Notes on Measure and integration

Elias Hernandis

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Readers are asked to report errata to eliashernandis@gmail.com.

Chapter 1

Introduction

1.1 What is a σ-algebra ?

Definition 1 (Sigma algebra)**.** Let X be a set and $\mathcal{A} \subset \wp(X)$ a collection of subsets of X. We say that $\mathcal{A}$ is a sigma algebra or σ-algebra (on X) if it satisfies these three properties:

1. It contains the set, $X \in \mathcal{A}$;
2. it is closed under set complement $A \in \mathcal{A} \implies X \setminus A = A^c \in \mathcal{A},\ \forall A \subset X$, and
3. it is closed under **countable**[a] union $(A_n)_{n\in\mathbb{N}} \in \mathcal{A} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

[a]In these notes we shall follow the Shilling convention and use countable to describe any set whose cardinality is less than or equal to that of the set of the natural numbers, i.e. $\#A \leq \#\mathbb{N}$. This means that countable does not necessarily mean infinite.

Definition 2 (Measurable space)**.** A measurable space is a pair $(X, \mathcal{A})$ where X is a set and $\mathcal{A}$ is a σ-algebra .

From now on we shall assume X is a set and $\mathcal{A}$ is a σ-algebra on X.

Remark 1.

- From properties 1 and 2 we have $\varnothing = X^c \in \mathcal{A}$.
- From properties 2 and 3 we have that $\mathcal{A}$ is also closed under **countable** intersection. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then, using De Morgan's laws we have

$$\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c \in \mathcal{A}$$

because $A_n^c \in \mathcal{A}$ because of 2, the union is included because of 3 and its complement is also included because of 2.

Example 1 (First examples of σ-algebras).

1. The power set $\wp(X)$ is the largest σ-algebra on X.
2. $\{\varnothing, X\}$ is the smallest σ-algebra on X.
3. Given a subset $A \subset X$, the smallest σ-algebra which contains information about A is $\{\varnothing, X, A, A^c\}$. More on this later.
4. Let X be an uncountable set. Then $\mathcal{A} = \{A \subset X \mid A \text{ is countable or } A^c \text{ is coutable }\}$ is a σ-algebra .

   Proof. We shall prove the three properties of a σ-algebra

   (a) $X \in \mathcal{A}$ because $X^c = \varnothing$ is countable.

   (b) If $A \in \mathcal{A}$ and is countable the $A^c \in \mathcal{A}$ because $(A^c)^c$ is countable. If $A \in \mathcal{A}$ but A is not countable then A^c must be so, hence $A^c \in \mathcal{A}$.

   (c) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. To see if the union is included we distinguish to cases:

      i. If A_n is countable $\forall n \in N$ then $\bigcup_{n\in\mathbb{N}} A_n$ is also countable and therefore in $\mathcal{A}$.[1]

      ii. If there is any A_k that is uncountable, then the union is uncountable but

      $$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subset A_k^c$$

      Recall that A_k^c must be countable because $A_k \in \mathcal{A}$ and therefore $\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c$ must also be countable. Thus, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

   □

5. (The preimage[2] σ-algebra). Let X, X' be sets, let $\mathcal{A}'$ be a σ-algebra on X' and let $f : X \to X'$ be a function between the two sets. We claim that[3] $\mathcal{A} = \{f^{-1}(A') \mid A' \in \mathcal{A}'\}$ is a σ-algebra .

   Proof. (a) $X = f^{-1}(X') \in \mathcal{A}$ because $X' \in \mathcal{A}$.

[1] Recall that the countable union of countable sets is still a countable set.

[2] Recall some properties of preimages (that are not always true for images): Set difference and union (and therefore complement and intersection) are well defined and behave as expected. Moreover, union and intersection behave as expected even for countable arities.

[3] Here f^{-1} denotes the preimage of a set, not the inverse function. Recall that the preimage of a function $f : X \to X'$ is defined as $f^{-1}(A') = \{x \in X \mid f(x) \in A' \subset X'\}$

(b) If $A \in \mathcal{A}$ then $A^c = X \setminus A = f^{-1}(X') \setminus f^{-1}(A') = f^{-1}(X' \setminus A') \in \mathcal{A}$ because $X' \setminus A' = A'^c \in \mathcal{A}'$.

