EE6150: Stochastic modeling and the theory of queues
Assignment 1

Instructor: Prashanth L.A.

Due: Oct 5, 2020

Problem 1. A rope consists of n strands. When a strand is carrying a load of x tons, its failure rate is λx. At time zero all strands are working and equally share a combined load of L tons. When a strand fails, the remaining strands share the load equally. This process continues until all strands break, at which point the rope breaks. Compute the distribution of the lifetime of the rope. (5 marks)

Problem 2. Customers arrive according to a $PP(\lambda)$ to a service facility with $s \geq 1$ identical servers in parallel. The service times are i.i.d. $\exp(\mu)$ random variables. At time 0, all the servers are busy serving one customer each, and no customers are waiting.

Answer the following: (1+2+2 marks)

1. Compute the probability that the next arriving customer finds all servers busy.
2. Let N be the number of arrivals before the first service completion. Compute the pmf of N.
3. Compute the probability that the next arriving customer finds at least two servers idle.

Problem 3. A machine is up at time zero. It then alternates between two states: up or down. (When an up machine fails it goes to the down state, and when a down machine is repaired it moves to the up state.) Let U_n be the n^{th} up-duration, followed by the n^{th} down-duration, D_n. Suppose $\{U_n, n \geq 0\}$ is a sequence of iid $\exp(\lambda)$ random variables. The n^{th} down-duration is proportional to the n^{th} up-duration, i.e., there is a constant $c > 0$ such that $D_n = cU_n$.

Answer the following: (2+2 marks)

1. Let $F(t)$ be the number of failures up to time t. Is $\{F(t), t \geq 0\}$ a Poisson process? Justify your answer.
2. Let $R(t)$ be the number of repair completions up to time t. Is $\{R(t), t \geq 0\}$ a Poisson process? Justify your answer.

Problem 4. Two individuals, say 1 and 2, need kidney transplants. Without a transplant the remaining lifetime of person i is an $\exp(\mu_i)$ random variable, the two lifetimes being independent. Kidneys become available according to a Poisson process with rate λ. The first available kidney is supposed to go to person 1 if she is still alive when the kidney becomes available; else it will go to person 2. The next kidney will go to person 2, if she is still alive and has not already received a kidney. Compute the probability that person i receives a new kidney ($i = 1, 2$). (5 marks)

Problem 5. Let $\{N_i(t), t \geq 0\}(i = 1, 2)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. At time 0 a coin is flipped that turns up heads with probability p. Define

$$N(t) = \begin{cases} N_1(t) & \text{if the coin turns up heads,} \\ N_2(t) & \text{if the coin turns up tails.} \end{cases}$$

Answer the following: (1+2+2 marks)

(a) Is $\{N(t), t \geq 0\}$ a PP?

(b) Does $\{N(t), t \geq 0\}$ have stationary increments?

(c) Does $\{N(t), t \geq 0\}$ have independent increments?

Problem 6. Let $\{N(t), t \geq 0\}$ be a $NPP(\lambda(.))$ and $0 \leq t_1 \leq ... \leq t_n$ be given real numbers and $0 \leq k_1 \leq ... \leq k_n$ be given integers. Then, show that (3 marks)

$$P(N(t_1 = k_1, N(t_2) = k_2, \ldots, N(t_n) = k_n) =$$
$$e^{-\Lambda(t_n)} \frac{(\Lambda(t_1))^{k_1}}{k_1!} \frac{(\Lambda(t_2) - \Lambda(t_1))^{k_2-k_1}}{(k_2 - k_1)!} \cdots \frac{(\Lambda(t_n) - \Lambda(t_{n-1}))^{k_n-k_{n-1}}}{(k_n - k_{n-1})!}.$$

Problem 7. Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. Suppose all odd events (i.e. events occurring at time S_{2n+1} for $n \geq 0$) in this process are classified as type 1 events, and all even events are classified as type 2 events. Let $N_i(t)$ be the number of type i $(i = 1, 2)$ events that occur during $(0, t]$.

Answer the following: (1+1+1 marks)

1. Is $\{N_1(t), t \geq 0\}$ a PP?
2. Is $\{N_2(t), t \geq 0\}$ a PP?
3. Are $\{N_i(t), t \geq 0\}(i = 1, 2)$ independent of each other?

Problem 8. ***(Simulation Exercise)***

Suppose incoming calls on the cellphones in IITM campus follow a Poisson process with rate 10 per second. Suppose that the call duration follows a log-normal distribution with parameters $\mu = 4$ and $\sigma^2 = 1$.

Write a program in your favorite language to simulate the process described above.

Let $N(t)$ denote the number of calls in the campus at time t. Do multiple simulation runs, and plot the histogram of $N(t)$, for the following values of t: $10^3, 10^4, 10^6$. Tabulate the empirical mean value of $N(t)$ at the time instants given above.

Interpret the numerical results, and summarize your findings. (5 marks)