HIGH-ORDER CORRECTED TRAPEZOIDAL RULE FOR A CLASS OF SINGULAR INTEGRALS

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This is to certify that **Kunal Kishore**, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the thesis entitled **'High-order corrected Trapezoidal rule for a class of singular integrals''** under my supervision and guidance.

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Kunal Kishore

ABSTRACT

We review the Trapezoidal rule for the computation of definite integral of functions lacking simple anti-derivatives. We then look at complex singular functions and analyze some popular techniques for the computation of its integrals. Furthermore, we look at Trapezoidal rule generalization for various class of complex singular functions.

Then, we see applications of pre-corrected Trapezoidal rule for Direct acoustic obstacle scattering problem in $\mathbb{R}^2$ and quadrature rules in determining the numerical solution of Fredholm integral equation of the second kind.

LIST OF SYMBOLS OR ABBREVIATIONS

$\mathcal{C}(I)$	:	Space of all continuous function on an interval I
$\mathcal{B}_n(x)$	:	Bernoulli polynomial of degree n
$\mathcal{O}(n)$	:	Big O notation
$\mathbb{R}$	:	Set of all Real number
$\mathcal{T}^n(f)$	:	(n+1) point composite Trapezoidal rule approximation of f

n is a positive integer

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1. INTRODUCTION

Integral equations arise in numerous scientific and engineering problems. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, acoustical and waves, can be interpreted by integral equations. Unfortunately, in most cases, the underlying integrand does not have simple antiderivatives, and the kernel function may even be singular; therefore, in such cases, if the computation of a definite integral is needed, it will have to be computed numerically.

Numerical schemes for the approximation of definite integrals are generally referred to as a quadrature rule. Typically, a quadrature rule for the approximation of definite integral of a function $f \in C[a, b]$ is of the form

$$I_n(f) = \sum_{i=i}^{n} w_i f\left(x_i\right),$$

where $x_1, \cdots, x_n$ and $w_1, \cdots, w_n$ are referred as the quadrature nodes and the weights, respectively. Quadrature points x_i and weights w_i is crucial for the performance and accuracy of quadrature rules, and different quadrature schemes use different rules for defining the quadrature weights and points. By now, many efficient and high-order convergent quadrature rules are known; for instance, Gaussian Quadrature [1] ,Newton–Cotes formulas [2] ,Tanh-sinh quadrature [2].

Perhaps, among all existing Quadratures, the Trapezoidal rule is the most simple and numerically stable and offer several other advantages in terms of speed of computations in many algorithms [3]. It is well known that if a function f is smooth and periodic in the integration interval $[a, b]$, then trapezoidal rule approximation exhibits super algebraic convergence. Moreover, if f is periodic and analytic, then it will yield exponential convergence. On the other hand, if f is failing to be periodic in the integration region, the trapezoidal rule gives only second-order accuracy [4] even f is analytic in the integration interval $[a, b]$. To enhance the convergence rate of the trapezoidal rule for non-periodic functions, a 'Pre-corrected trapezoidal rule' is introduced by Rokhlin in [5]. Since then, a variant of the Pre-corrected trapezoidal rule for singular and non-singular integrands have been presented in the literature. For instance, see, [4, 6, 3, 7], and references therein.

The main idea of the pre-corrected trapezoidal rule is to modify the quadrature weights in the vicinity of the singularity or at the boundaries utilizing Euler–Maclaurin summation formula. An essential feature of this quadrature scheme is that its highly accurate version can be constructed even for singular integrands by adjusting the weights locally. The necessary weights can Pre-computed and tabulated for future applications. Owing to the capability of dealing with singular integrands, the Pre-corrected trapezoidal rule is heavily used across all scientific computing, including algorithms related to inverse Laplace transforms, special functions, wave scattering problem to name a few...

In this project, we study a high-order quadrature scheme for singular and non-singular function utilizing a modification of the classical trapezoidal rule known as the “Pre-corrected trapezoidal rule”. Therefore, in the following sections, we briefly discuss the trapezoidal rule followed by the Pre-corrected trapezoidal rule in the next chapter.

1.1 Trapezoidal Rule

There are several classes of quadrature rules, such as Interpolatory Quadrature, Gaussian Quadrature, among others. Interpolatory Quadrature refers to Quadrature rules which gives values of integral for some interpolation of the integrand. Newton–Cotes quadrature is an interpolatory quadrature that provides integral values for Lagrange polynomial interpolation of the given integrand.

1.1. Lagrange polynomial. *For any given set of* $n+1$ *points* $\{(x_j, y_j)\}$ *where* $j = 0, 1, 2....n$ *such that* $x_m \neq x_n \quad if \quad m \neq n$. *The* Lagrange polynomial *or* Lagrange interpolation formula *of these points is the lowest degree polynomial* $\mathcal{L}_n$ *such that*

$$\mathcal{L}_n(x_j) = y_j \quad for \quad j = 0, 1, ..., n$$

Such a polynomial is given by

$$\begin{aligned} \mathcal{L}_n(x) &= \sum_{j=0}^{n} y_j l_j(x) \\ l_j(x) &= \sum_{\substack{0 \leq m \leq n \\ m \neq j}} \frac{x - x_m}{x_j - x_m} \end{aligned} \tag{1}$$

where $l_j(x)$ *are* lagrange basis polynomials.

Consider an smooth function f to be integrated over $[a, b]$, approximate it via Lagrange interpolating polynomial at $n+1$ points $\{(x_i, f(x_i))\}$ for $i = 0, 1..., n$ where $x_i = a + i\frac{b-a}{n}$.

$$f(x) = \sum_{j=0}^{n} f(x_j) l_j(x) + E(f)$$

where the $E(f)$ is the error due to interpolation of integrand.
Now, integrating over $[a, b]$

$$\int_a^b f(x)dx \approx \int_a^b \sum_{j=0}^{n} f(x_j) l_j(x) dx = \sum_{j=0}^{n} f(x_j) \int_a^b l_j(x) dx$$

$$\implies \int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{2}$$

where $w_i = \int_a^b l_i(x)dx$.

Here we have taken Lagrange interpolating polynomial at n+1 points. We can get different Newton–Cotes quadrature formula by taking a different number of points. For instance, by taking 2 points, we get the Trapezoidal Rule.

1.2. Trapezoidal Rule. *Consider integral of a smooth function $f(x)$ over interval $[a, b]$. By taking 2 points $(a, f(a))$, $(b, f(b))$ and interpolating by Lagrange polynomial. we get,*

$$\mathcal{L}_1(x) = l_0(x)f(a) + l_1(x)f(b)$$

$$\textit{where } l_0(x) = \tfrac{x-x_1}{x_0-x_1} \qquad l_1(x) = \tfrac{x-x_0}{x_1-x_0}$$

weights are

$$w_0 = \int_a^b l_0(x)dx = \int_a^b \frac{x-x_1}{x_0-x_1}dx = \frac{b-a}{2}$$

$$w_1 = \int_a^b l_1(x)dx = \int_a^b \frac{x-x_0}{x_1-x_0}dx = \frac{b-a}{2}$$

by putting the weights we get ,

$$\mathcal{T}^1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{3}$$

and as $\mathcal{L}_1(x) \approx f(x)$

$$\mathcal{T}^1(f) \approx \int_a^b f(x)dx$$

Equation-(3) is known as Trapezoidal Rule.

However, taking Higher Degree Lagrange polynomials can cause Runge's phenomenon [1] which will reduce the accuracy of the Quadrature rule constructed from Higher degree Lagrange polynomials. To avoid such, we avoid taking higher Degree Lagrange polynomial. We divide the intervals into sub-intervals, and in each sub-intervals, we take an interpolating polynomial of small finite Degree. Quadrature rules constructed from such a process is called composite quadrature rules.

