frac{N+1}{N-1}, & d_N &= \frac{N^2+1}{(N-1)^2}, \\
 e_N &= c_N \frac{3N^2-7}{3(N-1)^2}
 \end{aligned}$$

all approach to unity in the limit of large number of points N . Gluckstern [1] has derived equivalent formulae for $\sigma_{\rho\rho}$, $\sigma_{\rho\phi}$ and $\sigma_{\phi\phi}$ at the beginning of the track ($q = -0.5$).

For comparison we give also the 2×2 covariance matrix for the straight line fit ($B = 0$) with equal spacing and precision of points:

$$\begin{aligned}
 \sigma_{\phi\phi} &= \frac{12}{N} \frac{\varepsilon^2}{L^2} f_N, & \sigma_{\phi d} &= \frac{12}{N} \frac{\varepsilon^2}{L} q f_N, \\
 \sigma_{dd} &= \frac{1}{N} \varepsilon^2 (1 + 12q^2 f_N), \quad (17)
 \end{aligned}$$

where $f_N = (N-1)/(N+1)$. Notice that the variance of the impact parameter increases quadratically as a function of the extrapolation length q . This is to be compared with the case of non-zero magnetic field where the increase goes as fourth power in q .

5.2. Best curvature precision

Suppose we have N detectors with equally good point resolution ε . Then one may ask what kind of detector spacing gives the best momentum resolution. In the following we consider the problem for both unconstrained and vertex constrained fits.

5.2.1. Unconstrained fit

We calculate $\sigma_{\rho\rho}$ in the frame of reference where $\langle s \rangle = 0$ so that according to Eq. (15) we have

$$\sigma_{\rho\rho} = \frac{4\langle s^2 \rangle}{S_w[\langle s^2 \rangle \sigma_{s^2 s^2} - \langle s^3 \rangle^2]}. \quad (18)$$

It turns out that for minimal $\sigma_{\rho\rho}$ the points must be placed symmetrically about $\langle s \rangle = 0$ so that $\langle s^3 \rangle$ vanishes. Then the curvature variance is

$$\sigma_{\rho\rho} = \frac{4}{S_w(\langle s^4 \rangle - \langle s^2 \rangle^2)}. \quad (19)$$

Minimizing Eq. (19) with respect to all s_i with $|s_i| < 0.5L$ (inner points) implies $s_i(s_i^2 - \langle s_{in} \rangle^2) = 0$. This is possible only if $s_i = 0$, $i = 1, \dots, M$, i.e. all $M < N$ inner points must be placed at the centre of track. Inserting this result into Eq. (19) we find that the best curvature precision is

$$\sigma_{\rho\rho} = \frac{64\varepsilon^2}{L^4} \frac{N}{M(N-M)}. \quad (20)$$

To optimize Eq. (20) with respect to the number of inner points one has to determine M such that the product $M(N-M)$ becomes maximal for fixed N . For example, if the total number of points N is divisible by 4, the smallest possible curvature variance is obtained when $N/2$ measurements are at the centre of track and $N/4$ at both ends. Then the curvature variance is

$$\sigma_{\rho\rho} = \frac{256\varepsilon^2}{NL^4}. \quad (21)$$

This result was obtained by Gluckstern [1] with a different approach. The more general formula (20) should be used, however, if N is not a multiple of 4. For instance if $N = 11$, the best distribution of the number of measurements is 3-5-3 at the beginning-centre-end positions.

5.2.2. Vertex constrained fit

In this case we calculate $\sigma_{\rho\rho}$ with the origin as the reference point. Using the path length coordinate s we get from Eq. (9):

$$\sigma_{\rho\rho} = \frac{4\langle s^2 \rangle}{S_w[\langle s^2 \rangle \langle s^4 \rangle - \langle s^3 \rangle^2]}. \quad (22)$$

Minimizing this with respect to the inner points $0 < s_i < L$ we find that

$$s_i = s_0 = 0.5\langle s^3 \rangle / \langle s^2 \rangle \quad (23)$$

which implies that for the best momentum precision the inner measurements should be concentrated at the same position: $s_i = \beta L$, $i = 1, \dots, M$ with $0 < \beta < 1$ and $M < N$. Inserting $s_0 = \beta L$ in Eq. (23) we derive the following 3rd order equation for the optimal position β for any given M :

$$\frac{M}{N-M}\beta^3 + 2\beta - 1 = 0. \quad (24)$$

One can find the optimal fraction $P = M/N$ by considering P as a continuous variable (large N) and minimizing Eq. (22) with respect to it. A brief calculation gives

$$P = \frac{1}{\sqrt{2}} \quad \text{and} \quad \beta = \sqrt{2} - 1 \quad (25)$$

giving the best curvature variance of

$$\sigma_{\rho\rho} = (68 + 48\sqrt{2}) \frac{\varepsilon^2}{NL^4} \simeq \frac{136\varepsilon^2}{NL^4}. \quad (26)$$

This means that the best transverse momentum precision in the case of vertex constrained fit is obtained by concentrating about 71% of the measurements near the position $s_0 = 0.41L$ and the remaining 29% at the end of the trajectory.

