Problem 1.13. Prove that fib(n) is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$ by proving:

$$fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Definitions

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & \text{otherwise} \end{cases}$$

Golden ratio property

$$\varphi^2 = \varphi + 1$$

Proof. I will induct on n

Base case (n = 0):

$$fib(0) = 0 = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{0}{\sqrt{5}} = 0$$

Base case (n = 1):

$$fib(1) = 1 = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Base case (n = 2):

$$fib(2) = fib(1) + fib(0) = 1 + 0 = 1 = \frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Inductive Hypothesis: Assume that for some $k \in \mathbb{N}_0$

$$fib(k) = \frac{\varphi^k - \psi^k}{\sqrt{5}}$$

Inductive Step: Show that

$$fib(k+1) = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

$$fib(k+1) = fib(k-1+1) + fib(k-2+1) \qquad \text{[By definition of series]}$$

$$= fib(k) + fib(k-1) = \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} \qquad \text{[By I.H]}$$

$$= \frac{\varphi^k + \varphi^{k-1} - \psi^k - \psi^{k-1}}{\sqrt{5}}$$

$$= \frac{\frac{\varphi^{k+1}+\varphi^k}{\varphi} - \frac{\psi^{k+1}+\psi^k}{\psi}}{\sqrt{5}} = \frac{\frac{\varphi^k(\varphi+1)}{\varphi} - \frac{\psi^k(\psi+1)}{\psi}}{\sqrt{5}}$$

$$= \frac{\frac{\varphi^k\varphi^2}{\varphi} - \frac{\psi^k\psi^2}{\psi}}{\sqrt{5}} = \frac{\varphi^k\varphi - \psi^k\psi}{\sqrt{5}} \qquad \text{[By G.R property]}$$

$$= \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

$\therefore$ By the principle of induction, the claim holds for all $n \in \mathbb{N}_0$ $\square$