Notes on Mathematical Modelling

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December 29, 2019

0.1 Acknowledgements

These notes are based on the lectures of Carolin Kreisbeck (c.kreisbeck@uu.nl) during Fall of 2019 at Universiteit Utrecht. The lectures were intended to serve as a preparation for the reading of the texbook of the course [1].

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Contents

Chapter 1

Basic tools for mathematical modelling

Teaching for this chapter started on Monday, 2019.11.11 (week 46a) and ended on Monday, 2019.11.18. This chapter corresponds to part of chapter 1 in [1].

In this introductory chapter we will introduce the mindset that we should have when trying to **translate a specific problem** from the natural sciences, the social sciences or technology into a **well-defined mathematical problem**[1].

TODO: some context and general pointers would probably look good here.

1.1 Case study: population dynamics

Suppose we want to model the change in population (i.e. number of individuals) in an environment over a period of time. First thing we need is to make some assumptions about what's really happening here. We might, for example, make the following assumptions[2]

1. growth rate independent of population size (unlimited growth possible, neglecting e.g. limited resources)
2. growth rate independent of time (neglecting time-dependence due to e.g. influence of enemies, economical or cultural changes)
3. population within closed systems (neglecting e.g. migration)
4. assuming an equal distribution of male and female, age distribution not considered
5. continuous model with non-integer solutions (idealization reasonable for very large populations, for small populations stochastic effects have to be taken into account)

[1] This is the definition of *mathematical modelling* given in [1, p. 1] with Kreisbeck's emphasis.

[2] Stolen from [2].

After this, we name the quantities that intervene in our problem. We will use t for time, $x(t)$ for the number of individuals (population) at time t and $\frac{dx}{dt}(t)$ or $x'(t)$ for the rate of change in population. To model the change we introduce the quantities

- $b(t, \Delta t)$ for the increase of population during the time interval $(t, \Delta t)$, and
- $d(t, \Delta t)$ for the decrease of population during the time interval $(t, \Delta t)$.

Therefore the population at time $t + \Delta t$ is given by

$$x(t + \Delta t) = x(t) + b(t, \Delta t) - d(t, \Delta t).$$

That Δt desperately wants us to take the limit as $\Delta t \to 0$ and so we do

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{b(t, \Delta t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{d(t, \Delta t)}{\Delta t}.$$

Note that here we're assuming the limit really does exist, which is quite a big assumption... Rename,

$$B(t) = \lim_{\Delta t \to 0} \frac{b(t, \Delta t)}{\Delta t} \text{ and } D(t) = \lim_{\Delta t \to 0} \frac{d(t, \Delta t)}{\Delta t}$$

and use the definition of derivative to get

$$\frac{dx}{dt}(t) = x'(t) = B(t) - D(t)$$

where $B(t)$ and $D(t)$ being the rates at which the population increasing, resp. decreases at time t. Recall that we assumed that the rates of change in the population were independent of time and population size. This is equivalent to saying that $B(t)$ and $D(t)$ are really constants which gives us the final model

$$x'(t) = \beta - \delta \implies x(t) = (\beta - \delta)t.$$

The previous is a not-particularly-interesting ODE with solution[3]

$$x(t) = (\beta - \delta)x + C.$$

This model has a lot of shortcomings, first of all, it does not account for the size of the population in the rates of change. But, one might argue that the more individuals there are in a population the greater the rates of change are. We can go back and restate assumption one as "population increase, resp. decrease in the time interval $(t, \Delta t)$ is directly proportional to the population at time t and the time passed". This in turn gives us

$$b(t, \Delta t) = \beta x(t)\Delta t \text{ and } d(t, \Delta t) = \delta x(t)\Delta t.$$

Taking the limit as before leads us to the model

$$x'(t) = (\beta - \delta)x(t) = px(t).$$

[3] For help with solving differential equations see [3].

From now on, we shall let $p = \beta - \delta$ since we don't really need to distinguish between changes in the population because of births and deaths—we just care about the overall evolution of the population. This is another ODE, this time a bit more interesting, with solution

$$x(t) = Ce^{pt}.$$

Although a bit better, you can probably see that this model explodes as time passes since it does not include any provisions for when the population turns stupidly large. Anyhow, it is common enough that it deserves its own name:

the **exponential growth model**.

A small step in the right direction would be to account for a population limit in the system, i.e. number of individuals that flips the rate of growth. More precisely, let's change assumption one to "there is a number x_M that is the maximum population in the system (sometimes called the *carrying capacity* and that the rate of change in population $p(x(t))$ is positive if $x(t) > x_M$ and negative if $x(t) > x_M$". The easiest way to model this is with a linear ansatz for $p(x)$, i.e. something of the form $p(x) = q(x_M - x)$ with a parameter[4] $q > 0$. Notice how

$$\begin{cases} p(x) > 0 & \text{if } x < x_M, \text{ and} \\ p(x) < 0 & \text{if } x > x_M \end{cases}.$$

Plugging this into our previous model to get

$$x'(t) = q(x_M - x(t))x(t) = qx_Mx(t) - qx^2(t),$$

which is our final model for now and is called the **logistic growth model**

This ODE can be solved exactly and the solution is

$$x(t) = \frac{x_M x_0}{x_0 + (x_M - x_0)e^{-x_M q(t-t_0)}},$$

where t_0 is the initial time and $x_0 = x(t_0)$ is the initial population.

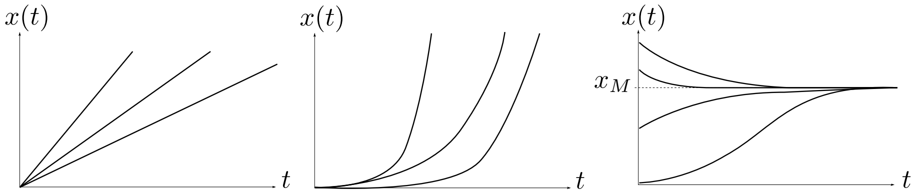

Figure 1.1: Solutions for the three iterations of the model, for different values for the parameters on each version.