It should be borne in mind, however, that efficient pattern recognition requires more uniform spacing of detected points. A possible scenario to optimize detector performance and cost would be to place precise ‘momentum analyzer’ detector layers as discussed above and less precise (and less expensive) ‘pattern analyzer’ layers in the intermediate positions.

6. Discussion

6.1. Examples

In the following we illustrate the usage of the formulae derived above with a couple of examples. The transverse momentum is a function of curvature and magnetic field: $p_t[\text{GeV}] = 0.3B/\rho[\text{T}][\text{m}]$. It follows that the relative error on transverse momentum reads:

$$\frac{\Delta p_t}{p_t} = \frac{\Delta \rho}{0.3B} p_t, \quad (27)$$

where $\Delta \rho$ depends on the detector resolutions and spacing. Utilizing Eq. (15) and taking the high p_t limit so that we can use the radial coordinates r instead of s we get

$$\frac{\Delta p_t}{p_t} = \frac{[S_w(\sigma_{rr^2} - (\sigma_{rr})^2/\sigma_{rr})]^{-1/2}}{0.3B} p_t. \quad (28)$$

Let us assume that there are 5 inner points measured with 20 μm precision and 6 outer points measured with 50 μm precision in a 4 T field. We assume further that the first point is at 0.05 m from the origin and that the point spacing is constant 0.1 m. The resulting precisions are then:

$$\begin{aligned} \frac{\Delta p_t}{p_t} &= 0.17 p_t[\text{TeV}], \quad \Delta \phi = 0.1 \text{ mrad}, \\ \Delta d &= 20 \mu\text{m}, \end{aligned} \quad (29)$$

where $\Delta \phi$ and Δd are also calculated from Eq. (15). Adding the vertex point and assuming 10 μm precision gives: $\Delta p_t/p_t = 0.13 p_t[\text{TeV}]$, $\Delta \phi = 60 \mu\text{rad}$ and $\Delta d = 9 \mu\text{m}$.

6.2. Full covariance matrix

So far we have considered the projection of the particle trajectory on the bending plane. The trajectory in 3D space is described by five parameters for which we choose ρ , ϕ and d and the following two parameters: λ the angle between the trajectory and the xy plane (‘dip’ angle) and z_0 the z -coordinate of the d.c.a. point.

With this parametrization the 3D coordinates of the d.c.a. point on the trajectory are $(d \sin \phi, -d \cos \phi, z_0)$. The analytic calculation of the full

5×5 covariance matrix is not quite feasible. Instead, it is a straightforward calculation to derive the five parameters covariance matrix as a composition of diagonal blocks of a 3×3 matrix (for ρ , ϕ , d) derived above and a 2×2 matrix (for λ , z_0). In this representation one ignores the possible correlation between the xy and z measurements, but the result is normally a very good approximation. For completeness we give the explicit covariance terms of the fitted λ and z_0 in the limit of large p_t :

$$\begin{aligned}\sigma_{\lambda\lambda} &= (S_v \sigma_{rr})^{-1} [1 + (\sigma_{zr}/\sigma_{rr})^2]^{-2}, \\ \sigma_{\lambda z_0} &= -(S_v \sigma_{rr})^{-1} [1 + (\sigma_{zr}/\sigma_{rr})^2]^{-1} \langle r \rangle, \\ \sigma_{z_0 z_0} &= (S_v \sigma_{rr})^{-1} \langle r^2 \rangle,\end{aligned}\quad (30)$$

where S_v is the sum of z weights: $S_v = \sum \Delta z_i^{-2}$ and r is the radial coordinate as before. For fitted λ and z_0 we have $\tan \lambda = \sigma_{zr}/\sigma_{rr}$ and $z_0 = \langle z \rangle - \langle r \rangle \tan \lambda$.

6.3. Multiple scattering

To a good approximation the multiple scattering (m.s.) contribution to the trajectory errors can be added quadratically to the errors due to detector resolution to obtain the total error. A more precise treatment of the m.s. contribution should be done numerically using e.g. the method described in Ref. [5].

Here we just take an example for curvature variation due to m.s. which is given by [6]

$$\Delta\rho(\text{m.s.}) \simeq \frac{0.016[\text{GeV}][\text{m}]}{\beta p_t L_s \cos \lambda} \sqrt{L_s/L_0} \quad (31)$$

where L_0 is the radiation length, L_s is the trajectory length in space and β is the particle velocity. Taking the above example with no vertex point and assuming that the detector ‘thickness’ in units of radiation lengths is $L_s/L_0 = 20\%$ at $\lambda = 0$ we get (for $\beta \simeq 1$):

$$\frac{\Delta p_t}{p_t} \simeq 0.006 \oplus 0.17 p_t [\text{TeV}]. \quad (32)$$

We see that in this case multiple scattering becomes significant for $p_t < 30 \text{ GeV}$ and the relative transverse momentum error levels off to a constant value at lower transverse momenta.