(c) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. By definition there must be a collection $(A'_n)_{n\in\mathbb{N}} \subset \mathcal{A}'$ for which $A_n = f^{-1}(A'_n),\ \forall n \in \mathbb{N}$. Recall that

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} f^{-1}(A'_n) = f^{-1}(\bigcup_{n\in\mathbb{N}} A'_n) \in \mathcal{A}$$

because $\mathcal{A}'$ is a σ-algebra hence $\bigcup_{n\in\mathbb{N}} A'_n \in \mathcal{A}'$.

□

Now we will explore a result which will allow us to generate more examples from existing ones and clarify operations between σ-algebras .

Theorem 1. Given a set X. The arbitrary intersection $\bigcap_{\alpha\in I} \mathcal{A}_\alpha$ of σ-algebras is again a σ-algebra .

Proof. We shall go over the three properties of σ-algebras .

1. $X \in \mathcal{A}_\alpha,\ \forall \alpha \in I$ hence $X \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$
2. If $A \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$ then in particular $A \in \mathcal{A}_\alpha,\ \forall \alpha \in I$. Because each $\mathcal{A}_\alpha$ is a σ-algebra , then $A^c \in A_n,\ \forall \alpha \in I \implies A_n^c \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$.
3. If $(A_n)_{n\in\mathbb{N}} \subset \bigcap_{\alpha\in I} \mathcal{A}_\alpha$ then, as before, we have that $(A_n)_{n\in\mathbb{N}} \subset A_n,\ \forall \alpha \in I$. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_\alpha \implies \bigcup_{n\in\mathbb{N}} A_n \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$.

□

Definition 3 (σ-algebra generated by a collection of subsets)**.** Let X be a set and $\mathcal{G}$ a collection of subsets of X. We denote by $\sigma(\mathcal{G})$ the smallest σ-algebra which contains $\mathcal{G}$ and we define it by

$$\sigma(\mathcal{G}) := \bigcap \mathcal{C} \text{ where } \mathcal{G} \subset \mathcal{C} \wedge \mathcal{C} \text{ is a } \sigma\text{-algebra}$$

We also say that $\mathcal{G}$ is a generator of $\sigma(\mathcal{G})$ or that $\sigma(\mathcal{G})$ is generated by $\mathcal{G}$.

Theorem 2. For any $\mathcal{G} \subset \wp(X)$, $\sigma(\mathcal{G})$ exists and is a the smallest σ-algebra containing $\mathcal{G}$.

Proof. Let $\mathcal{A} = \sigma(\mathcal{G})$. Since $\wp(X) \supset \mathcal{G}$ and $\wp(X)$ is a σ-algebra , the intersection $\mathcal{A}$ is non-empty and contains $\mathcal{G}$. Because of the previous result, $\mathcal{A}$ itself is also a σ-algebra . (So we have existence). Furthermore, $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{G}$ because if there were another σ-algebra $\mathcal{A}'$ containing $\mathcal{G}$ it would be included in the intersection hence $\mathcal{A} \subseteq \mathcal{A}'$. □

Remark 2.

1. If $\mathcal{G}$ is a σ-algebra , then $\sigma(\mathcal{G}) = \mathcal{G}$.
2. For any $A \in X$, $\sigma(\{A\}) = \{\emptyset, X, A, A^c\}$.
3. Let $\mathcal{G}, \mathcal{F} \subset \wp(X)$. If $\mathcal{G} \subseteq \mathcal{F}$ then $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$.

   Proof. $\mathcal{G} \subseteq \mathcal{F} \subseteq \sigma(\mathcal{F})$. So $\sigma(\mathcal{F})$ is a σ-algebra containing $\mathcal{G}$. But $\sigma(\mathcal{G})$ is the smallest σ-algebra containing $\mathcal{G}$ therefore $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$. □

1.2 A canonical σ-algebra on $\mathbb{R}^n$

In what follows we will use some basic topology concepts. Lets give some names

- $\mathcal{O}^n := \{A \subset \mathbb{R}^n \mid A \text{ is open }\}$,
- $\mathcal{C}^n := \{A \subset \mathbb{R}^n \mid A \text{ is closed }\}$, and
- $\mathcal{K}^n := \{A \subset \mathbb{R}^n \mid A \text{ is compact }\}$.

