Notes on Measure and integration

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0.1 Acknowledgements

These notes are based on the lectures of Karma Dajani (k.dajani1@uu.nl) during Fall of 2019 at Universiteit Utrecht. These lectures are in turn based on the book Measures, Integrals and Martingales by René L. Schiling (2nd edition).

Readers are asked to report errata to eliashernandis@gmail.com.

0.2 Why these notes?

As mentioned before, these notes are based on lectures which are themselves based on a book. In fact, the lectures are almost one-to-one to the book, and there are handwritten notes available online.

However, the process of writing these, as opposed to simply taking notes in class allows me to make sure I understand the connections between the material. In other courses, where the contents were not as well organised as in this one, I had to reorganise the contents of the corresponding version of the notes for that course. Even though this is mostly unnecessary in this course, the handwritten lecture notes allow for more formal proofs or verifications where there was no time to complete them. Also, the book suggest that the reader verify some assumptions made during the proofs of some remarks or lemmas. Hence this notes are most valuable as a complement to the book.

Another valuable thing about these notes are the solutions to the assignments that we were given during the course. These have been compiled from corrections given to me by the TAs[1].

0.3 Recommendations

The official course page is [1]. In addition, we were offered exercise sets and more material through Blackboard. Grades were also published there.

Always trust the book when there is a discrepancy. Though there may be errata (in fact, see [2] for a list of them), it is way more likely that I've made a mistake.

The solutions to problems of the second edition are available online, see [3]. All problems from the first edition are included in the second edition, albeit with a different numbering. Order is preserved, though.

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[1] If you want to see what a disaster it was when I started this subject, look at the scanned copies of submitted exercises...

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Chapter 1

σ-algebras

1.1 What is a σ-algebra?

Definition 1.1 (Sigma algebra)**.** Let X be a set and $\mathcal{A} \subset \wp(X)$ a collection of subsets of X. We say that $\mathcal{A}$ is a sigma algebra or σ-algebra (on X) if it satisfies these three properties:

1. It contains the set, $X \in \mathcal{A}$;
2. it is closed under set complement $A \in \mathcal{A} \implies X \setminus A = A^c \in \mathcal{A},\ \forall A \subset X$, and
3. it is closed under **countable**[a] union $(A_n)_{n\in\mathbb{N}} \in \mathcal{A} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

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[a] In these notes we shall follow the convention from [4, bottom of p. 7] and use countable to describe any set whose cardinality is less than or equal to that of the set of the natural numbers, i.e. $\#A \leq \#\mathbb{N}$. This means that countable does not necessarily mean infinite.

Definition 1.2 (Measurable space)**.** A measurable space is a pair $(X, \mathcal{A})$ where X is a set and $\mathcal{A}$ is a σ-algebra .

From now on we shall assume X is a set and $\mathcal{A}$ is a σ-algebra on X.

Remark 1.1.

- From properties 1 and 2 we have $\varnothing = X^c \in \mathcal{A}$.[1]
- From properties 2 and 3 we have that $\mathcal{A}$ is also closed under **countable** intersection. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then, using De Morgan's laws we have

$$\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c \right)^c \in \mathcal{A}$$

---

[1] In fact, sometimes we substitute property one for $\varnothing \in \mathcal{A}$. This is equivalent as long as still take complements with respect to the universe X, i.e. $A^c = X \setminus A$.

because $A_n^c \in \mathcal{A}$ because of 2, the union is included because of 3 and its complement is also included because of 2. When a set has this property, we sometimes say $\mathcal{A}$ is $\cap$-stable(see definition 3.3).

- Clearly, finite unions and intersections of sets in the σ-algebra end up in the σ-algebra . For the union of M sets take $A_n = \emptyset, \forall n > M$ and apply 3 (or the equivalent for the intersection).

Example 1.1. The power set $\wp(X)$ is the largest σ-algebra on X.

Example 1.2. $\{\emptyset, X\}$ is the smallest σ-algebra on X.

Example 1.3. Given a subset $A \subset X$, the smallest σ-algebra which contains information about A is $\{\emptyset, X, A, A^c\}$. More on this later.

Let X be an uncountable set. Then $\mathcal{A} = \{A \subset X \mid A \text{ is countable or } A^c \text{ is coutable }\}$ is a σ-algebra .

Proof. We shall prove the three properties of a σ-algebra

1. $X \in \mathcal{A}$ because $X^c = \emptyset$ is countable.

2. If $A \in \mathcal{A}$ and is countable the $A^c \in \mathcal{A}$ because $(A^c)^c$ is countable. If $A \in \mathcal{A}$ but A is not countable then A^c must be so, hence $A^c \in \mathcal{A}$.

3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. To see if the union is included we distinguish to cases:

   (a) If A_n is countable $\forall n \in N$ then $\bigcup_{n\in\mathbb{N}} A_n$ is also countable and therefore in $\mathcal{A}$.[2]

   (b) If there is any A_k that is uncountable, then the union is uncountable but

   $$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subset A_k^c$$

   Recall that A_k^c must be countable because $A_k \in \mathcal{A}$ and therefore $\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c$ must also be countable. Thus, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

□

Example 1.4 (Trace σ-algebra)**.** Let $(X, \mathcal{A})$ be a measurable space and let $E \subset X$. Then

$$\mathcal{A}_E = \{A \cap E \mid A \in \mathcal{A}\} \tag{1.1}$$

is a σ-algebra on E.

Proof.

1. Clearly, $E = X \cap E \in \mathcal{A}_E$ since $X \in \mathcal{A}$.

---

[2] Recall that the countable union of countable sets is still a countable set.

2. For the second property we need to take care to take the complement relative to E, as we want $\mathcal{A}_E$ to be a σ-algebra on E. Here we reserve A^c to denote the complement relative to X and just write $E \setminus A$ if we mean relative to E.

   Let $B \in \mathcal{A}_E$. Then there exists an $A \in \mathcal{A}$ such that $B = A \cap E$. We have that the complement of B relative to E is

   $$E \setminus B = E \cap B^c = E \cap (A \cap E)^c = E \cap (A^c \cup E^c) = A^c \cap E \in \mathcal{A}_E,$$

   since $A^c \in \mathcal{A}$.

3. Finally, let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}_E$. Then there exists $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $B_n = A_n \cap E$, $\forall n \in \mathbb{N}$. Then,

   $$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} (A_n \cap E) = E \cap \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_E,$$

   since $\mathcal{A}$ is closed under countable union.

□

Example 1.5 (Preimage σ-algebra)**.** Recall some properties of preimages (that are not always true for images): Set difference and union (and therefore complement and intersection) are well defined and behave as expected. Moreover, union and intersection behave as expected even for countable arities.

Let X, X' be sets, let $\mathcal{A}'$ be a σ-algebra on X' and let $f : X \to X'$ be a function between the two sets. We claim that[3] $\mathcal{A} = \{f^{-1}(A') \mid A' \in \mathcal{A}'\}$ is a σ-algebra on X[4]

Proof.

1. $X = f^{-1}(X') \in \mathcal{A}$ because $X' \in \mathcal{A}$.

2. If $A \in \mathcal{A}$ then $A^c = X \setminus A = f^{-1}(X') \setminus f^{-1}(A') = f^{-1}(X' \setminus A') \in \mathcal{A}$ because $X' \setminus A' = A'^c \in \mathcal{A}'$.

3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. By definition there must be a collection $(A'_n)_{n\in\mathbb{N}} \subset \mathcal{A}'$ for which $A_n = f^{-1}(A'_n)$, $\forall n \in \mathbb{N}$. Recall that

   $$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} f^{-1}(A'_n) = f^{-1}(\bigcup_{n\in\mathbb{N}} A'_n) \in \mathcal{A}$$

   because $\mathcal{A}'$ is a σ-algebra hence $\bigcup_{n\in\mathbb{N}} A'_n \in \mathcal{A}'$.

□

Now we will explore a result which will allow us to generate more examples from existing ones and clarify operations between σ-algebras .

---

[3] Here f^{-1} denotes the preimage of a set, not the inverse function. Recall that the preimage of a function $f : X \to X'$ is defined as $f^{-1}(A') = \{x \in X \mid f(x) \in A' \subset X'\}$

[4] When we write $f : X \to X'$ we mean that f is surjective, namely that $f^{-1}(X') = X$. Without this, the first property of the σ-algebra cannot be guaranteed. However, this is not usually a problem, as we can always redefine f as $f : X \to f(X)$ and take the trace σ-algebra $\mathcal{A}_{f(X)}$ (c.f. example 1.4).

Lemma 1.2. Given a set X. The arbitrary intersection $\bigcap_{\alpha\in I}\mathcal{A}_\alpha$ of σ-algebras is again a σ-algebra .

Proof. We shall go over the three properties of σ-algebras .

1. $X \in \mathcal{A}_\alpha,\ \forall\alpha\in I$ hence $X \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha$
2. If $A \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha$ then in particular $A\in\mathcal{A}_\alpha,\ \forall\alpha\in I$. Because each $\mathcal{A}_\alpha$ is a σ-algebra , then $A^c \in A_n,\ \forall\alpha\in I \implies A_n^c \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha$.
3. If $(A_n)_{n\in\mathbb{N}} \subset \bigcap_{\alpha\in I}\mathcal{A}_\alpha$ then, as before, we have that $(A_n)_{n\in\mathbb{N}} \subset A_n,\ \forall\alpha\in I$. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_\alpha \implies \bigcup_{n\in\mathbb{N}} A_n \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha$.

□

Definition 1.3 (σ-algebra generated by a collection of subsets)**.** Let X be a set and $\mathcal{G}$ a collection of subsets of X. We denote by $\sigma(\mathcal{G})$ the smallest σ-algebra which contains $\mathcal{G}$ and we define it by

$$\sigma(\mathcal{G}) := \bigcap \mathcal{C} \text{ where } \mathcal{G}\subset\mathcal{C}\wedge\mathcal{C} \text{ is a } \sigma\text{-algebra}$$

We also say that $\mathcal{G}$ is a generator of $\sigma(\mathcal{G})$ or that $\sigma(\mathcal{G})$ is generated by $\mathcal{G}$.

Lemma 1.3. For any $\mathcal{G}\subset\wp(X)$, $\sigma(\mathcal{G})$ exists and is a the smallest σ-algebra containing $\mathcal{G}$.

Proof. Let $\mathcal{A}=\sigma(\mathcal{G})$. Since $\wp(X)\supset\mathcal{G}$ and $\wp(X)$ is a σ-algebra , the intersection $\mathcal{A}$ is non-empty and contains $\mathcal{G}$. Because of the previous result, $\mathcal{A}$ itself is also a σ-algebra . (So we have existence). Furthermore, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{G}$ because if there were another σ-algebra $\mathcal{A}'$ containing $\mathcal{G}$ it would be included in the intersection hence $\mathcal{A}\subseteq\mathcal{A}'$. □

Remark 1.4.

1. If $\mathcal{G}$ is a σ-algebra , then $\sigma(\mathcal{G})=\mathcal{G}$.
2. For any $A\in X$, $\sigma(\{A\})=\{\varnothing,X,A,A^c\}$.
3. Let $\mathcal{G},\mathcal{F}\subset\wp(X)$. If $\mathcal{G}\subseteq\mathcal{F}$ then $\sigma(\mathcal{G})\subseteq\sigma(\mathcal{F})$.

   Proof of 3. $\mathcal{G}\subseteq\mathcal{F}\subseteq\sigma(\mathcal{F})$. So $\sigma(\mathcal{F})$ is a σ-algebra containing $\mathcal{G}$. But $\sigma(\mathcal{G})$ is the smallest σ-algebra containing $\mathcal{G}$ therefore $\sigma(\mathcal{G})\subseteq\sigma(\mathcal{F})$. □

1.1.1 Final remarks and the good set principle

The definition of a σ-algebra is more axiomatic than constructive. This is not coincidental, σ-algebras are normally really hard to describe constructively since they are huge sets. This last notion of generators will prove very helpful in the coming chapters: most of the time we will be able to get theorems that only depend on a generator of the σ-algebra at hand.

In particular, part 3 of this last remark will be used extensively throughout the course. Also, it gives rise to a common proof technique in measure theory: the **good set principle**.

Remark 1.5 (Good set principle)**.** The good set principle is a proof technique that comes in handy when we want to show that every member of a σ-algebra $\mathcal{A}$ satisfies some property P. In general, this proof technique works as follows (cf. [5]).

Define

$$\mathcal{B} := \{A \in \mathcal{A} \mid A \text{ has property } P\} \tag{1.2}$$

We will show that

1. $\mathcal{B}$ is a σ-algebra
2. There is a collection $\mathcal{G} \subset \mathcal{B}$ such that $\sigma(\mathcal{G}) = \mathcal{A}$

Then, by remark 1.4 we have that $\mathcal{A} = \sigma(\mathcal{G}) \subset \sigma(\mathcal{B})$, but since $\mathcal{B}$ is already a σ-algebra , we have $\mathcal{A} \subset \mathcal{B}$. Additionally, by definition of $\mathcal{B}$, we already have that $\mathcal{A} \subset \mathcal{B}$, therefore $\mathcal{A} = \mathcal{B}$, i.e. all the members of $\mathcal{A}$ satisfy property P.

1.2 The Borel σ-algebra on $\mathbb{R}^n$

In what follows we will use some basic topology concepts. Let us write

- $\mathcal{O}^n := \{A \subset \mathbb{R}^n \mid A \text{ is open }\}$,
- $\mathcal{C}^n := \{A \subset \mathbb{R}^n \mid A \text{ is closed }\}$, and
- $\mathcal{K}^n := \{A \subset \mathbb{R}^n \mid A \text{ is compact }\}$.

The collection $\mathcal{O}^n$ is a topology, meaning it satisfies the following properties:

1. $\varnothing, \mathbb{R}^n \in \mathcal{O}^n$
2. It is closed under finite intersections, i.e. $V, W \in \mathcal{O}^n \implies V \cap W \in \mathcal{O}^n$.
3. It is closed under arbitrary unions, i.e.

$$(A_\alpha)_{\alpha \in I} \in \mathcal{O}^n \implies \bigcup_{\alpha \in I} A_\alpha \in \mathcal{O}^n.$$

We shall call the pair $(\mathbb{R}^n, \mathcal{O}^n)$ a topological space. We now consider the smallest σ-algebra containing $\mathcal{O}^n$.

Definition 1.4 (Borel σ-algebra)**.** The Borel σ-algebra on $\mathbb{R}^n$ is the smallest σ-algebra containing $\mathcal{O}^n$. We denote it by $\sigma(\mathcal{O}^n)$ or by $\mathcal{B}(\mathbb{R}^n)$.

Theorem 1.6 (Topological generators of the Borel σ-algebra)**.**

$$\mathcal{B}(\mathbb{R}^n) := \sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n) = \sigma(\mathcal{K}^n)$$

Proof. First, we prove the first equality, i.e. $\sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n)$ by proving mutual inclusion. To show $\sigma(\mathcal{C}^n) \subset \sigma(\mathcal{O}^n)$ it is enough to show that $\mathcal{C}^n \subset \sigma(\mathcal{O}^n)$ (recall remark 1.4). Let $C \in \mathcal{C}^n$ be any closed set in $\mathbb{R}^n$. By definition C^c is open hence $C^c \in \mathcal{O}^n \subset \sigma(\mathcal{O}^n)$. Because $\sigma(\mathcal{O}^n)$ is a σ-algebra it must be true that $(C^c)^c = C \in \sigma(\mathcal{O}^n)$. The same holds for the other inclusion.

Now we turn our attention to $\sigma(\mathcal{C}^n) = \sigma(\mathcal{K}^n)$. The inclusion $\sigma(\mathcal{K}^n) \subset \sigma(\mathcal{C}^n)$ is trivial because every compact set is closed in $\mathbb{R}^n$ (recall remark 1.4). For the other one, it is again enough to show that $\mathcal{C}^n \in \sigma(\mathcal{K}^n)$. Let $C \in \mathcal{C}^n$ and define $C_k := C \cap \overline{B_k(0)}$ which is[5] closed and bounded. By construction $C = \bigcup_{k\in\mathbb{N}} C_k \in \mathcal{K}^n$ thus $\mathcal{C}^n \in \sigma(\mathcal{K}^n)$. ☐

We would now like to find smaller sets of generators for the Borel σ-algebra on $\mathbb{R}^n$. Let us define the following collections (where × denotes de cartesian product of the intervals):

- The collection of open rectangles (or cubes or hypercubes)

$$\mathcal{J}^{o,n} = \{\bigtimes_{i=1}^{n}(a_i, b_i) \mid a_i, b_i \in \mathbb{R}\}$$

- The collection of (from the right) half-open rectangles

$$\mathcal{J}^{n} = \{\bigtimes_{i=1}^{n}[a_i, b_i) \mid a_i, b_i \in \mathbb{R}\}$$

Theorem 1.7 (Borel interval generators)**.** We have

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}^n_{rat}) = \sigma(\mathcal{J}^{o,n}_{rat}) = \sigma(\mathcal{J}^n) = \sigma(\mathcal{J}^{o,n})$$

Proof. Let's begin by proving $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}^{o,n}_{rat})$. Recall that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n)$, so to prove the previous equality it suffices to prove the following two mutual inclusions:

- $\sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{O}^n)$. From remark 1.4 we have that to prove this it suffices to say that every open rectangle is an open set and thus $\mathcal{J}^{o,n}_{rat} \subset \mathcal{O}^n \implies \sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{O}^n)$.

- $\sigma(\mathcal{O}^n) \subseteq \sigma(\mathcal{J}^{o,n}_{rat})$. To prove this we make the following claim:

$$U \in \mathcal{O}^n \implies U = \bigcup_{I\in\mathcal{J}^{o,n}_{rat},\ I\subseteq U} I$$

  Again we shall attack this by proving the mutual inclusion of the two sets:

  - It is clear that $\bigcup_{I\in\mathcal{J}^{o,n}_{rat},\ I\subset U} I \subseteq U$ because of the restriction on the union.

---

[5] Here $B_r(c_0)$ and $\overline{B_r(c_0)}$ denote the open and closed balls of radius r and centre c_0, respectively. Clearly these are both bounded sets.

  - For the reverse containment, we have that as U is open, for any $x \in U$ there is a ball $B_\varepsilon(x) \subseteq U$. Because the rationals $\mathbb{Q}^n$ are dense in the reals $\mathbb{R}^n$ we can chose a rectangle $I \subset B_\varepsilon(x)$ and hence U is contained in the union.

    It is clear that all the sets $I \subseteq U$ are also in $\sigma(\mathcal{J}^{o,n}_{rat})$. However, for the union of them to be inside de σ-algebra we must ensure that the number of sets that participate is countable. Each rectangle I can be fully determined by two of its corners, which in turn have coordinates in $\mathbb{Q}^n$. Therefore, the number of sets intervening in the union is $\#(\mathbb{Q}^n \times \mathbb{Q}^n) = \#\mathbb{N}$ and thus the union is again within the σ-algebra .

