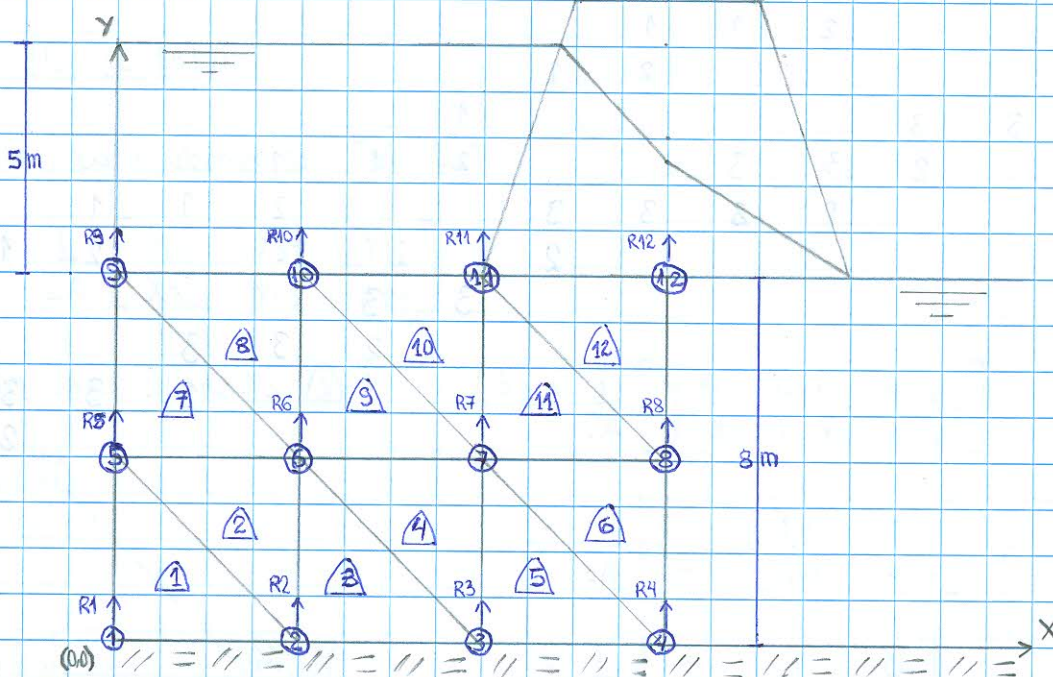


# Lista de Exercícios Nº 04

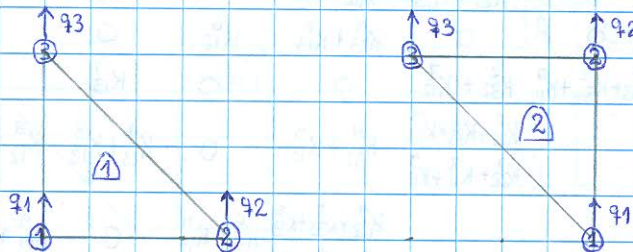
Estudante: Celso Salvador Villalobos

Materia: Métodos Numéricos em Engenharia Civil

COORDENADAS GLOBAIS



COORDENADAS LOCAIS



GLOBAL	LOCAL											
	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	2	1	1									
3			2	1	1							
4					2	1						
5	3	3					1					
6		2	3	3			2	1	1			
7				2	3	3			2	1	1	
8						2					2	1
9							3	3				
10								2	3	3		
11										2	3	3
12												2