The collection $\mathcal{O}^n$ is a topology, meaning it satisfies the following properties:

1. $\emptyset, \mathbb{R}^n \in \mathcal{O}^n$
2. It is closed under finite intersections, i.e. $V, W \in \mathcal{O}^n \implies V \cap W \in \mathcal{O}^n$.
3. It is closed under arbitrary unions, i.e.

$$(A_\alpha)_{\alpha \in I} \in \mathcal{O}^n \implies \bigcup_{\alpha \in I} A_\alpha \in \mathcal{O}^n.$$

We shall call the pair $(\mathbb{R}^n, \mathcal{O}^n)$ a topological space. We now consider the smallest σ-algebra containing $\mathcal{O}^n$.

Definition 4 (Borel σ-algebra)**.** The Borel σ-algebra on $\mathbb{R}^n$ is the smallest σ-algebra containing $\mathcal{O}^n$. We denote it by $\sigma(\mathcal{O}^n)$ or by $\mathcal{B}(\mathbb{R}^n)$.

Theorem 3.

$$\mathcal{B}(\mathbb{R}^n) := \sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n) = \sigma(\mathcal{K}^n)$$

Proof. First, we prove the first equality, i.e. $\sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n)$ by proving mutual inclusion. To show $\sigma(\mathcal{C}^n) \subset \sigma(\mathcal{O}^n)$ it is enough to show that $\mathcal{C}^n \subset \sigma(\mathcal{O}^n)$ (recall remark 2). Let $C \in \mathcal{C}^n$ be any closed set in $\mathbb{R}^n$. By definition C^c is open hence $C^c \in \mathcal{O}^n \subset \sigma(\mathcal{O}^n)$. Because $\sigma(\mathcal{O}^n)$ is a σ-algebra it must be true that $(C^c)^c = C \in \sigma(\mathcal{O}^n)$. The same holds for the other inclusion.

Now we turn our attention to $\sigma(\mathcal{C}^n) = \sigma(\mathcal{K}^n)$. The inclusion $\sigma(\mathcal{K}^n) \subset \sigma(\mathcal{C}^n)$ is trivial because every compact set is closed in $\mathbb{R}^n$ (recall remark 2). For the other one, it is again enough to show that $\mathcal{C}^n \in \sigma(\mathcal{K}^n)$. Let $C \in \mathcal{C}^n$ and define $C_k := C \cap \overline{B_k(0)}$ which is[4] closed and bounded. By construction $C = \bigcup_{k\in\mathbb{N}} C_k \in \mathcal{K}^n$ thus $\mathcal{C}^n \in \sigma(\mathcal{K}^n)$. □

We would now like to find smaller sets of generators for the Borel σ-algebra on $\mathbb{R}^n$. Let us define the following collections (where $\times$ denotes de cartesian product of the intervals):

- The collection of open rectangles (or cubes or hypercubes)

$$\mathcal{J}^{o,n} = \{\bigtimes_{i=1}^{n}(a_i, b_i) \mid a_i, b_i \in \mathbb{R}\}$$

- The collection of (from the right) half-open rectangles

$$\mathcal{J}^{n} = \{\bigtimes_{i=1}^{n}[a_i, b_i) \mid a_i, b_i \in \mathbb{R}\}$$

Theorem 4. We have

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}^n_{rat}) = \sigma(\mathcal{J}^{o,n}_{rat}) = \sigma(\mathcal{J}^n) = \sigma(\mathcal{J}^{o,n})$$

Proof. Let's begin by proving $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}^{o,n}_{rat})$. Recall that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n)$, so to prove the previous equality it suffices to prove the following two mutual inclusions:

- $\sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{O}^n)$. From remark 2 we have that to prove this it suffices to say that every open rectangle is an open set and thus $\mathcal{J}^{o,n}_{rat} \subset \mathcal{O}^n \implies \sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{O}^n)$.