Because $\mathcal{J}^{o,n}_{rat} \subset \mathcal{J}^{o,n} \subset \mathcal{O}^n$ we get for free that $\sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{J}^{o,n}) \subseteq \sigma(\mathcal{O}^n)$. We therefore conclude that

$$\sigma(\mathcal{J}^{o,n}_{rat}) = \sigma(\mathcal{J}^{o,n}) = \sigma(\mathcal{O}^n)$$

Now we would like to prove that half open sets also yield the same Borel σ-algebra for $\mathbb{R}^n$.

- We begin by noticing that we can write open sets as infinite unions of half open ones

  $$\bigtimes_{i=1}^{n}(a_i, b_i) = \bigcup_{n\in\mathbb{N}} \bigtimes_{i=1}^{n}[a_i + \frac{1}{n}, b_i)$$

  for both rectangles with rational and real endpoints. Thus, we have

  $$\begin{aligned} \mathcal{J}^{o,n}_{rat} \subseteq \sigma(\mathcal{J}^{n}_{rat}) &\implies \sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{J}^{n}_{rat}) \\ \mathcal{J}^{o,n} \subseteq \sigma(\mathcal{J}^{n}) &\implies \sigma(\mathcal{J}^{o,n}) \subseteq \sigma(\mathcal{J}^{n}) \end{aligned}$$

  (remember that σ-algebras are closed under countable unions).

- For the reverse containment we must notice that we can write (right) half-open sets as intersections of open ones

  $$\bigtimes_{i=1}^{n}[a_i, b_i] = \bigcap_{n\in\mathbb{N}} \bigtimes_{i=1}^{n}(a_i - \frac{1}{n}, b_i)$$

  for rectangles with both rational and real endpoints. Similarly, we have

  $$\begin{aligned} \mathcal{J}^{n}_{rat} \subseteq \sigma(\mathcal{J}^{o,n}_{rat}) &\implies \sigma(\mathcal{J}^{n}_{rat}) \subseteq \sigma(\mathcal{J}^{o,n}_{rat}) \\ \mathcal{J}^{n} \subseteq \sigma(\mathcal{J}^{o,n}) &\implies \sigma(\mathcal{J}^{n}) \subseteq \sigma(\mathcal{J}^{o,n}) \end{aligned}$$

- We conclude that

  $$\begin{aligned} \sigma(\mathcal{J}^{o,n}_{rat}) &= \sigma(\mathcal{J}^{n}_{rat}) \\ \sigma(\mathcal{J}^{o,n}) &= \sigma(\mathcal{J}^{n}) \end{aligned}$$

With the previous equalities the theorem has been proved. □

To recap, there are a few important points on this proof:

- First, we would like to have a more tangible generator for the Borel σ-algebra on $\mathbb{R}^n$.
- We choose rectangles because they are easier to work with and have direct application on probability theory (we could also have chosen balls, for instance).
- They key to proving that two σ-algebras are the equal is to prove that each is contained in the other. To do this, we use the generators: if the generator of $\mathcal{A}$ is contained in $\sigma(\mathcal{A}')$ and the generator of $\mathcal{A}'$ is contained in $\sigma(\mathcal{A})$, we are done.
- To prove this containments we have had to write sets from the generator as unions of sets from the other σ-algebra . It is key to make sure that these unions only iterate over a countable number of elements.

Example 1.6 (Another characterisation of the Borel σ-algebra on $\mathbb{R}^n$)**.** In this example we shall see that $\mathbb{B} = \{B_r(x) \mid x \in \mathbb{R}^n, r \in \mathbb{R}^+\}$ is also a generator of $\mathcal{B}(\mathbb{R}^n)$.

Proof. We proceed as before, first defining an auxiliary collection where the radii are all rational and the centres have rational coordinates:

$$\mathbb{B}' = \{B_r(x) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$$

It is trivial that $\mathbb{B}' \subset \mathbb{B} \subset \mathcal{O}^n$. Therefore we have that

$$\sigma(\mathbb{B}') \subseteq \sigma(\mathbb{B}) \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n)$$

For the reverse inclusions, we will focus on proving that $\mathcal{O}^n \subseteq \sigma(\mathbb{B}')$. For this we claim that any open set $U \in \mathcal{O}^n$ can be written as

$$U = \bigcup_{B \in \mathbb{B}', B \subseteq U} B$$

We need to verify two things. That the previous equality is true and that the number of sets that intervene in the union is countable.

1. It is clear that any set $B \subset \mathbb{B}'$ is also in U by the definition of the union. For the reverse inclusion, we shall choose a point $q \in \mathbb{Q}^n$ such that $\|x - q\| < r/3$. This is possible because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Next we will choose a radius $r' \in \mathbb{Q}$ such that $r' < r$. This is also possible for the same reason. Now we consider $B = B_{r'}(q) \in \mathbb{B}'$ which is assured to contain x.
2. Moreover, each of these balls is fully determined by an $(n+1)$ tuple of rationals (namely the center coordinates and the radius). Hence, the number of sets in the union is $\#(\mathbb{Q}^n \times \mathbb{Q}) = \#\mathbb{N}$.

$\square$

Chapter 2

Measures

2.1 Definition. Properties.

Definition 2.1 (Measure)**.** Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to [0, \infty)$ is called a measure (on X) if

1. $\mu(\varnothing) = 0$ and
2. **(σ-aditivity)** if $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is a pairwise disjoint (i.e. $A_i \cap A_j = \varnothing$ if $i \neq j$) sequence, then

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

And, analogously to σ-algebras

Definition 2.2 (Measure space)**.** Let $(X, \mathcal{A})$ be a measurable space and μ a measure. The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

In the definition of measure (2.1), the symbol $\dot{\bigcup}$ indicates that the union is disjoint.

Some special measures get cool names, for instance:

- If $\mu(X) < \infty$ we say that μ is **finite** and that $(X, \mathcal{A}, \mu)$ is a **finite measure space**.
- IF $\mu(X) = 1$ then $(X, \mathcal{A}, \mu)$ is a probability space and we usually denote it by $(\Omega, \mathcal{A}, \mathbb{P})$.

From the two conditions on 2.1, one can derive many properties, but before, let us introduce some notation.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. We say that (A_n) is an **increasing**, resp. **decreasing**, **sequence** and denote it by $A_n \uparrow A$, resp. $A_n \downarrow A$ according to the following definition.

$$A_n \uparrow A \iff A_1 \subseteq A_2 \subseteq \dots \text{ and } A = \bigcup_{n\in\mathbb{N}} A_n \tag{2.1}$$

$$A_n \downarrow A \iff A_1 \supseteq A_2 \supseteq \dots \text{ and } A = \bigcap_{n\in\mathbb{N}} A_n \tag{2.2}$$

We say that an increasing sequence $A_n \uparrow A$ is an **exhausting sequence** if $A_n \subseteq X$ and $A = X$. We write $A_n \uparrow X$ for an exhausting sequence.

Definition 2.3 (σ-finite measure)**.** A measure μ is called σ-finite if there exists an exhausting sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\mu(A_n) < \infty,\ \forall n \in \mathbb{N}$. A measure space with this kind of measure is called a σ-finite measure space.

Remark 2.1. Finiteness is stronger than σ-finiteness[1]. Namely, any finite measure is σ-finite since one can choose the sequence $A_n = X,\ \forall n \in \mathbb{N}$ which is an exhausting sequence $A_n \uparrow X$ and $\mu(A_n) < \infty,\ \forall n \in \mathbb{N}$. On the other hand, an example of a σ-finite measure which is not finite is the Lebesuge measure (see example 2.4).

Theorem 2.2 (Properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $A, B, A_n, B_n \in \mathcal{A},\ \forall n \in \mathbb{N}$. Then,

1. **(finite additivity)** $A \cap B = \varnothing \implies \mu(A \uplus B) = \mu(A) + \mu(B)$,
2. **(monotonicity)** if $A \subseteq B$ then $\mu(A) \le \mu(B)$,
3. if $A \subset B$ and $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$,
4. **(strong additivity)** $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,
5. **(finite subadditivity)** $\mu(A \cup B) \le \mu(A) + \mu(B)$,
6. **(continuity from below)** if $A_n \uparrow A$ then $\mu(A) = \mu(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \mu(A_n) = \lim_{n\to\infty} \mu(A_n)$,
7. **(continuity from above)** if $\mu(A) < \infty, \forall A \in \mathcal{A}$ and $A_n \downarrow A$ then $\mu(A) = \mu(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \mu(A_n) = \lim_{n\to\infty} \mu(A_n)$, and
8. **(sigma subadditivity)** $\mu(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n\in\mathbb{N}} \mu(A_n)$.

Proof.

1. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $A_1 = A,\ A_2 = B$ and $A_i = \varnothing$ for $i > 2$. It is clear that A_n are disjoint since $A \cap B = \varnothing$ and $A_n \cap \varnothing = \varnothing,\ \forall n \in \mathbb{N}$.
2. Write $B = (B \setminus A) \uplus A$. Then, because of σ-aditivity we have

$$\mu(B) = \mu(B \setminus A) + \mu(A) \ge \mu(A) \text{ since } \mu(B \setminus A) \ge 0$$

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[1] By σ-finiteness we mean that a measure satisfies definition 2.3, not that it is s-finite, which we wont see in this course. See [6] for more details.

3. As previously write $\mu(B) = \mu(B \setminus A) + \mu(A)$. Because $\mu(A) < \infty$ we can subtract it on both sides to get $\mu(B) - \mu(A) = \mu(B \setminus A)$.
4. Write $A \cup B = A \setminus B \uplus B \setminus A \uplus A \cap B$ hence $\mu(A \cup B) = \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B)$. Add $\mu(A \cap B)$ on both sides and group terms:

$$\mu(A \cup B) + \mu(A \cap B) = \underbrace{\mu(A \setminus B) + \mu(A \cap B)}_{\mu(A)} + \underbrace{\mu(B \setminus A) + \mu(A \cap B)}_{\mu(B)}$$

5. $\mu(A \cap B) \leq \mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B)$
6. Define the sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ by $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for $n > 1$. It is clear that B_n is pairwise disjoint and that $\biguplus_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n = A$. Hence

$$\begin{aligned} \mu(A) &= \mu(\bigcup_{n \in \mathbb{N}} A_n) = \mu(\biguplus_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n) = \lim_{m \to \infty} \sum_{n=1}^{m} \mu(B_n) \\ &= \lim_{m \to \infty} \mu(\biguplus_{n=1}^{m} B_n) = \lim_{m \to \infty} \mu(A_m) = \sup_{n \in \mathbb{N}} \mu(A_n) \end{aligned}$$

since $\mu(A_n)$ is an increasing sequence. The introduction of the limits in the previous chain of equalities has to be done carefully, as we are building on the definition of limits for sequences of numbers. The equality between the first and the second lines comes from σ-additivity.

7. Let $D_n = A_1 \setminus A_n$, $\forall n \in \mathbb{N}$. Then D_N is an increasing sequence with

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} (A_1 \setminus A_n) = \bigcup_{n \in \mathbb{N}} (A_1 \cap A_n^c) &= A_1 \cap \bigcup_{n \in \mathbb{N}} A_n^c \\ &= B_1 \cap \left(\bigcap_{n \in \mathbb{N}} A_n \right)^c = A \setminus \bigcap_{n \in \mathbb{N}} A_n \end{aligned}$$

Thus,

$$\begin{aligned} &\mu(A \setminus \bigcap_{n \in \mathbb{N}} A_n) \overset{(3)}{=} \mu(A) - \mu(\bigcap_{n \in \mathbb{N}} A_n) = \mu(\bigcup_{n \in \mathbb{N}} D_n) \\ &\overset{(4)}{=} \lim_{n \to \infty} \mu(D_n) = \lim_{n \to \infty} \mu(A_1 \setminus A_n) \overset{(3)}{=} \lim_{n \to \infty} (\mu(A) - \mu(A_n)) = \mu(A) - \lim_{n \to \infty} \mu(A_n) \end{aligned}$$

Substracting $\mu(A) < \infty$ from both sides we have

$$\mu(\bigcap_{n \in \mathbb{N}} A_n) = \lim_{n \to \infty} \mu(A_n)$$

8. Let $(A_n)_{n \in \mathbb{N}}$ be any countable subcollection of $\mathcal{A}$. Define

$$E_n = \bigcup_{m=1}^{n} A_m \in \mathcal{A}$$

Then $E_n \uparrow \bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} A_n$. Thus, by (6) we have

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \mu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \mu\left(\bigcup_{m=1}^{n} A_m \right) \overset{(5)}{\leq} \lim_{n \to \infty} \sum_{m=1}^{n} \mu(A_m)$$

$\square$

2.2 Examples

Throughout this section, let $(X, \mathcal{A})$ be a measurable space.

Example 2.1 (Dirac measure)**.** We define the **Dirac measure** for a given $x_0 \in X$ as follows:

$$\delta_{x_0}(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A \end{cases}, \qquad \forall A \in \mathcal{A} \tag{2.3}$$

Clearly δ_{x_0} is a measure since it satisfies the two properties. First, $\delta_{x_0}(\varnothing) = 0$ since $\forall x_0 \in X,\ x_0 \notin \varnothing$. Second, for any pairwise disjoint collection of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ we have two possibilities:

- If $x_0 \notin \biguplus_{n\in\mathbb{N}} A_n$ then clearly $x_0 \notin A_n,\ \forall n \in N$ so
$$0 = \mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) = 0$$
- Otherwise, if $x_0 \in \biguplus_{n\in\mathbb{N}} A_n$ then there must be only one $n_0 \in \mathbb{N}$ such that $x_0 \in A_{n_0}$ since (A_n) is a pairwise disjoint collection. Thus,
$$1 = \mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A_{n_0}) + \sum_{n\neq n_0} \mu(A_n) = 1 + 0$$

Example 2.2 (Counting measure)**.** Let $X = \mathbb{R}$ and choose $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable }\}$. We already saw on the Chapter 1 that $\mathcal{A}$ is a σ-algebra . Now define $\mu : \mathcal{A} \to [0, \infty)$ as

$$\mu(A) = \begin{cases} 0 & \text{if } \#A \leq \#\mathbb{N} \\ 1 & \text{if } \#A^c \leq \#\mathbb{N} \end{cases} \tag{2.4}$$

We have that $\mu(\varnothing) = 0$ since $\#\varnothing$ is countable. As for σ-additivity we must recall from set theory that the union of countable sets is also countable, so:

- If $\#A_n \leq \#\mathbb{N},\ \forall n \in \mathbb{N}$ then
$$0 = \mu(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = 0$$
- If there exists an $n_0 \in \mathbb{N}$ such that $A^c_{n_0}$ is countable then
$$\left(\biguplus_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A^c_n \subseteq A^c_{n_0}$$
and hence $\left(\biguplus_{n\in\mathbb{N}} A_n\right)^c$ is countable. Furthermore, since the collection is pairwise disjoint, $\forall n \in \mathbb{N}, n \neq n_0$ we have $A^c_n \subseteq A_{n_0}$ so $\#A^c_n,\ \forall n \neq n_0$. Thus
$$1 = \mu(\biguplus_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A_{n_0}) + \sum_{n\neq n_0} \mu(A_n) = 1 + 0$$

Example 2.3 (Discrete probability measure)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a measure space where $\Omega = \{\omega_1, \omega_2, \dots\}$ is a countable set, $\mathcal{A} = \wp(\Omega)$ (which is of course a σ-algebra) and let $(p_1, p_2, \dots)$ be a probability vector where $\sum_{n \in \mathbb{N}} p_n = 1$ (and p_i is the probability of ω_i). Define the measure $\mathbb{P} : \mathcal{A} \to [0, \infty)$ where

$$\mathbb{P}(A) = \sum_{\omega_i \in A} p_i = \sum_{i=1}^{\infty} p_i \delta_{\omega_i}(A)$$

Let's verify that $\mathbb{P}$ is a measure. First, $\mathbb{P}(\varnothing) = 0$ since $\varnothing$ cannot contain any ω_i. As for σ-additivity, we have

$$\begin{aligned} \mathbb{P}\left(\dot{\bigcup_{n=1}^{\infty}} A_n \right) &= \sum_{i=1}^{\infty} p_i \delta_{x_i} \left(\dot{\bigcup_{n=1}^{\infty}} A_n \right) = \sum_{i=1}^{\infty} p_i \sum_{n=1}^{\infty} \delta_{x_i}(A_n) \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} p_i \delta_{x_i} A_n \right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) \end{aligned}$$

Nota that here we can exchange the summations because all the terms are non-negative so the convergence problems (oscillating convergence, that is) are eliminated.

Example 2.4 (Lebesgue measure)**.** For now we shall define this measure only on n-dimensional rectangles. The generalisation to arbitrary Borel sets will come in chapter 4.

Consider the measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$ where $\mathbb{R}^n$ and $\mathcal{B}(\mathbb{R}^n)$ are the usual suspects and $\lambda^n : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ is defined as:

$$\lambda^n \left(\bigtimes_{i=1}^{n} [a_i, b_i) \right) = \prod_{i=1}^{n} (b_i - a_i) \tag{2.5}$$

or, more informally, the hypervolume of the rectangle in question.

We are not ready to verify that λ^n is a measure over $\mathcal{B}(\mathbb{R}^n)$ yet. In fact, we will need to wait until the end of Chapter 4, where, once we have the generalisation to arbitrary Borel sets, we shall prove that it is a measure.

We are however, in a position to prove that λ^n is σ-finite. Let $A_i = [-i, i)^n$ for all $n \in \mathbb{N}$. Clearly, $(A_i)_{n \in \mathbb{N}}$ is an exhausting sequence since $A_i \uparrow \mathbb{R}^n$. Also, $\lambda^n(A_i) = (2i)^n < \infty, \forall n \in \mathbb{N}$. Therefore λ^n is σ-finite.

Note that λ^n is not finite, since $\lambda(R^n) \nless \infty$.

Chapter 3

Uniqueness of measures

In this chapter we shall introduce some technicalities to be able to extend the Lebesgue measure to arbitrary Borel sets in $\mathcal{B}(\mathbb{R}^n)$. The first tool is a generalized version of a σ-algebra , where union closure is only required for disjoint unions. We will explore the properties of this new construct and try to draw similarities to what we know about σ-algebras . Finally we will give a result on the conditions that a σ-algebra must meet to be able to uniquely define a measure on it by defining the measure on the elements of a generator.

3.1 Preliminaries

Definition 3.1 (Dynkin system)**.** A family $\mathcal{D} \subset \wp(X)$ is called a Dynkin system if it satisfies these three properties:

1. $X \in \mathcal{D}$
2. $\forall A \in \wp(X),\ A \in \mathcal{D} \implies A^c \in \mathcal{D}$
3. For any countable collection of pairwise disjoint sets (A_n)

$$(A_n)_{n\in\mathbb{N}} \subset \mathcal{D} \implies \dot{\bigcup_{n\in\mathbb{N}}} A_n \in \mathcal{D}$$

Note that every σ-algebra is a Dynkin system but the converse need not be true.

We have an analogue version to the generator theorem for σ-algebras in the case of Dynkin systems:

Definition 3.2. Let $\mathcal{G} \subset \wp(X)$ be a collection of subsets of X. We define the Dynkin system generated by $\mathcal{G}$ as

$$\delta(\mathcal{G}) = \bigcap_{\mathcal{G}\subset\mathcal{C},\mathcal{C}\text{Dynkin}} \mathcal{C}$$

Lemma 3.1. Let $\mathcal{G} \subset \wp(X)$ be a collection of subsets of X. Then $\delta(\mathcal{G})$ is a Dynkin system and it is the smallest.

