Notes on Mathematical Modelling

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0.1 Acknowledgements

These notes are based on the lectures of Carolin Kreisbeck (c.kreisbeck@uu.nl) during Fall of 2019 at Universiteit Utrecht. The lectures were intended to serve as a preparation for the reading of the texbook of the course [1].

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Contents

Chapter 1

Basic tools for mathematical modelling

Teaching for this chapter started on Monday, 2019.11.11 (week 46a) and ended on Monday, 2019.11.18. This chapter corresponds to part of chapter 1 in [1].

In this introductory chapter we will introduce the mindset that we should have when trying to **translate a specific problem** from the natural sciences, the social sciences or technology into a **well-defined mathematical problem**[1].

TODO: some context and general pointers would probably look good here.

1.1 Case study: population dynamics

Suppose we want to model the change in population (i.e. number of individuals) in an environment over a period of time. First thing we need is to make some assumptions about what's really happening here. We might, for example, make the following assumptions[2].

1. growth rate independent of population size (unlimited growth possible, neglecting e.g. limited resources)

2. growth rate independent of time (neglecting time-dependence due to e.g. influence of enemies, economical or cultural changes)

3. population within closed systems (neglecting e.g. migration)

4. assuming an equal distribution of male and female, age distribution not considered

5. continuous model with non-integer solutions (idealization reasonable for very large populations, for small populations stochastic effects have to be taken into account)

[1]This is the definition of *mathematical modelling* given in [1, p. 1] with Kreisbeck's emphasis.

[2]Stolen from [2].

After this, we name the quantities that intervene in our problem. We will use t for time, $x(t)$ for the number of individuals (population) at time t and $\frac{dx}{dt}(t)$ or $x'(t)$ for the rate of change in population. To model the change we introduce the quantities

- $b(t, \Delta t)$ for the increase of population during the time interval $(t, \Delta t)$, and
- $d(t, \Delta t)$ for the decrease of population during the time interval $(t, \Delta t)$.

Therefore the population at time $t + \Delta t$ is given by

$$x(t + \Delta t) = x(t) + b(t, \Delta t) - d(t, \Delta t).$$

That Δt desperately wants us to take the limit as $\Delta t \to 0$ and so we do

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{b(t, \Delta t}{\Delta t} - \lim_{\Delta t \to 0} \frac{d(t, \Delta t}{\Delta t}.$$

Note that here we're assuming the limit really does exist, which is quite a big assumption… Rename,

$$B(t) = \lim_{\Delta t \to 0} \frac{b(t, \Delta t}{\Delta t} \text{ and } D(t) = \lim_{\Delta t \to 0} \frac{d(t, \Delta t}{\Delta t}$$

and use the definition of derivative to get

$$\frac{dx}{dt}(t) = x'(t) = B(t) - D(t)$$

where $B(t)$ and $D(t)$ being the rates at which the population increasing, resp. decreases at time t. Recall that we assumed that the rates of change in the population were independent of time and population size. This is equivalent to saying that $B(t)$ and $D(t)$ are really constants which gives us the final model

$$x'(t) = \beta - \delta \implies x(t) = (\beta - \delta)t.$$

The previous is a not-particularly-interesting ODE with solution

$$x(t) = (\beta - \delta)x + C.$$

This model has a lot of shortcomings, first of all, it does not account for the size of the population in the rates of change. But, one might argue that the more individuals there are in a population the greater the rates of change are. We can go back and restate assumption one as "population increase, resp. decrease in the time interval $(t, \Delta t)$ is directly proportional to the population at time t and the time passed". This in turn gives us

$$b(t, \Delta t) = \beta x(t)\Delta t \text{ and } d(t, \Delta t) = \delta x(t)\Delta t.$$

Taking the limit as before leads us to the model

$$x'(t) = (\beta - \delta)x(t).$$

This is another ODE, this time a bit more interesting, with solution

$$x(t) = Ce^{(\beta - \delta)t}.$$

Although a bit better, you can probably see that this model explodes as time passes since it does not include any provisions for when the population turns stupidly large. Anyhow, it is common enough that it deserves its own name: the **exponential growth model**.

A small step in the right direction would be to account for a population limit in the system, i.e. number of individuals that flips the rate of growth. More precisely, let's change assumption one to "there is a number x_M that is the maximum population in the system (sometimes called the *carrying capacity* and that the rate of change in population $q = \beta - \delta$ is positive if $q < x_M$ and negative if $p > x_M$." Now we get

$$b(t, \Delta t) - d(t, \Delta t) = q(x_M - x(t))\delta t, \qquad \text{for } q,\ x_M > 0.$$

Taking the limit again we get or final model for now, the **logistic growth model**

$$x'(t) = q x_M x(t) - q x^2(t).$$

This ODE can be solved explicitly and the solution is

$$x(t) = \frac{x_M x_0}{x_0 + (x_M - x_0)e^{-x_M q(t-t_0)}},$$

where t_0 is the initial time and $x_0 = x(t_0)$ is the initial population.