- $\sigma(\mathcal{O}^n) \subseteq \sigma(\mathcal{J}^{o,n}_{rat})$. To prove this we make the following claim:

$$U \in \mathcal{O}^n \implies U = \bigcup_{I\in\mathcal{J}^{o,n}_{rat},\ I\subseteq U} I$$

  Again we shall attack this by proving the mutual inclusion of the two sets:

  - It is clear that $\bigcup_{I\in\mathcal{J}^{o,n}_{rat},\ I\subset U} I \subseteq U$ because of the restriction on the union.

[4] Here $B_r(c_0)$ and $\overline{B_r(c_0)}$ denote the open and closed balls of radius r and centre c_0, respectively. Clearly these are both bounded sets.

  - For the reverse containment, we have that as U is open, for any $x \in U$ there is a ball $B_\varepsilon(x) \subseteq U$. Because the rationals $\mathbb{Q}^n$ are dense in the reals $\mathbb{R}^n$ we can chose a rectangle $I \subset B_\varepsilon(x)$ and hence U is contained in the union.

    It is clear that all the sets $I \subseteq U$ are also in $\sigma(\mathcal{J}^{o,n}_{rat})$. However, for the union of them to be inside de σ-algebra we must ensure that the number of sets that participate is countable. Each rectangle I can be fully determined by two of its corners, which in turn have coordinates in $\mathbb{Q}^n$. Therefore, the number of sets intervening in the union is $\#(\mathbb{Q}^n \times \mathbb{Q}^n) = \#\mathbb{N}$ and thus the union is again within the σ-algebra .

Because $\mathcal{J}^{o,n}_{rat} \subset \mathcal{J}^{o,n} \subset \mathcal{O}^n$ we get for free that $\sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{J}^{o,n}) \subseteq \sigma(\mathcal{O}^n)$. We therefore conclude that

$$\sigma(\mathcal{J}^{o,n}_{rat}) = \sigma(\mathcal{J}^{o,n}) = \sigma(\mathcal{O}^n)$$

Now we would like to prove that half open sets also yield the same Borel σ-algebra for $\mathbb{R}^n$.

- We begin by noticing that we can write open sets as infinite unions of half open ones

  $$\bigtimes_{i=1}^{n}(a_i, b_i) = \bigcup_{n\in\mathbb{N}} \bigtimes_{i=1}^{n}[a_i + \frac{1}{n}, b_i)$$

  for both rectangles with rational and real endpoints. Thus, we have

  $$\begin{aligned} \mathcal{J}^{o,n}_{rat} \subseteq \sigma(\mathcal{J}^{n}_{rat}) &\implies \sigma(\mathcal{J}^{o,n}_{rat}) \subseteq \sigma(\mathcal{J}^{n}_{rat}) \\ \mathcal{J}^{o,n} \subseteq \sigma(\mathcal{J}^{n}) &\implies \sigma(\mathcal{J}^{o,n}) \subseteq \sigma(\mathcal{J}^{n}) \end{aligned}$$

  (remember that σ-algebras are closed under countable unions).

- For the reverse containment we must notice that we can write (right) half-open sets as intersections of open ones

  $$\bigtimes_{i=1}^{n}[a_i, b_i] = \bigcap_{n\in\mathbb{N}} \bigtimes_{i=1}^{n}(a_i - \frac{1}{n}, b_i)$$

  for rectangles with both rational and real endpoints. Similarly, we have

  $$\begin{aligned} \mathcal{J}^{n}_{rat} \subseteq \sigma(\mathcal{J}^{o,n}_{rat}) &\implies \sigma(\mathcal{J}^{n}_{rat}) \subseteq \sigma(\mathcal{J}^{o,n}_{rat}) \\ \mathcal{J}^{n} \subseteq \sigma(\mathcal{J}^{o,n}) &\implies \sigma(\mathcal{J}^{n}) \subseteq \sigma(\mathcal{J}^{o,n}) \end{aligned}$$