Proof. $\delta(\mathcal{G})$ is a Dynkin system because of the restriction on the intersection. Note that the intersection is non-empty because $\wp(X)$ is a Dynkin system and that the intersection of Dynkin systems is also a Dynkin system (the proof is the same as the one for σ-algebras but using disjoint unions).

Now let's look at why it is the smallest. Suppose there is another Dynkin system $\mathcal{D}$ that contains the collection $\mathcal{G}$. Then $\mathcal{D}$ is a Dynkin system and therefore intervenes in the intersection. Therefore $\delta(\mathcal{G}) \subset \mathcal{D}$. ☐

So, we already saw that every σ-algebra is a Dynkin system and that Dynkin systems can be generated from a collection of subsets. A natural question to ask is, when is a Dynkin system a σ-algebra ? Let us introduce some terminology first.

Definition 3.3 (Stable under finite intersection)**.** Let $\mathcal{D}$ be a collection of sets. We say that $\mathcal{D}$ is stable under finite intersection, or closed under finite intersection or $\cap$-stable if

$$C, D \in \mathcal{D} \implies C \cap D \in \mathcal{D}$$

Some sources call such a collection a Π-system.

Lemma 3.2. Let $\mathcal{D}$ be a Dynkin system. $\mathcal{D}$ is $\cap$-stable $\iff$ $\mathcal{D}$ is a σ-algebra .

Proof. It is clear that going from right to left is true, since all σ-algebras are $\cap$-stable.

For the implication from left to right we proceed as follows. Assuming $\mathcal{D}$ is a $\cap$-stableDynkin system, to prove that $\mathcal{D}$ is a σ-algebra we only need to generalise the last property to countable unions of arbitrary collections of sets in $\mathcal{D}$, not just disjoint ones. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ be a sequence of sets in $\mathcal{D}$. We define a new sequence $(E_n)_{n\in\mathbb{N}}$ by

$$E_1 = D_1 \text{ for } n = 1, \qquad E_n = D_n \setminus E_{n-1} \text{ for } n > 1.$$

Rewriting the set difference as an intersection we have that

$$E_n = D_n \setminus E_{n-1} = D_n \setminus \bigcup_{i=1}^{n-1} D_i = D_n \cap \left(\bigcap_{i=1}^{n-1} D_i^c \right) \in \mathcal{D}$$

since $\mathcal{D}$ is $\cap$-stable. It is clear that E_n are pairwise disjoint, i.e. $E_i \cap E_j = \varnothing$ for $i \neq j$. We also have that

$$\dot{\bigcup_{n\in\mathbb{N}}} E_n = \bigcup_{n\in\mathbb{N}} D_n \in \mathcal{D}$$

Hence, $\mathcal{D}$ is a σ-algebra . ☐

Remark 3.3. As a consequence we have that if $\mathcal{G} \subset \wp(X)$ and $\delta(\mathcal{G})$ is $\cap$-stablethen $\delta(\mathcal{G})$ is a σ-algebra containing $\mathcal{G}$ therefore $\sigma(\mathcal{G}) \subseteq \delta(\mathcal{G})$. Since $\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$ always holds, we have that

$$\text{if } \delta(\mathcal{G}) \text{ is } \cap\text{-stable then } \delta(\mathcal{G}) = \sigma(\mathcal{G})$$

This is nice, but it would even be nicer if we could just argue about the generators, since Dynkin systems and σ-algebras are hard to reason about. We'll do just that.

Lemma 3.4. Let $\mathcal{G}$ be a collection of subsets of X. If $\mathcal{G}$ is $\cap$-stablethen $\delta(\mathcal{G}) = \sigma(\mathcal{G})$.

Proof. We only need to show that if $\mathcal{G}$ is $\cap$-stablethen $\delta(\mathcal{G})$ also is. Then, because of the previous remark we have $\delta(\mathcal{G}) = \sigma(\mathcal{G})$.

We shall proceed in steps.

1. **Claim 1.** For each $E \in \delta(\mathcal{G})$, the collection $\mathcal{D}_E = \{F \subseteq X \mid F \cap E \in \delta(\mathcal{G})\}$ is a Dynkin system. We prove the three properties of a Dynkin system.

   (a) Clearly $\varnothing = \varnothing \cap E \in \mathcal{D}_E$

   (b) For any $F \in \mathcal{D}_E$ we have $F^c = X \setminus F \subseteq X$ and

   $$\begin{aligned} F^c \cap E =& (F^c \cup E^c) \cap E \\ &= (F \cap E)^c \cap E \\ &= (F \cap E) \uplus E^c \in \delta(\mathcal{G}), \end{aligned}$$

   since $F \cap E \in \delta(\mathcal{G})$ by hypothesis and $E^c \in \delta(\mathcal{G})$ since $\delta(\mathcal{G})$ is a Dynkin system.

   (c) Finally, for any collection of disjoint subsets $(F_n)_{n\in\mathbb{N}} \subset \mathcal{D}_E$ we need to show that the disjoint union is still in $\mathcal{D}_E$. We have that for all $n \in \mathbb{N}$, $F_n \cap E \in \delta(\mathcal{G})$ by hypothesis so

   $$\biguplus_{n\in\mathbb{N}} F_n \cap E = \biguplus_{n\in\mathbb{N}} (F_n \cap E) \in \delta(\mathcal{G}).$$

2. **Claim 2.** $\mathcal{G} \subset \mathcal{D}_G$, $\forall G \in \mathcal{G}$. Let $G' \in \mathcal{G}$. Since $\mathcal{G}$ is $\cap$-stablewe have that $G' \cap G \in \mathcal{G}$ and hence $G' \cap G \in \delta(\mathcal{G}) \implies G' \in \mathcal{D}_G$.

As a consequence of these claims we have that, since $\mathcal{G} \subset \mathcal{D}_G$ then $\delta(\mathcal{G}) \subset \delta(\mathcal{D}_G) = \mathcal{D}_G$. Therefore, for any $E \in \delta(\mathcal{G})$ and any $G \in \mathcal{G}$ we have that $E \cap G \in \delta(\mathcal{G})$. This also shows that $\delta(\mathcal{G}) \subset \mathcal{D}_E$, for any $E \in \delta(\mathcal{G})$. In other words we have that for any $E, F \in \delta(\mathcal{G})$, $E \cap F \in \delta(\mathcal{G})$. Thus, $\delta(\mathcal{G})$ is $\cap$-stableimplying that $\delta(\mathcal{G})$ is a σ-algebra and hence $\delta(\mathcal{G}) = \sigma(\mathcal{G})$. ☐

3.2 Uniqueness of measures

Now we move on to define the requirements needed to be able to guarantee uniqueness of a measure over a σ-algebra given a definition of the measure on a generator of that σ-algebra .

Theorem 3.5 (Uniqueness of measures)**.** Let $(X, \mathcal{A})$ be a measurable space where $\mathcal{A} = \sigma(\mathcal{G})$ for some collection $\mathcal{G}$ of subsets of X where $\mathcal{G}$ satisfies the following:

1. $\mathcal{G}$ is $\cap$-stable(so $\delta(\mathcal{G}) = \sigma(\mathcal{G})$), and
2. there exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ such that $G_n \uparrow X$ (so $X = \bigcup_{n \in \mathbb{N}} G_n$).

If μ, ν are measures on $\mathcal{A} = \sigma(\mathcal{G})$ such that $\mu(G) = \nu(G) < \infty, \ \forall G \in \mathcal{G}$ then $\mu = \nu$, i.e. $\mu(A) = \nu(A), \ \forall A \in \mathcal{A}$.

Proof. TODO ☐

Suppose that we don't have an exhausting sequence on the generator. A neat trick (that doesn't always work) is to extend the generator to $\mathcal{G} \cup \{X\}$ and define the trivial exhausting sequence $G_n = X, \ \forall n \in \mathbb{N}$. If it holds that $\mu(X) < \infty$, i.e., if μ is a finite measure, then we are in a position to apply this theorem. See exercise 5.9 for an opportunity to apply this trick.

Lemma 3.6.

1. The n-dimensional Lebesgue measure λ^n is invariant under translations, i.e.

$$\lambda^n(x + B) = \lambda^n(B), \quad \forall x \in \mathbb{R}‘n, B \in \mathcal{B}(\mathbb{R}^n),$$

where $x + B = \{x + y \mid y \in B\}$.

2. Every measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ which is invariant under translations and satisfies $\kappa = \mu([0,1]^n) < \infty$ is a multiple of the Lebesgue measure $\mu = \kappa\lambda^n$.

Proof.

1. First of all, let us check that $\lambda(x + B)$ is well defined, i.e. that $B \in \mathcal{B}(\mathbb{R}^n) \implies x + B \in \mathcal{B}(\mathbb{R}^n)$. There is a clever way to do this. Define

$$\mathcal{A}_x = \{B \in \mathcal{B}(\mathbb{R}^n) \mid x + B \in \mathcal{B}(\mathbb{R}^n)\} \subset \mathcal{B}(\mathbb{R}^n).$$

It is clear that $\mathcal{A}_x$ is a σ-algebra on $\mathbb{R}^n$ and that $\mathcal{J} \subset \mathcal{A}_x$ since translations of half-open intervals are still half-open intervals and hence in $\mathcal{J}$ and thus in $\mathcal{B}(\mathbb{R}^n)$. Therefore $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}) \subset \mathcal{A}_x \subset \mathcal{B}(\mathbb{R}^n)$. Now we can start with the meat of the proof.

Define $\nu(B) = \lambda^n(x+B)$ for any $B \in \mathcal{B}(\mathbb{R}^n)$ and some fixed $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. ν is a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}‘n))$ since

$$\nu(\varnothing) = \lambda^n(x + \varnothing) = \lambda^n\left(\bigtimes_{i=1}^{n}[x_i + a, x_i + a)\right) = \prod_{i=1}^{n}(x_i + a - (x_i + a)) = 0$$

$$\text{and } \nu\left(\dot{\bigcup_{j=1}^{\infty}} A_j\right) = \lambda^n\left(\dot{\bigcup_{j=1}^{\infty}} x + A_j\right) = \sum_{j=1}^{\infty}\lambda^n(x + A_j) = \sum_{j=1}^{\infty}\nu(A_j).$$

Now take $I = \bigtimes_{i=1}^{n}[a_i, b_i] \in \mathcal{J}$ and note that

$$\begin{aligned}\nu(I) = \lambda^n(x+I) = \lambda^n\left(\bigtimes_{i=1}^{n}[a_i + x_i, b_i + x_i]\right)\\ = \prod_{i=1}^{n}(b_i + x - (a_i + x)) = \prod_{i=1}^{n}(b_i - a_i) = \lambda^n(I)\end{aligned}$$

This means that, if we restrict ourselves to the generator $\mathcal{J}$, we have that $\nu\mid_{\mathcal{J}} = \lambda^n\mid_{\mathcal{J}}$.[1] Recall that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J})$ and that the generator $\mathcal{J}$ admits the exhausting sequence $[-k,k)_{k\in\mathbb{N}} \subset \mathcal{J}$ with $\nu([-k,k)) = \lambda^n([-k,k)) = (2k)^n < \infty$. Hence, using the previous theorem, the measures ν and λ^n must coincide in every $A \subset \mathcal{B}(\mathbb{R}^n)$.

2. TODO

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[1] Where $f \mid A$ denotes the restriction of $f : X \to Y$ to the new domain $A \subset X$.

Chapter 4

Existence of measures

4.1 Preliminaries

In this chapter we shall explore a new structure, the semi ring and a new function, the premeasure. They are analogous to a σ-algebra and a measure, respectively, but weaker. Then we shall prove that under some conditions, premeasures can be extended to measures and semirings to σ-algebras . The most important implication of this result, called Caratheodory's theorem, is that the Lebesgue measure that we have so far only defined in the open n-dimensional intervals can be extended to any Borel set in $\mathcal{B}(\mathbb{R}^n)$ and thus is a proper measure.

Definition 4.1 (Semi-ring)**.** Let $\mathcal{S} \subset X$ be a collection of subsets of a set X. We say that $\mathcal{S}$ is a semi-ring if the following are satisfied:

1. $\emptyset \in \mathcal{S}$,
2. $S, T \in \mathcal{S} \implies S \cap T \in \mathcal{S}$ (or $\mathcal{S}$ is $\cap$-stable), and
3. if $S, T \in \mathcal{S}$ then there exists a finite collection of pairwise disjoint sets $S_1, \dots, S_M \in \mathcal{S}$ such that $S \setminus T = \dot{\bigcup}_{j=1}^{M} S_j$ (so $S \setminus T$ is the disjoint union of a finite collection in $\mathcal{S}$).

We will see that $\mathcal{J}$ and $\mathcal{J}_{rat}$ are semi-rings.

Definition 4.2 (Premeasure)**.** Let X be a set, $\mathcal{S}$ a semiring of subsets of X, and $\mu : \mathcal{S} \to [0, \infty)$ be a function. We say that μ is a premeasure if the following are satisfied:

1. $\mu(\emptyset) = 0$,
2. if $(S_n)_{n \in \mathbb{N}}$ is a pairwise disjoint collection of sets in $\mathcal{S}$ then

$$\mu(\dot{\bigcup_{j \in \mathbb{N}}} S_j) = \sum_{j \in \mathbb{N}}^{n} \mu(S_j),$$

in other words, σ-additivity.

What is missing for a premeasure to become a measure is the fact that it is not defined on a σ-algebra on X, but rather on a weaker structure, the semi-ring $\mathcal{S}$.

4.2 The Caratheodory theorem

Theorem 4.1 (Caratheodory)**.** Let X be a set, $\mathcal{S}$ a semi-ring on X and μ a premeasure defined on $\mathcal{S}$. Then μ has an extension to a measure μ defined on $\sigma(\mathcal{S})$. If $\mathcal{S}$ contains an exhausting sequence $S_n \uparrow X$ with $\mu(S_n) < \infty$ then the extension is unique.

And that's it. We could end the chapter here or make it much longer by proving Caratheodory's theorem. We may do so, if I manage to find the time to write it down, but otherwise look it up.

What we will do is apply the theorem to the Lebesgue measure and give an outline of the proof.

Remark 4.2. The n-dimensional Lebesgue measure satisfies the hypothesis for the premeasure in Caratheodory's theorem.

Chapter 5

Measurable mappings

In mathematics mappings between sets are a central topic. Moreover, when sets have a specific structure, we want mappings that preserve the that same structure between the two sets. For instance we have

- groups and homomorphisms, which hold the group operation: $f(a \cdot b) = f(a) \cdot f(b)$;
- topological spaces and continuous functions, which hold the topology of the spaces in question: for any open set V and continuous function f, $f^{-1}(V)$ is also an open set;
- and naturally we wish that between measurable spaces, there are appropriate mappings that preserve the measurable structure (σ-algebra).

Let's dive right into it.

5.1 Definition. Properties.

Definition 5.1 (Measurable map)**.** Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be measurable spaces. A map $T : X \to X'$ is said to be $\mathcal{A}/\mathcal{A}'$-measurable (or measurable unless this is too ambiguous) if

$$T^{-1}(A') = \{x \in X \mid T(x) \in A'\} \in \mathcal{A}, \qquad \forall A' \in \mathcal{A}',$$

i.e. if the preimage of every measurable set in $\mathcal{A}'$ is a measurable set in $\mathcal{A}$.

In the next chapter we will particularise this to mappings $T : X \to \mathbb{R}$ and measurable spaces $(X, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ to pave the way for integration.

Remark 5.1.

1. In probability theory, a measurable map is usually called a random variable.

2. Consider the collection $T^{-1}(\mathcal{A}') = \{T^{-1}(A') \mid A' \in \mathcal{A}'\}$. Recall example 1.5, in which we proved that the preimage of a σ-algebra is a σ-algebra . We can rephrase the definition of measurability as
$$T \text{ is measurable } \iff T^{-1}\mathcal{A}' \subset \mathcal{A}$$
3. If we write $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ then we usually mean that T is $\mathcal{A}/\mathcal{A}'$-measurable.
4. If $T : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then we simply say T is Borel-measurable.

Lemma 5.2 (Measurability can be checked only on the generators)**.** Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces with $\mathcal{A}' = \sigma(\mathcal{G}')$. A map $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable $\iff T^{-1}(G') \in \mathcal{A},\ \forall G' \in \mathcal{G}' \iff T^{-1}(\mathcal{G}') \subseteq \mathcal{A}$.

Proof. If T is $\mathcal{A}/\mathcal{A}'$ measurable then we have that $T^{-1}(A') \in \mathcal{A},\ \forall A' \in \mathcal{A}$ so, in particular, $T^{-1}(G') \in \mathcal{A},\ \forall G' \in \mathcal{G} \subset \mathcal{A}'$.

For the converse let us define
$$\Sigma' := \{A' \subset X' \mid T^{-1}(A') \in \mathcal{A}\}$$
As usual we will check that Σ' is itself a σ-algebra on X' and therefore
$$A' = \sigma(\mathcal{G}') \subseteq \sigma(\Sigma') = \Sigma' \implies T^{-1}(A') \in A,\ \forall A' \in \mathcal{A}'.$$

Let's verify that Σ' is indeed a σ-algebra on X'.

1. $X' \in \Sigma'$ since $T^{-1}(X') = X \in \mathcal{A}$
2. For any $A' \in \Sigma'$ we have that $T^{-1}(A'^c) = T^{-1}(X' \setminus A') = T^{-1}(X') \setminus T^{-1}(A') = X \setminus T^{-1}(A') \in \mathcal{A}$ since $T^{-1}(A') \in \mathcal{A}$ by hypothesis.
3. For any collection $(A'_n)_{n\in\mathbb{N}} \subset \Sigma'$ we have that
$$T^{-1}\left(\bigcup_{n\in\mathbb{N}} A'_n\right) = \bigcup_{n\in\mathbb{N}} T^{-1}(A'_n) \in \mathcal{A},$$
since $T^{-1}(A'_n) \in \mathcal{A}$ by hypothesis.

□

So it is enough to check measurability on the generators only. This leads us to the following remark.

Remark 5.3. Any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a measurable mapping.

Proof. Recall that f is continuous if and only if for any open set $V \subset \mathbb{R}^m$ then $f^{-1}(V) \in \mathbb{R}^n$ is also open. Recall that $\mathcal{B}(\mathbb{R}^m)$ is also generated by the open sets $\mathcal{O}^m$ hence if f is continuous, $f^{-1}(V) \in \mathcal{O}^n = \mathcal{B}(\mathbb{R}^n) \implies f$ is a measurable map. □

Keep in mind that not every measurable map is continuous.

Example 5.1 (A non-continuous measurable map.)**.** Let $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a map defined by $f(x) = \mathbb{1}_A$ for some $A \subset \mathbb{R}$. This map is clearly not continuous as $\mathbb{1}_A$ takes only two values, 0 and 1. However, it is indeed measurable. Take any $a \in \mathbb{R}$. Then,

$$\mathbb{1}_A{}^{-1}((a,\infty)) = \{x \in \mathbb{R} \mid \mathbb{1}_A(x) > a\} = \begin{cases} \varnothing & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ \mathbb{R} & \text{if } a < 0 \end{cases} \in \mathcal{B}(\mathbb{R})$$

By lemma 5.2 we have that f is Borel measurable.