We'll stop here for now, but keep in mind that we're missing the second half of the solution—we still need to apply this models to the real world. This

[4]The meaning of q is a bit more complex to explain, but at heart it is just a proportionality constant.

means *fitting* those curves to the specific problem at hand, in this case, getting some data (at least two data points) to calculate the constants that are present in our solutions. Also, recall that we made a lot of assumptions, there are more population dynamics models that account for changes in the environment, migrations, etc.

1.2 Dimensional analysis and non-dimensionalisation

The previous models had two or three parameters each, but as we work our way to more complex examples the number of parameters will increase. Moving around all those constants is cumbersome and draws our attention away from really understanding the problem at hand. In addition, as we apply our models to specific problems we will need to take into account the units of the quantities we are dealing with.

1.2.1 Dimensional analysis

In addition to being a prerequisite to doing non-dimensionalisation, dimensional analysis provides a sanity check for us to *make sure we're not adding apples to oranges*. When coming up with a model, we generally need to specify the *physical dimension* of the quantities involved, but not necessarily want to specify the particular units that quantity is expressed in. For this we will denote by $[c]$ the physical dimension of a quantity c. For example if t denotes time, when we write $[t]$ we mean the *dimension of time* or *some units of time* but do not specify which.

Revisiting our population model we can define the characteristic units

$$T := \text{ time }, \qquad \text{and} \qquad N := \# \text{ of individuals}$$

so that we get the following dimensions for the involved quantities

$$[t] = T,\ [x(t)] = N,\ [x'(t)] = \frac{N}{T}.$$

We can also do this for the parameters by solving for them in the equation for the model[5]

$$[x_M] = N,\ [t_0] = T,\ [x_0] = N,\ [q] = \frac{[x']}{[x_M - x][x]} = \frac{N/T}{N \cdot N} = \frac{1}{T}.$$

1.2.2 Non-dimensionalisation

Once we know the physical units of all involved quantities in our model we are ready to choose actual units for our model. For instance, for time we might choose years, days or hours but most of the time it is better to choose appropriate units for our problem. Non-dimensionalisation[6] is a recipe for choosing the most appropriate units.

From the dimensional analysis of our population dynamics examples we now that there are two physical dimensions and therefore we will choose two characteristic quantities $\bar{t}$ and $\bar{x}$.

[5]Notice how $[x_M - x] = N$ and not something weird like $N - N = 0$, since when you subtract apples from apples you still get apples.

[6]Yes, this is an accepted spelling although not very common in the literature.

1.2.3 Non-dimensionalisation when there are several options. The projectile problem.

1.3 Asymptotic expansion method

1.3.1 Error estimation

1.4 Exercises

Exercise (Recap, part a)**.** Solve the IVP

$$\begin{cases} y'(t) = \frac{2t}{y(t)-1},\ t > 0 \\ y(0) = 2 \end{cases}$$

Proof. We ommit the t in $y(t)$ and just write $y = y(t)$ for short. Via separation of variables we get

$$(y-1)y' = 2t.$$

Integrating on both sides with respect to t we get,

$$\int (y-1)y'\ dt = \int 2t\ dt$$

and with the rule for substitution on integrals we rewrite it as

$$\int (y-1)\ dy = \int 2t\ dt.$$

Solving the integrals we arrive at the equality

$$\frac{y^2}{2} - y = t^2 + c$$

and solving for y yields

$$y_{\pm} = 1 \pm \sqrt{1 - 2(t^2 + c)}.$$

Substituting the initial condition $y(0) = 2$ we get

$$1 \pm \sqrt{1 - 2c} = 2$$

which can only be satisfied with y_+ and picking $c = 0$. Therefore, the final solution to the IVP is

$$y(t) = 1 + \sqrt{1 - 2t^2}.$$

□

Exercise (Recap, part b)**.** Compute the solution to the second-order IVP

$$\begin{cases} y''(t) = y(t) + e^t,\ t > 0, \\ y(0) = 1,\ y'(0) = 1. \end{cases}$$

Proof.

1. Step 1: we find the general solution y_h to the homogeneous ODE $y'' = y$. It follows from setting the characteristic polynomial $P(\lambda) = \lambda^2 - 1 = 0 \iff \lambda = \pm 1$. Therefore,
$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

2. Step 2: we find a particular solution $y_p(t)$ to the ODE $y'' = y + e^t$. One way to do it is with the method of indeterminate coefficients. We need a solution wich is independent from the homogeneous solution $y_h(t)$ so we make the ansatz
$$y_p(t) = ate^t.$$
Computing the first and second derivatives of $y_p(t)$ and substituting them into the original ODE (without boundary conditions) yields the equality
$$ae^t + ae^t + ate^t = ate^t + e^t \implies a = \frac{1}{2}.$$

3. Step 3: the solution to the IVP is given by
$$y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2} e^t.$$
Pluging in the initial values for y and y' we arrive at the system of linear equations
$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 + \frac{1}{2} = 1 \end{cases} \iff \begin{cases} c_1 = \frac{3}{4} \\ c_2 = \frac{1}{4}. \end{cases}$$
Therefore, the solution to the IVP is given by the function
$$y(t) = \frac{3}{4} e^t + \frac{1}{4} e^{-t} + \frac{1}{2} te^t.$$

$\square$

Exercise (1.4)**.** We consider the model of limited growth of populations
$$x'(t) = qx_M x(t) - qx^2(t), \ x(0) = x_0.$$

1. Nondimensionalize the model using appropriate units for t and x. Which possibilities exist?

2. What nondimensionalization is appropriate for $x_0 \ll x_M$ (x_0 "much smaller than" x_M) in the sense that omitting small terms leads to a reasonable model?