1.3. Composite Trapezoidal Rule. *To get Higher order approximation from Trapezoidal Rule ,we can partition $[a, b]$ into uniform n sub-intervals*

$$[x_i, x_{i+1}]_{i=1,2,\cdots,n} \quad , \textit{where } h = \frac{b-1}{n+1}, \; x_i = a + ih$$

and apply the trapezoidal rule to each sub-intervals,i.e

$$\mathcal{T}^n(f) = \sum_{k=1}^{n+1} \frac{x_k - x_{k-1}}{2} \left[f(x_{k-1}) + f(x_k) \right] \tag{4}$$

As Lagrange interpolating polynomial are interpolation of $f(x)$.

$$\mathcal{T}^{n-1}(f) \approx \int_a^b f(x)dx$$

Equation-(4) is known as Composite Trapezoidal Rule.

1.4. Order of Convergence. *An sequence of quadrature rule $\{Q_n\}_{n\geq 0}$ is said to converge to a point α with* order of convergence *p if*

$$|\alpha - Q_{n+1}| \leq C|\alpha - Q_n|^p \qquad , n \geq 0$$

for some $C > 0$. Smallest C at which equality holds is called **Rate of Convergence.** or **Asymptotic error constant**

1.5 Example. Consider a function $f(x) = x^3 + sin(x)$ over interval $[0, 1]$,by analytical integration we get

$$I(f) = \int_0^1 [x^3 + sin(x)]dx = 7.096976941318602 \cdot 10^{-1} \tag{5}$$

using the n-point Trapezoidal rule, .we get

N	$\mathcal{T}^N(f)$	Error	Relative Error
4	$7.22925937571502 \cdot 10^{-1}$	$1.32282434396419 \cdot 10^{-2}$	1.00
8	$7.13005223491722 \cdot 10^{-1}$	$3.30752935986134 \cdot 10^{-3}$	$2.50 \cdot 10^{-1}$
16	$7.10524605712215 \cdot 10^{-1}$	$8.26911580354528 \cdot 10^{-4}$	$6.25 \cdot 10^{-2}$
32	$7.09904423853836 \cdot 10^{-1}$	$2.06729721975552 \cdot 10^{-4}$	$1.56 \cdot 10^{-2}$
64	$7.09749376676525 \cdot 10^{-1}$	$5.16825446643665 \cdot 10^{-5}$	$3.90 \cdot 10^{-3}$
128	$7.09710614775162 \cdot 10^{-1}$	$1.29206433016060 \cdot 10^{-5}$	$9.76 \cdot 10^{-4}$
256	$7.09700924293132 \cdot 10^{-1}$	$3.23016127146136 \cdot 10^{-6}$	$2.44 \cdot 10^{-4}$
512	$7.09698501672206 \cdot 10^{-1}$	$8.07540345815205 \cdot 10^{-7}$	$6.10 \cdot 10^{-5}$
1024	$7.09697896016948 \cdot 10^{-1}$	$2.01885088202403 \cdot 10^{-7}$	$1.52 \cdot 10^{-5}$

Table 1.1
Error report of Trapezoidal rule approximation for Integral-(5)

Here $N = n - 1$, $\mathcal{T}^N(f)$ is Integral value calculated via n-point Trapezoidal rule. Error is the absolute difference in values between analytic Integration and n-point Trapezoidal rule approximation ($\mathcal{T}_N$). We get an approximation of $I(f)$ as required by the n-point Trapezoidal rule. Also, we can see that with an increase in quadrature points, accuracy increases.

We have defined the Trapezoidal rule and shown its applications for the Approximation of Integrals involving smooth functions. In the next chapter, we will extend the Trapezoidal rule to approximate certain classes of singular functions while also improving its order of convergence.

2. SINGULAR INTEGRATION & PRE-CORRECTED TRAPEZOIDAL RULE

Consider an function that is not as smooth as required by Trapeziodal rules. For example if we take $f(x) = \log(1/x)$ to be integrated over interval $[0, 1]$ as $f(x)$ is not smooth at $x = 0$ and as a result Trapezoid quadrature won't converge to High order. This is because, For optimal via trapezoid rule, we require integrand to be twice continuously differentiable [2] ,whereas $f(x) = \log(\frac{1}{x})$ doesn't even have first continuous derivative over the interval $[0, 1]$.

2.1. Singularity. *are points/regions at which an mathematical object is not defined or takes an infinite value.*

2.2. Singular integrand. *are functions containing singularities in the interval to be integrated, and the specific point at which Singularity integrand does not have value is called* **Points of Singularity**.

For instance $f(x) = \log(\frac{1}{x})$ is a singular functions with $x = 0$ as one of its point of Singularity. This chapter focuses on various integration techniques to deal with singular functions.

2.1 Change of variable

Change of variable [1] is a prevalent technique to deal with the integral of the form.

$$I(f) = \int_a^b f(x)dx$$

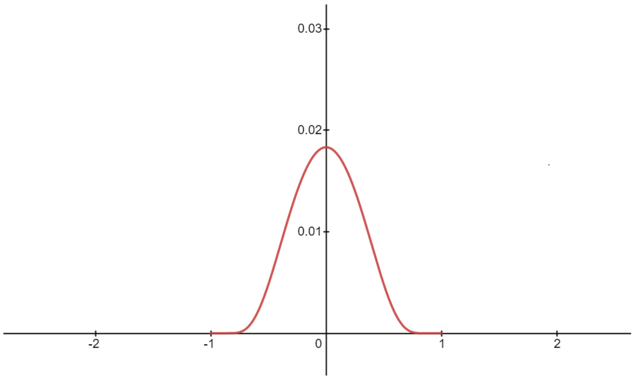
Where function $f(x)$ is smooth over the open interval (a, b), but has a Singularity point at one or both endpoints. We can rid of the Singularity by introducing the change of variable i.e

$$x = \phi(t), \quad \phi(t) = a + \frac{b-a}{\gamma}\Psi(t), \quad \Psi(t) = \int_{-1}^{t} \psi(u)du \tag{6}$$

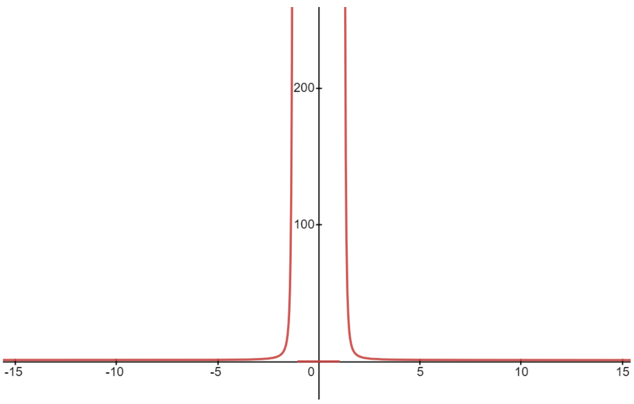
where

$$\psi(t) = \exp\left(\frac{-c}{1-t^2}\right) \quad \text{and} \quad \gamma = \Psi(1)$$

Here, $c > 0$ is arbitrary.



Graph of $\psi(t)$ showing it's behavior as $t \to \pm 1$ over smaller y axis units
Fig 2.1



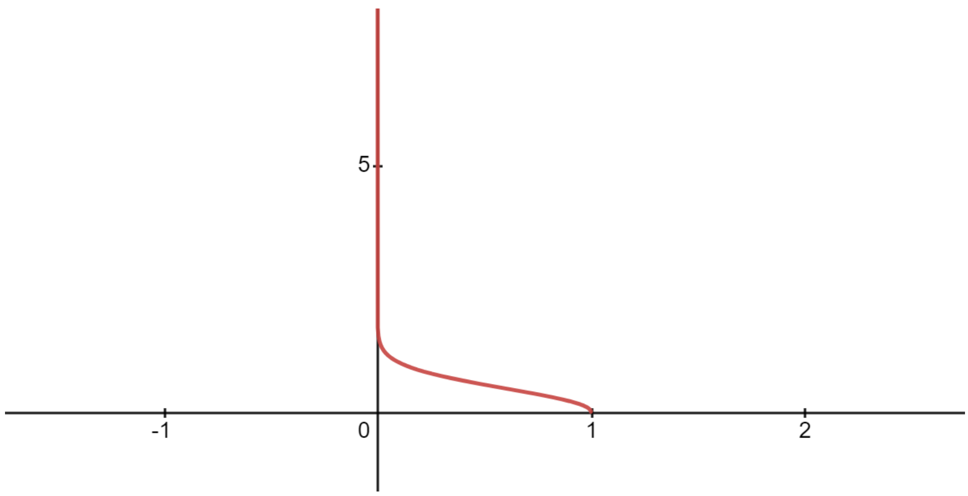
Graph of $\psi(t)$ showing it's behavior as $t \to \pm 1$ over larger y axis units
Fig 2.2