Before we move on, lets prove that the composition of measurable maps is another measurable map, as we would do with any other structure-preserving map in any other field of mathematics.

Lemma 5.4. Let $(X_i, \mathcal{A}_i)$ be measurable spaces for $i = 1,2,3$ and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be two $\mathcal{A}_1/\mathcal{A}_2$- resp. $\mathcal{A}_2/\mathcal{A}_3$-measurable maps. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_4$-measurable.

Proof. We need to show that $(S \circ T)^{-1}(A) \in \mathcal{A}_1$ for any $A \in \mathcal{A}_3$. Now $(S \circ T)^{-1}(A) = T^{-1}(S^{-1}(A))$ with $S^{-1}(A) \in \mathcal{A}_2$ and hence $T^{-1}(S^{-1}(A)) \in \mathcal{A}_1$. ☐

5.2 σ-algebras in relation to measurable maps. Image measures.

When dealing with measurable maps, we often have a σ-algebra on the codomain but none is given for the domain. Naturally, we want to know which σ-algebras on X render a map $T : X \to X'$ measurable when $(X', \mathcal{A}')$ is the destination measurable space.

It is clear that if we take $\mathcal{A} = \wp(X)$ to be the σ-algebra on X then any map is measurable since

$$T^{-1}(A') \subset X \implies T^{-1}(A') \in \wp(X), \ \forall A' \in \mathcal{A}'.$$

Also, from the examples on chapter 1 know that the preimage of a σ-algebra is another σ-algebra . This would then be the smallest σ-algebra on X that makes T measurable but we cannot remove any sets from $T^{-1}(\mathcal{A}')$ without endangering the measurability of T.

What if we have many mappings to different measurable spaces that share the same origin set? We cannot guarantee that taking the union of all the σ-algebras which individually render each mapping measurable is again a σ-algebra . Let us formalise this.

Definition 5.2. Let $(T_i)_{i\in I}$ be arbitrarily many mappings $T_i : X \to X'$ from the same set X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right).$$

We say that $\sigma(T_i : i \in I)$ is generated by the family $T_i^{-1}(\mathcal{A}_i)$.

Lemma 5.5. The previous definition makes sense, i.e. $\mathcal{A} = \sigma(T_i \mid i \in I)$ is indeed a σ-algebra which renders all T_i simultaneously $\mathcal{A}/\mathcal{A}_i$ measurable.

Proof. Let $A_i \in \mathcal{A}_i$ for any $i \in I$. Then, clearly, $T_i^{-1}(A_i) \in T^{-1}(\mathcal{A}_i)$, so any T_i is measurable. But is $\mathcal{A}$ a σ-algebra ? Well, that is the reason why we included the σ-hull in the above definition, to guarantee that the union does not break the σ-algebra structure. ☐

To tie it back to measures we will give the following definition:

Definition 5.3 (Image measure)**.** Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. Then, for every measure μ on $(X, \mathcal{A})$, we define the image measure of μ under T, denoted by $T(\mu)$ or $\mu \circ T^{-1}$, by

$$T(\mu)(A') = \mu(T^{-1}(A')), \quad \forall A' \in \mathcal{A}'$$

Lemma 5.6. The image measure is indeed a measure.

Proof. We just have to check the two properties of measures.

1. Clearly $T(\mu)(\varnothing) = \mu(T^{-1}(\varnothing)) = \mu(\varnothing) = 0$.
2. For any pairwise disjoint collection $(A'_n)_{n\in\mathbb{N}} \subset \mathcal{A}'$ we have

$$\begin{aligned} T(\mu)\left(\dot\bigcup_{n\in\mathbb{N}} A'_n\right) &= \mu\left(T^{-1}\left(\dot\bigcup_{n\in\mathbb{N}} A'_n\right)\right) \\ &= \mu\left(\dot\bigcup_{n\in\mathbb{N}} T^{-1}(A'_n)\right) \\ &= \sum_{n\in\mathbb{N}} \mu(T^{-1}(A'_n)) = \sum_{n\in\mathbb{N}} T(\mu)(A'_n) \end{aligned}$$

☐

5.3 Exercises

Example 5.2. This is a version of exercise 7.6 that I think is more precisely formulated, partly inspired by [7].

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be a surjective, $\mathcal{A}/\mathcal{A}'$-measurable map. Then $T(\mathcal{A})$ is a σ-algebra $\iff$ T^{-1} is $\mathcal{A}'/\mathcal{A}$ measurable.

Exercise 5.3.1. Let $T : (X, \mathcal{A}) \to (X, \mathcal{A}')$ be a measurable map. Under which circumstances is the family of sets $T(\mathcal{A})$ a σ-algebra ?.

Proof. We will see that the first and third properites of σ-algebras hold more or less trivially. It is the closing under complements that gives us problems.

We claim that $T(\mathcal{A})$ is a σ-algebra $\iff$ $T^{-1} : X' \to X$ is a measurable map. Let us assume that T is surjective, as when we define $T^{-1} : X' \to X$ we are implicitly denoting that. Otherwise, we can just redefine $X' = T(X)$ and move on.

For the reverse implication let's verify the properties of a σ-algebra over X' on $T(\mathcal{A})$.

1. First, we need to show that $X' \in T(\mathcal{A})$. This is clear from the fact that $X \in \mathcal{A}$ and T is surjective.
2. For any $A' \in T(\mathcal{A})$ we need to show that $A'^c \in T(\mathcal{A})$. Since T is $\mathcal{A}/\mathcal{A}'$ measurable, $A = T^{-1}(A') \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra , we have that A^c is in $\mathcal{A}$. Moreover, since T^{-1} is measurable then $T^{-1}(A^c) = T^{-1}(T^{-1}(A)^c) = T^{-1}(T^{-1}(A^c)) = T(A^c) \in T(\mathcal{A})$.
3. Finally, for any collection $(A'_n)_{n\in\mathbb{N}} \subset T(\mathcal{A})$ we need to show that $\bigcup_{n\in\mathbb{N}} A'_n \in T(\mathcal{A})$. For each $A'_n \in T(\mathcal{A})$ we have a set $A_n \in \mathcal{A}$ such that $A'_n = T(A_n)$. Thus,
$$\bigcup_{n\in\mathbb{N}} A'_n = \bigcup_{n\in\mathbb{N}} T(A_n) = T\left(\bigcup_{n\in\mathbb{N}} A_n\right) \in T(\mathcal{A}),$$
since $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

□

Chapter 6

Measurable functions

In this chapter we restrict our attention to measurable maps whose domain is any measurable space $(X, \mathcal{A})$ but whose domain is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. To distinguish them from the general case, we shall use the same terminology as [4], and call them measurable functions.

The following result is trivial but important, as the intervals we introduce in the following lemma are easy to work with.

Lemma 6.1. Let $(X, \mathcal{A})$ be a measurable space, and $u : X \to \mathbb{R}$. Then, u is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable

$$\begin{aligned}
&\iff u^{-1}((a,\infty)) = \{x \in X \mid u(x) > a\} = \{u > a\} \in \mathcal{A} \\
&\iff u^{-1}([a,\infty)) = \{x \in X \mid u(x) > a\} = \{u \geq a\} \in \mathcal{A} \\
&\iff u^{-1}((-\infty,a)) = \{x \in X \mid u(x) > a\} = \{u < a\} \in \mathcal{A} \\
&\iff u^{-1}((-\infty,a]) = \{x \in X \mid u(x) > a\} = \{u \leq a\} \in \mathcal{A}
\end{aligned}$$

Notice that we have introduced some notation here, namely, for a function $u : X \to \mathbb{R}$ we denote by $\{u > a\}$ the set $\{x \in X \mid u(x) > a\}$. We define analogous notations for $\geq, <, \leq, =, \neq, \in, \notin$ and more.

Proof. The proof follows immediately from lemma 5.2. Recall that all the families of intervals of the lemma are generators of the Borel σ-algebra on $\mathbb{R}$. ☐

6.1 The extended real line $\overline{\mathbb{R}}$

Throughout this chapter, we will deal with the concepts of $\lim_n, \limsup_n, \liminf_n, \sup_n$ and $\inf_n$ which will often be infinite. If we agree that $-\infty < x < \infty,\ \forall x \in \mathbb{R}$ it makes sense to include the values $\pm\infty$ in $\mathbb{R}$ to build the extended space $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$. We would like $\overline{\mathbb{R}}$ to inherit as much as possible from the algebraic, topological and measurable structures of $\mathbb{R}$.

6.1.1 Extension of the algebraic structure

We extend the algebraic structure by extending the addition and multiplication tables as follows:

$+$	$x \in \mathbb{R}$	$+\infty$	$-\infty$
$y \in \mathbb{R}$	$x + y$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	not defined
$-\infty$	$-\infty$	not defined	$-\infty$

$\cdot$	0	$x \in \mathbb{R} \setminus \{0\}$	$+\infty$	$-\infty$
0	0	0	0^*	0^*
$y \in \mathbb{R} \setminus \{0\}$	0	$x \cdot y$	$\operatorname{sgn}(y) \cdot \infty$	$-\operatorname{sgn}(y) \cdot \infty$
$+\infty$	0^*	$\operatorname{sgn}(x) \cdot \infty$	$+\infty$	$-\infty$
$-\infty$	0^*	$-\operatorname{sgn}(x) \cdot \infty$	$-\infty$	$+\infty$

Caution: here we understand ± as limits but 0 only as bona-fide 0 (i.e. not as a limit, which would cause convergence problems). Conventions are tricky. Expresions of the form

$$\infty - \infty \text{ or } \frac{\pm\infty}{\pm\infty}$$

should be avoided.

6.1.2 Extension of the topological structure

Definition 6.1 (Neighbourhoods in $\overline{\mathbb{R}}$)**.** For some $x \in \overline{\mathbb{R}}$ we say that a neighbourhood of x is a set of the form

$$\begin{aligned} (x - \varepsilon, x + \varepsilon) &\text{ if } x \in \mathbb{R} \\ (a, +\infty] &\text{ if } x = +\infty \\ [-\infty, a) &\text{ if } x = -\infty \end{aligned}$$

for some $a, \varepsilon \in \mathbb{R}$.

Definition 6.2 (Open set in $\overline{\mathbb{R}}$)**.** We say that a set $U \subseteq \overline{\mathbb{R}}$ is open if, for every point $x \in U$ there exists a neighbourhood $B(x)$ of x such that $x \in B(x) \subseteq \overline{\mathbb{R}}$.

6.1.3 Extension of the measurable structure

Definition 6.3 (Borel σ-algebra on $\overline{\mathbb{R}}$)**.** The Borel σ-algebra on $\overline{\mathbb{R}}$ is defined by

$$\mathcal{B}(\overline{\mathbb{R}}) := \{B^* = B \cup S \mid B \in \mathcal{B}(\mathbb{R}) \wedge S \in \mathcal{S}\},$$

where $\mathcal{S} = \{\varnothing, \{+\infty\}, \{-\infty\}, \{-\infty, +\infty\}\}$.

The reason this extension is still called a Borel σ-algebra is justified by the above definition of $\mathcal{B}(\overline{\mathbb{R}})$, by the extension of the topological structure and by the following lemma.

Lemma 6.2. $\mathcal{B}(\overline{\mathbb{R}})$ is generated by sets of the form $[a, \infty]$ (or $(a, \infty]$ or $[-\infty, a]$ or $[-\infty, a)$), where $a \in \mathbb{R}$ or $a \in \mathbb{Q}$ (and hence those intervals are subsets of $\overline{\mathbb{R}}$ or $\overline{\mathbb{Q}}$, resp.)

Proof. It is analogous to the one given for theorem 1.7. ☐

6.1.4 Final remarks

Definition 6.4 (Numerical function)**.** A function $u : X \to \overline{\mathbb{R}}$ that takes values on $\overline{\mathbb{R}}$ is called a numerical function.

Definition 6.5 (Set of measurable functions)**.** Let $(X, \mathcal{A})$ be a measurable space. We write

$$\begin{aligned} \mathcal{M} := \mathcal{M}(\mathcal{A}) &:= \{u : X \to \mathbb{R} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\mathbb{R})\text{-measurable } \}, \\ \mathcal{M}_{\overline{\mathbb{R}}} := \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A}) &:= \{u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})\text{-measurable } \} \end{aligned}$$

for the families of real-values and numerical-valued measurable functions on X.

6.2 Simple functions

Now we will see some important (yet simple) examples. Throughout this section, $(X, \mathcal{A})$ will be a measurable space.

Example 6.1 (Indicator functions)**.** Let $A \in \mathcal{A}$ and define $\mathbb{1}_A : X \to \mathbb{R}$ by

$$\mathbb{1}_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}.$$

Then $\mathbb{1}_A$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable since

$$\mathbb{1}_A{}^{-1}((-\infty, a)) = \{\mathbb{1}_A < a\} = \begin{cases} \varnothing & \text{if } a \leq 0 \\ A^c & \text{if } 0 < a \leq 1 \in \mathcal{A} \\ \mathbb{R} & \text{if } a > 1 \end{cases}$$

In fact, from the proof we can see that $\mathbb{1}_A$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable if and only if $A \in \mathcal{A}$. So measurability of $\mathbb{1}_A$ as a **function** is equivalent to the measurability of A as a **set** (recall that a set is measurable if it is part of the $\mathcal{A}$ which is the domain of the measure at hand).

The following example is so important that we will give it as a definition.

Definition 6.6 (Simple function)**.** Let $A_1, \dots, A_M \in \mathcal{A}$ be pairwise disjoint subsets of X and $y_1, \dots, y_M \in \mathbb{R}$. A function $f : X \to \mathbb{R}$ of the form

$$f(x) = \sum_{i=1}^{M} y_i \mathbb{1}_{A_i}$$

is called a simple function.

Note that

$$f(x) = \begin{cases} y_i & \text{if } x \in A_i \\ 0 & \text{if } x \notin \dot{\bigcup}_{i=1}^{M} A_i \end{cases}$$

is an alternative definition. The i that makes $x \in A_i$ hold is unique or non-existent since, (A_i) are pairwise disjoint, but do not necessarily cover the whole X.

This small inconvenience (of the possibility that i does not exist) is easily fixed by extending the collection of sets to be a partition by defining

$$A_0 = X \setminus \dot{\bigcup_{i=1}^{M}} A_i \text{ and } y_0 = 0.$$

Then one can write $f(x) = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i}$. This is called the **standard representation** of the simple function f.

Notice that a simple function takes only finitely many values. The converse is also true.

Lemma 6.3. Any measurable function g which takes only finitely many values $\{y_0, y_1, \dots, y_M\}$ can be expressed as a linear combination of simple functions.

Proof. Define $A_i := \{g = y_i\} = \{x \in X \mid g(x) = y_i\} = g^{-1}(\{y_i\})$. $A_i \in \mathcal{A}, \forall i = 0, \dots, M$ since $\{y_i\} = (-\infty, y_i] \setminus (-\infty, y_i) \in \mathcal{B}(\mathbb{R})$ and g is assumed to be measurable.

Since $y_0, \dots y_M$ are all distinct, then $A_i \cap A_j = \varnothing$ if $i \neq j$. Furthermore, g takes the value y_i on A_i, and thus

$$g = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i},$$

which is in fact the standard representation of g. □

In general, the representation of a simple function as a linear combination of simple functions is not unique. Consider the function

$$f(x) = \mathbb{1}_{[0,1]}(x) + \mathbb{1}_{[0,\frac{2}{3}]}(x) + \mathbb{1}_{[0,\frac{1}{3}]}(x) = \begin{cases} 0 & \text{if } x \notin [0,1] \\ 3 & \text{if } x \in [0,\frac{1}{3}] \\ 2 & \text{if } x \in (\frac{1}{3},\frac{2}{3}] \\ 1 & \text{if } x \in (\frac{2}{3},1] \end{cases}$$

which can also be given by its standard representation

$$f(x) = 0 \cdot \mathbb{1}_{\mathbb{R}\setminus[0,1]} + 1 \cdot \mathbb{1}_{(\frac{2}{3},1]} + 2 \cdot \mathbb{1}_{(\frac{1}{3},\frac{2}{3}]} + 3 \cdot \mathbb{1}_{[0,\frac{1}{3}]}$$

Simple functions are the building blocks of all measurable functions (in the sense that any measurable function is the limit of a sequence of simple functions).

To end, we will honour simple functions with their family's own definition.

Definition 6.7 (Family of simple functions)**.** We denote by $\mathcal{E}(\mathcal{A}) = \mathcal{E} \subseteq \mathcal{M}$ the family of all simple functions $f : (X, \mathcal{A}) \to (X, \mathcal{B}(\mathbb{R}))$.

6.2.1 Properties of simple functions

Theorem 6.4 (Properties of simple functions)**.** Let $f, g \in \mathcal{E}(\mathcal{A})$. Then the following hold:

1. $f \pm g \in \mathcal{E}(\mathcal{A})$ and $f \cdot g \in \mathcal{E}(\mathcal{A})$
2. $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are in $\mathcal{E}(\mathcal{A})$
3. $|f| \in \mathcal{E}(\mathcal{A})$

Proof. Let

$$f = \sum_{i=0}^{M} a_i \mathbb{1}_{A_i}, \qquad g = \sum_{j=0}^{N} b_j \mathbb{1}_{B_j}$$

be the standard representations of f and g, resp. Then,

1. $$f \pm g = \sum_{i=0}^{M}\sum_{j=0}^{N}(a_i \pm b_j)\mathbb{1}_{A_i \cap B_j}, \qquad \sum_{i=0}^{M}\sum_{j=0}^{N}(a_i \cdot b_j)\mathbb{1}_{A_i \cap B_j}$$

   are the standard representations of $f \pm g$ and $f \cdot g$, respectively and thus $f \pm g, f \cdot g \in \mathcal{E}(\mathcal{A})$.

2. $$f^+ = \sum_{i | a_i \geq 0} a_i \mathbb{1}_{A_i} \qquad f^- = \sum_{i | a_i \leq 0} -a_i \mathbb{1}_{A_i}$$

   are the standard representations of f^+ and f^-, resp. and hence $f^+, f^- \in \mathcal{E}(\mathcal{A})$.

3. $|f| = f^+ + f^- \in \mathcal{E}(\mathcal{A})$ by the first two properties.

□

6.3 Sequences of simple functions. The sombrero lemma.

Theorem 6.5 (Sombrero lemma)**.** Let $(X, \mathcal{A})$ be a measurable space and $u : X \to [0, \infty]$ a non-negative $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function. Then, there exists an increasing sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}(\mathcal{A})$ of non-negative simple functions such that for any $x \in X$

$$u(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} f_n(x).$$

Note that in the previous statement, f_n is a sequence of real-valued (as opposed to numerical) functions. Also, the limit is to be understood point-wise (as opposed to uniformly).

Proof. TODO □

Corollary 6.6. Let $(X, \mathcal{A})$ be a measurable space. Then, for any numerical and $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function $u : X \to \overline{\mathbb{R}}$ there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}(\mathcal{A})$ such that $|f_n| \leq |u|$ and

$$u(x) = \lim_{n\to\infty} f_n(x)$$

Moreover, if u is bounded then convergence is uniform (as opposed to point-wise).

