Lagrange Interpolation for Equally-Spaced abscissa

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Theorem (Lagrange Interpolation for Equally-Spaced abscissa)**.** *Suppose* a, b *are a pair of real number such that* $a < b$, $m \geq 3$ *be an integer and* $h = \frac{b-a}{m-1}$. *let* $f \in C^m[a-mh, b+mh]$ *and equispaced points* x_k *be defined as* $x_k = \frac{b-a}{2} + kh$. *Then, for any real number* p *there exists a real number* $\xi, -mh < \xi < mh$, *such that*

$$f(x_0 + ph) = \sum_k \mathcal{A}_k^m f(x_k) + \mathcal{R}_{m-1,p} \tag{1}$$

where k *varies from*

$$-\frac{1}{2}(m-2) \leq k \leq \frac{1}{2}m \quad \textit{for even } m$$

$$-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1) \quad \textit{for odd } m$$

with

$$\mathcal{A}_k^m(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \sum_t (p-t)$$

and

$$\mathcal{R}_{m-1,p} = \frac{h^m}{m!} f^{(m)}(\xi) \sum_n (p-n)$$

here both t *and* n *varies same as* k.

Proof. Assuming m to be odd. For a function $f \in C^m[a-mh, b+mh]$,its Lagrange interpolation polynomial constructed from equispaced points

$$x_{-\frac{1}{2}(m-1)}, x_{-\frac{1}{2}(m-1)+1}, \ldots\ldots, x_{\frac{1}{2}(m-1)-1}, x_{\frac{1}{2}(m-1)}$$

is given by

$$\mathcal{L}_{m-1}(x) = \sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} l_j(x_j) f(x_j) \tag{2}$$

and,

$$f(x) = \sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} l_j(x_j) f(x_j) + R_{m-1}(x) \tag{3}$$

with,

$$l_j(x) = \sum_{i=-\frac{1}{2}(m-1), i \neq j}^{\frac{1}{2}(m-1)} \frac{x - x_i}{x - x_j}$$

$$R_{m-1}(x) = \frac{f^{(m)}(\xi)}{m!} \Bigg[\sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} x - x_j \Bigg]$$

We can write x as $x = x_0 + ph$ for some real number p.

$$\begin{aligned}
R_{m-1}(x) = R_{m-1}(x_0 + ph) &= \frac{f^{(m)}(\xi)}{m!} \Bigg[\sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} x - x_j \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} \Bigg[\Big(x - x_{-\frac{1}{2}(m-1)}\Big) \Big(x - x_{-\frac{1}{2}(m-1)+1}\Big) \ldots\ldots. \Big(x - x_{\frac{1}{2}(m-1)}\Big) \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} \Bigg[\Big(x_0 + ph - x_0 + \frac{1}{2}(m-1)h\Big) \ldots. \Big(x_0 + ph - x_0 - \frac{1}{2}(m-1)h\Big) \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} \Bigg[\Big(ph + \frac{1}{2}(m-1)h\Big) \ldots. \Big(ph - \frac{1}{2}(m-1)h\Big) \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} \Bigg[h^m \Big(p + \frac{1}{2}(m-1)\Big) \ldots. \Big(p - \frac{1}{2}(m-1)\Big) \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} \Bigg[h^m \Big(p + \frac{1}{2}(m-1)\Big) \ldots. \Big(p - \frac{1}{2}(m-1)\Big) \Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} h^m \sum_{n=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p - n) \\
&= \mathcal{R}_{m-1,p}
\end{aligned} \tag{4}$$

$$\begin{aligned}
l_k(x) = l_k(x_0 + ph) &= \sum_{i=-\frac{1}{2}(m-1), i \neq k}^{\frac{1}{2}(m-1)} \frac{x - x_i}{x_k - x_i} \\
&= \Bigg[\frac{\big(x - x_{-\frac{1}{2}(m-1)}\big) \ldots\ldots \big(x - x_{k-1}\big) \big(x - x_{k+1}\big) \ldots\ldots. \big(x - x_{\frac{1}{2}(m-1)}\big)}{\big(x_k - x_{-\frac{1}{2}(m-1)}\big) \ldots\ldots \big(x_k - x_{k-1}\big) \big(x_k - x_{k+1}\big) \ldots\ldots. \big(x - x_{\frac{1}{2}(m-1)}\big)} \Bigg]
\end{aligned}$$