- We conclude that

  $$\begin{aligned} \sigma(\mathcal{J}^{o,n}_{rat}) &= \sigma(\mathcal{J}^{n}_{rat}) \\ \sigma(\mathcal{J}^{o,n}) &= \sigma(\mathcal{J}^{n}) \end{aligned}$$

With the previous equalities the theorem has been proved. □

To recap, there are a few important points on this proof:

- First, we would like to have a more tangible generator for the Borel σ-algebra on $\mathbb{R}^n$.
- We choose rectangles because they are easier to work with and have direct application on probability theory (we could also have chosen balls, for instance).
- They key to proving that two σ-algebras are the equal is to prove that each is contained in the other. To do this, we use the generators: if the generator of $\mathcal{A}$ is contained in $\sigma(\mathcal{A}')$ and the generator of $\mathcal{A}'$ is contained in $\sigma(\mathcal{A})$, we are done.
- To prove this containments we have had to write sets from the generator as unions of sets from the other σ-algebra . It is key to make sure that these unions only iterate over a countable number of elements.

Example 2 (Another characterisation of the Borel σ-algebra on $\mathbb{R}^n$)**.** In this example we shall see that $\mathbb{B} = \{B_r(x) \mid x \in \mathbb{R}^n, r \in \mathbb{R}^+\}$ is also a generator of $\mathcal{B}(\mathbb{R}^n)$.

Proof. We proceed as before, first defining an auxiliary collection where the radii are all rational and the centres have rational coordinates:

$$\mathbb{B}' = \{B_r(x) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$$

It is trivial that $\mathbb{B}' \subset \mathbb{B} \subset \mathcal{O}^n$. Therefore we have that

$$\sigma(\mathbb{B}') \subseteq \sigma(\mathbb{B}) \subseteq \sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n)$$

For the reverse inclusions, we will focus on proving that $\mathcal{O}^n \subseteq \sigma(\mathbb{B}')$. For this we claim that any open set $U \in \mathcal{O}^n$ can be written as

$$U = \bigcup_{B \in \mathbb{B}', B \subseteq U} B$$

We need to verify two things. That the previous equality is true and that the number of sets that intervene in the union is countable.

1. It is clear that any set $B \subset \mathbb{B}'$ is also in U by the definition of the union. For the reverse inclusion, we shall choose a point $q \in \mathbb{Q}^n$ such that $\|x - q\| < r/3$. This is possible because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Next we will choose a radius $r' \in \mathbb{Q}$ such that $r' < r$. This is also possible for the same reason. Now we consider $B = B_{r'}(q) \in \mathbb{B}'$ which is assured to contain x.
2. Moreover, each of these balls is fully determined by an $(n+1)$ tuple of rationals (namely the center coordinates and the radius). Hence, the number of sets in the union is $\#(\mathbb{Q}^n \times \mathbb{Q}) = \#\mathbb{N}$.

□

Chapter 2

Measures

Definition 5 (Measure)**.** Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to [0, \infty)$ is called a measure (on X) if

1. $\mu(\varnothing) = 0$ and
2. **(σ-aditivity)** if $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is a pairwise disjoint (i.e. $A_i \cap A_j = \varnothing$ if $i \neq j$) sequence, then

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

In the previous definition the symbol $\dot\cup$ indicates that the union is disjoint. Also, let's introduce some more notation. Given a sequence $(A_n)_{n\in\mathbb{N}}$ we shall say

$$A_n \uparrow A \iff A_1 \subseteq A_2 \subseteq \dots \qquad \text{and } A = \bigcup_{n\in\mathbb{N}} A_n$$
$$A_n \downarrow A \iff A_1 \supseteq A_2 \supseteq \dots \qquad \text{and } A = \bigcap_{n\in\mathbb{N}} A_n$$

And some more terminology:

Definition 6 (Exhausting sequence)**.** Within a measurable space $(X, \mathcal{A})$ sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is called an exhausting sequence if $A_n \uparrow X$.