Proof. Write $u = u^+ - u^-$, where u^+, u^- are both non-negative functions. We first show that u^+, u^- are $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable. We shall proceed by using lemma 6.1, with generators of the form $(a, \infty])$. For any $a \in \overline{\mathbb{R}}$ we have

$$\{u^+ > a\} = \begin{cases} X & \text{if } a < 0 \\ \{u \geq a\} & \text{if } a \geq 0 \end{cases} \in \mathcal{A}$$

Similarly,

$$\{u^- > a\} = \begin{cases} X & \text{if } a < 0 \\ \{-u \geq a\} = \{u < a\} & \text{if } a \geq 0 \end{cases} \in \mathcal{A}.$$

Since u^+, u^- are non-negative measurable functions, theorem 6.5 applies and there exist two sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset \mathcal{E}(\mathcal{A})$ such that $f_n \uparrow u^+$ and $g_n \uparrow u^-$. Hence

$$\lim_{n\to\infty} f_n - \lim_{n\to\infty} g_n = \lim_{n\to\infty} (f_n - g_n) = \lim_{n\to\infty} h_n = u^+ - u^- = u.$$

Furthermore, $|f_n - g_n| \leq |f_n| - |g_n| = f_n - g_n \leq u^+ + u^- = |u|$. Finally, if u is bounded then $\exists N \in \mathbb{N}$ such taht $u(x) \leq N, \forall x \in X$. Then from the proof of theorem 6.5 (applied to u^+ and u^-) we see that $\forall n \geq N$ and $\forall x \in X$

$$|f_n(x) - g_n(x) - u(x)| \leq |f_n - u^+(x)| + |g_n - u^-(x)| \leq \frac{1}{2^{n-1}}.$$

This implies uniform convergence. □

Convention Given a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, by $\sup_{n\in\mathbb{N}} u_n$, $\inf_{n\in\mathbb{N}} u_n$, $\limsup_{n\in\mathbb{N}} u_n$ and $\liminf_{n\in\mathbb{N}}$ we mean the point-wise defined functions $(\sup_{n\in\mathbb{N}} u_n)(x) = \sup_{n\in\mathbb{N}} u_n(x) = \sup\{u_n(x) \mid n \in \mathbb{N}\}$ and similarly for others.

Recall that

$$\liminf_{n\to\infty} u(x) = \sup_{n\geq 1} \inf_{m\geq n} u_m(x) \tag{6.1}$$

and

$$\limsup_{n\to\infty} u(x) = \inf_{n\geq 1} \sup_{m\geq n} u_m(x). \tag{6.2}$$

Also,

$$v_n(x) = \inf_{m\geq n} u_m(x) \uparrow \liminf_{n\to\infty} u(x),$$
$$w_m(x) = \sup_{m\geq n} u_m(x) \downarrow \limsup_{n\to\infty} u_n(x)$$

and

$$\inf_{n\in\mathbb{N}} u_n(x) \leq \liminf_{n\to\infty} u_n \leq \limsup_{n\to\infty} u_n(x) \leq \sup_{n\in\mathbb{N}} u_n(x).$$

If $\liminf_{n\to\infty} u_n(x) = \limsup_{n\to\infty} u_n(x)$, then $\lim_{n\to\infty} u_n(x)$ exists and equals the common value.

Corollary 6.7. Let $(X, \mathcal{A})$ be a measurable space and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ a sequence of $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable functions. Then, $\inf_{n\geq 1} u_n, \liminf_{n\to\infty} u_n, \limsup_{n\to\infty} u_n, \sup_{n\geq 1}$ are all $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable functions.

Proof. First we prove that sup and inf are both $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable.

$$\{\sup_{n\geq 1} u_n \leq a\} = \bigcap_{n=1}^{\infty} \{u_n \leq a\} \in \mathcal{A}$$
$$\{\inf_{n\geq 1} u_n \geq a\} = \bigcap_{n=1}^{\infty} \{u_n \geq a\} \in \mathcal{A}$$

Now we use these to prove the rest.

$\liminf_{n\to\infty} u_n = \sup_{n\geq 1} \inf_{m\geq n} u_m$. Set $v_n = \inf_{m\geq n} u_m$ as before and because of the above v_n is $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable. Since $v_n \uparrow \liminf_{n\to\infty} u_n$ we have $\liminf_{n\to\infty} = \sup_{n\geq 1} v_n \in \mathcal{A}$. Hence $\liminf_{n\to\infty} u_n$ is also $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable.

Similarly, write $w_n = \sup_{m\geq n} u_n \in \mathcal{A}$ because of the above. Then, since $w_n \downarrow \limsup_{n\to\infty} u_n$ we have $\limsup_{n\to\infty} u_n = \inf_{n\geq 1} w_n \in \mathcal{A}$ and hence $\limsup_{n\to\infty} u_n$ is $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable. ☐

In fact, we can deduce more.

Corollary 6.8. Let $u, v \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ be two $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable functions. Then $u \pm v$, $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$ are $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable.

Proof. By theorem 6.5 there exist two sequences of non-negative simple functions (f_n) and (g_n) such that $f_n \uparrow u$ and $g_n \uparrow v$. Since f_n and g_n are simple functions, by theorem 6.4 we have that $f_n \pm g_n$ is also a simple function with $\lim_{n\to\infty} f_n + g_n = u + v$. Moreover, $f_n + g_n$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable (since f_n, g_n are) and thus it is also $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable and hence $u + v$ is also $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable.

Similarly, we can prove the corollary for $u \vee v$ and $u \wedge v$. ☐

Remark 6.9. Applying the above to u^+, u^- we see that u is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable if and only if u^+ and u^- are $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable.

Corollary 6.10. If $u, v \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, then the following sets are all measurable:

$$\{u \leq v\}, \{u < v\}, \{u = v\}, \{u > v\}, \{u \geq v\}$$

Proof. TODO ☐

Chapter 7

Integrals of non-negative functions

We are now ready to define the integral of a non-negative function in terms of a measure.

7.1 Integral of a non-negative simple function

Since each $u \in \mathcal{M}^+(\mathcal{A})$ or $\mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ is a limit of an increasing sequence of simple functions (theorem 6.5), we concentrate first on the collection $\mathcal{E}^+ = \mathcal{E}^+(\mathcal{A})$ of all $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable non-negative simple functions.

Let $f(x) = \sum_{i=0}^N a_i \mathbb{1}_{A_i}$ be the standard representation of $f \in \mathcal{E}^+$. Recall that this means that (A_i) is a collection of sets in $\mathcal{A}$ which define a partition of X (i.e. $X = \dot\bigcup_{i=0}^N A_i$). Furthermore, on A_i, f takes the value a_i.

Let

$$I_\mu(f) := \sum_{i=0}^N a_i \mu(A_i). \tag{7.1}$$

We want to interpret $I_m u(f)$ as $\int f d\mu$, but there might be a small problem, namely f might have more than one standard representation so we need to check that the definition of $I_\mu(f)$ is independent of the specific representation, i.e. they all give the same answer.

Lemma 7.1 (Integrals of simple functions are representation-invariant)**.** If $f = \sum_{i=0}^M a_i \mathbb{1}_{A_i} = \sum_{j=0}^N b_j \mathbb{1} B_j$ are two standard representations of a simple function then $\sum_{i=0}^M a_i \mu(A_i) = \sum_{j=0}^n b_j \mu(B_j)$.

Proof. Recall that since (A_j) and (B_j) are partitions we may write

$$A_i = A_i \cap \dot\bigcup_{j=0}^M B_j = \dot\bigcup_{j=0}^M A_i \cap B_j,$$

$$B_j = B_j \cap \dot\bigcup_{i=0}^N A_i = \dot\bigcup_{i=0}^N A_i \cap B_j.$$

Thus,

$$\begin{aligned}\sum_{i=0}^{M} a_i\mu(A_i) = \sum_{i=0}^{M} a_i\mu\left(\biguplus_{j=0}^{M} A_i \cap B_j\right) &= \sum_{i=0}^{M} a_i \sum_{j=0}^{N} \mu(A_i \cap B_j) \\ &= \sum_{i=0}^{M}\sum_{j=0}^{N} a_i\mu(A_i \cap B_j) = \sum_{j=0}^{N}\sum_{i=0}^{M} a_i\mu(A_i \cap B_j).\end{aligned}$$

Now,

- if $A_i \cap B_j = \varnothing$ then $a_i\mu(A_i \cap B_j) = 0 = b_j\mu(A_i \cap B_j)$, and
- if $A_i \cap B_j \neq \varnothing$ then $\exists x \in A_i \cap B_j$ so that $f(x) = a_i = b_j$. Thus $a_i\mu(A_i \cap B_j) = b_j\mu(A_i \cap B_j)$.

Therefore,

$$\begin{aligned}\sum_{i=0}^{m} a_i\mu(A_i) = \sum_{j=0}^{N}\sum_{i=0}^{M} a_i\mu(A_i \cap B_j) &= \sum_{j=0}^{N}\sum_{i=0}^{N} b_j\mu(A_i \cap B_j) \\ &= \sum_{j=0}^{N} b_j \sum_{i=0}^{M} \mu(A_i \cap B_j) = \sum_{j=0}^{N} b_j\mu(B_j)\end{aligned}$$

□

So $I_\mu(f)$ is well-defined regardless of the chosen standard representation. We are about to be ready to define the integral for arbitrary non-negative functions (once we verify that the properties of simple functions that have allowed us to state theorem 6.5 hold under integrals).

Theorem 7.2 (Properties of $I_\mu(f)$)**.** Let $f, g \in \mathcal{E}^+(\mathcal{A})$ and $\lambda \geq 0$, then

1. $I_\mu(\mathbb{1}_A) = \mu(A)$, for any $A \in \mathcal{A}$
2. $I_\mu(\lambda f) = \lambda I_\mu(f)$ (positive homogenous degree 1)
3. $I_\mu(f + g) = I_\mu(f) + I_\mu(f)$ (additive)
4. if $g \leq g$ then $I_\mu(f) \leq I_\mu(g)$ (monotone)

Proof. TODO □

7.2 Integral of a non-negative function

Definition 7.1 ((μ)-integral of a non-negative function)**.** Let $u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$, then the ($\mu$)-integral of u is defined by

$$\int u d\mu := \sup\{I_\mu(g) \mid g \leq u, g \in \mathcal{E}^+(\mathcal{A})\} \in [0, \infty] \tag{7.2}$$

Some alternative notations are

$$\int u d\mu = \int_X u d\mu = \int_X u(x) d\mu(x).$$

Note that here we are taking the supremum over all simple functions less than or equal to μ. We will eventually show that it is enough to find a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ such that $f_n \uparrow u$, since we will prove that

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu = \sup_{n\geq 1} \int f_n d\mu.$$

Before we move one, let's make sure that we did not break anything. Namely, does the definition of the integral above extend the definition of I_μ for non-negative simple functions, i.e. $I_\mu(f) = \int f d\mu$ if f is a non-negative simple function. The answer is yes!

Lemma 7.3. If $f \in \mathcal{E}^+(\mathcal{A})$ then $I_\mu(f) = \int f d\mu$.

Proof. Since $f \in \mathcal{E}^+(\mathcal{A})$ and $f \leq f$, then by definition of the supremum, we have $I\mu(f) \leq \int f d\mu$. Now, by monotonicity of I_μ, for any $g \in \mathcal{E}^+(\mathcal{A})$ such that $g \leq f$ one has $I_\mu(g) \leq I_\mu(f)$. Thus,

$$\int f d\mu = \sup\{I_\mu(g) \mid g \leq f,\ g \in \mathcal{E}^+(\mathcal{A})\} \leq I_\mu(f).$$

This shows that $I_\mu(f) = \int f d\mu$. ☐

Remark 7.4. It is easy to see that μ-integrals are monotone. Since if $u \leq v$, then $\{g \in \mathcal{E}^+(\mathcal{A}) \mid g \leq u\} \subset \{h \in \mathcal{E}^+(\mathcal{A}) \mid h \leq v\}$. Hence,

$$\int u d\mu = \sup\{I_\mu(g) \mid g \in \mathcal{E}^+(\mathcal{A}), g \leq u\} \leq \sup\{I_\mu(h) \mid h \in \mathcal{E}^+(\mathcal{A}), h \leq v\} = \int v d\mu$$

Now we come to the central theorem of this chapter.

Theorem 7.5 (Beppo-Lévi)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$. Then,

$$u := \sup_{n\geq 1} u_n = \lim_{n\to\infty} u_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A}),$$

and

$$\int u d\mu = \int \sup_{n\geq 1} u_n d\mu = \sup_{n\geq 1} \int u_n d\mu, \tag{7.3}$$

or, alternatively,

$$\int u d\mu = \int \lim_{n\to\infty} u_n d\mu = \lim_{n\to\infty} \int u_n d\mu \tag{7.4}$$

Notice that the assumption that u_n is an increasing sequence is necessary to be able to interchange limits.

Proof. TODO. Important ☐

The following corollary is really just a more concise statement of the previous theorem.

Corollary 7.6. Let $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ be a non-negative, numerical, $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ be a sequence of non-negative simple functions such that $f_n \uparrow u$. Then,

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu \tag{7.5}$$

Proof. Clearly the hypothesis of theorem 7.5 are satisfied. (Just take $u_n = f_n$). ☐

Now we extend theorem 7.2 to the case for integrals of arbitrary non-negative functions.

Theorem 7.7 (Properties of integrals of non-negative functions)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $u, v \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$. Then,

1. $\int \mathbb{1}_A d\mu = I_\mu(A) = \mu(A)$
2. $\int \alpha u d\mu = \alpha \int u d\mu, \ \forall \alpha \geq 0.$[a]
3. $\int (u+v) d\mu = \int u d\mu + \int v d\mu$
4. if $u \leq v$, then $\int u d\mu \leq \int v d\mu$

---

[a] Here we require $\alpha \geq 0$ to keep the functions non-negative. As soon as we generalise to arbitrary functions this assumption is not needed.

Proof.

1. $\mathbb{1}_A$ is clearly a simple function so, by definition $\int \mathbb{1}_A d\mu = I_\mu(\mathbb{1}_A) = \mu(A)$.
2. By theorem 6.5 there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+$ such that $f_n \uparrow u$. Then,
$$\int \alpha u d\mu \overset{7.6}{=} \lim_{n\to\infty} \int \alpha f_n d\mu \overset{7.3}{=} \lim_{n\to\infty} I_\mu(\alpha f_n) \overset{7.2}{=} \alpha \lim_{n\to\infty} I_\mu(f_n) \overset{7.3}{=} \alpha \int u d\mu$$
3. Once again by theorem 6.5 there exist sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+$ such that $f_n \uparrow u$ and $g_n \uparrow v$. Then, $(f_n + g_n) \uparrow (u+v)$ and hence,
$$\begin{aligned}\int (u+v) d\mu &\overset{7.6}{=} \lim_{n\to\infty} \int (f_n + g_n) d\mu \overset{7.3}{=} \lim_{n\to\infty} I_\mu(f_n + g_n) \\ &\overset{7.2}{=} \lim_{n\to\infty} I_\mu(f_n) + \lim_{n\to\infty} I_\mu(g_n) \overset{7.3}{=} \int u d\mu + \int v d\mu\end{aligned}$$
4. Already proven in remark 7.4.

☐

Corollary 7.8. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$. Then, $\sum_{n=1}^\infty u_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ and

$$\int \sum_{n=1}^\infty u_n d\mu = \sum_{n=1}^\infty \int u_n d\mu \tag{7.6}$$

Proof. Let $v_m = \sum_{n=1}^m u_n$. Then $v_m \uparrow \sum_{n=1}^\infty u_n$ and thus

$$\begin{aligned}\int \sum_{n=1}^{\infty} u_n d\mu = \int \lim_{m\to\infty} v_m d\mu \overset{7.6}{=} \lim_{m\to\infty} \int v_m d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} u_n d\mu \\ \overset{7.7}{=} \lim_{m\to\infty} \sum_{n=1}^{m} \int u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu\end{aligned}$$

☐

Theorem 7.9 (Fatou's lemma)**.** Let $(u_n)_{n\in\mathbb{N}}$ be **any** sequence in $\mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$. Then,

$$u := \liminf_{n\to\infty} u_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A}) \tag{7.7}$$

and

$$\int \liminf_{n\to\infty} u_n d\mu \leq \liminf_{n\to\infty} \int u_n d\mu \tag{7.8}$$

The great thing about theorem 7.9 is that it makes no assumptions of monotonicity and can be applied to any sequence of non-negative, numerical, measurable functions.

Proof. Equation 7.7 follows from corollary 6.7. ☐

Chapter 8

Integrals of measurable functions

In this chapter we will extend the notion of integral to the general case $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ (as opposed to just positive functions, which were discussed in the previous chapter).

We have already seen that any function $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ may be written as the difference of two non-negative functions $u = u^+ - u^-$, hence we have the tendency to define

$$\int u d\mu = \int u^+ d\mu - \int u^- d\mu$$

However, since $\int u^+ d\mu, \int u^- d\mu \in [0, \infty]$ and therefore both can be $+\infty$ we must be careful with the definition of the integral for a general u.

Definition 8.1 (μ-integrable)**.** Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, then u is said to be μ-integrable if

$$\max\left\{\int u^+ d\mu, \int u^- d\mu\right\} < \infty, \tag{8.1}$$

i.e. both are finite. In this case we define

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, +\infty). \tag{8.2}$$

Remark 8.1. If

$$\min\left\{\int u^+ d\mu, \int u^- d\mu\right\} < \infty,$$

i.e. at most one of the integrals is $+\infty$, then one can still define

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in [-\infty, +\infty].$$

We will denote the set of all μ-integrable functions by

$$\mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) := \{u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A}) \mid u \text{ is } \mu - \text{integrable}\}. \tag{8.3}$$

Similarly, if we wish to restrict ourselves to real-valued functions we will write

$$\mathcal{L}^1(\mu) := \{u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \mid u \text{ is } \mu-\text{integrable}\}. \tag{8.4}$$

Remark 8.2. Note that for $u \geq 0$, $\int u d\mu$ always exists, but it can have the value $+\infty$. So for $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ we have

$$u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \iff \int u d\mu < \infty.$$

Some authors still call a positive function μ-integrable if it takes the value $+\infty$. We will not use this convention. Take care to update your understanding by comparing definition 7.1 and definition 8.1.

As before, if we need to stress the integration variable (the space where the integral is defined), we write

$$\int u d\mu = \int_X u(x) d\mu(x).$$

The following theorem gives us for ways to check that a function $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ is μ-integrable.

Theorem 8.3 (Characterisation of μ-integrability)**.** Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, then the following are equivalent:

1. $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$;
2. $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$;
3. $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$;
4. $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ with $w \geq 0$ and $|u| \leq w$.

Proof.

- 1. $\iff$ 2. follows from the definition of μ-integrability.
- 2. $\iff$ 3. follows from $|u| = u^+ + u^-$ and $u^+, u^- \leq |u|$ and the monotonicity for non-negative measurable functions.
- 3. $\iff$ 4. follows from monotonicity of the integral of non-negative measurable functions.