Proof. The two main involved quantities in this model are t with dimension T and $x(t)$ with dimension N. We make the following change of variables:
$$\tau = \frac{t}{\overline{t}}, \qquad y(\tau) = \frac{x(t) - x_0}{\overline{x}} = \frac{x(\tau \overline{t}) - x_0}{\overline{x}}.$$
We solve for $x(t)$ and $x'(t)$ to be able to plug $y(\tau)$ into the model:
$$x(t) = y(\tau)\overline{x} + x_0, x^2(t) = y^2(\tau)\overline{x}^2 + x_0^2 + 2x_0\overline{x}y(\tau), x'(t) = \frac{d}{dt}\left(\overline{x}y(\tau) - x_0\right) = \overline{x}y'(\tau) \cdot \frac{1}{\overline{t}}.$$

Therefore,

$$\frac{\overline{x}}{\overline{t}}y'(\tau) = qx_M\left(\overline{x}y(\tau) + x_0\right) - q\left(\overline{x}^2y'(\tau) + x_0^2 + 2x_0\overline{x}y(\tau)\right),$$

or, equivalently,

$$y'(\tau) = qx_M\overline{t}y(\tau) + qx_M\frac{\overline{t}}{\overline{x}}x_0 - q\overline{t}\overline{x}y^t(\tau) - 2qx_0\overline{t}y(\tau).$$

As for the inital conditions we get

$$y(0) = \frac{x(0\cdot\overline{t}) - x_0}{\overline{x}} = 0$$

as expected, because of the way we chose the change $y(\tau)$.

There are many possibilities for the choice of $\overline{t}$ and $\overline{x}$. We will not discuss the now, but rather wait until part be. For an example of a detailed discussion without making extra assumptions see .

If we make the assumption that x_0 is very small, we may discard some terms in the above nondimensionalised model to get

$$\begin{cases} y'(\tau) = qx_M\overline{t}y(\tau) - q\overline{t}\overline{x}y^2(\tau), \\ y(0) = 0. \end{cases}$$

In this case, setting the cofficients for the function y to equal 0 we get the system

$$\begin{cases} qx_M\overline{t} = 1, \\ x\overline{t}\overline{x} = 1 \end{cases} \iff \begin{cases} \overline{t} = \frac{1}{qx_M}, \\ \overline{x} = \frac{x_M}{\overline{t}}. \end{cases}$$

In this case there is only this posibility, since we have two restrictions on two characteristic quantities.

□

Exercise. *(Nondimensionalization, scale analysis)* A body of mass m is thrown upwards in a vertical direction from the Earth's surface with a velocity v. The air resistance is supposed to be taken into account by Stokes' law $F_R = -cv$ for the flow resistance in viscous fluids, which is reasonable for small velocities. Here c is a coefficient depending on the shape and the size of the body. The motion is supposed to depend on the mass m, the velocity v, the gravitational acceleration g and the friction coefficient c with dimension $[c] = M/T$.

1. Determine the possible dimensionless parameters and reference values for height and time.

2. The initial value problem for the height is assumed to take the form

$$mx'' = cx' = -mg, \qquad x(0) = 0, \qquad x'(0) = v.$$

   Nondimensionalize the differential equation. Again different possibilities are available.

3. Discuss the different possibilities of a reduced model if $\beta := cv/(mg)$ is small.

Proof. TODO □

Exercise (1.6)**.** A model for the vertical throw on the Earth taking into account the air resistance is given by

$$mx''(t) = -mg - c\,|x'(t)|\,x'(t),\ \ x(t_0) = 0, x'(t_0) = v_0.$$

In this model the gravitational force is approximated by $F = -mg$, the air resistance for a given velocity v is described by $-c\,|v|\,v$ with a proportionality constant c depending on the shape and size of the body and the density of the air. This law is reasonable for high velocities.

1. Nondimensionalize the model. What possibilities exist?

2. Compute the maximal height of the throw for the data $m = 0.1\,\mathrm{kg},\ g = 10\,\mathrm{m/s^2},\ v_0 = 10\,\mathrm{m/s}, c = 0.01\,\mathrm{kg/m}$ and compare the result with the corresponding result for the model without air resistance.

Proof. We make the change of variables

$$\tau = \frac{t - t_0}{\bar{t}}, \qquad y = \frac{x}{\overline{x}}$$

and plug it into the IVP to get

$$y''(\tau) = -\frac{\bar{t}^2 g}{\overline{x}} - \frac{c\overline{x}}{m}\,|y'(\tau)|\,y'(\tau), \qquad y(0) = 0, \qquad y'(0) = \frac{\bar{t}}{\overline{x}}v_0.$$

We would like as many coefficients as possible to equal 1 but we have 3 coefficients and 2 characteristic quantities so we will have to make a compromise. There are three possibilities for the compromise:

1. Set $\frac{\bar{t}^2 g}{\overline{x}} = \frac{c\overline{x}}{m} = 1$ and leave $\frac{\bar{t}}{\overline{x}}v_0$ as is, which would yield $\bar{t} = \sqrt{\frac{m}{gc}},\ \overline{x} = \frac{m}{c}$, and

$$y'' = -1 - |y'|\,y', \qquad y(0) = 0, \qquad y'(0) = v_0\sqrt{\frac{c}{gm}}.$$

2. Set $\frac{\bar{t}^2 g}{\overline{x}} = \bar{t}/\overline{x}v_0 = 1$ and leave $\frac{c\overline{x}}{m}$ as is, which would yield $\bar{t} = \frac{v_0}{g},\ \overline{x} = \frac{v_0^2}{g}$, and

$$y'' = -1 - \frac{cv_0^2}{mg} - |y'|\,y', \qquad y(0) = 0, \qquad y'(0) = 1.$$

3. Set $\frac{c\overline{x}}{m} = \bar{t}/\overline{x}v_0 = 1$ and leave $\frac{\bar{t}^2 g}{\overline{x}}$ as is, which would yield $\bar{t} = \frac{m}{cv_0}, \overline{x} = \frac{m}{c}$, and

$$y'' = -\frac{mg}{cv_0^2} - |y'|\,y, \qquad y(0) = 0, \qquad y'(0) = 1.$$

The second part is left to the reader. □

Exercise (1.8)**.** *(Formal aymptotic expansion)*

1. For the initial value problem

$$x''(t) + \varepsilon x'(t) = -1, \qquad x(0) = 0, \qquad x'(0) = 1$$

compute the formal asymptotic expansion of the solution $x(t)$ up to the second order in ε.