$K_{11}^1$	$K_{12}^2$	0	0	$K_{13}^1$	0	0	0	0				
	$K_{22}^1 + K_{11}^2 + K_{11}^3$	$K_{12}^3$	0	$K_{23}^1 + K_{13}^2$	$K_{12}^2 + K_{13}^3$	0	0	0				
		$K_{22}^3 + K_{11}^4 + K_{11}^5$	$K_{12}^5$	0	$K_{23}^3 + K_{13}^4$	$K_{12}^4 + K_{13}^5$	0	0				
			$K_{22}^5 + K_{11}^6$	0	0	$K_{23}^5 + K_{13}^6$	$K_{12}^6$	0				
				$K_{33}^1 + K_{33}^2 + K_{11}^7$	$K_{32}^2 + K_{12}^7$	0	0	$K_{13}^7$				
					$K_{22}^2 + K_{33}^3 + K_{33}^4$	$K_{32}^4 + K_{12}^9$	0	$K_{23}^7 + K_{13}^8$	$K_{12}^8 + K_{13}^9$			
					$K_{22}^7 + K_{11}^8 + K_{11}^9$							
						$K_{22}^4 + K_{33}^5 + K_{33}^6$	$K_{32}^6 + K_{12}^{11}$	0	$K_{23}^9 + K_{13}^{10}$	$K_{12}^{10} + K_{13}^{11}$		
						$K_{22}^9 + K_{11}^{10} + K_{11}^{11}$						
							$K_{22}^6 + K_{22}^{11} + K_{11}^{12}$	0	0	$K_{23}^{11} + K_{13}^{12}$	$K_{12}^{12}$	
								$K_{33}^7 + K_{33}^8$	$K_{32}^8$	0	0	
									$K_{22}^8 + K_{33}^9 + K_{33}^{10}$	$K_{32}^{10}$	0	
										$K_{22}^{10} + K_{33}^{11} + K_{33}^{12}$	$K_{32}^{12}$	
											$K_{32}^{12}$	



Se sabe que:

$$[K] = \int_V [B]^T [C] [B] dV \quad dV = 1 \times dA$$

$$[K] = 1 \int_A [B]^T [C] [B] dA$$

$$[K] = [B]^T [C] [B] A$$

tambem; para elemento T3o CST

$$[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

Então para, Elemento 1:

$$[B] = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$(x_3, y_3)$   
 $(0, 4)$

$(x_1, y_1)$   
 $(0, 0)$

$(x_2, y_2)$   
 $(4, 0)$

$$= \frac{1}{2A} \begin{bmatrix} 0-4 & 4-0 & 0-0 \\ 0-4 & 0-0 & 4-0 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

Para, Elemento 2:

$$[B] = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$(x_3, y_3)$   
 $(0, 4)$

$(x_2, y_2)$   
 $(4, 4)$

$(x_1, y_1)$   
 $(4, 0)$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

Para, Elemento 3:

$$[B] = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$(x_3, y_3)$   
 $(4, 4)$

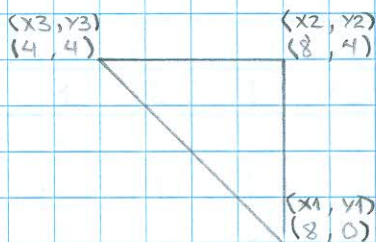
$(x_1, y_1)$   
 $(4, 0)$

$(x_2, y_2)$   
 $(8, 0)$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$



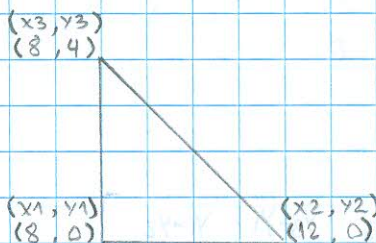
Para, elemento ④ :



$$[B] = \frac{1}{2A} \begin{bmatrix} 4-4 & 4-0 & 0-4 \\ 4-8 & 8-4 & 8-8 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

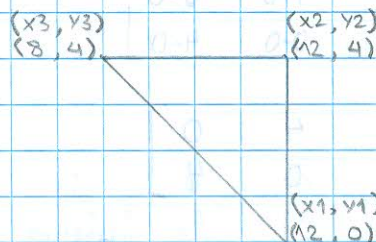
Para, elemento ⑤ :



$$[B] = \frac{1}{2A} \begin{bmatrix} 0-4 & 4-0 & 0-0 \\ 8-12 & 8-8 & 12-8 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