Here ψ is constructed to be smooth at $t = \pm 1$. As the argument in the exponential goes to $-\infty$ as $t \to \pm 1$ as shown in Fig 2.1 and Fig 2.2. Hence ψ and it's derivatives will vanish at the endpoints. Thus ψ and therefore ϕ will be very smooth at endpoints ,which will compensate for almost any non-smooth behavior in f. Also change in behavior from x to t will map original interval $[a, b]$ to $[-1, 1]$. Now our new integral becomes

$$I(f) = \int_a^b f(x)dx = \int_{-1}^1 f(\phi(t))\phi'(t)dt \tag{7}$$

2.3 Example. Take integral of function $f(x) = \sqrt{-log(x)}$ over the interval $[0, 1]$.

$$I(f) = \int_0^1 \sqrt{-log(x)}.dx$$



Graph of $\sqrt{-log(x)}$
Fig 2.3

If we try to approximate its integral by using the n-point Trapezoidal rule. We do not get any helpful result as $f(x)$ has a singularity at one of its endpoints at $x = 0$ as shown in Fig 2.3, and the Trapezoidal rule depends on endpoints. To overcome these problems, one can use the change of variables formula as defined in equation-6, and then our integration will change as per equation-7.
By using change of variables (equation-6), our Integral becomes

$$I(f) = \int_0^1 \sqrt{-log(x)}dx = \int_{-1}^1 \sqrt{-log(\phi(t))}\phi'(t)dt \tag{8}$$

If we use the n-point Trapezoidal rule via the change of variables Integral as defined above, then.

N	$\mathcal{T}_n(\sqrt{-log(x)})$	Error	Relative Error
4	$8.92023451508737 \cdot 10^{-1}$	$5.79652605597925 \cdot 10^{-3}$	1.00
8	$8.85813283262624 \cdot 10^{-1}$	$4.13642190133623 \cdot 10^{-4}$	$7.13 \cdot 10^{-2}$
16	$8.86218508617344 \cdot 10^{-1}$	$8.41683541430438 \cdot 10^{-6}$	$1.45 \cdot 10^{-3}$
32	$8.86226921208637 \cdot 10^{-1}$	$4.24412083255277 \cdot 10^{-9}$	$7.32 \cdot 10^{-7}$
64	$8.86226930386919 \cdot 10^{-1}$	$4.93416141278402 \cdot 10^{-9}$	$8.51 \cdot 10^{-7}$

Table 2.1
Error report of Trapezoidal rule approximation with Change of variables formula for Integral-(8)

Here $N = n - 1$, $\mathcal{T}_n(\sqrt{-log(x)})$ is Integral value calculated via (n+1)-point Trapezoidal rule. Error is the absolute difference in values between analytic Integation and Trapezoidal rule approximation $\mathcal{T}_n$. We get an approximation of $I(f)$ as required by (n+1)-point Trapezoidal rule by using change of variable formula (equation-6).

2.2 Analytic treatment of Singularity

Consider Integral of the form.

$$I(f) = \int_a^b f(x)dx$$

divide the interval $[a, b]$ into parts, one containing the singular point and treat it analytically, i.e. if $h \in [a, b]$ is the singular point, then

$$I(f) = \int_a^b f(x)dx = \left(\int_a^{h-\epsilon} + \int_{h-\epsilon}^{h+\epsilon} + \int_{h+\epsilon}^{b} \right) f(x)dx = I_1 + I_2 + I_3$$

where

$$I_1 = \int_a^{h-\epsilon} f(x)dx, \quad I_2 = \int_{h-\epsilon}^{h+\epsilon} f(x)dx \quad \text{and} \quad I_3 = \int_{h+\epsilon}^{b} f(x)dx$$

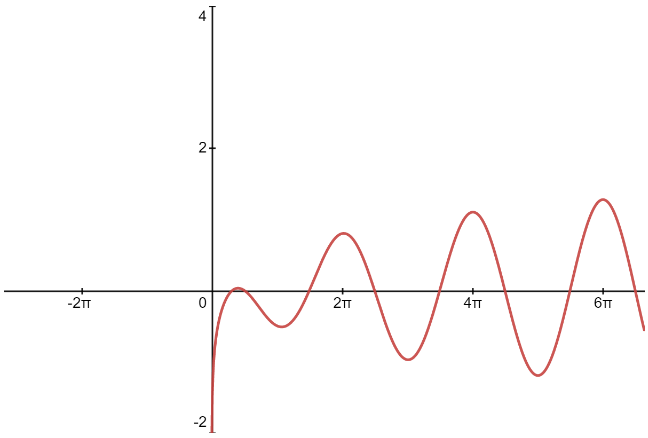
we can use any standard quadrature rules on I_1 and I_3 as $f(x)$ is smooth in $[a, h - \epsilon]$ and $[h + \epsilon, b]$. Coming to I_2 , assuming $f(x)$ has a convergent Taylor series, $f(x) = \sum_{j=0}^{\infty} a_j x^j$. We can evalute by replacing $f(x)$ by first few terms of its taylor expansion ,to get fairly accurate results.

$$I_2 = \int_{h-\epsilon}^{h+\epsilon} f(x)dx = \int_{h-\epsilon}^{h+\epsilon} \sum_{j=0}^{\infty} a_n x^j dx \approx \int_{h-\epsilon}^{h+\epsilon} \sum_{j=0}^{k} a_n x^j dx$$

for some k , as

$$k \to \infty \qquad \Longrightarrow \qquad \int_{h-\epsilon}^{h+\epsilon} \sum_{j=0}^{k} a_n x^j dx \to I_2$$

2.4 Example. Consider the function $f(x) = cos(x)log(x)$ having singularity at $x = 0$ as shown in Fig 2.4, to be integrated over interval $[0, 4\pi]$



Graph of $cos(x)log(x)$
Fig 2.4

$$\mathcal{I} = \int_0^{4\pi} cos(x)log(x)dx = \left(\int_0^{.1} + \int_{.1}^{4\pi}\right) cos(x)log(x)dx$$
$$= I_1 + I_2$$

$$I_1 = \int_0^{k} cos(x)log(x)dx \qquad\qquad k = 0.1$$
$$= k\left[log(k) - 1\right] - \frac{k^3}{6}\left[log(k) - \frac{1}{3}\right] + \frac{k^5}{600}\left[log(k) - \frac{1}{5}\right] - ....$$

we can get a very accurate value of I_1 by taking the first three terms only. I_2 can be calculated by standard quadrature rules or analytically.

2.3 Error in Trapezoidal rule

2.5. Bernoulli polynomials *$\mathcal{B}(x)$ are defined as*

$$\mathcal{B}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{n-k}(x^k) \tag{9}$$

for $n \geqslant 0$ with $\mathcal{B}_0(x) = 1$, with properties

- $\mathcal{B}_1(x) = x - \frac{1}{2}$,
- $\mathcal{B}'(x) = n\mathcal{B}_{n-1}(x)$ *for* $n \geqslant 2$
- $\mathcal{B}(0) = \mathcal{B}_n(1) = 0$ *for* $n = 3, 5, 7..$

also, Bernoulli polynomials can be generated by a generating function given by Euler, i.e

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x) \frac{t^n}{n!}$$

at $x = 0$, it becomes

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n \frac{t^n}{n!}$$

2.6. Euler-Maclaurin Theorem. *Let a, b with $a < b$ be two real number and let $p \geq 1$ be an integer. Then for function $f \in \mathcal{C}_p[a, b]$, there exist a real number ξ in (a, b) such that*

$$\int_a^b f(x)dx - \mathcal{T}_n(f) = \sum_{j=1}^{\lfloor p/2 \rfloor - 1} h^{2j} \frac{\mathcal{B}_{2j}}{2j!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] - E_n(f) \tag{10}$$

$$E_n(f) = \frac{h^p \mathcal{B}_p}{p!} f^{(p)}(\xi)$$

where $\mathcal{T}_n(f)$ is $n+1$ point Trapezoidal rule, $\mathcal{B}_j$ is jth Bernoulli number.