2. Compute the formal asymptotic expansion for the instance of time $t^* > 0$, for which $x(t^*) = 0$ holds true, up to first order in ε, by substituting the series expansion $t^* \sim t_0 + \varepsilon t_1 + O(\varepsilon^2)$ into the approximation obtained for x leading to a determination of t_0 and t_1.

Proof. TODO □

Exercise (1.9)**.** A model already nondimensionalised for the vertical throw with *small* air resistance is given by

$$x''(t) = -1 - \varepsilon(x'(t))^2, \qquad x(0) = 0, \qquad x'(0) = 1.$$

The model describes the throw up to the maximal height.

1. Compute the first two coefficientes $x_0(t)$ and $x_1(t)$ in the asymptotic expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

for small ε.

2. Compute the maximal height of the throw up to terms of order ε using asymptotic expansion.

3. Compare the results from 2. for the data of with the exact result and the result neglecting the air resistance.

Proof. TODO □

Exercise (1.10)**.** *(Multiscale approach)* The function $y(t)$ is supposed to solve the initial value problem

$$y''(t) + 2\varepsilon y'(t) + (1 + \varepsilon^2)y(t) = 0, \qquad y(0) = 0, \qquad y'(0) = 1,$$

for $t > 0$ and a small parameter $\varepsilon > 0$.

1. Compute the approximation of the solution by means of formal asymptotic expansion up to first order in ε.

2. Compare the function obtained in 1. with the exact solution

$$y(t) = e^{-\varepsilon t} \sin t.$$

For which times t the approximation from 1. is good?

3. To get a better approximation one can try the approach

$$y \sim y_0(t,\tau) + \varepsilon y_1(t,\tau) + \varepsilon^2 y_2(t,\tau) + \dots,$$

here $\tau = \varepsilon t$ is a slow time scale. Substitute this ansatz in the differential equation and compute y_0 such that the approximation becomes better. *Hint:* The equation of lowest order does not determine y_0 uniquely and coefficient functions in τ apper. Choose them in a clever way such that y_1 is easily computable.

Chapter 2

Linear systems of equations

2.1 Modelling electrical netwroks

Chapter 3

Ordinary differential equations

Teaching started on Monday 2019.11.25 (week 48a).

This chapter corresponds to part of chapter 4 in [1].

3.1 Quantitative analysis of models in population dynamics

Recall from week 46 (chapter 1) that we had two models for population dynamics.

- The first one, the exponential model was described by

  $$x'(t) = px(t), \quad p \in \mathbb{R},$$

  where p was the growth rate.

- The second, the constrained model was described by

  $$x'(t) = qx_M x(t) - qx^2(t), \quad q, x_M \in \mathbb{R}$$

  where $q > 0$ was the growth rate and x_M was the maximum carrying capacity of the environment in number of individuals.

Both of these models share the common mathematical structure of an autonomous equation, i.e. an equation of the form

$$x'(t) = f(x(t)), \tag{3.1}$$

where f (read x') does not depend explicitly[1] on t.

In this section we will focus on the qualitative aspects of the model, i.e. what information can we get from it without explicitly solving the equations (which in this case we can, but in the next examples we won't).

[1]That is, f cannot *unwrap* t out of $x(t)$ and do anything with it alone, it has to work on $x(t)$ as its variable.

Recall that a **stationary solution** of an ODE is one that stays constant in time, i.e. of the form $x(t) = c$. How can we find them? Easy, if x is constant then we must have $x'(t) = f(x(t)) = 0$. For our previous models this means

- $x(t) = 0$ for the exponential model, and
- $x(t) = 0$ or $x(t) = x_M$ for the second model. We get these two solutions from solving
$$x'(t) = qx_M x(t) - qx^(t) = 0$$
for $x(t)$ using the well known quadratic formula.

We are interested in these solutions because they are predictable and *don't blow up* as time passes. Later in this chapter we will formally define the concept of stability and quantify how stable solutions are based on how close they are to the stationary solutions.

3.1.1 Introduction to linear stability analysis

For now we will settle with something called **linear stability analysis**. The main idea is to linearise the solution (i.e. Taylor expand up to degree 1) a stationary solution. Let x^* be a stationary solution to an autonomous problem of the form 3.1. The linear expansion we are talking about is

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + O(|x - x^*|).$$

To make things easier, let us take $y(t) = x(t) - x^*(t)$. We shall ignore the error term $O(|x - x^*|) = O(y(t))$ and thus we get

$$y'(t) = f'(x^*)y(t). \tag{3.2}$$

Now, 3.2 is trivial to solve explicitly—it is a linear homogeneous equation with constant coefficients

$$y(t) = ce^{f'(x^*)t}.$$

Intuitively, as $t \to \infty$ we have

$$|y(t)| \to 0 \implies |x(t) - x^*(t)| \to 0 \iff x(t) \to x^*(t),$$

i.e. the linearised solution $y(t)$ converges to the stationary solution $x^*(t)$.

More on this later.