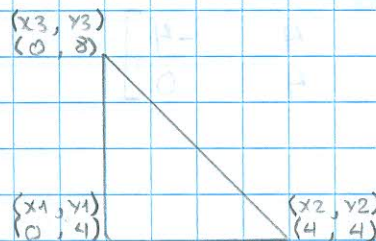
Para, elemento ⑥ :



$$[B] = \frac{1}{2A} \begin{bmatrix} 4-4 & 4-0 & 0-4 \\ 8-12 & 12-8 & 12-12 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

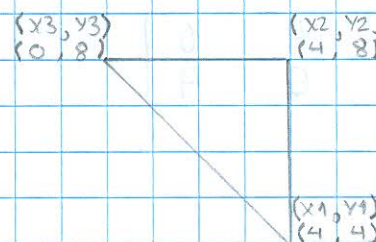
Para, elemento ⑦ :



$$[B] = \frac{1}{2A} \begin{bmatrix} 4-8 & 8-4 & 4-4 \\ 0-4 & 0-0 & 4-0 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

Para, elemento ⑧ :



$$[B] = \frac{1}{2A} \begin{bmatrix} 8-8 & 8-4 & 4-8 \\ 0-4 & 4-0 & 4-4 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$



Para elemento (9):

$(x_3, y_3)$   
(4, 8)

$(x_1, y_1)$   
(4, 4)

$(x_2, y_2)$   
(8, 4)

$$[B] = \frac{1}{2A} \begin{bmatrix} 4-8 & 8-4 & 4-4 \\ 4-8 & 4-4 & 8-4 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

Para elemento (10):

$(x_3, y_3)$   
(4, 8)

$(x_2, y_2)$   
(8, 8)

$(x_1, y_1)$   
(8, 4)

$$[B] = \frac{1}{2A} \begin{bmatrix} 8-8 & 8-4 & 4-8 \\ 4-8 & 8-4 & 8-8 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

Para elemento (11):

$(x_3, y_3)$   
(8, 8)

$(x_1, y_1)$   
(8, 4)

$(x_2, y_2)$   
(12, 4)

$$[B] = \frac{1}{2A} \begin{bmatrix} 4-8 & 8-4 & 4-4 \\ 8-12 & 8-8 & 12-8 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

Para elemento (12):

$(x_3, y_3)$   
(8, 8)

$(x_2, y_2)$   
(12, 8)

$(x_1, y_1)$   
(12, 4)

$$[B] = \frac{1}{2A} \begin{bmatrix} 8-8 & 8-4 & 4-8 \\ 8-12 & 12-8 & 12-12 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

tem-se que:

$$[K] = [B]^T [C] [B] A \quad \text{onde:} \quad A = \frac{4 \times 4}{2} = 8 \text{ m}^2$$

$$K = 10^{-4} \text{ cm/s} = 10^{-6} \text{ m/s}$$

$$[C] = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = 10^{-6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Para o elemento tipo (1), tem-se que:

$$[K] = \frac{1}{2A} \begin{bmatrix} -4 & -4 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} 10^{-6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2A} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} A = 5 \times 10^{-7} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Para os elementos 3, 5, 7, 9, 11 é igual a 1.

$$[K]_{1,3,5,7,9,11} = 5 \times 10^{-7} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Para o elemento tipo (2), tem-se que:

$$[K] = \frac{1}{2A} \begin{bmatrix} 0 & -4 \\ 4 & 4 \\ -4 & 0 \end{bmatrix} 10^{-6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2A} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} A = 5 \times 10^{-7} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Para os elementos 4, 6, 8, 10, 12 é igual a 2.

