NUMERICAL APPROXIMATION OF SINGULAR INTEGRALS ARISING FROM TOPOLOGICAL DEFECTS IN AMBIENT REAL SPACE: THE HASIMOTO SOLITON

KATE HARVEY AND SCOTT STRONG

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, COLORADO SCHOOL OF MINES

INTRODUCTION

Imagine a dolphin blowing a bubble ring. The ring drifts through water, changing shape as it goes. We can think of this bubble as a vortex, and its autonomous nature along with the way it couples with its surrounding fluid, make it self-inducing, or responsible for its own movement. Now, take the bubble and compress it until all that remains is a 2-dimensional circle. This circle is a vortex filament.

Vorticity can be defined as $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$, where $\boldsymbol{\nabla}$ is the curl of the vortex and $\boldsymbol{v}$ is the velocity field the vortex lives in. We want to know what $\boldsymbol{v}$ is so we can accurately simulate the movement of the vortex filament. To retrieve $\boldsymbol{v}$ from the vorticity equation, we use the Biot-Savart integral (BSI)[1], which acts as a left-inverse operator and pulls out the velocity field. Reducing the BSI volume integral to an integral over the vortex line yields the following equation[2]:

$$\boldsymbol{v}(\mathbf{r}, t) \propto \int_0^L \frac{(\mathbf{r} - \boldsymbol{\gamma}) \times d\boldsymbol{\gamma}}{|\boldsymbol{r} - \boldsymbol{\gamma}|^3} \quad (1)$$

What we are most interested in is the velocity right next to the filament. In our BSI equation, $\mathbf{r}$ is the position of the observer and $\boldsymbol{\gamma}$ is the parametrized filament. We see that if $\mathbf{r} = \boldsymbol{\gamma}$, the BSI contains a singularity. For simulation purposes, a regularization technique is needed. Current literature explores approximation techniques and the introduction of an ad-hoc parameter. This project explores those solutions with the goal of stable simulation and visualization.

METHODOLOGY

We begin with the Local Induction Approximation (LIA)[3], the lowest order approximation of the BSI, and simulate binormal flow using a first-order finite differencing method:

$$\boldsymbol{\gamma}_t = \kappa \hat{b} \ (2)$$

Where κ represents curvature. As an example, the Hasimoto soliton was parametrized and evolved (Fig. 1 and 2).

The LIA is an old regularization technique. We add the next order approximation, which we call the Generalized Local Induction Equation (GLIE)[4]:

$$\boldsymbol{\gamma}_t = \kappa(1 + \varepsilon\kappa^2)\hat{b} \ (3)$$

(Fig. 3 and 4).

As a comparable to GLIE, we introduce a method which uses the integral representation of the non-circulatory binormal flow[4]:

$$\boldsymbol{\gamma}_B = \kappa\left(\int_0^L \frac{\cos\theta - 1}{(c_1 + c_2\cos\theta)^{3/2}}\, d\theta\right)\hat{b} \ (4)$$

Where $c_1 = \varepsilon^2 - 2\varepsilon\cos\phi + 2$ and $c_2 = 2\varepsilon\cos\phi - 2$. (Fig. 5 and 6).

SIMULATIONS AND VISUALIZATIONS

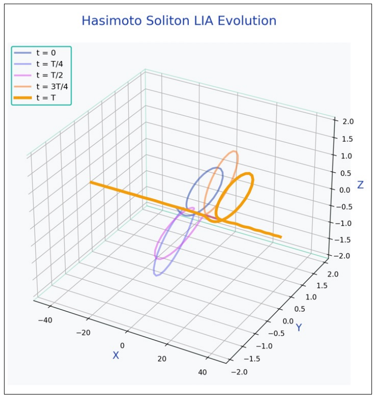

Figure 1. LIA evolution

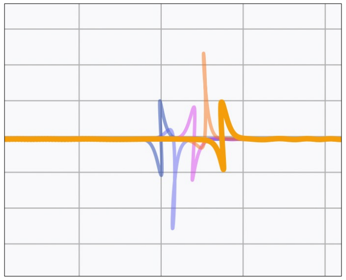

Figure 2. X-Z view of LIA evolution

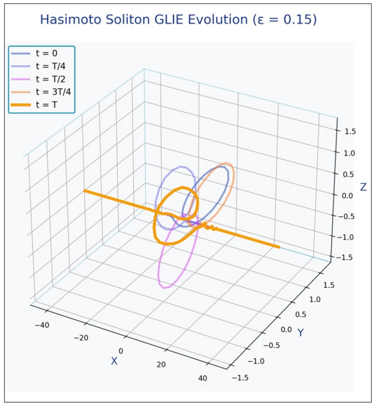

Figure 3. GLIE evolution

Figure 4. X-Z view of GLIE evolution

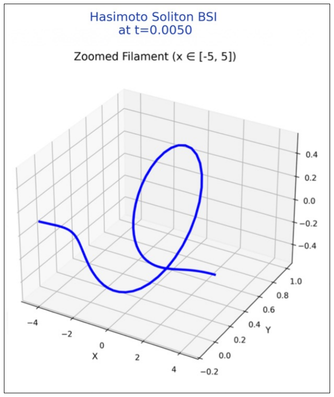

Figure 5. BSI evolution, ~50 time steps

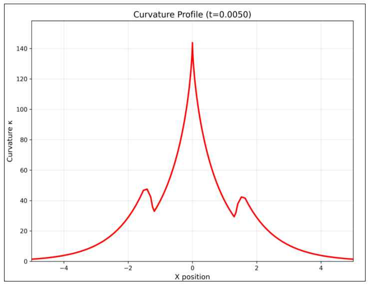

Figure 6. Curvature of BSI Evolution

DISCUSSION

Looking at Figures 4 and 6, we notice small spikes appearing with the introduction of a non-linear term. With curvature plots, this spike becomes more apparent and is not a result of computational instability.

Simulations for Figures 5 and 6 require a lot of computational power and run slowly. This is due to the complexity of the integral in equation 4 and computing across the entire mesh.

FUTURE WORK

Future work includes goals of:
- Deeper analysis of the spikes seen in curvature plots
- Handling instabilities and improving optimization for Equation 4
- Implementing the full integral from Equation 1

REFERENCES

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[3] L. S. Da Rios, "Sul moto dei filetti vorticosi di forma qualunque (On the motion of vortex filaments of any shape)," *Rendiconti del Reale Istituto Lombardo di Scienze e Lettere*, vol. 18, pp. 75–79, 1909.

[4] S. A. Strong and L. D. Carr, "Generalized local induction equation, elliptic asymptotics, and simulating superfluid turbulence," *Journal of Mathematical Physics*, vol. 53, no. 3, p. 033102, Mar. 2012, doi: 10.1063/1.3696689.