□

Theorem 8.4 (Properties of the μ-integral)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, $u, v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $\alpha \in \mathbb{R}$. Then, the following hold

1. $\alpha u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $\int \alpha u d\mu = \alpha \int u d\mu$.
2. $u + v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $\int (u+v) d\mu = \int u d\mu + \int v d\mu$.
3. $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$
4. if $u \le v$, then $\int u d\mu \le \int v d\mu$.
5. $$\left| \int u d\mu \right| \le \int |u| \, d\mu.$$

Proof. For 1. and 2., we first use theorem 8.3 to prove that αu and $u + v$ are in $\mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. Then, we rewrite each in terms of positive and negative parts to prove that the integrals coincide.

1. We will prove that $\alpha u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \iff |\alpha u| \, \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. Now,

   $$\int |\alpha u| \, d\mu = \int |\alpha| \, |u| \, d\mu \overset{7.7}{=} |\alpha| \int |u| \, d\mu < \infty.$$

   Thus, by theorem 8.3 we have that $\alpha u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. To check $\int \alpha u d\mu = \alpha \int u d\mu$ we consider two cases:

   - If $\alpha \ge 0$ then $(\alpha u)^+ = \alpha u^+$ and $(\alpha u)^- = \alpha u^-$ hence

     $$\begin{aligned} \int \alpha u d\mu &= \int \alpha u^+ d\mu - \int \alpha u^- d\mu = \alpha \int u^+ d\mu - \alpha \int u^- d\mu \\ &= \alpha \left(\int u^+ d\mu - \int u^- d\mu \right) = \alpha \int u d\mu. \end{aligned}$$

   - If $\alpha < 0$ then $(\alpha u)^+ = -\alpha u^-$ and $(\alpha u)^- = -\alpha u^+$. Therefore,

     $$\begin{aligned} \int \alpha u d\mu &= \int \left((\alpha u)^+ - (\alpha u)^- \right) d\mu \\ &= \int \underbrace{-\alpha}_{>0} u^- d\mu - \int \underbrace{-\alpha}_{>0} u^+ d\mu \\ &= (-\alpha) \int u^- d\mu - (-\alpha) \int u^+ d\mu \\ &= (-\alpha) \left(\int u^- d\mu - \int u^+ d\mu \right) \\ &= (-\alpha) \left(- \int u d\mu \right) = \alpha \int u d\mu \end{aligned}$$

2. $|u + v| \le |u| + |v|$ so by linearity for non-negative integrals we have

   $$\int |u + v| \, d\mu \le \int |u| \, d\mu + \int |v| \, d\mu < \infty.$$

Hence, by theorem 8.3, $u+v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. As before, $(u+v) = (u+v)^+ - (u+v)^- = u^+ - u^- + v^+ - v^-$ hence

$$(u+v)^+ + u^- + v^- = (u+v)^- = u^+ + v^+,$$

so

$$\int [(u+v)^+ + u^- + v^-]\, d\mu = \int [(u+v)^- = u^+ + v^+]\, d\mu$$

where all functions are non negative. Applying theorem 7.7 we have

$$\int (u+v)^+ d\mu + \int u^- d\mu + \int v^- d\mu = \int (u+v)^- d\mu + \int u^+ d\mu + \int v^+ d\mu.$$

Rewriting again (every term is finite since $u, v, u+v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ so subtraction is not a problem), we have

$$\int (u+v)^+ d\mu - \int (u+v)^- d\mu = \int u^+ d\mu - \int u^- d\mu + \int v^+ d\mu - \int v^- d\mu,$$

or,

$$\int (u+v) d\mu = \int u d\mu + \int v d\mu.$$

3. Observe that both $\max(u,v), \min(u,v) \le |u| + |v|$ so

$$\int \max(u,v) d\mu \le \int |u|\, d\mu + \int |v|\, d\mu < \infty$$
$$\int \min(u,v) d\mu \le \int |u|\, d\mu + \int |v|\, d\mu < \infty$$

by theorem 8.3. Thus, $\max(u,v), \min(u,v) \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

4. Suppose $u \le v$ then $u^+ \le v^+$ and $u^- \ge v^-$, so

$$\int u d\mu = \int u^+ d\mu - \int v^- d\mu \le \int v^+ d\mu - \int v^- d\mu = \int v d\mu.$$

5. Recall that one can defined $|a| = \max(a, -a)$. Then

$$\begin{aligned} \left|\int u d\mu\right| &= \max\left\{\int u d\mu, -\int u d\mu\right\} \\ &= \max\left\{\int u d\mu, \int -u d\mu\right\} \\ &= \max\left\{\int |u|\, d\mu, \int |u|\, d\mu\right\} = \int |u|\, d\mu. \end{aligned}$$

□

Remark 8.5. Note that in property 2. we did had the assumption that $(u+v)$ does not take the values ±∞. Hence we cannot say that $\mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ is a linear space.

However, $\mathcal{L}^1(\mu)$, the set of all μ-integrable real-valued functions is a linear space.

8.1 Restricting the domain

Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. For any $A \in \mathcal{A}$, the function $\mathbb{1}_A \cdot u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ and $|\mathbb{1}_A \cdot u| \leq |u|$. Hence, by theorem 8.3 (4.) we have that $\mathbb{1}_A \cdot u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mathcal{A})$.

Similarly, if $(X, \mathcal{A}, \mu)$ is a measure space and $u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$, for any $A \in \mathcal{A}$ we can define the function $\mathbb{1}_A \cdot u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$ by corollary 6.8 since $\mathbb{1}_A \cdot u = \min(\mathbb{1}_A, u)$ in this case (since u is non-negative).

When we wish to restrict the domain of integration, we write,

$$\int_A u d\mu := \int \mathbb{1}_A u d\mu. \tag{8.5}$$

Note that if $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ $\int_A u d\mu < \infty$ but if $u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$ then $\int_A u d\mu$ is well defined but can take the value $+\infty$.

We do we keep coming back to the case $u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$? Well, because in this case we can associate a new measure on $(X, \mathcal{A})$ as follows:

Lemma 8.6. Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) := \int_A u d\mu = \int \mathbb{1}_A u d\mu. \tag{8.6}$$

Then ν is a measure on $(X, \mathcal{A})$. Moreover, if $\int u d\mu < \infty$, i.e. u is, **additionally**, in $\mathcal{L}^1_{\overline{\mathbb{R}}}(\mathcal{A})$, then ν is a finite measure.

Proof. As usual,

1.

$$\nu(\varnothing) = \int_{\varnothing} u d\mu = \int 0 \cdot u d\mu = 0$$

2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$. Then

$$\mathbb{1}_{\dot\bigcup_{n\in\mathbb{N}} A_n} = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n},$$

and thus,

$$\begin{aligned}
\nu\left(\dot\bigcup_{n\in\mathbb{N}} A_n\right) &= \int_{\dot\bigcup_{n\in\mathbb{N}} A_n} u d\mu \\
&= \int \mathbb{1}_{\dot\bigcup_{n\in\mathbb{N}} A_n} u d\mu \\
&= \int \left(\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n}\right) u d\mu \\
&= \int \left(\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} u\right) d\mu \\
&\overset{7.8}{=} \sum_{n\in\mathbb{N}} \int \mathbb{1}_{A_n} u d\mu = \sum_{n\in\mathbb{N}} \nu(A_n).
\end{aligned}$$

□

8.2 Examples

Example 8.1. Let $(X, \mathcal{A}, \delta_A)$ be a measure space where δ_A is the Dirac measure for a fixed $A \in \mathcal{A}$ (see example 2.1). Then for $u : \mathcal{A} \to \mathbb{R}, u \geq 0$ we have

$$\int u d\delta_A = \int u(x)\delta_A(dx) = u(A).$$

Then, if we generalise $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ we have

$$\int u d\delta_A = \int (u^+ - u^-) d\delta_A = \int u^+ d\delta_A - \int u^- d\delta_A = u^+(A) - u^-(A) = u(A).$$

We conclude that $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\delta_A) \iff |u(A)| < \infty$.

Example 8.2. Consider the measure space $(\mathbb{N}, \wp(\mathbb{N}), \mu)$ where μ is the discreet measure given by

$$\mu = \sum_{n \in \mathbb{N}} \alpha_n \delta_{\{n\}}.$$

We have seen that

$$\int |u| \, d\mu = \sum_{n \in \mathbb{N}} \alpha_n |u(n)|.$$

Thus, $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \iff \sum_{n \in \mathbb{N}} \alpha_n |u(n)| < \infty$. Note that we can write

$$u(m) = \sum_{n \in \mathbb{N}} u(n) \mathbb{1}_{\{n\}}(m),$$

so if $\alpha_1 = \alpha_2 = \cdots = 1$ then $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \iff \sum_{n \in \mathbb{N}} |u(n)| < \infty$. In particular, it is necessary that $u(n) \in \mathbb{R}, \ \forall n \in \mathbb{N}$. We can in this case express

$$\mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) = \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \wedge \sum_{n \in \mathbb{N}} |x_n| < \infty \right\} := {}^1(\mathbb{N}).$$

We call ${}^1(\mathbb{N})$ the set of all summable sequences. This space is important in functional analysis.

Example 8.3. If $(X, \mathcal{A}, \mu)$ is a finite measure space, then any bounded function $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ is μ-integrable. That is, if for any given $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ there exists $M \in \mathbb{R}, \ M > 0$ such that $|u(x)| \leq M, \ \forall x \in \mathcal{A}$ then u is μ-integrable.

$$\int |u| \, d\mu = \int |u| \, \mathbb{1}_X d\mu \leq \int M \int_X d\mu = M \int \mathbb{1}_X d\mu = M\mu(X) < \infty.$$

Chapter 9

Null sets and the notation almost everywhere

In this chapter we formally introduce the concept of a null set and the handy notation almost everywhere, which gives us a way to formalise ideas that we get intuitively.

Definition 9.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. The collection of μ-null sets is defined by

$$\mathcal{N}_\mu = \{A \in \mathcal{A} \mid \mu(A) = 0\} \tag{9.1}$$

Note that $\mathcal{N}_\mu$ is a σ-algebra , not necessarily on X, contained in $\mathcal{A}$.

Definition 9.2. We say that a property $\Pi = \Pi(x)$ holds μ almost everywhere, in short μ-a.e. or a.e., if there exits

$$N \in \mathcal{N}_\mu \text{ such that } \{x \in X \mid \Pi(x) \text{ fails }\} \subset N. \tag{9.2}$$

Note that we do not require that we do not require that $\{x \in X \mid \Pi(x) \text{ fails }\}$ be a measurable set, it only has to be a subset of a measurable set. In practice, this set is indeed a μ-null set itself.

Example 9.1. Let Π be the property that $u(x) = v(x)$ for $u, v \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$. Then both $\{x \in X \mid u(x) = v(x)\}$ and $\{x \in X \mid u(x) \neq v(x)\}$ are measurable (since $u, v : \mathcal{A} \to \overline{\mathbb{R}}$). In this case, saying that Π holds a.e. means that $\{x \in X \mid u(x) \neq v(x)\} \in \mathcal{N}_\mu$.

Before moving on, we will prove a very handy inequality.

Theorem 9.1 (Markov inequality)**.** Let $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$, $A \in \mathcal{A}$ and $c > 0$. Then,

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| \, d\mu. \tag{9.3}$$

In particular if $A = X$ then

$$\mu(\{|u| \geq c\}) \leq \frac{1}{c} \int |u| \, d\mu.$$

Informally, this means that an integrable function does not blow up (too much), i.e. an integrable function only attains large values on a set with a small measure.

Proof.

$$\begin{aligned} \mu(\{u \geq c\} \cap A) &= \int 1_{\{|u| \geq c\} \cap A} d\mu \\ &= \frac{c}{c} \int \mathbb{1}_{\{|u| \geq c\}} \cdot \mathbb{1}_A d\mu \\ &= \frac{1}{c} \int_A c \mathbb{1}_{\{|u| \geq c\}} d\mu \\ &\leq \frac{1}{c} \int_A \underbrace{|u| \, \mathbb{1}_{\{|u| \geq c\}}}_{\geq |u|} d\mu \leq \frac{1}{c} \int_A |u| \, d\mu \end{aligned}$$

□

Theorem 9.2. Let $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ be a numerical integrable function on a measure space $(X, \mathcal{A}, \mu)$. Then,

1.

$$\int |u| \, d\mu = 0 \iff |u| = 0 \ \mu\text{-a.e.} \iff \mu\left(\{x \in X \mid u(x) \neq 0\}\right) = 0 \tag{9.4}$$

2. For any $N \in \mathcal{N}_\mu$, we have

$$\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \text{ and } \int_N u d\mu = 0. \tag{9.5}$$

Proof. Let us start with 2. Define $f_n = \min\{|u|, n\}$. Then, $f_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ and $f_n \uparrow |u|$ and $\mathbb{1}_N \cdot f_n \uparrow \mathbb{1}_N \cdot |u|$. By theorem 7.5,

$$\begin{aligned} \int_N |u| \, d\mu &= \int \mathbb{1}_N \, |u| \, d\mu \\ &= \int \sup_{n \in \mathbb{N}} \mathbb{1}_N f_n d\mu \\ &= \sup_{n \in \mathbb{N}} \int \mathbb{1}_N f_n d\mu \\ &\leq \sup_{n \in \mathbb{N}} \int n \mathbb{1}_N d\mu = \sup_{n \in \mathbb{N}} n\mu(N) = \sup_{n \in \mathbb{N}} 0 = 0. \end{aligned}$$

Thus, $\int \mathbb{1}_N |u| \, d\mu = 0 \implies \mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. Now,

$$0 \le \left| \int_n u d\mu \right| \le \int_N |u| \, d\mu = 0,$$

thus $\int_n u d\mu = 0$. ☐

Now we move on to 1. Assume $\int |u| \, d\mu = 0$. Then,

$$\begin{aligned} \mu(\{u \ne 0\}) &= \mu(\{|u| > 0\}) \\ &= \mu\left(\bigcup_{n=1}^{\infty} \{|u| \ge \frac{1}{n}\} \right) \\ &\overset{2.2}{\le} \sum_{n=1}^{\infty} \mu(\{|u| \ge \frac{1}{n}\}) \\ &\overset{9.1}{\le} \sum_{n=1}^{\infty} n \int |u| \, d\mu \\ &= \sum_{n=1}^{\infty} 0 = 0. \end{aligned}$$

Thus $|u| = 0$ a.e.
Conversely, assume $|u| = 0$ a.e. Then $\{|u| \ne 0\} \in \mathcal{N}_\mu$ and

$$\begin{aligned} \int |u| \, d\mu &= \int \left(|u| \, \mathbb{1}_{\{|u|=0\}} + |u| \, \mathbb{1}_{\{|u|=0\}} \right) d\mu \\ &= \int \underbrace{|u|}_{=0} \, \mathbb{1}_{\{|u|=0\}} d\mu + \underbrace{\int |u| \, \mathbb{1}_{\{|u|=0\}} d\mu}_{\text{=0 by part 2.}} = 0 \end{aligned}$$

This theorem is very important and used widely as it allows us to change the value of a function on a μ-null set without changing the value of the integral.

Corollary 9.3. Let $u, v \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ be such that $u = v$ a.e. Then,

1. if $u, v \ge 0$, then $\int u d\mu = \int v d\mu$ (though it is possible that both sides are $+\infty$).
2. $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mathcal{A}) \iff v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mathcal{A})$ and, also in this case $\int u d\mu = \int v d\mu$.

Proof. Part 1. Assume $u, v \in \mathcal{M}^{+}_{\overline{\mathbb{R}}}(\mathcal{A})$ with $u = v$ a.e.

$$\begin{aligned} \int u d\mu &= \int \left(u \mathbb{1}_{\{u=v\}} + u \mathbb{1}_{\{u \ne v\}} \right) d\mu \\ &= \int \underbrace{u \mathbb{1}_{\{u=v\}}}_{=v} d\mu + \underbrace{\int u \mathbb{1}_{\{u \ne v\}} d\mu}_{\text{0 by theorem 9.2}} \\ &= \int v \mathbb{1}_{\{u=v\}} d\mu + \underbrace{\int v \mathbb{1}_{\{u \ne v\}} d\mu}_{\text{0 by theorem 9.2}} \\ &= \int \left(v \mathbb{1}_{\{u=v\}} + v \mathbb{1}_{\{u \ne v\}} \right) d\mu = \int v d\mu. \end{aligned}$$

Part 2 follows from part 1. Note that if $u = v$ a.e. then $u^+ = v^+$ and $u^+ = v^-$ a.e. So if $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$, then by part 1,

$$\max\left(\int u^+ d\mu, \int u^- d\mu\right) = \max\left(\int v^+ d\mu, \int v^- d\mu\right),$$

which implies that $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and

$$\int u d\mu = \int u^+ d\mu - \int u^- d\mu \overset{\text{part 1}}{=} \int v^+ d\mu - \int v^- d\mu = \int v d\mu.$$

☐

Another consequence of theorem 9.2 is that integrable functions take values in $\mathbb{R}$ a.e.

Corollary 9.4. Let $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. Then u takes values in $\mathbb{R}$ μ-a.e. , i.e.

$$\mu(\{x \in X \mid |u(x)| = +\infty\}) = \mu(\{x \in X \mid |u| = \infty\}) = 0.$$

In particular, we can find $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ such that $u = v$ μ-a.e. and $\int u d\mu = \int v d\mu$.

Proof. TODO. ☐

As a consequence of this corollary, we can now restrict our attention to $\mathcal{L}^1(\mu)$ without loss of generality.

Finally, we give a result that is helpful for verifying that two functions are equal μ-a.e.

Corollary 9.5. Let $\mathcal{B} \subseteq \mathcal{A}$ be a σ-algebra .

1. If $u, w \in \mathcal{L}^1(\mathcal{B})$ (so $u, w \in \mathcal{M}(\mathcal{B})$) and $\int_B u d\mu = \int_B w d\mu$ for all $B \in \mathcal{B}$, then $u = w$ μ-a.e.
2. If $u, w \in \mathcal{M}^+(\mathcal{B})$ and $\int_B u d\mu = \int_B w d\mu$ for all $B \in \mathcal{B}$ and $\mu \mid_{\mathcal{B}}$ is σ-finite then $u = w$ μ-a.e.

Proof. TODO ☐

Chapter 10

Convergence theorems

The purpose of this chapter is to generalise the results that were discussed in chapter 7, namely theorem 7.5, theorem 7.9 and the reverse Fatou lemma. By generalisation we mean that we will remove the restrictions about positive functions and/or increasing sequences.

10.1 Convergence theorems

First, we extend theorem 7.5 to negative functions.

Theorem 10.1 (Monotone convergence)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be an increasing sequence of measurable functions such that

$$u_n \leq u_{n+1}, \ \forall n \in \mathbb{N} \text{ and define } u = \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A}).$$

Then,

$$u \in \mathcal{L}^1(\mu) \iff \sup_{n\in\mathbb{N}} \int u_n d\mu < \infty, \tag{10.1}$$

and in that case

$$\int u d\mu = \int \lim_{n\to\infty} u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n = \sup_{n\in\mathbb{N}} \int u_n d\mu = \lim_{n\to\infty} \int u_n d\mu. \tag{10.2}$$

Here we're writting both the lim and the sup versions of the equalities to be extra clear at the expense of being too verbose. This is not the case in [4, p-88].