$$[K]_{2,4,6,8,10,12} = 5 \times 10^{-7} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Sabemos que:

$$[K] \{q\} = \{Q\} \quad \text{como não existe fluxo prescrito nas nós.}$$

$$[K] \{q\} = \{0\}$$



$$5 \times 10^{-7} \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -2 & 8 & -2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -2 & 8 & -2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \\ h_{11} \\ h_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Condições de contorno:

$$h_1 = h_5 = h_9 = h_{10} = h_{11} = 13 \text{ m}$$

$$h_4 = h_8 = h_{12} = 10,5 \text{ m}$$

$$\begin{bmatrix} 4 & -1 & -2 & 0 \\ -1 & 4 & 0 & -2 \\ -2 & 0 & 8 & -2 \\ 0 & -2 & -2 & 8 \end{bmatrix} \begin{Bmatrix} h_2 \\ h_3 \\ h_6 \\ h_7 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 10,5 \\ 52 \\ 47 \end{Bmatrix}$$

$$\begin{Bmatrix} h_2 \\ h_3 \\ h_6 \\ h_7 \end{Bmatrix} = \begin{Bmatrix} 12,48 \\ 11,72 \\ 12,61 \\ 11,96 \end{Bmatrix}$$

Cálculo das velocidades:

$$\{g\} = \begin{Bmatrix} dh/dx \\ dh/dy \end{Bmatrix} = [B] \{q\}$$

$$\{v\} = -[R] \{g\} = -[R][B] \{q\} \quad \text{onde } [R] \text{ é } \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$$



Elemento ①:

$$\{V_1\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 13 \\ 12,48 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 1,275 \cdot 10^{-7} \\ 0,0 \end{Bmatrix} \text{ m/s}$$

Elemento ②:

$$\{V_2\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 12,48 \\ 12,61 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 9,75 \cdot 10^{-8} \\ -3,0 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento ③:

$$\{V_3\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 12,48 \\ 11,72 \\ 12,61 \end{Bmatrix} = \begin{Bmatrix} 1,9 \cdot 10^{-7} \\ -3,0 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento ④:

$$\{V_4\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 11,72 \\ 10,50 \\ 12,61 \end{Bmatrix} = \begin{Bmatrix} 1,625 \cdot 10^{-7} \\ -5,75 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento ⑤:

$$\{V_5\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 11,72 \\ 10,50 \\ 11,96 \end{Bmatrix} = \begin{Bmatrix} 3,075 \cdot 10^{-7} \\ -5,75 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento ⑥:

$$\{V_6\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 10,5 \\ 10,5 \\ 11,96 \end{Bmatrix} = \begin{Bmatrix} 3,65 \cdot 10^{-7} \\ 0,0 \end{Bmatrix} \text{ m/s}$$



Elemento 7:

$$\{V_7\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 13 \\ 12,61 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 9,75 \cdot 10^{-8} \\ 0,0 \end{Bmatrix} \text{ m/s}$$

Elemento 8:

$$\{V_8\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 12,61 \\ 13 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 0,0 \\ -9,75 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento 9:

$$\{V_9\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 12,61 \\ 11,96 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 1,625 \cdot 10^{-7} \\ -9,75 \cdot 10^{-8} \end{Bmatrix} \text{ m/s}$$

Elemento 10:

$$\{V_{10}\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 11,96 \\ 13 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 0,0 \\ -2,6 \cdot 10^{-7} \end{Bmatrix} \text{ m/s}$$

Elemento 11:

$$\{V_{11}\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 11,96 \\ 10,5 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 3,65 \cdot 10^{-7} \\ -2,6 \cdot 10^{-7} \end{Bmatrix} \text{ m/s}$$

Elemento 12:

$$\{V_{12}\} = - \frac{10^{-6}}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -4 & 4 & 0 \end{bmatrix} \begin{Bmatrix} 10,5 \\ 10,5 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 6,25 \cdot 10^{-7} \\ 0,0 \end{Bmatrix} \text{ m/s}$$