3.2 Predator–prey models

3.2.1 Derivation of the Lotka–Volterra equations

Now we turn our attention to environments where there are two species and one eats/hunts/harvests the other. Let us model this from scratch to get yet another example of how things work in Mathematical modelling. For this derivation we shall use the following assumptions.

1. The prey population has unlimited resources available for its growth all the time.
2. The predator population feeds exclusively on the prey population.
3. TODO: Something i cant remember.
4. The rate of growth of the populations is proportional to their size.
5. The environment is stable over time.

We now proceed with the standard recipe for deriving models.

1. **Name the quantities involved in the problem.** We are trying to model how two populations change over time so we need

$$\begin{aligned} t &:= \text{time} \\ x_1(t) &:= \text{size of prey population at time } t \\ x_2(t) &:= \text{size of predator population at time } t \end{aligned}$$

2. **Find relations between the quantities.** From assumption X we now that the growth of both species is directly proportional to the size of the populations, in other words

$$x_1' = p_1 x_1 \text{ and } x_2' = p_2 x_2.$$

A priori, we don't know if p_1 depends only on t or on $x_2(t)$ or on both. The possibility that p_1 depends on $x_1(t)$ is ruled out by the assumption that growth is proportional to size. Looking at the assumptions once more we find that p_1 cannot depend on t since "the environment is stable over time". There fore it must be that p_1 is a function only of $x_2(t)$, which really makes sense, since the size of the prey species depends on how many individuals are being eaten by the predator species. A similar argument for p_2 yields

$$p_1(x_2(t)) \text{ and } p_2(x_1(t)).$$

But what do these functions p_1 and p_2 look like? Well, the prey population naturally grows since we assumed unlimited resources but at the same time it is being eaten at some rate by the predator population. Similarly, the predator population naturally dies unless they can feed on the prey population. We introduce the parameters $\alpha, \beta, \gamma, \delta > 0$ and formalise these relations with

$$p_1(x_2(t)) = -\beta x_2(t) + \alpha \qquad \text{and} \qquad p_2(x_1(t)) = \delta x_1(t) - \gamma.$$

Finally we get our model, commonly referred to as the Lotka–Volterra equations, derived independently by both authors from around 1920 to around 1925 [4].

$$\begin{cases} x_1' &= (\alpha - \beta x_2)x_1 \\ x_2' &= (\delta x_1 - \gamma)x_2 \end{cases}. \tag{3.3}$$

The mathematical structure of this problem is that of an planar system of autonomous ODEs. We may rewrite it as

$$\begin{cases} \mathbf{x}' &= f(\mathbf{x}) \\ \mathbf{x}(t_0) &= \mathbf{x_0} \end{cases} \tag{3.4}$$

with $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. If f is sufficiently nice (i.e. locally Lipschitz) then an initial value problem of the form in 3.4 has locally unique solutions. However, it is not the explicit solutions that interest us right now, but rather the qualitative aspects of their behaviour.

3.2.2 Qualitative analysis of the Lotka–Volterra equations

We look into the stationary solutions to later look at stability. Once more, setting $x_1, x_2 = 0$ in 3.3 gives us the stationary solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha/\beta \\ \gamma/\delta \end{pmatrix}.$$

There are too many parameters to work comfortably with this solutions. Let us non-dimensionalise before moving on to get rid of as many parameters as we can. Very quickly, we choose the fundamental dimensions T for time and N for number of individuals and carry out a dimensional analysis to get

$$\begin{matrix} [t] &= T & [x_1] = [x_2] &= N \\ [\alpha] &= [\gamma] = \frac{1}{T} & [\beta] = [\delta] &= \frac{1}{NT} \end{matrix}.$$

We choose the characteristic quantities $\overline{x_1}, \overline{x_2}$ and $\bar{t}$ and set up the change of variables

$$z_1 = \frac{x_1}{\overline{x_1}}, \qquad z_2 = \frac{x_2}{\overline{x_2}} \qquad \text{and} \qquad \tau = \frac{t}{\bar{t}}.$$

Substitute with care in 3.3 (careful with the derivatives) to get

$$\begin{cases} z_1' &= \bar{t}\alpha z_1 - \beta\overline{x_2}\bar{t}z_1z_2 \\ z_2' &= \delta\overline{x_1}\bar{t}z_1z_2 - \gamma\bar{t}z_2 \end{cases}.$$

Notice how we have four different coefficients for z_1 and z_2 but only have three characteristic quantities. This means we'll need to make a compromise. Which one to make is dictated by our taste and the mathematical or biological interpretation of the parameters we choose. We will not do all four options here but the Lotka–Volterra equations often come with

$$\bar{t}\alpha = 1, \qquad \beta\overline{x_2}\bar{t} = 1 \qquad \text{and} \qquad \delta\overline{x_1}\bar{t} = \gamma\bar{t},$$

which in turn give us the **non-dimensionalised version of the Lotka–Volterra equations**

$$\begin{cases} z_1' &= (1 - z_2)z_1 \\ z_2' &= a(z_1 - 1)z_2 \end{cases}, \tag{3.5}$$

where there is only one parameter $a = \gamma/\alpha$.

In this form, the stationary solutions are given by

$$\mathbf{z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Directional field of the Lotka-Volterra equations

Plotting the directional field is a tool that proves useful when we want to get an overall idea of how the system behaves. Recall that given an autonomous ODE of the form $\mathbf{x}' = f(\mathbf{x})$ the directional field is a vector field of the form $x \mapsto f(x)$. In two dimensions this corresponds to a graph where the axes represent the values that the functions x_1 and x_2 can take and the arrows point in the direction $(f_1(x), f_2(x))$.

It is hard to draw these diagrams by hand but there are a number of steps we can take to get an idea of what they look like. Let's go back to the Lotka-Volterra model in the non-dimensionalised form. We plot the directional field by following these steps

1. **Find the stationery solutions.** These become points in the vector field since $f(x) = 0$ by definition of stationary solution (and therefore there is no arrow to draw).