The error of the composite trapezoidal rule ,$\mathrm{E}_n(f)$ is the difference between the value of the analytical integral and the numerical result via trapezoidal rule.

$$\begin{aligned}
\mathrm{E}_n(f) &= \left| \int_a^b f(x)dx - \mathcal{T}_n(f) \right| \\
&= \left| \sum_{j=1}^{\lfloor p/2 \rfloor - 1} h^{2j} \frac{\mathcal{B}_{2j}}{2j!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] - \frac{h^p \mathcal{B}_p}{p!} f^{(p)}(\xi) \right|
\end{aligned}$$

2.7 Lemma. *Consider a periodic integrand* $f \in \mathcal{C}_p[0,T]$ *having period* T. *Then* $f(0) = f(T)$. *Then error in integration of* f *over interval* $[0,T]$ *will be given by:*

$$\mathrm{E}_n(f) = \left| \sum_{j=1}^{\lfloor p/2 \rfloor -1} h^{2j} \frac{\mathcal{B}_{2j}}{2j!} \left[f^{(2j-1)}(T) - f^{(2j-1)}(0) \right] - \frac{h^p \mathcal{B}_p}{p!} f^{(p)}(\xi) \right|$$

A derivative of a periodic function is always periodic with the same period. $f(T) = f(0) \implies f^{(j-1)}(T) = f^{(j-1)}(0)$.

$$\mathrm{E}_n(f) = \left| -\frac{h^p \mathcal{B}_p}{p!} f^{(p)}(\xi) \right|$$

$$\mathcal{T}_n(f) = \int_0^T f(x)dx + \mathcal{O}(h^p)$$

Trapezoidal rule converges to high order for smooth and periodic functions as illustrated above, whereas, for non-periodic functions approximations, we have to do Endpoints (Boundary) corrections to get high order convergence.

2.4 Endpoints correction

Most of the results of this section are taken from [4, 5]. Before we get to the Endpoint correction, we need the following results for its construction.

2.8 Theorem (Lagrange Interpolation for Equally-Spaced abscissa)**.** *Suppose* a, b *are a pair of real number such that* $a < b$, $m \geq 3$ *be an integer and* $h = \frac{b-a}{m-1}$.
let $f \in C^m[a - mh, b + mh]$ *and equispaced points* x_k *be defined as* $x_k = \frac{b-a}{2} + kh$. *Then, for any real number* p *there exists a real number* ξ, $-mh < \xi < mh$, *such that*

$$f(x_0 + ph) = \sum_k \mathcal{A}_k^m f(x_k) + \mathcal{R}_{m-1,p} \tag{11}$$

where k *varies from*

$$-\frac{1}{2}(m-2) \leq k \leq \frac{1}{2}m \quad \textit{for even } m$$

$$-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1) \quad \textit{for odd } m$$

with

$$\mathcal{A}_k^m(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \sum_t (p-t)$$

and

$$\mathcal{R}_{m-1,p} = \frac{h^m}{m!} f^{(m)}(\xi) \sum_n (p-n)$$

here both t and n varies same as k.

The proof of this theorem can be found in Appendix A.

2.9 Lemma. *Let $f;[a,b] \to \mathbb{R}$ is a function satisying the conditions of theorem 2.8 , and the coefficents $\mathcal{D}_{i,k}^m$ are defined as*

$$\mathcal{D}_{i,k}^m = \frac{\partial^{2i-1}}{\partial^{2i-1}} \left[\mathcal{A}_k^m(p)\right]^{p=0}$$

where $\mathcal{A}_k^m(p), k, m$ is as same as defined in theorem 2.8 and i is a positive integer.
then

$$f^{(2i-1)}(x_0) = \sum_{k=\frac{-(m-1)}{2}}^{\frac{m-1}{2}} \frac{\mathcal{D}_{i,k}^m}{h^{2i-1}} f(x_k) + \mathcal{O}(h^m) \tag{12}$$

2.10 Theorem. *Suppose m, l, k are integers and coefficients $a_{k,l}^m$ are defined by recussive relations*

$$a_{1,1}^3 = 1 \tag{13a}$$
$$a_{1,2}^3 = 1 \tag{13b}$$
$$a_{k,l}^{2k+1} = (k-k^2)a_{k-1,l}^{2k-1} + a_{k-1,l}^{2k-1} + a_{k-1,l-2}^{2k-1} \tag{13c}$$
$$a_{k,l}^{m+2} = a_{k,l-2}^m - \big(\frac{m+1}{2}\big)^2 a_{k,l}^m \tag{13d}$$

with $a_{k,l}^m = 0 \ \forall \ k \leq 0$ or $l \leq 0$ or $m \leq 1$.
then

$$\mathcal{A}_k^m(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{\big(\frac{m-1}{2}+1\big)!\big(\frac{m-1}{2}-1\big)!} \sum_{l=1}^{\frac{m-1}{2}} a_{k,l}^m p^l$$

The following lemma is a corollary of this theorem.

2.11 Lemma. *Let $m \geq 3$ be an odd integer.Then,*

$$\mathcal{D}_{i,k}^{m} = \frac{(-1)^{\frac{m-1}{2}+k}}{\left(\frac{m-1}{2}+1\right)!\left(\frac{m-1}{2}-1\right)!} a_{k,2i-1}^{m}(2i-1)!$$

for any k, i such that $-\frac{m-1}{2} \geq k \geq \frac{m-1}{2}$, and $1 \geq i \geq \frac{m-1}{2}$, with the coefficients $a_{k,l}^{m}$ defined by the recurrence relation in Lemma 2.9 and $\mathcal{D}_{i,k}^{m}$ from lemma 2.9.

Consider a pair of integers n, m such tha $m \geq 3$ and odd ,while $n \geq 2$. Also let $a \leq b$ be a pair of real numbers , $h = \frac{b-a}{n-1}$, and $f : [a-mh, b+mh] \to \mathbb{R}^1$ be an integrable function. Then *Endpoint corrected Trapezoidal rule* $\mathcal{T}_{\alpha^m}^{n}$ (or $\mathcal{T}_{\alpha}^{n}$) for nonsingular functions is defined by formula:

$$\mathcal{T}_{\alpha^m}^{n} = \mathcal{T}^{n}(f) + h \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \big(f(b+kh) - f(b-kh)\big)\alpha_k^m \tag{14}$$

where correction coefficients α_k^m is defined as

$$\alpha_k^m = \sum_{l=1}^{\frac{m-1}{2}} \frac{\mathcal{D}_{i,k}^{m}\mathcal{B}_{2l}}{(2l)!} \tag{15}$$

where $\mathcal{D}_{i,k}^{m}$ are defined in 2.9 (and also 2.11) and $\mathcal{B}_{2l}$ are Bernoulli numbers.

2.12 Theorem. *Let $m \geq 3$ and $n \geq 2$ are a pair of integer such that m is an odd integer, then for any k such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$. Then*

$$|\alpha_k^m| < 1 \tag{16}$$

where the coefficients α_k^m are defined in 15.
Further, suppose $a < b$ be two real number. Then, the Endpoint corrected Trapeziodal rule $\mathcal{T}_{\alpha^m}^{n}$(given by 14) is of order m;i.e , for any $f \in \mathcal{C}^m[a-mh, b+mh]$, there exists $c > 0$ such that

$$\left|\mathcal{T}_{\alpha^m}^{n}(f) - \int_a^b f(x)dx\right| < \frac{c}{n^{m+1}} \tag{17}$$

The proofs of theorems 2.10 and 2.12 can be found Appendix A.