Definition 7 (Measure space)**.** Let $(X, \mathcal{A})$ be a measurable space and μ a measure. The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

- If $\mu(X) < \infty$ we say that $(X, \mathcal{A}, \mu)$ is a finite measure space.
- IF $\mu(X) = 1$ then $(X, \mathcal{A}, \mu)$ is a probability space.

Definition 8 (σ-finite measure)**.** A measure μ is called σ-finite if there exists am exhausting sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\mu(A_n) < \infty$, $\forall n \in \mathbb{N}$. A measure space with this kind of measure is called a σ-finite measure space.

Theorem 5 (Properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $A, B, A_n, B_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$. Then,

1. **(finite additivity)** $A \cap B = \varnothing \implies \mu(A \uplus B) = \mu(A) + \mu(B)$,
2. **(monotonicity)** if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,
3. if $A \subset B$ and $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$,
4. **(strong additivity)** $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,
5. **(finite subadditivity)** $\mu(A \cup B) \leq \mu(A) + \mu(B)$,
6. **(continuity from below)** if $A_n \uparrow A$ then $\mu(A) = \mu(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \mu(A_n) = \lim_{n\to\infty} \mu(A_n)$,
7. **(continuity from above)** if $\mu(A) < \infty, \forall A \in \mathcal{A}$ and $A_n \downarrow A$ then $\mu(A) = \mu(\bigcup_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \mu(A_n) = \lim_{n\to\infty} \mu(A_n)$, and
8. **(sigma subadditivity)** $\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$.

Proof.

1. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $A_1 = A$, $A_2 = B$ and $A_i = \varnothing$ for $i > 2$. It is clear that A_n are disjoint since $A \cap B = \varnothing$ and $A_n \cap \varnothing = \varnothing$, $\forall n \in \mathbb{N}$.

2. Write $B = (B \setminus A) \uplus A$. Then, because of σ-aditivity we have
$$\mu(B) = \mu(B \setminus A) + \mu(A) \geq \mu(A) \text{ since } \mu(B \setminus A) \geq 0$$

3. As previously write $\mu(B) = \mu(B \setminus A) + \mu(A)$. Because $\mu(A) < \infty$ we can subtract it on both sides to get $\mu(B) - \mu(A) = \mu(B \setminus A)$.

4. Write $A \cup B = A \setminus B \uplus B \setminus A \uplus A \cap B$ hence $\mu(A \cup B) = \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B)$. Add $\mu(A \cap B)$ on both sides and group terms:
$$\mu(A \cup B) + \mu(A \cap B) = \underbrace{\mu(A \setminus B) + \mu(A \cap B)}_{\mu(A)} + \underbrace{\mu(B \setminus A) + \mu(A \cap B)}_{\mu(B)}$$

5. $\mu(A \cap B) \leq \mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B)$

6. Define the sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for $n > 1$. It is clear that B_n is pairwise disjoint and that $\dot\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n = A$. Hence

$$\begin{aligned}\mu(A) &= \mu(\bigcup_{n\in\mathbb{N}} A_n) = \mu(\dot{\bigcup_{n\in\mathbb{N}}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n) = \lim_{m\to\infty} \sum_{n=1}^{m} \mu(B_n) \\ &= \lim_{m\to\infty} \mu(\dot{\bigcup_{n=1}^{m}} B_n) = \lim_{m\to\infty} \mu(A_m) = \sup_{n\in\mathbb{N}} \mu(A_n)\end{aligned}$$

since $\mu(A_n)$ is an increasing sequence. The introduction of the limits in the previous chain of equalities has to be done carefully, as we are building on the definition of limits for sequences of numbers. The equality between the first and the second lines comes from σ-additivity.