Of course, we can state the same for decreasing sequences and infima, as taking $u_n = -v_n$ for some decreasing sequence (v_n) is enough to fulfill the assumptions of the theorem 10.1. Anyway, we can state the result.

Corollary 10.2. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be a sequence of decreasing integrable functions such that

$$u_n \geq u_{n+1}, \ \forall n \in \mathbb{N} \text{ and define } u = \inf_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A}).$$

Then,

$$u \in \mathcal{L}^1(\mu) \iff \inf_{n\in\mathbb{N}} \int u_n d\mu > -\infty, \tag{10.3}$$

in which case

$$\int u d\mu = \int \lim_{n\to\infty} u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu = \inf_{n\in\mathbb{N}} \int u_n d\mu = \lim_{n\to\infty} \int u_n d\mu. \tag{10.4}$$

Now we move on to a very important result. This time, not only we drop non-negativity, but also, monotonicity. Of course, there is a price to pay for this.

Theorem 10.3 (Lebesgue Dominated Convergence)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be a sequence of functions such that $|u_n| \leq w$, $\forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^1_+(\mu)$. If $u(x) = \lim_{n\to\infty} u_n(x)$ exists for almost every $x \in X$ then $u \in \mathcal{L}^1(\mu)$ and we have

$$\lim_{n\to\infty} \int |u_n - u| d\mu = 0, \tag{10.5}$$

$$\lim_{n\to\infty} \int u_n d\mu = \int \lim_{n\to\infty} u_n d\mu = \int u d\mu. \tag{10.6}$$

10.2 Applications to parameter dependent-integrals

Theorem 10.4 (Continuity lemma)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $\varnothing \neq (a,b) \subset \mathbb{R}$ a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ be a function satisfying

1. $x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$;
2. $t \mapsto u(t,x)$ is continuous for every fixed $x \in X$; and
3. $|u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b) \times X$ and some $w \in \mathcal{L}^1_+(\mu)$.

Then the function $v : (a,b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int u(t,x)\mu(dx) \tag{10.7}$$

is continuous.

Theorem 10.5 (Differentiability lemma)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, $\varnothing \neq (a, b) \subset \mathbb{R}$ a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ be a function satisfying

1. $x \mapsto u(t, x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$;
2. $t \mapsto u(t, x)$ is differentiable for every fixed $x \in X$; and
3. $|\partial_t u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1_+(\mu)$.

Then the function $v : (a, b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int u(t, x) \mu(dx) \tag{10.8}$$

is differentiable and its derivative is

$$\partial_t v(t) = \int \partial_t u(t, x) \mu(dx). \tag{10.9}$$

Chapter 11

Exercise sets

11.1 Exercise set 1

Due September 20th, 2019.

Exercise 11.1.1. Let X be a nonempty set and $\mathcal{A} = \{A_1, A_2, \dots\}$ a collection of disjoint subsets of X such that $X = \bigcup_{n=1}^{\infty} A_n$. Show that each element $A \in \sigma(\mathcal{A})$ is a union of at most a countable subcollection of elements of $\mathcal{A}$. (3 pts)

The technique used here is similar to the good set principle (remark 1.5), only that it is not necessary to show that the good set is inside the σ-algebra .

Proof. The idea is to prove that $\sigma(\mathcal{A})$ only contains countable unions of sets of $\mathcal{A}$.

Let us define

$$\mathcal{B} = \left\{ \bigcup_{i \in I} A_i \mid A_i \in \mathcal{A} \wedge I \subset \mathbb{N} \right\}.$$

We shall prove that $\mathcal{B}$ is a σ-algebra on $\mathcal{A}$ and thus $\sigma(\mathcal{A}) \subseteq \mathcal{B}$, i.e. $\sigma(\mathcal{A})$ is made up of unions of at most a countable subcollection of $\mathcal{A}$.

1. Firstly, $\varnothing \in \mathcal{B}$ since $\varnothing = \bigcup_{i \in I} A_i$ by choosing $I = \varnothing \subset \mathbb{N}$.
2. Secondly, for any set $B = \bigcup_{i \in I} A_i \in \mathcal{B}$ we have that

   $$B^c = \left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c = \bigcap_{i \in I} \bigcup_{j \neq i} A_j = \bigcup_{j \notin I} A_j = \bigcup_{j \in I^c} A_j \in \mathcal{B},$$

   since $I^c = \mathbb{N} \setminus I \subset \mathbb{N}$.
3. Finally, for any countable collection $(B_n)_{n \in \mathbb{N}} \in \mathcal{B}$ we need to show that the union is also in $\mathcal{B}$.

   $$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in I_n} A_i = \bigcup_{j \in J} A_j \in \mathcal{B},$$

   since $J = \bigcup_{n \in \mathbb{N}} I_n \subset \mathbb{N}$.

Since we have shown that $\mathcal{B}$ is a σ-algebra on $\mathcal{A}$ and thus $\sigma(\mathcal{A}) \subset \mathcal{B}$, we conclude that the elements of $\sigma(\mathcal{A})$ must be all unions of at most countable subcollections of elements in $\mathcal{A}$, which are the only elements of $\mathcal{B}$. □

Exercise 11.1.2. Let $(X, \mathcal{D}, \mu)$ be a measure space, and let $\overline{\mathcal{D}}^{\mu}$ be the completion of the σ-algebra $\mathcal{D}$ with respect to the measure μ (see exercise 4.15). We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{D}}^{\mu}$. Supose $f : X \to X$ is a function such that $f^{-1}(B) \in \mathcal{D}$ and $\mu(\inf f(B)) = \mu(B)$ for each $B \in \mathcal{D}$. Show that $\inf f(\overline{B}) \in \overline{\mathcal{D}}^{\mu}$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{D}}^{\mu}$. (3 pts)

Proof. First we show that $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}^{\mu}$, for all $\overline{B} \in \overline{\mathcal{D}}^{\mu}$. Recall from the definition of the completion $\overline{\mathcal{D}}^{\mu}$ of the σ-algebra $\mathcal{D}$ that any set $\overline{B} \in \overline{\mathcal{D}}^{\mu}$ can be written as $\overline{B} = B \cap M$ for some subset M of a μ-measurable null set N in $\mathcal{D}$. Therefore

$$f^{-1}(\overline{B}) = f^{-1}(B \cup M) = f^{-1}(B) \cup f^{-1}(M)$$

Because $N \supset M$ we also have that $f^{-1}(N) \subset f^{-1}(M)$ and $\mu(f^{-1}(N)) = \mu(N) = 0$ by the definition of f. This means that $f^{-1}(M)$ is also a subset of a μ-measurable null set in $\mathcal{D}$ and since $f^{-1}(B) \in \mathcal{D}$ by definition of f we have

$$f^{-1}(\overline{B}) = \underbrace{f^{-1}(B)}_{\in \mathcal{D}} \cup \underbrace{f^{-1}(M)}_{\subset f^{-1}(N),\ \mu(f^{-1}(N))=0} \in \overline{\mathcal{D}}^{\mu}$$

Now we need to verify that $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{D}}^{\mu}$. Recall that the extension $\overline{\mu}$ is well-defined in $\overline{\mathcal{D}}^{\mu}$ with $\overline{\mu}(\overline{B}) := \mu(B)$ for any $\overline{B} = B \cup M \in \overline{\mathcal{D}}^{\mu}$. Hence,

$$\begin{aligned}
\overline{\mu}(f^{-1}(\overline{B})) &= \overline{\mu}(f^{-1}(B \cup M)) \\
&= \overline{\mu}(f^{-1}(B) \cup f^{-1}(M)) \\
&= \mu(f^{-1}(B)) \\
&= \mu(B) =: \overline{\mu}(\overline{B})
\end{aligned}$$

□

Exercise 11.1.3. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty)$ a function satisfying

1. μ is finitely additive
2. μ is σ-subadditive

Show that μ is σ-additive. (4 pts)

Proof. The plan for the proof is to sandwich $\mu(\bigcup_{n\in\mathbb{N}} A_n)$ between two sums that are the same when taking the limit.

First of all, because of σ-subadditivity we have that, for any countable collection $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$,

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n), \tag{11.1}$$

which in particular holds for pairwise disjoint unions, which we will assume from here on. We can rewrite the union as

$$\dot{\bigcup_{n\in\mathbb{N}}} A_n = \dot{\bigcup_{n=1}^{N}} \mathbin{\dot\cup} \dot{\bigcup_{n=N+1}^{\infty}} A_n.$$

Because of finite additivity we can introduce the measure as

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \mu\left(\dot{\bigcup_{n=1}^{N}}\right) + \mu\left(\dot{\bigcup_{n=N+1}^{\infty}} A_n\right).$$

And applying finite additivity again we get

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n=1}^{N} \mu(A_n) + \mu\left(\dot{\bigcup_{n=N+1}^{\infty}} A_n\right).$$

Since $\mu \geq 0$ we can rearrange the previous expression to obtain

$$\sum_{n=1}^{N} \mu(A_n) \leq \sum_{n=1}^{N} \mu(A_n) + \mu\left(\dot{\bigcup_{n=N+1}^{\infty}} A_n\right) = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) \tag{11.2}$$

By combining 11.1 and 11.2 we get

$$\sum_{i=1}^{N} \mu(A_n) \leq \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$$

Taking the limit as $N \to \infty$, which we can do since N is not inside the arguments to μ (in that case it would require that μ was already a measure, which it isn't, yet), we have

$$\sum_{n\in\mathbb{N}} \mu(A_n) \leq \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n) \implies \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n),$$

or that μ is σ-additive. ☐

11.2 Exercise set 2

Due September 27th, 2019.

Exercise 11.2.1. Let $\mathbb{Q}$ be the set of all real rational numbers and let $\mathcal{I}_{\mathbb{Q}} = \{[a,b)_{\mathbb{Q}} \mid a,b \in \mathbb{Q}\}$ where $[a,b)_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid a \leq q < b\}$.

1. Prove that $\sigma(\mathcal{I}_{\mathbb{Q}}) = \wp(\mathbb{Q})$ where $\wp(\mathbb{Q})$ is the collection of all subsets of $\mathbb{Q}$. (1.5 pts.)
2. Let μ be the counting measure on $\wp(\mathbb{Q})$ and let $\nu = 2\mu$. Show that $\nu(A) = \mu(A)$ for all $A \in \mathcal{I}_{\mathbb{Q}}$, but $\nu \neq \mu$ on $\sigma(\mathcal{I}_{\mathbb{Q}}) = \wp(\mathbb{Q})$. Why doesn't this contradict Theorem 5.7 in your book? (1.5 pts.)

Proof.

1. We shall prove the double containment. First, recall that $\wp(\mathbb{Q})$ is a σ-algebra on $\mathbb{Q}$. Also, by Remark 3.5 we have that $\mathcal{I}_{\mathbb{Q}} \subseteq \wp(\mathbb{Q}) \implies \sigma(\mathcal{I}_{\mathbb{Q}}) \subseteq \sigma(\wp(\mathbb{Q})) \subseteq \wp(\mathbb{Q})$. The last inclusion comes from the fact that $\wp(\mathbb{Q})$ is also a σ-algebra on $\mathbb{Q}$ so it must contain the smallest σ-algebra on $\mathbb{Q}$ that contains information about $\wp(\mathbb{Q})$. For the reverse containment, we shall prove that any subset $A \in \wp(\mathbb{Q})$ is also in $\sigma(\mathcal{I}_{\mathbb{Q}})$. For any $A \subset \mathbb{Q}$ define $(q_n)_{n\in\mathbb{N}}$ to be an enumeration of the rationals in A. This is posible since $\#\mathbb{Q} = \#\mathbb{N}$. Therefore, we can write

   $$A = \bigcup_{n\in\mathbb{N}} \{q_n\}, \text{ where } \{q_n\} \in \sigma(\mathcal{I}_{\mathbb{Q}})$$

   Therefore, $A \in \sigma(\mathcal{I}_{\mathbb{Q}})$ because σ-algebras are closed under countable union.

2. It is clear that any interval $A \in \mathcal{I}_{\mathbb{Q}}$ contains infinitely many rationals, except if the interval is empty, i.e. $a = b \implies [a, b) = \varnothing$. Therefore,

   $$\mu(A) = \nu(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{otherwise} \end{cases}$$

   But, if we consider $A \in \sigma(\mathcal{I}_{\mathbb{Q}}) = \wp(\mathbb{Q})$ then we have some finite sets where $\mu(A) = \#A$, and clearly $\nu(A) = 2\#A$. The equality between μ and ν only holds when the set is either empty or infinite, but not for finite sets such as $A = \{1, 2\}$ where $\mu(A) = 2$ but $\nu(A) = 4$.

   Why doesn't this contradict Theorem 5.7? Even though there is an exhausting sequence in the generator, namely $(A_n)_{n\in\mathbb{N}}$ where $A_n = [-n, n)_{\mathbb{Q}}$, the measure is not finite for any A_n. Moreover, there cannot be any exhausting sequence in $\mathcal{I}_{\mathbb{Q}}$ with a finite measure because we already saw that $\mu(A) = \infty, \ \forall A \in \mathcal{I}_{\mathbb{Q}}, \ A \neq \varnothing$.

□

Exercise 11.2.2. Let X be a set and $\mu, \nu : \wp(X) \to [0, \infty)$ two outer measures on X. Define $\rho : \wp(X) \to [0, \infty)$ by $\rho(A) = \max(\mu(A), \nu(A))$. Show that ρ is another outer measure on X.

Proof. Firstly, from the definition of ρ we can see that the domain and codomain are compatible with the definition of an outer measure. Next, we prove each of the properties of an outer measure.

1. $\rho(\varnothing) = \max(\mu(\varnothing), \nu(\varnothing)) = \max(0, 0) = 0$

2. For any $A, B \in \wp(X), \ A \subseteq B$ we have

   $$\rho(A) = \max(\mu(A), \nu(A)) \leq \max(\mu(B), \nu(B)) = \rho(B)$$

3. For any sequence $(A_n)_{n\in\mathbb{N}} \subset \wp(X)$ we must verify that $\rho(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \rho(A_n)$.

   Let us prove the following first. For any function $f : X \times Y \to \mathbb{R}$ we have

   $$\max_{x\in X} \sum_{y\in Y} f(x, y) \leq \sum_{y\in Y} \max_{x\in X} f(x, y)$$

Choose any $x_0 \in X$ and any $y_0 \in Y$ and we have that $f(x_0, y_0) \leq \max f(x_0, y_0)$. Hence $\sum_{y \in Y} f(x_0, y) \leq \sum_{y \in Y} f(x, y)$. Because this is true for all $x_0 \in X$ we have $\max_{x \in X} \sum_{y \in Y} f(x, y) \leq \sum_{y \in Y} \max_{x \in X} f(x, y)$.

Using this, i.e. choosing $Y = \mathbb{N}$, $X = \{\mu, \nu\}$ and defining $f(\mu, n) = \mu(A_n)$ and $f(\nu, n) = \nu(A_n)$ we have

$$\begin{aligned}\rho(\bigcup_{n \in \mathbb{N}} A_n) = \max\left\{\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right), \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right)\right\} &= \max\left\{\sum_{n \in \mathbb{N}} \mu(A_n), \sum_{n \in \mathbb{N}} \nu(A_n)\right\} \\ &\leq \sum_{n \in \mathbb{N}} \max\{\mu(A_n), \nu(A_n)\} = \sum_{n \in \mathbb{N}} \rho(A_n)\end{aligned}$$

☐

Exercise 11.2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. For $A \in \mathcal{A}$ let

$$S(A) := \{B \in \mathcal{A} \mid B \subset A, \mu(B) < \infty\}.$$

Define $\nu : \mathcal{A} \to [0, \infty]$ by $\nu(A) := \sup\{\mu(B) \mid B \in S(A)\}$.

1. Show that ν is monotone, i.e. if $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subseteq A_2$, then $\nu(A_1) \leq \nu(A_2)$. (0.5 pts).
2. Show that if $A \in \mathcal{A}$ with $\mu(A) < \infty$, then $\nu(A) = \mu(A)$. (1 pt.)
3. Show that ν is a measure on $\mathcal{A}$. (2.5 pts.)
4. Show that if μ is σ-finite, the $\mu = \nu$. (1 pt.)

Proof.

1. We first show that if $A_1 \subseteq A_2$, then $S(A_1) \subseteq A_2$. Let $C \in S(A_1)$, then $C \in \mathcal{A} \wedge C \subset A_1 \wedge \mu(C) < \infty$. Clearly $C \subseteq A_2$ since $A_1 \subseteq A_2$ so $C \in S(A_2)$. Finally,

$$\nu(A_1) = \sup\{\mu(B) \mid B \in S(A_1)\} \leq \sup\{\mu(B) \mid B \in S(A_2)\} = \nu(A_2).$$

2. Clearly the domain and range of ν are compatible with the definition of a measure, i.e. $\nu : \mathcal{A} \to [0, \infty]$. We need to show that the two properties from definition 2.1 hold.
   (a) $\nu(\varnothing) = \sup\{\nu(B) \mid B \in S(\varnothing)\} = \mu(\varnothing) = 0$ since $S(\varnothing) = \{\varnothing\}$.
   (b) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be a pairwise disjoint sequence of sets in $\mathcal{A}$. Then,

$$\nu\left(\biguplus_{n \in \mathbb{N}} A_n\right) =$$

☐

11.3 Exercise set 3

Due October 4th, 2019.

Exercise 11.3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mathcal{G} = \{A_1, A_2, \dots\}$ a countable partition of X with $A_k \in \mathcal{A}$, $\forall k \in \mathbb{N}$. Define a function $u : X \to \mathbb{R}$ by $u(x) = \sum_{k=1}^{\infty} k \cdot \mathbb{1}_{A_k}$.

1. Show that u is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable. (1.5 pts)
2. Show that $\sigma(\mu) = \sigma(\mathcal{G})$, where $\sigma(\mu)$ is the smallest σ-algebra making u measurable. (2.5 pts)
3. Suppose that $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Define ν on $\mathcal{A}$ by
$$\nu(B) = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu(B \cap A_n)}{\mu(A_n)}.$$
Show that ν is a **finite** measure on $(X, \mathcal{A})$. (1.5 pts)
4. Under the assumptions of part 3, prove that if $B \in \mathcal{A}$, then $\mu(B) = 0$ if and only if $\nu(B) = 0$. (1 pt)

Proof.

1. Observe that the function $u(x)$ always evaluates to a natural number. In particular, $u(x) = k$ for the $k \in \mathbb{N}$ that satisfies $x \in A_k$. u is well-defined since $\mathcal{G}$ is a countable partition of X.