2.5 Correction for singular functions

Suppose a $\mathcal{C}^\infty$ function $g : (0,1] \to R$, defined as

$$g(x) = \tau(x)s(x) + \upsilon(x) \tag{18}$$

where $\upsilon(x), \tau(x) \in \mathcal{C}^k[a,b]$,$k \geq 1$ and $s(x) \in \mathcal{C}(0,1]$ an integrable function with a singularity at zero.
Consider real numbers a, b such that $a < 0$ and $b > 0$. Then, Trapeziodal rule can't be used to approximate functions like $g(x)$ over interval $[a,b]$. As $g(x)$ has singularity at 0 and depending on number of quadrature points taken, we may not get approximations.

For example consider $\int_{-\pi}^{\pi} e^{2cos(2x)+sin(3x)} log(\sqrt{2}(1 - cos(x)))$.

$$\mathcal{T}^{32}(e^{2cos(2x)+sin(3x)} log(\sqrt{2}(1 - cos(x)))) = \infty$$

Also, even for standard Endpoint-corrected Trapezoidal rule.

$$\mathcal{T}^{32}_{\alpha^m}(e^{2cos(2x)+sin(3x)} log(\sqrt{2}(1 - cos(x)))) = \infty$$

One way to overcome this via defining a new function $\overline{g}(x)$ as

$$\overline{g}(x) = \begin{cases} g(x) & : \quad x \neq 0 \\ 0 & : \quad x = 0 \end{cases}$$

and we define the punched endpoint corrected trapezoidal rule, as

$$\mathcal{T}^n_{0,\alpha^m}(g(x)) = \mathcal{T}^n_{\alpha^m}(\overline{g}(x)) \tag{19}$$

As $g(x)$ has singularity at 0, the punched endpoint corrected trapezoidal rule $\mathcal{T}^n_{0,\alpha^m}$ gives a low order approximation to the integrand (18). For a High order approximation, a proper Singularity correction is needed.

2.5.1 Right-end and left-end corrected Trapeziodal rule

We define Right-end corrected Trapeziodal rule $\mathcal{T}^n_{R\alpha^m}$ for any integral $\int_0^b f(x)dx$by the formula:

$$\mathcal{T}^n_{R\alpha^m}(f) = h\left[\sum_{i=1}^{n-2} f(x_i) + \frac{f(x_{n-1})}{2}\right]` + h\sum_{k=1}^{\frac{m-1}{2}}\left[f(b+kh) - f(b-kh)\right]\alpha^m_k$$

$$h = \frac{b}{n-1} \qquad , \qquad x_i = ih \tag{20}$$

2.13 Lemma. *Let $m \geq 3$ is an integer. Suppose $f \in \mathcal{C}^{m+1}[0, b+mh]$ such that $f(0) = f^{(1)}(0) = ... = f^{(m)} = 0$. Then, the Right-end corrected Trapeziodal rule $\mathcal{T}^n_{R\alpha^m}$ is of order m , i.e there exists a real number $c > 0$ such that:*

$$\left| \mathcal{T}^n_{R\alpha}(f(x)) - \int_0^b f(x)dx \right| < \frac{c}{n^{m+1}} \tag{20}$$

$\forall n = 1, 2, 3....$

Proof. This follows from Theorem-2.12 . □

Also we can define left-end corrected Trapeziodal rule $\mathcal{T}^n_{L\alpha}$ as

$$\mathcal{T}^n_{L\alpha^m}(f) = h\left[\sum_{i=1}^{n-2} f(x_{-i}) + \frac{f(x_{-(n-1)})}{2}\right]` + h\sum_{k=1}^{\frac{m-1}{2}}\left[f(-b+kh) - f(-b-kh)\right]\alpha^m_k \tag{21}$$

and also similar to Lemma-2.13, as a corollary of theorem-2.12. we say $\mathcal{T}^n_{L\alpha^m}$ has left-end order of $m \geq 3$ if for any $f \in \mathcal{C}^{m+1}[-b-mh, 0)$ having $f(0) = f^{(1)}(0) = ... = f^{(m)} = 0$,satisfies

$$\left| \mathcal{T}^n_{L\alpha}(f(x)) - \int_{-b}^0 f(x)dx \right| < \frac{c}{n^m} \tag{22}$$

for $c > 0$ and $\forall n = 1, 2, 3....$.

2.5.2 Endpoints correction for singular functions

Suppose a $\mathcal{C}^\infty$ function $g : [-kh, b+mh] \to \mathbb{R}$, defined as

$$g(x) = \tau(x)s(x) + \upsilon(x) \tag{23}$$

where $\upsilon(x), \tau(x) \in \mathcal{C}^k[-kh, b+mh]$ and $s(x) \in \mathcal{C}[-b, 0) \cup (0, b]$ an integrable function with a singularity at zero. For a finite sequence $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$. Defining the mapping $\mathcal{T}^n_{\alpha,\beta}$ as

$$\begin{gathered} \mathcal{T}^n_{\alpha,\beta}(g) = \mathcal{T}^n_{R\alpha}(g) + h\sum_{j=-k, j\neq 0}^{k} \beta_j f(x_j) \\ h = \frac{b}{n-1} \quad , \quad x_j = jh \end{gathered} \tag{24}$$

This mapping is the expression for **Endpoint corrected Trapeziodal rule for Singular function** if it has order of k i.e if it satisfies

$$\left| \mathcal{T}^n_{\alpha,\beta}(g(x)) - \int_0^b g(x)dx \right| < \frac{c}{n^k} \tag{25}$$

For any constant $c > 0$. Here we have taken interval $(0, b]$, but it can be used for any interval. We can generate β from given α (which can calculated via construction given in section-2.4) which has right-end order of k.

For a pair of natural numbers k, n with $k \geq 1$ and $m \geq 3$. Consider the following system of linear equations with respect to the unknowns $\{\beta^n_j\}$ with $j = \pm 1, \pm 2..., \pm k$
For $i = 1, 2, 3...., k$.

$$\sum_{j=-k, j\neq 0}^{k} x_j^{i-1} \beta^n_j = \frac{1}{h}\left[\int_0^b x^{i-1}dx - \mathcal{T}^n_{R\alpha^m}(x^{i-1}) \right] \tag{26}$$

and, for i =k+1,k+2,...,2k equations are:

$$\sum_{j=-k, j\neq 0}^{k} x_j^{i-k-1} s(x_j) \beta^n_j = \frac{1}{h}\left[\int_0^b x^{i-k-1}s(x)dx - \mathcal{T}^n_{R\alpha^m}(x^{i-k-1}s(x)) \right] \tag{27}$$

By using this linear system (formed by equations 26 ,27) a matrix can be construced, denoting it by $\mathbb{A}^{nk}_s$, its right side by $\mathbb{Y}^{nk}_s$ and its solution by $\beta_n = (\beta^n_{-k}, \beta^n_{-(k-1)}, ..., \beta^n_{(k-1)}, \beta^n_k)$.

2.14 Theorem. *Suppose that the function $g : [-kh, b+mh] \to \mathbb{R}^1$ as defined by 23. Furthermore suppose the linear system formed by equation 26 and 26 have solutions $\beta_n = (\beta^n_{-k}, \beta^n_{-(k-1)}, ..., \beta^n_{(k-1)}, \beta^n_k)$*
for all sufficiently large n, and that the sums

$$\sum_{j=1}^{2k} (\beta^n_j)^2 \tag{28}$$

are bounded uniformly with respect to n. Then there exists a real $c > 0$ such that

$$\left| \mathcal{T}^n_{\alpha^n,\beta^m}(g) - \int_0^b g(x)dx \right| < \frac{c}{n^k} \tag{29}$$

for all sufficiently large n.

The use of $\mathcal{T}^n_{\alpha^n,\beta^m}$ (24) as Endpoint corrected Trapeziodal rule for Singular function of form (23) is an collorary of the previous theorem.

