EE6150: Stochastic modeling and the theory of queues
Assignment 3

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Problem 1. Let $\{X_n, n \geq 0\}$ be a DTMC on state-space $\{0, 1, 2, \ldots\}$ with the following transition probabilities

$$p_{i,j} = \begin{cases} \frac{1}{i+2} \; 0 \leq j \leq i+1, \; i \geq 0, \text{ and} \\ 0 \;\; \text{otherwise} . \end{cases}$$

Is the above DTMC positive recurrent? If so, compute its limiting distribution.

Problem 2. A shuttle bus with finite capacity B stops at bus stops numbered $0, 1, 2, \ldots$ on an infinite route. Let Y_n be the number of riders waiting to ride the bus stop at stop n. Assume that $\{Y_n, n \geq 0\}$ is a sequence of iid random variables with common pmf

$$\alpha_k = \mathrm{P}\left(Y_n = k\right), k = 0, 1, 2, \ldots.$$

Every passenger who is on the bus alights at a given bus stop with probability p.The passengers behave independently of each other. After the passengers alight, as many of the waiting passengers board the bus as there is room on the bus.

Numerically compute the limiting distribution of the number of passengers on the bus. Assume that the bus capacity is 20, the number of passengers waiting for the bus at any stop is a Poisson random variable with mean 10, and that a passenger on the bus alights at any stop with probability .4.

Problem 3. Let S be a set of all permutations of the integers $1, 2, \ldots, N$.
Let $X_0 = (1, 2, \ldots, N)$ be the initial permutation. Construct X_{n+1} from X_n by interchanging the I^{th} and J^{th} component of X_n, where I and J are independent and uniformly distributed over $\{1, 2, \ldots, N\}$.
(If $I = J$, then $X_{n+1} = X_n$). Show that $X_n, n \geq 0$ is a DTMC. Compute its limiting distribution.

Problem 4. Consider a service system where customers arrive at times $n = 0, 1, 2, \ldots,$ and form a queue for service. The departures from this queue also occur at times $n = 0, 1, 2, \ldots$. At each time n, a customer is removed (a departure occurs) from the head of the queue (if it is not empty) with probability q, then a customer is added (an arrival occurs) at the tail end of the queue with probability p $(0 \leq p, q \leq 1)$. All the arrivals and departures are independent of everything else. Assume that the queue is positive recurrent, and compute its limiting distribution.

Problem 5. A DTMC is said to be tree if between any two distinct states i and j there is exactly one sequence of distinct states $i_1, i_2, \ldots, i_r$ such that

$$p_{i,i_1}, p_{i_1,i_2}, \ldots, p_{i_r,j} \geq 0.$$

Show that a positive recurrent tree DTMC is reversible.

Problem 6. Let P be a diagonalizable $N \times N$ transition probability matrix. Suppose a square matrix A is *diagonalizable* such that

$$A = XDX^{-1}$$

Then

$$A^n = XD^nX^{-1} = \sum_{j=1}^{m} \lambda_j^n x_j y_j \tag{1}$$

Using the representation in the above equation (1) study the limiting behavior of P^n and $\frac{M^{(n)}}{(n+1)}$ in terms of the eigenvectors. Consider the following cases

1. $\lambda = 1$ is an eigenvalue of multiplicity one, and it is the only eigenvalue with $|\lambda|= 1$.
2. $\lambda = 1$ has multiplicity k, and these are the only eigenvalues with $|\lambda|= 1$.
3. There are eigenvalues with $|\lambda|= 1$ other than $\lambda = 1$.

Problem 7. Consider a DTMC space with state space $\{1, 2, \ldots, N\}$ with the following $N \times N$ transition probability matrix

$$\begin{bmatrix}
q & p & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 \\
q & 0 & p & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 \\
0 & q & 0 & p & 0 & . & . & . & . & . & 0 & 0 & 0 \\
. \\
. \\
. \\
0 & . & . & . & 0 & 0 & q & 0 & p & 0 & . & . & 0. \\
. \\
. \\
0 & 0 & 0 & 0 & . & . & . & . & . & 0 & q & 0 & p \\
0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & q & p
\end{bmatrix}$$

Answer the following

1. Is the DTMC irreducible ?
2. Is the DTMC periodic ?
3. Does the DTMC have stationary distribution. If yes, provide the same .

Problem 8. Consider a DTMC with the following transition probability matrix:

$$\begin{bmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{bmatrix} \quad \text{* denotes a non-zero entry}$$

Find the communicating classes and determine whether they are

i) transient/recurrent

ii) aperiodic

Problem 9. Consider a DTMC with the following transition probability matrix:

$$\begin{bmatrix} 0.4 & 0 & 0.6 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

Find $\lim_{n\to\infty} p_{i,j}^{(n)}$ for $i, j \in \{1, 2, 3, 4, 5\}$.

Problem 10. (**Simulation Exercise**)

Refer to this book here **Frankenstein or, The Modern Prometheus**. Build a Markov chain and learn the transition probabilities using empirical frequencies.
Solve

$$\pi = \pi\, \hat{p}_n, \text{ for } n \gg 1,$$

where $\hat{p}_n$ is the learned transition probability matrix. Check whether the π obtained via solving the equation tallies with the time average of the letters

i) halfway through the book

ii) end of the book