2. **Find the isoclines**, i.e. the curves[2]

$$N_1 = \{f_1 = 0\} = \{\mathbf{z} \mid f_1(\mathbf{z}) = 0\} = \{(z_1, z_2) \mid f_1(z_1, z_2) = 0\}$$
$$N_2 = \{f_2 = 0\} = \{\mathbf{z} \mid f_2(\mathbf{z}) = 0\} = \{(z_1, z_2) \mid f_2(z_1, z_2) = 0\}$$

3. **Find the areas of monotonicity**, i.e. the regions of the plane given by[3]

$$D_{++} = \{f_1 > 0\} \cap \{f_2 > 0\}$$
$$D_{+-} = \{f_1 > 0\} \cap \{f_2 < 0\}$$
$$D_{-+} = \{f_1 < 0\} \cap \{f_2 > 0\}$$
$$D_{--} = \{f_1 < 0\} \cap \{f_2 < 0\}$$

   Keep in mind that the isoclines tell us where there is a change from one area of monotonicity to another[4].

4. **Draw the vector field**, i.e. some arrows.

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[2]Here we mean *curve* in the most general sense—isoclines can be straight lines or curves and they can be disjoint as is the case in this example.

[3]It is useful to write them as an intersection since therefore we can reuse the sets $\{f_n < 0\}$ and $\{f_n > 0\}$ to calculate all the regions depending on how they overlap.

[4]The reason for this is that we normally ask that the functions involved in the ODE are *nice enough*, i.e. at least C^2 which means that the derivatives are continuous.

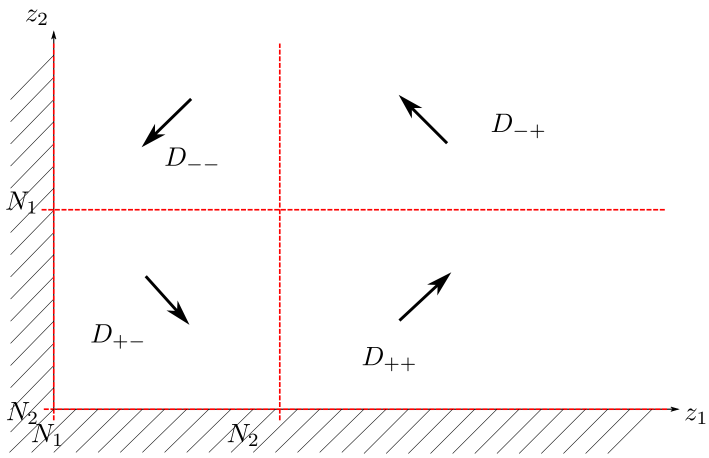

Figure 3.1: Sketch of the direction field for the Lotka-Volterra equations.

In the computer generated picture it is probably easier to tell what is happening with the solutions. If one were to choose a point in the plane, a solution going through that point would move in the direction of the vector in that point. Repeating this process can give us a rough idea of what the solutions look like: orbits around the stationary solution $(1,\ 1)$.

3.2.3 Phase diagram of the Lotka-Volterra equations

In our qualitative analysis of the Lotka-Volterra model we have found the stationary solutions and more or less characterised the rest of the solutions using the direction field. We would, however, like to have a more precise idea of what the solutions look like. We will devote the rest of this section to that.

Definition 3.1 (First integral)**.** TODO. we really need a definition here :(

3.3 Stability theory

Consider the general autonomous system

$$\mathbf{x}' = f(\mathbf{x}) \tag{3.6}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n,\ \Omega$ is open and $f \in C^2(\Omega)$.

Definition 3.2 (Lyapunov stability)**.** Let $\mathbf{x}^*$ be a stationary solution to 3.6. We say $\mathbf{x}^*$ is Lyapunov stable (or just stable for short) if for every open neighbourhood U of $\mathbf{x}^*$ there exists an open neighbourhood V of $\mathbf{x}^*$ such that any solution $\mathbf{x}$ of 3.6 with $\mathbf{x}(0) \in B$ satisfies $\mathbf{x}(t) \in U$ for all $t > 0$.

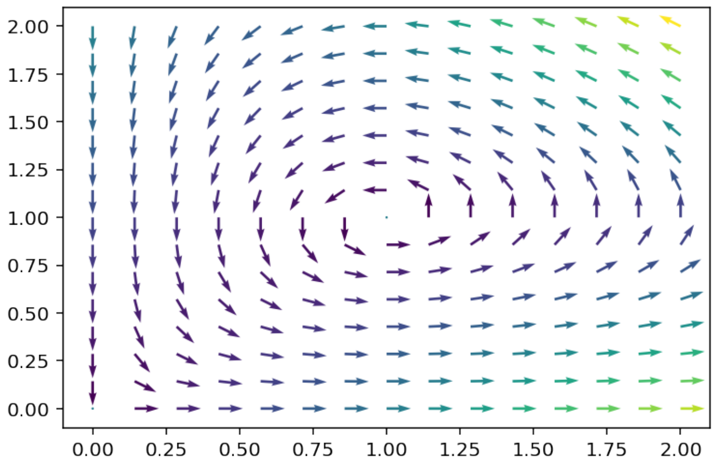

Figure 3.2: Direction field for the Lotka-Volterra model with $a = 0.5$ generated by a computer.

Definition 3.3 (Asymptotic stability)**.** Let $\mathbf{x}^*$ be a stationary solution to 3.6. We say $\mathbf{x}^*$ is asymptotically stable if there exists an open neighbourhood W of $\mathbf{x}^*$ such that for any solution $\mathbf{x}$ of 3.6 with $\mathbf{x}(0) \in W$ it holds that

$$\|\mathbf{x}(t) - \mathbf{x}^*\| \to 0 \text{ as } t \to \infty.$$

Remark 3.4. If $\mathbf{x}^*$ is asymptotically stable then it is Lyapunov stable.