Remark. *Note that the weights at, $\alpha_1, \alpha_2, \cdots, \alpha_m$ in equation* (14) *are independent of $s(x)$, since the function $g(x)$ (from* (23)*) is non-singular at $x = 1$. Thus, the expression $h \sum \big(f(b + kh) - f(b - kh)\big)\alpha_k^m$ is a standard end-point correction to the trapezoidal rule. On the other hand, the expression $h \sum \beta_j f(x_j)$ can be viewed as an endpoint correction to a function singular at the end where the correction is being applied. Thus, the coefficients $\beta_1, \beta_2, \cdots, \beta_m$ do depend on the function $s(x)$. So Coefficents $\alpha_1, \alpha_2, \cdots, \alpha_m$ don't differ for different $s(x)$ where as $\beta_1, \beta_2, \cdots, \beta_m$ differ for different $s(x)$[5].*

Remark. *Now that we have constructed Endpoint corrected Trapezoidal rule for Singular function i.e $\mathcal{T}^n_{\alpha,\beta}$. By looking its expression (equation-*(24)*) it can be seen that the values of constants α , β must be "precomputed" before its use for approximation. As a result, Both Endpoint corrected Trapezoidal rule for Singular function, and non-singular functions are called "Precorrected Trapezoidal rule".*

2.5.3 Singularites of the forms $|x|^\lambda$ and $log(|x|)$

Consider functions $\phi_1(x), \phi_2(x), ..., \phi_{2k}(x) : [-kh, b + mh] \in \mathbb{R}^1$ defined by the formulae

$$\phi_i(x) = \begin{cases} x^{i-1} & \text{for } i = 1, 2, ..., k \\ x^{i-k-1}s(x) & \text{for } i = k+1, k+2, ..., 2k \end{cases} \tag{30}$$

where $s(x)$ is as defined in (23).

2.15 Lemma. *if $s(x) = x^\lambda$ where $\lambda \in (-1, 1)/\{0\}$, then the functions $\phi_1(x), \phi_2(x), ..., \phi_{2k}$ constitute a Chebyshev system on the interval $[-kh, b + mh]$ (i.e the determinent of the $2k \times 2k$ matrix B_{ij} defined by the formula $B_{ij} = \phi_i(t_j)$ which is nonzero for any 2k distinct points on the interval $[-kh, b + mh]$)*

One of the important result which follows from lemma-2.15 is that the matrix of linear equations-26,27 is non-singular.

2.16 Theorem. *For funtion of type (23) ,if $s(x) = |x|^\lambda$ with $0 < |\lambda| \leq 1$ and $m > k$, then the convergence rate of the quadrature rule $\mathcal{T}^n_{\alpha^n,\beta^m}(g)$ is at k.*

Proof. From lemma-2.15 it follows that the matrix of system-26,27 is non-singular.
taking the system26,27

$$\sum_{j=-k,j\neq 0}^{k} x_j^{i-1}\beta_j^n = \frac{1}{h}\left[\int_0^b x^{i-1}dx - \mathcal{T}_{R\alpha^m}^n(x^{i-1})\right]$$

$$\sum_{j=-k,j\neq 0}^{k} x_j^{i-k-1}s(x_j)\beta_j^n = \frac{1}{h}\left[\int_0^b x^{i-k-1}s(x)dx - \mathcal{T}_{R\alpha^m}^n(x^{i-k-1}s(x))\right]$$

now we multiply the i^{th} equation by $\frac{1}{h^{i-1}}$ for $i = 1, 2, \cdots, k$ and by $\frac{1}{h^{i-1-k+\lambda}}$ for $i = k+1, k+2, \cdots, 2k$ obtaining the system of equations
For $i = 1, 2, \cdots, k$

$$\sum_{j=-k,j\neq 0}^{k} j^{i-1}\beta_j^n = \frac{1}{h^i}\left[\int_0^b x^{i-1}dx - \mathcal{T}_{R\alpha^m}^n(x^{i-1})\right] \tag{31}$$

and for $i = k+1, k+2, \cdots, 2k$

$$\sum_{j=-k,j\neq 0}^{k} j^{i-k-1+\lambda}\beta_j^n = \frac{1}{h^{i-k+\lambda}}\left[\int_0^b x^{i-k-1+\lambda}dx - \mathcal{T}_{R\alpha^m}^n(x^{i-k-1+\lambda})\right] \tag{32}$$

Denoting the matrix formed by linear system of equations-31 and 32 by $\mathbb{B}_k$ and its righ-hand side by $\mathbb{Z}_k^n$. Here $\mathbb{B}_k$ is independent of n. The rapid convergence of right-hand sides of 31 and 32 is assured by lemma-2.13.
If $m > k$, then $|\mathbb{Z}_k^n|$ is bounded uniformly w.r.t to n. Now, due to Theorem-2.14 the convergence rate of $\mathcal{T}_{\alpha^n,\beta^m}^n$ is at atleast k.

□

Now, as we have Constructed the "Pre-corrected" Trapezoidal rule, the next chapter focuses on its applications to solve real-world problems.

3. APPLICATIONS OF PRE-CORRECTED TRAPEZOIDAL RULE

3.1 Nyström method

A Fredholm integral equation of the second kind is given as

$$\varphi(t) = f(t) + \int_a^b K(t,x)\varphi(x)dx \tag{33}$$

where $K(t,x)$ is the integral kernal and $f(t)$ is a function. The problem revolves around finding $\varphi(t)$.let (A) be an integral operator given by:

$$(A[\varphi])(x) = \int_a^b K(x,y)\varphi(y)dy \qquad x \in [a,b]$$

where K is a continuous integral kernel. Now let $\{\mathcal{Q}_n\}_{n\geq 0}$ be a sequence of quadrature rules. Approximating (A) by $\{\mathcal{Q}_n\}_{n\geq 0}$.

$$(A_n[\varphi])(x) = \sum_{k=1}^{n} w_k K(x,x_k)\varphi(x_k) \qquad x \in [a,b]$$

where $x_k = a + k\frac{b-a}{n}$, w_k are weights of quadrature rule $\mathcal{Q}_n$.
By using (A) operator, Fredholm integral equation of the second kind equation-(33) can be written as

$$\varphi - A\varphi = f$$

Furthermore, by using $\{\mathcal{Q}_n\}_{n\geq 0}$ be a sequence of quadrature rules. The solution to the Fredholm integral equation of the second kind is approximated by

$$\varphi_n - A_n\varphi_n = f \tag{34}$$

For $x_j, j = 0, 1..., n-1$ with $x_j = a + \frac{b-a}{n}j$, equation-(34) can be written as:

$$\varphi(x_j) - \sum_{k=1}^{n} w_k K(x_j, x_k)\varphi(x_k) = f(x_j)$$

expanding the summation

$$\implies \varphi(x_j) - w_1 K(x_j, x_1)\varphi(x_1) - ...... - w_n K(x_j, x_n)\varphi(x_n) = f(x_j)$$

$$\implies -w_1 K(x_j, x_1)\varphi(x_1) - ...... - w_j K(x_j, x_j)\varphi(x_j) + \varphi(x_j)... \\ ... - w_n K(x_j, x_n)\varphi(x_n) = f(x_j)$$

$$\implies -w_1 K(x_j, x_1)\varphi(x_1) - ...... - \left[w_j K(x_j, x_j) - 1\right]\varphi(x_j)... \\ ... - w_n K(x_j, x_n)\varphi(x_n) = f(x_j)$$

For $j = 0, 1, 2, ..., n-1$
$\left[-(w_1 K(x_1, x_1) - 1)\varphi(x_1)\right] ...... - w_n K(x_j, x_n)\varphi(x_n) = f(x_1)$.
.
.
.
$-w_1 K(x_j, x_1)\varphi(x_1) - ...... \left[-(w_j K(x_j, x_j) - 1)\varphi(x_j)\right] ...... - w_n K(x_j, x_n)\varphi(x_n) = f(x_j)$.
.
.
.
$-w_1 K(x_j, x_1)\varphi(x_1) - ...... \left[-(w_n K(x_n, x_n) - 1)\varphi(x_n)\right] = f(x_n)$
These equations form a linear system, which can be represented by:

$$\begin{bmatrix} 1 - w_1 K(x_1, x_1) & .... & -w_n K(x_1, x_n) \\ . & & \\ . & & \\ -w_1 K(x_n, x_1) & .... & 1 - w_n K(x_n, x_n) \end{bmatrix} \begin{bmatrix} \varphi(x_1) \\ . \\ . \\ \varphi(x_n) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ . \\ . \\ f(x_n) \end{bmatrix}$$