1.2 Dimensional analysis and non-dimensionalisation

1.2.1 Non-dimensionalisation when there are several options. The projectile problem.

1.3 Asymtotic expansion method

1.3.1 Error estimation

Chapter 2

Linear systems of equations

2.1 Modelling electrical netwroks

Chapter 3

Ordinary differential equations

Teaching started on Monday 2019.11.25 (week 48a).

This chapter corresponds to part of chapter 4 in [1].

3.1 Quantitative analysis of models in population dynamics

Recall from week 46 (chapter 1) that we had two models for population dynamics.

- The first one, the exponential model was described by

  $$x'(t) = px(t), \quad p \in \mathbb{R},$$

  where p was the growth rate.

- The second, the constrained model was described by

  $$x'(t) = qx_M x(t) - qx^2(t), \quad q, x_M \in \mathbb{R}$$

  where $q > 0$ was the growth rate and x_M was the maximum carrying capacity of the environment in number of individuals.

Both of these models share the common mathematical structure of an autonomous equation, i.e. an equation of the form

$$x'(t) = f(x(t)), \tag{3.1}$$

where f (read x') does not depend explicitly[1] on t.

In this section we will focus on the qualitative aspects of the model, i.e. what information can we get from it without explicitly solving the equations (which in this case we can, but in the next examples we won't).

[1]That is, f cannot *unwrap* t out of $x(t)$ and do anything with it alone, it has to work on $x(t)$ as its variable.

Recall that a **stationary solution** of an ODE is one that stays constant in time, i.e. of the form $x(t) = c$. How can we find them? Easy, if x is constant then we must have $x'(t) = f(x(t)) = 0$. For our previous models this means

- $x(t) = 0$ for the exponential model, and
- $x(t) = 0$ or $x(t) = x_M$ for the second model. We get these two solutions from solving
$$x'(t) = qx_M x(t) - qx^(t) = 0$$
for $x(t)$ using the well known quadratic formula.

We are interested in these solutions because they are predictable and *don't blow up* as time passes. Later in this chapter we will formally define the concept of stability and quantify how stable solutions are based on how close they are to the stationary solutions.

3.1.1 Introduction to linear stability analysis

For now we will settle with something called **linear stability analysis**. The main idea is to linearise the solution (i.e. Taylor expand up to degree 1) a stationary solution. Let x^* be a stationary solution to an autonomous problem of the form 3.1. The linear expansion we are talking about is

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + O(|x - x^*|).$$

To make things easier, let us take $y(t) = x(t) - x^*(t)$. We shall ignore the error term $O(|x - x^*|) = O(y(t))$ and thus we get

$$y'(t) = f'(x^*)y(t). \tag{3.2}$$

Now, 3.2 is trivial to solve explicitly—it is a linear homogeneous equation with constant coefficients
$$y(t) = ce^{f'(x^*)t}.$$

Intuitively, as $t \to \infty$ we have

$$|y(t)| \to 0 \implies |x(t) - x^*(t)| \to 0 \iff x(t) \to x^*(t),$$

i.e. the linearised solution $y(t)$ converges to the stationary solution $x^*(t)$.

More on this later.

3.2 Predator–prey models

3.2.1 Derivation of the Lotka–Volterra equations

Now we turn our attention to environments where there are two species and one eats/hunts/harvests the other. Let us model this from scratch to get yet another example of how things work in Mathematical modelling. For this derivation we shall use the following assumptions.

1. The prey population has unlimited resources available for its growth all the time.

2. The predator population feeds exclusively on the prey population.

3. TODO: Something i cant remember.

4. The rate of growth of the populations is proportional to their size.

5. The environment is stable over time.

We now proceed with the standard recipe for deriving models.