$$= \left[\frac{\Big(x_0 + ph - x_0 + \frac{1}{2}(m-1)h\Big)....\Big(x_0 + ph - x_0 - (k-1)h\Big)\Big(x_0 + ph - x_0 - (k+1)h\Big) \\ ....\Big(x_0 + ph - x_0 - \frac{1}{2}(m-1)h\Big)}{\Big(x_0 + kh - x_0 + \frac{1}{2}(m-1)h\Big)....\Big(x_0 + kh - x_0 - (k-1)h\Big)\Big(x_0 + kh - x_0 - (k+1)h\Big) \\ ....\Big(x_0 + kh - x_0 - \frac{1}{2}(m-1)h\Big)} \right]$$

$$= \left[\frac{\Big(p + \frac{1}{2}(m-1)\Big)....\Big(p - (k-1)\Big)\Big(p - (k+1)\Big)....\Big(p - \frac{1}{2}(m-1)\Big)}{\Big(k + \frac{1}{2}(m-1)\Big)....\Big(k - (k-1)\Big)\Big(k - (k+1)\Big)....\Big(k - \frac{1}{2}(m-1)\Big)} \right] \frac{\Big(p-k\Big)}{\Big(p-k\Big)}$$

$$= \left[\frac{\Big(p + \frac{1}{2}(m-1)\Big)....\Big(p - k + 1\Big)\Big(p - k\Big)\Big(p - k - 1\Big)....\Big(p - \frac{1}{2}(m-1)\Big)}{\Big(k + \frac{1}{2}(m-1)\Big)....\Big(k - k + 1\Big)\Big(k - k - 1\Big)....\Big(k - \frac{1}{2}(m-1)\Big)} \frac{1}{\Big(p-k\Big)} \right]$$

$$= \frac{\alpha}{\beta} \tag{5}$$

where $\alpha = \big(p + \frac{1}{2}(m-1)\big)....\big(p-k+1\big)\big(p-k\big)\big(p-k-1\big)....\big(p - \frac{1}{2}(m-1)\big)$,
$\beta = \big(k + \frac{1}{2}(m-1)\big)....\big(k-k+1\big)\big(k-k-1\big)....\big(k - \frac{1}{2}(m-1)\big)\big(p-k\big)$
Now,

$$\begin{aligned}
\beta &= \big(k + \frac{1}{2}(m-1)\big)....\big(k-k+1\big)\big(k-k-1\big)....\big(k - \frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \big(k + \frac{1}{2}(m-1)\big)....\big(2\big)\big(1\big)\big(-1\big)\big(-2\big)....\big(k - \frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \Big(k + \frac{1}{2}(m-1)\Big)!\big(-1\big)\big(-2\big)....\big(k - \frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \Big(k + \frac{1}{2}(m-1)\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\big(1\big)\big(2\big)....\big(\frac{1}{2}(m-1) - k\big)\big(p-k\big) \\
&= \Big(\frac{1}{2}(m-1) + k\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\Big(\frac{1}{2}(m-1) - k\Big)!\Big(p-k\Big)
\end{aligned}$$

$$\alpha = \prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} \Big(p - t\Big)$$

putting α, β back in equation 5

$$\begin{aligned}
l_k(x) &= \frac{\alpha}{\beta} \\
&= \frac{\prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)}(p-t)}{\Big(\frac{1}{2}(m-1) + k\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\Big(\frac{1}{2}(m-1) - k\Big)!\Big(p-k\Big)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Big(-1\Big)^{m-1}}{\Big(-1\Big)^{m-1}} \frac{\Big(-1\Big)^{k-\frac{m-1}{2}} \prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p-t)}{\Big(\frac{1}{2}(m-1)+k\Big)!\Big(\frac{1}{2}(m-1)-k\Big)!\Big(p-k\Big)} \\
&= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p-t) \\
&= \mathcal{A}_k^m(p)
\end{aligned} \tag{6}
$$

By using equations 6 , 3 , 4

$$
f(x) = f(x_0 + ph) = \sum_k \mathcal{A}_k^m f(x_k) + \mathcal{R}_{m-1,p}
$$

where k varies from $-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1) \quad , \quad m \textit{ is odd.}$

Similarly, for even m, we can get the result by constructing Lagrange interpolating polynomial from equispaced points

$$
x_{-\frac{1}{2}(m-2)}, x_{-\frac{1}{2}(m-2)+1}, \ldots\ldots, x_{\frac{1}{2}m-1}, x_{\frac{1}{2}m}
$$

$\square$

References

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