We can get $\varphi(x_i)$ values by

$$\begin{bmatrix} \varphi(x_1) \\ . \\ . \\ \varphi(x_n) \end{bmatrix} = \begin{bmatrix} 1 - w_1 K(x_1, x_1) & .... & -w_n K(x_1, x_n) \\ . & & \\ . & & \\ -w_1 K(x_n, x_1) & .... & 1 - w_n K(x_n, x_n) \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ . \\ . \\ f(x_n) \end{bmatrix}$$

3.1 Theorem. *For a uniquely solvable Fredholm integral equation of the second kind with a continuous kernel and a continuous right-hand side, the Nystrom method with a convergent sequence of quadrature formulas is uniformly convergent.*
Proof:

$$\|(A - A_n)\varphi\|_\infty = \max_{x\in[a,b]} \left[\int_a^b K(x,y)\varphi(y)dy - \sum_{k=1}^{n} w_k K(x,x_k)\varphi(x_k) \right]$$

as the sequence of quadrature formulas are uniformly convergent, as $n \to \infty$ $\left[\int_a^b K(x,y)\varphi(y)dy - \sum_{k=1}^{n} w_k K(x,x_k)\varphi(x_k)\right] \to 0.$

3.2 Example. Consider the integral equation.

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)} \quad ; \quad x \in [0.1] \qquad (35)$$

Using Nyström method by Trapezoidal rule, we get , value of $\varphi(x)$ for $x = 0, 0.25, 0.5, 0.75, 1.0$

N	x =0	x=0.25	x=0.5	x=0.75	x=1.0
4	0.007146	0.008878	0.010816	0.013007	0.015479
8	0.001788	0.002224	0.002711	0.003261	0.003882
16	0.000447	0.000556	0.000678	0.000816	0.000971
32	0.000112	0.000139	0.000170	0.000204	0.000243

Table 3.1
Error report of Nyström method solved via Trapezoidal rule for Integral-(35)

3.2 Correction weights

3.2.1 Quadrature weights α_k^m for nonsingular functions

Here we list Quadrature weights α_k^m for Endpoint corrected Trapeziodal rule for nonsingular functions, $\mathcal{T}_{\alpha_k^m}$ (see 14 in chapter-2). Equispaced endpoint corrections transform the trapezoidal rule into a high-order quadrature for functions with several continuous derivatives. The quadrature rule $\mathcal{T}_{\alpha_k^m}$ is given by the formula:

$$\mathcal{T}^n_{\alpha^m} = \mathcal{T}^n(f) + h \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \big(f(b+kh) - f(b-kh)\big)\alpha^m_k \tag{36}$$

m=3	m=5	m=9
$0.4166666666666667 \cdot 10^{-1}$	$0.5694444444444444 \cdot 10^{-1}$	$0.6965636022927689 \cdot 10^{-1}$
	$-0.7638888888888889 \cdot 10^{-2}$	$-0.1877177028218695 \cdot 10^{-1}$
		$0.3643353174603175 \cdot 10^{-2}$
		$-0.3440531305114639 \cdot 10^{-03}$

m=17	m=33	m=43
$0.7836226334784645 \cdot 10^{-1}$	$0.8356586223906431 \cdot 10^{-1}$	$0.8490582345073516 \cdot 10^{-1}$
$-0.2965891540255508 \cdot 10^{-1}$	$-0.3772568901686391 \cdot 10^{-1}$	$-0.4001723785254229 \cdot 10^{-1}$
$0.1100166460634853 \cdot 10^{-1}$	$0.1891730418359046 \cdot 10^{-1}$	$0.2156339227395192 \cdot 10^{-1}$
$-0.3464763345380610 \cdot 10^{-2}$	$-0.9296840793733075 \cdot 10^{-2}$	$-0.1173947578371037 \cdot 10^{-1}$
$0.8560837610996298 \cdot 10^{-03}$	$0.4266725355474016 \cdot 10^{-2}$	$0.6165108551649839 \cdot 10^{-2}$
$-0.1531936403942661 \cdot 10^{-03}$	$-0.1781711570625946 \cdot 10^{-2}$	$-0.3051271143145484 \cdot 10^{-2}$
$0.1753039202853559 \cdot 10^{-04}$	$0.6648868875120770 \cdot 10^{-03}$	$0.1403005122150106 \cdot 10^{-2}$
$-0.9595026156320693 \cdot 10^{-6}$	$-0.2183589125884841 \cdot 10^{-3}$	$-0.5931791433462842 \cdot 10^{-3}$
	$0.6214890604453148 \cdot 10^{-4}$	$0.2286250628123645 \cdot 10^{-3}$
	$-0.1506576957395117 \cdot 10^{-4}$	$-0.7968542809070158 \cdot 10^{-4}$
	$0.3044582263327824 \cdot 10^{-5}$	$0.2490991825767152 \cdot 10^{-4}$
	$-0.4984930776384444 \cdot 10^{-6}$	$-0.6921164516465828 \cdot 10^{-5}$
	$0.6348092751221161 \cdot 10^{-7}$	$0.1691476513287747 \cdot 10^{-5}$
	$-0.5895566482845523 \cdot 10^{-8}$	$-0.3590633248885163 \cdot 10^{-6}$
	$0.3550460453274996 \cdot 10^{-9}$	$0.6517156577922871 \cdot 10^{-7}$
	$-0.1040273372883201 \cdot 10^{-10}$	$-0.9908863655077215 \cdot 10^{-8}$
		$0.1227209060809220 \cdot 10^{-8}$
		$-0.1188834746888414 \cdot 10^{-9}$
		$0.8447408532519018 \cdot 10^{-11}$
		$-0.3914655644778233 \cdot 10^{-12}$
		$0.8806394737861057 \cdot 10^{-14}$

Table 3.2
Quarature weights α^m_k

3.2.2 Quadrature weights β^m_k for singular functions

Given below are the values of the correction weights of the corrected trapezoidal rule $\mathcal{T}_{\alpha^n,\beta^m}$ (see (24) in chapter 2). This rule is used for the approximation of integrals of functions of the form:

$$g(x) = \tau(x)s(x) + \upsilon(x) \tag{37}$$

where $\upsilon(x), \tau(x) \in \mathcal{C}^k[-kh, b+mh]$ and $s(x) \in \mathcal{C}[-b,0) \cup (0,b]$ an integrable function with a singularity at zero. Endpoint corrected Trapeziodal rule for

Singular function , $\mathcal{T}^n_{\alpha,\beta}$ is given by:

$$\mathcal{T}^n_{\alpha,\beta}(g) = \mathcal{T}^n_{R\alpha}(g) + h \sum_{j=-k, j\neq 0}^{k} \beta_j f(x_j) \tag{38}$$