3.3.1 Linear stability analysis

Definition 3.5. Let $\mathbf{x}^*$ be a stationary solution to 3.6. The linear system

$$\mathbf{y}' = Df(\mathbf{x}^*)\mathbf{y} \tag{3.7}$$

is called a linearisation of 3.6 in $\mathbf{x}^*$.

Moreover, $\mathbf{x}^*$ is called linearly unstable, resp. stable / asymptotically stable if $\mathbf{0}$ is unstable, resp. stable / asymptotically stable for .

Theorem 3.6 (Principle of linearised stability)**.** If $\mathbf{x}^*$ is linearly asymptotically stable, resp. unstable, then $\mathbf{x}^*$ is asymptotically stable, resp. unstable.

Remark 3.7. The previous theorem does not work for Lyapunov stability, i.e.

$$\mathbf{x}^* \text{ linearly stable } \not\Rightarrow \mathbf{x}* \text{ stable.}$$

Chapter 4

Calculus of variations

Definition 4.1 (Variational problem)**.** Let X be a real vector space (of functions, possibly infinite dimensional), $\mathcal{A} \subset X$ a set of admisible functions and $\mathcal{I} : X \to \mathbb{R}$ a functional that assigns a real number for each $u \in X$.

A variational problem is the task

$$\text{minimise } \mathcal{I}(u), \quad u \in \mathcal{A}.$$

In particular we are concerned with variational problems of integral form, i.e. those where

$$\mathcal{I}(u) = \int_a^b f(x, u(x), u'(x))dx, \tag{4.1}$$

where $u : [a, b] \to \mathbb{R}^m$ and $f : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$.

As mathematicians, we are immediately concerned about the exitence and uniqueness of solutions. Until 1850, it was thought that minimisers for integral problems always existed but Weierstrass gave a counter example. From that moment on, a new theory for solving variational problems, now known as the direct method was developed. In this section we will mostly concentrate on the classical method for finding minimisers which basically replicates the process of finding minima for functions in vector calculus. The reason for this is that a formal treatment of the direct method requires advanced mathematical tools from functional analysis, which is not a prerequisite for this course.

4.1 The classical method

This method was developed by Euler and Lagrange during the 18th century. The strategy for dealing with variational problems of the form (4.1) is to just go ahead and find the minima of the function $\mathcal{I}(u)$. For this, let us recall the approach taken in vector calculus to minimise a function $f : \mathbb{R}^m \to \mathbb{R}^n$.

1. Find $\overline{x} \in \mathbb{R}^m$ such that $Df(\overline{x}) = \mathbf{0}$,
2. Find $D^2 f(\overline{x})$, and
3. Check that $D^2 f(\overline{x})$ is positive definite.

The problem with this strategy is that we don't know if Df or D^2f will exist for our functional $f = \mathcal{I}$. Therefore, we will introduce a weaker notion of derivative, the variation, which is easier to work with in the context of these problems.

Definition 4.2 (Admissible perturbation)**.** Let X and $\mathcal{A}$ be a function space and a set of admissible functions, resp. Let $u \in \mathcal{A}$. For any $\varphi \in X$ we say φ is an admissible perturvation of u iff there exists a $\varepsilon_0 > 0$ such that

$$u + \varepsilon\varphi \in A, \qquad \text{for every } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Definition 4.3 (Variation)**.** Let $X = \{f : A \to B\}$ be a function space, $u, \varphi \in X$ and $x \in A$. We define the variation of u at a in the direction of φ as

$$\delta u(a)(\varphi) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} u(a + \varepsilon\phi). \tag{4.2}$$

4.2 Exercises

Exercise (Dido's problem)**.** Let $L > 0$ be a given length. We consider the maximisation problem

$$\text{maximise} \int_0^L u(s)\sqrt{1 - u'(s)^2}ds, \text{ for } u \in \mathcal{A},$$

where $\mathcal{A} = \{C^1((0,L)) \cap C^0([0,L]) : u(0) = 0, u(L) = 0, |u'(s)| \text{ for } s \in (0,L)\}$, which emerges from modeling Dido's problem.

1. Determine the corresponding Euler-Lagrange equation and find a non-negative solution $\overline{u} \in \mathcal{A} \cap C^2((0,L))$.

2. Sketch the curve $\{(\varphi(s), \overline{u}(s)) : s \in [0,L]\}$ with $\varphi(s) = \int_0^s \sqrt{1 - \overline{u}'(\tau)}d\tau$ for $s \in [0,L]$.

3. Interpret b) in the context of Dido's problem.

Hint for a): [1]Prove and use the following statement: Let $a, b \in \mathbb{R}$ with $a < b$, $u \in C^2((a,b))$ and $f \in C^2(\mathbb{R} \times \mathbb{R})$ such that $\partial_p f(u,u') \in C^1(a,b)$. Then

$$\frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u') \text{ in } (a,b)$$

if, and only if, there exists $c \in \mathbb{R}$ such that

$$f(u,u') - u'\partial_p f(u,u') = c \text{ in } (a,b).$$

[1] In the original statement for this problem, the hint was only a simple implication, but by proving it one realised that it was a double implication.