7. Summary

We have derived the explicit covariance matrix (8) of particle trajectory curvature, azimuth angle and impact parameter valid for full range of curvatures at the limit of negligible multiple scattering. The formulae are valid for hybrid detector systems with variable measurement precisions. We consider a number of interesting special cases: homogeneous detector system, inclusion of the vertex constraint and the case of zero magnetic field. We also consider the Gluckstern ‘straightening hypothesis’ and show that it is an approximation good for small curvatures. We derive simplified covariance formulae (15) valid for high transverse momenta and discuss some practical examples. We also discuss briefly the effect of multiple scattering.

Appendix A. Inversion of the weight matrix

In the following we show the main steps in the calculation of the covariance matrix elements (8). With derivatives (7) the weight matrix (4) becomes

$$W = S_w \times \begin{pmatrix} \frac{1}{2}\langle r^4 \rangle & -\frac{1}{2}\langle xr^2 \rangle & \frac{1}{2}\langle r^2 \rangle + \frac{1}{2}\rho\langle yr^2 \rangle \\ -\frac{1}{2}\langle xr^2 \rangle & \langle x^2 \rangle & -\langle x \rangle - \rho\langle xy \rangle \\ \frac{1}{2}\langle r^2 \rangle + \frac{1}{2}\rho\langle yr^2 \rangle & -\langle x \rangle - \rho\langle xy \rangle & 1 + 2\rho\langle y \rangle + \rho^2\langle y^2 \rangle \end{pmatrix}. \quad (\text{A.1})$$

where we use the notation $S_w = \sum w_i$. Using the identities

$$\begin{aligned}\frac{1}{2}\rho\langle r^2 \rangle &= -\langle y \rangle, & \frac{1}{2}\rho\langle r^4 \rangle &= -\langle yr^2 \rangle, \\ \frac{1}{2}\langle xr^2 \rangle &= -\langle xy \rangle\end{aligned} \quad (\text{A.2})$$

which follow from Eqs. (3) and (5), the matrix (A.1) transforms to

$$W = S_w \times \begin{pmatrix} \frac{1}{2}\langle r^4 \rangle & -\frac{1}{2}\langle xr^2 \rangle & \frac{1}{2}\langle r^2 \rangle - \frac{1}{2}\rho^2\langle r^4 \rangle \\ -\frac{1}{2}\langle xr^2 \rangle & \langle x^2 \rangle & -\langle x \rangle + \frac{1}{2}\rho^2\langle xr^2 \rangle \\ \frac{1}{2}\langle r^2 \rangle - \frac{1}{2}\rho^2\langle r^4 \rangle & -\langle x \rangle + \frac{1}{2}\rho^2\langle xr^2 \rangle & 1 - \rho^2\langle x^2 \rangle \end{pmatrix}. \quad (\text{A.3})$$

Notice that Eq. (3) is valid for the transformed coordinates with solutions (5).

In order to invert (A.3) we have to calculate $\det W$. After a brief manipulation of the rows and columns the determinant becomes

$$\det W = S_w^3 \begin{vmatrix} \frac{1}{4}\langle r^4 \rangle & -\frac{1}{2}\langle xr^2 \rangle & \frac{1}{2}\langle r^2 \rangle \\ -\frac{1}{2}\langle xr^2 \rangle & \langle x^2 \rangle & -\langle x \rangle \\ \frac{1}{2}\langle r^2 \rangle & -\langle x \rangle & 1 + \Delta \end{vmatrix} \quad (\text{A.4})$$

with $\Delta = \rho^2 \langle y^2 \rangle - \frac{1}{2} \rho^4 \langle r^4 \rangle \simeq (\rho \bar{\epsilon})^2$ so that Δ is highly negligible and we can set $\Delta \simeq 0$ ($\bar{\epsilon}$ is the average point measurement error). Then with little further manipulation of rows and columns we get a remarkably simple expression for the determinant:

$$\begin{aligned} \det W &= S_w^3 \begin{vmatrix} \frac{1}{4}\sigma_{r^2 r^2} & -\frac{1}{2}\sigma_{xr^2} & 0 \\ -\frac{1}{2}\sigma_{xr^2} & \sigma_{xx} & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \frac{1}{4} S_w^3 [\sigma_{xx} \sigma_{r^2 r^2} - (\sigma_{xr^2})^2]. \end{aligned} \quad (\text{A.5})$$

The explicit expressions (8) of the covariance matrix elements are then readily derived from Eqs. (A.3) and (A.5).

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