1. **Name the quantities involved in the problem.** We are trying to model how two populations change over time so we need

$$\begin{aligned} t &:= \text{time} \\ x_1(t) &:= \text{size of prey population at time } t \\ x_2(t) &:= \text{size of predator population at time } t \end{aligned}$$

2. **Find relations between the quantities.** From assumption X we now that the growth of both species is directly proportional to the size of the populations, in other words

$$x_1' = p_1 x_1 \text{ and } x_2' = p_2 x_2.$$

A priori, we don't know if p_1 depends only on t or on $x_2(t)$ or on both. The possibility that p_1 depends on $x_1(t)$ is ruled out by the assumption that growth is proportional to size. Looking at the assumptions once more we find that p_1 cannot depend on t since "the environment is stable over time". There fore it must be that p_1 is a function only of $x_2(t)$, which really makes sense, since the size of the prey species depends on how many individuals are being eaten by the predator species. A similar argument for p_2 yields

$$p_1(x_2(t)) \text{ and } p_2(x_1(t)).$$

But what do these functions p_1 and p_2 look like? Well, the prey population naturally grows since we assumed unlimited resources but at the same time it is being eaten at some rate by the predator population. Similarly, the predator population naturally dies unless they can feed on the prey population. We introduce the parameters $\alpha, \beta, \gamma, \delta > 0$ and formalise these relations with

$$p_1(x_2(t)) = -\beta x_2(t) + \alpha \qquad \text{and} \qquad p_2(x_1(t)) = \delta x_1(t) - \gamma.$$

Finally we get our model, commonly referred to as the Lotka–Volterra equations, derived independently by both authors from around 1920 to around 1925 [3].

$$\begin{cases} x_1' &= (\alpha - \beta x_2) x_1 \\ x_2' &= (\delta x_1 - \gamma) x_2 \end{cases}. \tag{3.3}$$

The mathematical structure of this problem is that of an autonomous planar system of ODEs. We may rewrite it as

$$\begin{cases} \mathbf{x}' & = f(\mathbf{x}) \\ \mathbf{x}(t_0) & = \mathbf{x_0} \end{cases} \tag{3.4}$$

with $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. If f is sufficiently nice (i.e. locally Lipschitz) then an initial value problem of the form in 3.4 has locally unique solutions. However, it is not the explicit solutions that interest us right now, but rather the qualitative aspects of their behaviour.

3.2.2 Qualitative analysis of the Lotka–Volterra equations

We look into the stationary solutions to later look at stability. Once more, setting $x_1, x_2 = 0$ in 3.3 gives us the stationary solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha/\beta \\ \gamma/\delta \end{pmatrix}.$$

There are too many parameters to work comfortably with this solutions. Let us non-dimensionalise before moving on to get rid of as many parameters as we can. Very quickly, we choose the fundamental dimensions T for time and N for number of individuals and carry out a dimensional analysis to get

$$\begin{matrix} [t] & = T & & [x_1] = [x_2] & = N \\ [\alpha] & = [\gamma] = \frac{1}{T} & & [\beta] = [\delta] & = \frac{1}{NT} \end{matrix}.$$

We choose the characteristic quantities $\overline{x_1}, \overline{x_2}$ and $\bar{t}$ and set up the change of variables

$$z_1 = \frac{x_1}{\overline{x_1}}, \qquad z_2 = \frac{x_2}{\overline{x_2}} \qquad \text{and} \qquad \tau = \frac{t}{\bar{t}}.$$

Substitute with care in 3.3 (careful with the derivatives) to get

$$\begin{cases} z_1' & = \bar{t}\alpha z_1 - \beta\overline{x_2}\bar{t}z_1 z_2 \\ z_2' & = \delta\overline{x_1}\bar{t}z_1 z_2 - \gamma\bar{t}z_2 \end{cases}.$$

Notice how we have four different coefficients for z_1 and z_2 but only have three characteristic quantities. This means we'll need to make a compromise. Which one to make is dictated by our taste and the mathematical or biological interpretation of the parameters we choose. We will not do all four options here but the Lotka–Volterra equations often come with

$$\bar{t}\alpha = 1, \qquad \beta\overline{x_2}\bar{t} = 1 \qquad \text{and} \qquad \delta\overline{x_1}\bar{t} = \gamma\bar{t},$$

which in turn give us the **non-dimensionalised version of the Lotka–Volterra equations**

$$\begin{cases} z_1' & = (1 - z_2)z_1 \\ z_2' & = a(z_1 - 1)z_2 \end{cases}, \tag{3.5}$$

where there is only one parameter $a = \gamma/\delta$.

Bibliography

[1] C. Eck, H. Garcke, and P. Knabner, *Mathematical modeling.* Springer, 2017. 0.1, 1, 1, 3

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[3] A. A. Berryman, “The orgins and evolution of predator-prey theory,” *Ecology*, vol. 73, no. 5, pp. 1530–1535, 1992. [Online]. Available: https://esajournals.onlinelibrary.wiley.com/doi/abs/10.2307/1940005 2