   We must check whether for any $B \in \mathcal{B}(\mathbb{R}), u^{-1}(B) \in \mathcal{A}$. Given a $B \in \mathcal{B}(\mathbb{R})$ we rewrite it as $B = B' \cup B_{nat}$ where B_{nat} contains all the naturals in B and $B' = B \setminus B_{nat}$. Then,
$$u^{-1}(B) = u^{-1}(B' \cup B_{nat}) = u^{-1}(B') \cup u^{-1}(B_{nat}) = \varnothing \cup \bigcup_{n \in B_{nat}} A_n \in \mathcal{A}.$$

2. By definition of $\sigma(u)$ we have
$$\sigma(u) := \sigma(u^{-1}(\mathcal{B}(\mathbb{R}))) = u^{-1}(\mathcal{B}(\mathbb{R})),$$
since the preimage of any σ-algebra is a σ-algebra .

   Also, recall from exercise set 1 that
$$\sigma(\mathcal{G}) = \left\{ \bigcup_{i \in I} A_i \mid A_i \in \mathcal{G},\ I \subset \mathbb{N} \right\}.$$
   Now we prove the double containment. Take any $A \in \sigma(u)$. Then there is a $B \in \mathcal{B}(\mathbb{R})$ such that $A = u^{-1}(B)$. By part one we already now that
$$A = \varnothing \cup \biguplus_{n \in B_{nat}} A_n \in \sigma(\mathcal{G}).$$
   For the reverse containment, let $G \in \sigma(\mathcal{G})$, therefore there is a set $I \subset \mathbb{N}$ such that $G = \bigcup_{i \in I} A_i$. Furthermore, $I \in \mathcal{B}(\mathbb{R})$ and thus, by part one,
$$\bigcup_{i \in I} A_i = u^{-1}(I) \in \sigma(u).$$

3. First let us check that ν is well defined. $\nu(B)$ is well defined for any $B \in \mathcal{A}$ since $B \cap A_n \in \mathcal{A}$ as $\mathcal{A}$ is $\cap$-stable. Also, ν is non-negative since it is computed as a sum of products of non-negative numbers.

   Now we check the two properties in the definition of measure:

   (a)

$$\nu(\varnothing) = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu(\varnothing \cap A_n)}{\mu(A_n)} = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu(\varnothing)}{\mu(A_n)} = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{0}{\mu(A_n)} = 0$$

   (b) For any pairwise disjoint collection $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ we have

$$\begin{aligned}\nu\left(\dot{\bigcup_{j\in\mathbb{N}}} B_j\right) &= \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu\left(\left(\dot{\bigcup}_{j\in\mathbb{N}} B_j\right) \cap A_n\right)}{\mu(A_n)} \\ &= \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu\left(\dot{\bigcup}_{j\in\mathbb{N}} (B_j \cap A_n)\right)}{\mu(A_n)} \\ &= \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\sum_{j\in\mathbb{N}} \mu\left(B_j \cap A_n\right)}{\mu(A_n)} \\ &= \sum_{j\in\mathbb{N}} \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu\left(B_j \cap A_n\right)}{\mu(A_n)} = \sum_{j\in\mathbb{N}} \nu(B_j)\end{aligned}$$

   Additionally, we must check that ν is finite. Since μ is a measure, it is monotone so $\mu(B \cap A_n) \le \mu(A_n)$ and thus

$$\nu(B) = \sum_{n=1}^{\infty} 3^{-n} \cdot \frac{\mu(B \cap A_n)}{\mu(A_n)} \le \sum_{n=1}^{\infty} 3^{-n} < \infty$$

4. It is clear that if $\mu(B) = 0$ then, by monotonicity, since $B \cap A_n \subset B$, we have that $0 \le \mu(B \cap A_n) \le \mu(B) = 0$ and therefore $\nu(B) = \sum_{n=1}^{\infty} 3^{-n} \cdot 0 = 0$.

   For the reverse, we have that if $\nu(B) = 0$, then it must be because $\mu(B \cap A_n) = 0,\ \forall n \in \mathbb{N}$ since $3^{-n} > 0,\ \forall n \in \mathbb{N}$. We can rewrite $\mu(B)$ as

$$\begin{aligned}\mu(B) = \mu(B \cap X) = \mu\left(B \cap \dot{\bigcup_{j\in\mathbb{N}}} A_j\right) &= \mu\left(\dot{\bigcup_{j\in\mathbb{N}}} B \cap A_j\right) \\ &= \sum_{j\in\mathbb{N}} \mu(B \cap A_j) = \sum_{j\in\mathbb{N}} 0 = 0.\end{aligned}$$

   Therefore $\mu(B) = 0 \iff \nu(B) = 0,\ \forall B \in \mathcal{A}$.

□

Exercise 11.3.2. Consider the measure space $([0,1], \mathcal{B}, \lambda)$ where $\mathcal{B} = \mathcal{B}(\mathbb{R}) \cap [0,1)$, i.e. the restriction of the Borel σ-algebra to the interval $[0,1)$, and λ denotes the Lebesgue measure restricted to $\mathcal{B}$. Define a map $T : [0,1) \to [0,1)$ by

$$T(x) = \begin{cases} 4x & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{4}{3}(x - \frac{1}{4}) & \text{if } \frac{1}{4} \le x < 1. \end{cases}$$

1. Show that T is $\mathcal{B}/\mathcal{B}$-measurable (in short, Borel measurable). (1.5 pts)
2. Consider the image measure $T(\lambda)$ defined by $T(\lambda)(B) = \lambda(T^{-1}(B))$, for all $B \in \mathcal{B}$. Show that $T(\lambda) = \lambda$. (2 pts)

Proof.

1. It is enough to check if $T^{-1}([a,b)) \in \mathcal{B}$, since $\mathcal{J} \cap [0,1) = \{[a,b) \mid 0 \le a \le b \le 1\}$ is a generator of $\mathcal{B}$.

$$\begin{aligned} T^{-1}([a,b)) &= \{x \in [0,1) \mid T(x) \in [a,b)\} \\ &= \left\{x \in [0,\frac{1}{4}) \mid T(x) \in [a,b)\right\} \cup \left\{x \in [\frac{1}{4},1) \mid T(x) \in [a,b)\right\} \\ &= \left\{x \mid 0 \le a \le x < b \le \frac{1}{4}\right\} \cup \left\{x \mid \frac{1}{4} \le a \le \frac{4}{3}(x-\frac{1}{4}) < b \le 1\right\} \\ &= \left[\frac{a}{4},\frac{b}{4}\right) \cup \left[\frac{3a}{4}+\frac{1}{4},\frac{3b}{4}+\frac{1}{4}\right) \in \mathcal{B}, \end{aligned}$$

since each of the intervals is itself in $\mathcal{J} \cap [0,1)$. (This is because $0 \le \frac{a}{4}, \frac{b}{4}, \frac{3a}{4}+\frac{1}{4}, \frac{3b}{4}+\frac{1}{4} < 1$ since $a,b \in [0,1)$.)

2. First we will prove that $T(\lambda)([a,b)) = \lambda([a,b)), \forall [a,b) \in \mathcal{J} \cap [0,1)$. Using the same as in part one we have

$$\begin{aligned} T(\lambda)([a,b)) &= \lambda(T^{-1}[a,b)) \\ &= \lambda\left(\left[\frac{a}{4},\frac{b}{4}\right) \uplus \left[\frac{3a}{4}+\frac{1}{4},\frac{3b}{4}+\frac{1}{4}\right)\right) \\ &= \frac{b}{4}-\frac{a}{4}+\frac{3b}{4}+\frac{1}{4}-\frac{3a}{4}-\frac{1}{4} \\ &= b-a = \lambda([a,b)). \end{aligned}$$

Recall that $\mathcal{B} = \sigma(\mathcal{J} \cap [0,1))$ and that there exists an exhausting sequence $B_n \uparrow [0,1)$ where $B_n = [0,1)$, $\forall n \in \mathbb{N}$ and that $\lambda([0,1)) < \infty$ and $T(\lambda)([0,1)) = \lambda(T^{-1}([0,1))) = \lambda([0,1)) < \infty$. Then, by uniqueness, we have that $T(\lambda) = \lambda$ on all $\mathcal{B}$.

□

11.4 Exercise set 4

Due October 25th, 2019.

Exercise 11.4.1. Let $(X,\mathcal{A},\mu)$ be a measure space and $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$. Define $B_n = \{x \in X \mid 2^{-n} \le |u(x)| < 2^n\}$, for $n \ge 1$. Set $B = \bigcup_{n=1}^{\infty} B_n$.

1. Show that $\int_X u d\mu = \int_B u d\mu$. (1.5 pts)
2. Prove that $\lim_{n\to\infty} \int_{B_n} u d\mu = \int_X u d\mu$. (1.5 pts)

3. Show that for every $\varepsilon > 0$, there exists a positive integer N, such that $\mu(B_N) < \infty$ and $\left|\int_{B_N^c} u d\mu\right| < \varepsilon$. (1.5 pts)

Exercise 11.4.2. Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of the one-dimensional Lebesgue measure to $[0,1]$.

1. Show that $\lim_{n\to\infty} \int_{[0,1]} \frac{x^n}{(1+x)^2} d\lambda(x) = 0$. (1.5 pts)

2. Show that $\lim_{n\to\infty} \int_{[0,1]} \frac{nx^{n-1}}{1+x} d\lambda(x) = \frac{1}{2}$. (1 pt)

Exercise 11.4.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty)$. For $n \geq 1$, define $u_n = \min\{\max(u, -n), n\}$.

1. Prove that $\lim_{n\to\infty} \|u_n\|_p = \|u\|_p$. (2 pts)

2. Prove that for any ε, there exists an integer $n \geq 1$ such that $\int |u - u_n|^p \leq \varepsilon$. (1 pt)

11.5 Practice Mid-Term, 2019-2020.

Exercise 11.5.1. Let $(X, \mathcal{A})$ be a measure space such that $\mathcal{A} = \sigma(\mathcal{G})$ where $\mathcal{G}$ is a collection of subsets of X such that $\varnothing \in \mathcal{G}$. Show that for any $A \in \mathcal{A}$ there exists a countable collection $\mathcal{G}_A \subseteq \mathcal{G}$ such that $A \in \sigma(\mathcal{G}_A)$.

Proof. This exercise presents a good opportunity to apply the good set principle (cf. remark 1.5).

Let

$$\mathcal{B} := \{A \in \mathcal{A} \mid \exists \mathcal{G}_A \subset \mathcal{G} \text{ countable, } A \in \sigma(\mathcal{G}_A)\}.$$

First, we will show that $\mathcal{B}$ is a σ-algebra .

1. Clearly $X \in \mathcal{A}$. Let $G_X = \{\varnothing\}$ which is clearly finite and $G_X \subset \mathcal{G}$ by hypothesis. Then $\sigma(\mathcal{G}_X) = \{\varnothing, \varnothing^c\} = \{\varnothing, X\} \implies X \in \mathcal{B}$.

2. Let $A \in \mathcal{B}$. Then $A \in \mathcal{A}$ by definition of $\mathcal{B}$ and there exists $\mathcal{G}_A \subset \mathcal{G}$ countable with $A \in \sigma(\mathcal{G}_A)$. Since $\sigma(\mathcal{G}_A)$ is a σ-algebra it contains A^c and clearly $A^c \in \mathcal{A}$. Let $\mathcal{G}_{A^c} = \mathcal{G}_A$ and therefore $A^c \in \mathcal{B}$.

3. Finally, let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$. Clearly, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and, for each $n \in \mathbb{N}$, there exists $\mathcal{G}_{A_n} \subset \mathcal{G}$ countable such that $A_n \in \sigma(\mathcal{G}_{A_n})$. Also, since $\mathcal{A}$ is a σ-algebra , we have that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. Let

$$\mathcal{G}_{\bigcup_{n\in\mathbb{N}} A_n} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_{A_n}.$$

$\mathcal{G}_{\bigcup_{n\in\mathbb{N}} A_n} \subset \mathcal{G}$ is countable since every $\mathcal{G}_{A_n} \subset \mathcal{G}$ is countable and $\mathcal{G}_{\bigcup_{n\in\mathbb{N}} A_n}$ is a countable union of countable sets. Therefore $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Now we show that $\mathcal{B} = \mathcal{A}$. Clearly, from the definition of $\mathcal{B}$ we see that $\mathcal{B} \subseteq \mathcal{A}$. For the reverse containment, we will show that $\mathcal{G} \subset \mathcal{B}$ and therefore $\mathcal{A} = \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{B}) = \mathcal{B}$. Let $A \in \mathcal{G} \subset \mathcal{A}$. Define $\mathcal{G}_A = \{A\} \subset \mathcal{G}$ and clearly finite and $A \in \sigma(\mathcal{G}_A)$. Therefore $A \in \mathcal{B}$, $\forall A \in \mathcal{G} \implies \mathcal{G} \subset \mathcal{B}$. ☐

Exercise 11.5.2. Let X be a set and $\mathcal{F}$ a collection of real-valued functions on X satisfying the following properties:

1. $\mathcal{F}$ contains the constant functions,
2. if $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f + g,\ fg, cf \in \mathcal{F}$.
3. if $f_n \in \mathcal{F}$, and $f = \lim_{n\to\infty} f_n$, then $f \in \mathcal{F}$.

For $A \subseteq X$, denote by $\mathbb{1}_A$ the indicator function of A, i.e.

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that the collection $\mathcal{A} = \{A \subseteq X \mid \mathbb{1}_A \in \mathcal{F}\}$ is a σ-algebra .

Proof. We shall prove that the three properties of a σ-algebra hold in $\mathcal{A}$:

1. $X \in \mathcal{A}$ since $\mathbb{1}_X(x) = 1$ constantly, and therefore $\mathbb{1}_X \in \mathcal{F}$.
2. if $A \in \mathcal{A}$, then $\mathbb{1}_A \in \mathcal{F}$. We can write $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$. Since $1, 1_A \in \mathcal{F}$ we have $\mathbb{1}_{A^c} \in \mathcal{F} \implies A^c \in \mathcal{A}$.
3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then $(\mathbb{1}_{A_n})_{n\in\mathbb{N}} \in \mathcal{F}$. We can write

   $$\mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} = \sum_{n=1}^{\infty} \mathbb{1}_{A_i}$$

   To prove $\mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_i} \in \mathcal{F}$ we construct the following sequence of functions $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}$ given by

   $$f_n = \sum_{i=1}^{n} \mathbb{1}_{A_i} = \mathbb{1}_{A_n} + f_{n-1}$$

   Clearly, $f_n \in \mathcal{F}$ for all n because of property 2. Then,

   $$f = \lim_{n\to\infty} f_n = \sum_{i=1}^{\infty} \mathbb{1}_{A_i} = \mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} \in \mathcal{F},$$

   and thus, $\bigcup_{n\in\mathbb{N}} A_i \in \mathcal{A}$.

□

Exercise 11.5.3. Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to to $[0,1]$, and λ is the restriction of the Lebesgue measure to $[0,1]$. Let $E_1, \ldots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \ldots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$.

Proof. Observe that if every $x \in [0,1]$ belongs to at least n sets in (E_j) then we have that

$$\sum_{j=1}^{m} \mathbb{1}_{E_j}(x) \geq n, \ \ \forall x \in [0,1].$$

We proceed by contradiction. Suppose $\lambda(E_j) < \frac{n}{m},\ \forall j \in \{1, \dots, m\}$. Then

$$n = \int_{[0,1]} n d\lambda \overset{7.4}{\leq} \int_{[0,1]} \sum_{j=1}^{m} \mathbb{1}_{E_j}(x) d\lambda = \sum_{j=1}^{m} \lambda(E_j) < m \cdot \frac{n}{m} = n.$$

This is a contradiction, so there must exists at least one $j \in \{1, \dots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$. ☐

Exercise 11.5.4. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is the one-dimensional Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n - 1} \frac{3k + 2^n}{2^n} \mathbb{1}_{[k/2^n, (k+1)/2^n]}(x), \ n \geq 1.$$

1. Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$, for all $x \in \mathbb{R}$.

2. Show that $\int \sup_{n \geq 1} f_n d\lambda = \frac{5}{2}$.

Proof.

1. For a given n and k we define

   $$a_k^n = \frac{3k + 2^n}{2^n} \text{ and } A_k^n = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right).$$

   Clearly, $A_k^n \in \mathcal{B}(\mathbb{R})$ for any n and k and $a_k^n > 0$. Therefore, f_n is a simple function and thus f_n is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable.

   Notice that

   $$\begin{aligned} A_k^n = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) &= \left[\frac{2k}{2 \cdot 2^n}, \frac{2k+2}{2 \cdot 2^n}\right) \\ &= \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) \cup \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) = A_{2k}^{n+1} \cup A_{2k+1}^{n+1}, \end{aligned}$$

   and

   $$a_k^n = \frac{3k + 2^n}{2^n} = \frac{6k + 2^{n+1}}{2^{n+1}} = a_{2k}^{n+1} \leq a_{2k}^{n+1} + a_{2k+1}^{n+1}.$$

   Thus,

   $$f_n = \sum_{k=0}^{2^n - 1} a_k^n \mathbb{1}_{A_k^n} \leq \sum_{k=0}^{2^n - 1} a_{2k}^{n+1} \mathbb{1}_{A_{2k}^{n+1}} + a_{2k+1}^{n+1} \mathbb{1}_{A_{2k+1}^{n+1}} = f_{n+1}.$$

2. Since (f_n) is an increasing sequence of simple functions we now that

   $$\int \sup_{n \geq 1} f_n d\mu = \sup_{n \geq 1} \int f_n d\mu = \sup_{n \geq 1} I_\mu(f_n) = \lim_{n \to \infty} I_\mu(f_n),$$

and

$$I_\mu(f_n) = \sum_{k=0}^{2^n-1} a_k^n \mu(A_k^n) = \sum_{k=0}^{2^n-1} \frac{3k+2^n}{2^n} \cdot \left(\frac{k+1}{2^n} - \frac{k}{2^n}\right)$$
$$= \sum_{k=0}^{2^n-1} \frac{3k}{2^n \cdot 2^n} + \frac{1}{2^n} = 1 + \frac{3}{2}\frac{2^n-1}{2^n}$$

. Thus,

$$\int \sup_{n\geq 1} f_n d\mu = \lim_{n\to\infty} I_\mu(f_n) = 1 + \frac{3}{2} = \frac{5}{2}.$$

☐

Exercise 11.5.5. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ sucht that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if f is any non-negative function on $(E, \mathcal{B})$, then $\int_E f d\mu \leq \int_E f d\nu$.

Proof. [1]By theorem 6.5 we now that there exists a sequence of non-negative simple functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathcal{B})$ such that $f_n \uparrow f$. Moreover, each of this simple functions has a representation of the form

$$f_n = \sum_{j=0}^{M_n} a_j \mathbb{1}_{A_j}$$

Thus,

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu = \lim_{n\to\infty} I_\mu(f_n)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=0}^{M_n} a_j \mu(A_j) \leq \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=0}^{M_n} a_j \nu(A_j)$$
$$= \lim_{n\to\infty} I_\nu(f_n) = \lim_{n\to\infty} \int_E f_n d\nu = \int f d\nu$$

☐

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[1] Now that the solutions to this practice midterm have been published, I noticed there is a much more cleaner (and modular!) way of doing this. First prove it for $f = \mathbb{1}_A$ for some $A \in \mathcal{A}$. Then, prove it for $f \in \mathcal{E}^+(\mathcal{A})$, for some simple function f. Finally, prove it for the general case. Each of these builds on the previous and you're left with something much more readable and useful.

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