Proof of the hint. Not very precise but something along the lines of

$$\begin{aligned}
&\frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u') \\
\iff &\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u') = 0 \\
\iff &\int \partial_z f(u,u') - \int \frac{d}{dx}\partial_p f(u,u') = \int 0 \\
\iff &f(u,u') - u'\partial_p f(u,u') = c.
\end{aligned}$$

Or, with derivatives,

$$\begin{aligned}
&f(u,u') - u'\partial_p f(u,u') = c \\
\iff &\frac{d}{dx}f(u,u') - \frac{d}{dx}u'\partial_p f(u,u') = \frac{d}{dx}c \\
\iff &\partial_z f(u,u')u' + \partial_p f(u,u')u'' - u''\partial_z f(u,u') - u'\frac{d}{dx}\partial_p f(u,u') = 0 \\
\iff &u'\left(\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u')\right) = 0.
\end{aligned}$$

In this case, if $u' \neq 0$ we have

$$\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u') = 0 \iff \frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u').$$

Otherwise, we go back to the original equation

$$f(u,u') - 0\partial_p f(u,u') = c \iff f(u,u') = c$$

which means that f is constant in u and therefore in x so it is trivial to see that

$$\partial_p f(u,u') = 0 = \partial_z f(u,u'),$$

which gives the first equation. □

Proof. For the purposes of determining the Euler-Lagrange equation we have $x = s,\ z = u(x),\ p = u'(s)$ and $f(x,z,p) = z\sqrt{1-p^2}$. The involved partial derivatives are

$$\partial_z f(x,z,p) = \sqrt{1-p^2} \text{ and } \partial_p f(x,z,p) = -\frac{zp}{\sqrt{1-p^2}},$$

and thus the Euler-Lagrange equation becomes

$$\frac{d}{dx}\frac{-uu'}{\sqrt{1-u'^2}} = \sqrt{1-u'^2}.$$

This looks complicated so we apply the hint:

$$\begin{aligned}
&u\sqrt{1-u'^2} + u'\frac{uu'}{\sqrt{1-u'^2}} = c \\
\iff &u\left(\sqrt{1-u'^2} + \frac{u'^2}{\sqrt{1-u'^2}}\right) = c \\
\iff &u\left(\frac{1-u'^2+u'^2}{\sqrt{1-u'^2}}\right) = c \\
\iff &u = c\sqrt{1-u'^2} \\
\iff &u' = \sqrt{1-\left(\frac{u}{c}\right)^2}.
\end{aligned}$$

That last equation desperately screams for separation of variables and a trigonometric change of variable:

$$\begin{aligned}
&\frac{du}{ds} = \sqrt{1-\left(\frac{u}{c}\right)^2} \\
\iff &\frac{du}{\sqrt{1-(u/c)^2}} = ds \\
\iff &\int \frac{1}{\sqrt{1-(u/c)^2}} du = \int ds.
\end{aligned}$$

Change $u/c = \sin y \implies u = c\sin y \implies du = c\cos y dy$ to get

$$\begin{aligned}
\int \frac{1}{\sqrt{1-(u/c)^2}} du &= \int \frac{1}{1-\sin^2 y} c\cos y dy \\
&= c\int \frac{\cos y}{\cos y} dy = cy \\
&= c \arcsin \frac{u}{c}
\end{aligned}$$

Plugging it back into the hint,

$$s + k = c\arcsin\frac{u}{c} \implies u(s) = c\sin\frac{k+s}{c},$$

which we rewrite picking different constants,

$$u(s) = k_1 \sin(k_2 s + k_3),$$

where the constants k_1, k_2, k_3 are obtained by enforcing $u \in \mathcal{A}$. More specifically, we require

$$u(0) = 0 \implies k_3 = 0 \text{ and } u(L) = 0 \implies k_2 L = n\pi.$$

Moreover, we require u to be non-negative, therefore $n = 1$ and thus $k_2 = \frac{\pi}{L}$ (otherwise the sine would go negative). Finally, we want

$$|u'(s)| < 1 \implies |k_1\cos(k_2 s + k_3)k_2| < 1 \implies k_1 k_2 < 1 \implies k_1 < \frac{L}{\pi},$$

since we require that k_1 is positive so that $u(s)$ also is. This final parameter is fixed by maximising $\mathcal{I}(u)$:

TODO □

Exercise (Geodesics in $\mathbb{R}^2$)**.** Let A and B be two points in the plane. What is the shortest connection between A and B?

1. Set up the variational problem to model the situation.

2. Solve the problem and interpret the result.

Proof. Let $X = C^1([0,1];\mathbb{R}^2)$ and $\mathcal{A} = \{u \in X \mid u(0) = A,\ u(1) = B\}$. We define our functional $\mathcal{I}$ as the length of the parametrised curve u as follows

$$\mathcal{I}(u) = \int_0^1 \|u'(t)\|\, dt.$$

Our variational problem is

$$\text{minimise } \mathcal{I}(u) \text{ for } u \in \mathcal{A}.$$

To solve it we use the Euler-Langrange method. We have $f(t, u(t), u'(t) = \|u'(t)\|$ and, in the form of the Euler-Langrange equations, we get $f(x, z, p) = \|p\|$. Therefore,

$$\delta_z f = \begin{pmatrix} 0 & 0 \end{pmatrix} \text{ and } \delta_p f = \begin{pmatrix} \frac{p_1}{\|p\|} & \frac{p_2}{\|p\|} \end{pmatrix}.$$

Notice how $p = u' : [0,1] \to \mathbb{R}^2$ so by $\delta_p f$ we really mean the last two numbers in Df (which is a row matrix since f is real valued). We arrive at the following Euler-Lagrange equation

$$\frac{d}{dt}\delta_p f = \delta_z \iff \frac{d}{dt}\begin{pmatrix} \frac{u_1}{\|u\|} & \frac{u_2}{\|u\|} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Instead of taking the derivative with respect to t, we may simply rewrite this as

$$\begin{pmatrix} \frac{u_1}{\|u\|} & \frac{u_2}{\|u\|} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

Therefore

$$u(t) = \int u'(t)dt = (c_1 t, c_2 t) + u_0,\ u_0 \in \mathbb{R}^2,$$

which is a parametrisation for a curve. The parameters c_1, c_2 and u_0 are determined by enforcing $u \in \mathcal{A}$:

$$u(0) = u_0 = A,\ u(1) = (c_1, c_2) + u_0 = B.$$

□

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