7. Let $D_n = A_1 \setminus A_n$, $\forall n \in \mathbb{N}$. Then D_N is an increasing sequence with

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} (A_1 \setminus A_n) = \bigcup_{n\in\mathbb{N}} (A_1 \cap A_n^c) = A_1 \cap \bigcup_{n\in\mathbb{N}} A_n^c \\ = B_1 \cap \left(\bigcap_{n\in\mathbb{N}} A_n\right)^c = A \setminus \bigcap_{n\in\mathbb{N}} A_n\end{aligned}$$

Thus,

$$\begin{aligned}&\mu(A \setminus \bigcap_{n\in\mathbb{N}} A_n) \overset{(3)}{=} \mu(A) - \mu(\bigcap_{n\in\mathbb{N}} A_n) = \mu(\bigcup_{n\in\mathbb{N}} D_n) \\ &\overset{(4)}{=} \lim_{n\to\infty} \mu(D_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n) \overset{(3)}{=} \lim_{n\to\infty} (\mu(A) - \mu(A_n)) = \mu(A) - \lim_{n\to\infty} \mu(A_n)\end{aligned}$$

Substracting $\mu(A) < \infty$ from both sides we have

$$\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$$

8. Let $(A_n)_{n\in\mathbb{N}}$ be any countable subcollection of $\mathcal{A}$. Define

$$E_n = \bigcup_{m=1}^{n} A_m \in \mathcal{A}$$

Then $E_n \uparrow \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} A_n$. Thus, by (6) we have

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu\left(\bigcup_{m=1}^{n} A_m\right) \overset{(5)}{\leq} \lim_{n\to\infty} \sum_{m=1}^{n} \mu(A_m)$$

□

Example 3 (First examples of measures)**.** Let $(X, \mathcal{A})$ be a measurable space

1. We define the **Dirac measure** for a given $x_0 \in X$ as follows:

$$\delta_{x_0}(A) = \begin{cases} 0 & \text{if } x_0 \notin A \\ 1 & \text{if } x_0 \in A \end{cases}, \qquad \forall A \in \mathcal{A}$$

   Clearly δ_{x_0} is a measure since it satisfies the two properties. First, $\delta_{x_0}(\varnothing) = 0$ since $\forall x_0 \in X,\ x_0 \notin \varnothing$. Second, for any pairwise disjoint collection of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ we have two possibilities:

   - If $x_0 \notin \dot{\bigcup}_{n\in\mathbb{N}} A_n$ then clearly $x_0 \notin A_n,\ \forall n \in N$ so

$$0 = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) = 0$$

   - Otherwise, if $x_0 \in \dot{\bigcup}_{n\in\mathbb{N}} A_n$ then there must be only one $n_0 \in \mathbb{N}$ such that $x_0 \in A_{n_0}$ since (A_n) is a pairwise disjoint collection. Thus,

$$1 = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A_{n_0}) + \sum_{n\neq n_0} \mu(A_n) = 1 + 0$$

2. Let $X = \mathbb{R}$ and choose $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable }\}$. We already saw on the Chapter 1 that $\mathcal{A}$ is a σ-algebra . Now define $\mu : \mathcal{A} \to [0, \infty)$ as

$$\mu(A) = \begin{cases} 0 & \text{if } \#A \leq \#\mathbb{N} \\ 1 & \text{if } \#A^c \leq \#\mathbb{N} \end{cases}$$

   We have that $\mu(\varnothing) = 0$ since $\#\varnothing$ is countable. As for σ-additivity we must recall from set theory that the union of countable sets is also countable, so:

   - If $\#A_n \leq \#\mathbb{N},\ \forall n \in \mathbb{N}$ then

$$0 = \mu(\dot{\bigcup_{n\in\mathbb{N}}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = 0$$

   - If there exists an $n_0 \in \mathbb{N}$ such that $A_{n_0}^c$ is countable then

$$\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subseteq A_{n_0}^c$$

     and hence $(\dot{\bigcup}_{n\in\mathbb{N}} A_n)^c$ is countable. Furthermore, since the collection is pairwise disjoint, $\forall n \in \mathbb{N}, n \neq n_0$ we have $A_n^c \subseteq A_{n_0}$ so $\#A_n^c,\ \forall n \neq n_0$. Thus

$$1 = \mu(\dot{\bigcup_{n\in\mathbb{N}}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A_{n_0}) + \sum_{n\neq n_0} \mu(A_n) = 1 + 0$$

3. **(Discrete probability measure)**

4. **(Lebesgue measure)**