	$s(x) = log(x)$	$s(x) = x^{\frac{1}{2}}$	$s(x) = x^{\frac{-1}{2}}$
k=2			
-1	$0.7518812338640025 \cdot 10^0$	$0.4911169802967502 \cdot 10^0$	$0.1635135941723353 \cdot 10^1$
-2	$-0.6032109664493744 \cdot 10^0$	$-0.3176980828356269 \cdot 10^0$	$-0.1533115151360971 \cdot 10^1$
1	$0.1073866830872157 \cdot 10^1$	$0.7141080571189234 \cdot 10^0$	$0.2143719446940490 \cdot 10^1$
2	$-0.7225370982867850 \cdot 10^0$	$-0.3875269545800468 \cdot 10^0$	$-0.1745740237302873 \cdot 10^1$
k=6			
-1	$0.2051970990601252 \cdot 10^1$	$0.1265469280121926 \cdot 10^1$	$0.4710262208645700 \cdot 10^1$
-2	$-0.7407035584542865 \cdot 10^1$	$-0.3802563634358600 \cdot 10^1$	$-0.2025763995934342 \cdot 10^2$
-3	$0.1219590847580216 \cdot 10^2$	$0.5639024206133662 \cdot 10^1$	$0.3690977699143199 \cdot 10^2$
-4	$-0.1064623987147282 \cdot 10^2$	$-0.4569107975444730 \cdot 10^1$	$-0.3458675005305701 \cdot 10^2$
-5	$0.4799117710681772 \cdot 10^1$	$0.1943368974038607 \cdot 10^1$	$0.1646218520818186 \cdot 10^2$
-6	$-0.8837770983721025 \cdot 10^0$	$-0.3411137981342110 \cdot 10^0$	$-0.3167334195084358 \cdot 10^1$
1	$0.2915391987686506 \cdot 10^1$	$0.1878261417316043 \cdot 10^1$	$0.6026290938505443 \cdot 10^1$
2	$-0.8797979464048396 \cdot 10^1$	$-0.4649333971499730 \cdot 10^1$	$-0.2274216675280301 \cdot 10^2$
3	$0.1365562914252423 \cdot 10^2$	$0.6444550155059975 \cdot 10^1$	$0.3978973181300623 \cdot 10^2$
4	$-0.1157975479644601 \cdot 10^2$	$-0.5048462684259424 \cdot 10^1$	$-0.3656337403895339 \cdot 10^2$
5	$0.5130987287355766 \cdot 10^1$	$0.2104363245869803 \cdot 10^1$	$0.1720419649716102 \cdot 10^2$
6	$-0.9342187797694916 \cdot 10^0$	$-0.3644552148433214 \cdot 10^0$	$-0.3285178657691059 \cdot 10^1$
k=10			
-1	$0.3256353919777872 \cdot 10^1$	$0.1953545360705999 \cdot 10^1$	$0.7677722423353747 \cdot 10^1$
-2	$-0.2096116396850468 \cdot 10^2$	$-0.1050311310076629 \cdot 10^2$	$-0.5894517227637276 \cdot 10^2$
-3	$0.6872858265408605 \cdot 10^2$	$0.3105516048922884 \cdot 10^2$	$0.2140398605114418 \cdot 10^3$
-4	$-0.1393153744796911 \cdot 10^3$	$-0.5850644296241638 \cdot 10^2$	$-0.4662332548976578 \cdot 10^3$
-5	$0.1874446431742073 \cdot 10^3$	$0.7437254291687940 \cdot 10^2$	$0.6631353162140867 \cdot 10^3$
-6	$-0.1715855846429547 \cdot 10^3$	$-0.6498918498319249 \cdot 10^2$	$-0.6351002576675097 \cdot 10^3$
-7	$0.1061953812152787 \cdot 10^3$	$0.3866979933460322 \cdot 10^2$	$0.4083227672169233 \cdot 10^3$
-8	$-0.4269031893958787 \cdot 10^2$	$-0.1502289586232686 \cdot 10^2$	$-0.1696285390723725 \cdot 10^3$
-9	$0.1009036069527147 \cdot 10^2$	$0.3445119980743215 \cdot 10^1$	$0.4126838241810020 \cdot 10^2$
-10	$-0.1066655310499552 \cdot 10^1$	$-0.3544413204640886 \cdot 10^0$	$-0.4476202232026015 \cdot 10^1$
1	$0.4576078100790908 \cdot 10^1$	$0.2895451608911961 \cdot 10^1$	$0.9675787330957780 \cdot 10^1$
2	$-0.2469045273524281 \cdot 10^2$	$-0.1277820188943208 \cdot 10^2$	$-0.6561769910673283 \cdot 10^2$
3	$0.7648830198138171 \cdot 10^2$	$0.3534092272477722 \cdot 10^2$	$0.2294242274362024 \cdot 10^3$
4	$-0.1508194558089468 \cdot 10^3$	$-0.6441908403427060 \cdot 10^2$	$-0.4907643918974356 \cdot 10^3$
5	$0.1996415730837827 \cdot 10^3$	$0.8029833065236247 \cdot 10^2$	$0.6906485447124722 \cdot 10^3$
6	$-0.1807965537141134 \cdot 10^3$	$-0.6926226351772149 \cdot 10^2$	$-0.6568499770824342 \cdot 10^3$
7	$0.1110467735366555 \cdot 10^3$	$0.4083390088012690 \cdot 10^2$	$0.4202275815793937 \cdot 10^3$
8	$-0.4438764193424203 \cdot 10^2$	$-0.1575467189373152 \cdot 10^2$	$-0.1739340651258045 \cdot 10^3$
9	$0.1044548196545488 \cdot 10^2$	$0.3593677332216888 \cdot 10^1$	$0.4219582451243715 \cdot 10^2$
10	$-0.1100328792904271 \cdot 10^1$	$-0.3681517162342983 \cdot 10^0$	$-0.4566454997023116 \cdot 10^1$

Table 3.3
Quadrature weights β^m_k

3.3 Numerical results

In this section, we test the performance of the pre-corrected trapezoidal quadrature rules (i.e. Endpoint corrected Trapezoidal rule, $\mathcal{T}^n_{\alpha,\beta}$) to approximate various integrals.

3.3.1 Direct acoustic obstacle scattering problem in $\mathbb{R}^2$

Corrected trapezoidal quadrature rules have been applied to approximate the numerical solution of integral equations representing scattering calculations (see [8, 9]). Consider Integral of the form :

$$J(v) = \int_{-\pi}^{\pi} v(x)log(w(1-cos(x)))dx \tag{39}$$

where $v(x)$ is a smooth function of period 2π and w is a positive real number. Integrals of the type (39) can be applied to the direct acoustic obstacle scattering problem in $\mathbb{R}^2$ (see [10]). Such problem involves calculations of integrals of the form

$$g(t) = \int_{0}^{2\pi} v(x)log(4sin^2(\frac{t-x}{2}))dx = \int_{-\pi}^{\pi} v(x)log(2(1-cos(t-x)))dx, \tag{40}$$

where $v(x)$ is a smooth function of period 2π and $t \in [0, 2\pi]$.

3.3 Example. Table shows the results of the accuracy of the Pre-corrected Trapezoidal rule to approximate

$$\int_{-\pi}^{\pi} e^{2cos(2x)+sin(3x)} log(-\sqrt{2}(1-cos(x))) \tag{41}$$

Using Pre-corrected Trapeziodal rule $\mathcal{T}^n_{\alpha,\beta}$ we get:

N	$\mathcal{T}^n_{\alpha,\beta}(I)$	Error	Relative Error
90	$-1.98017193567999 \cdot 10^1$	$1.60551555016042 \cdot 10^{-4}$	1.00
100	$-1.98017639074799 \cdot 10^1$	$1.16000874957223 \cdot 10^{-4}$	0.72
110	$-1.98017933316290 \cdot 10^1$	$8.65767259057293 \cdot 10^{-5}$	0.52
120	$-1.98018135589064 \cdot 10^1$	$6.63494484669513 \cdot 10^{-5}$	0.41

Table 3.4
Error report of Precorrected Trapezoidal rule approximation for Integral-(41)

Here $N = n - 1$, where n is the number of quadrature points used. $\mathcal{T}^n_{\alpha,\beta}(I)$ is Integral value calculated via n-point Pre-corrected Trapezoidal rule. Error is the absolute difference in values between analytic Integration and n-point Pre-corrected Trapezoidal rule approximation ($\mathcal{T}^n_{\alpha,\beta}$) of integrand. We get high order approximation of $I(f)$ as required by the Pre-corrected Trapezoidal rule.

3.4 Example. Table shows the results of the accuracy of the Pre-corrected Trapezoidal rule to approximate

$$I = \int_{-\pi}^{\pi} e^{2cos(2x)+sin(3x)} log(-\sqrt{2}(1 - cos(x))) \tag{42}$$

Using Pre-corrected Trapeziodal rule $\mathcal{T}^n_{\alpha,\beta}$ we get:

N	$\mathcal{T}^n_{\alpha,\beta}(I)$	Error	Relative Error
100	-9.01990790176420	0.000110944038631544	1.00
150	-9.02001887796678	$3.21639479494706 \cdot 10^{-8}$	$2.89 \cdot 10^{-4}$
200	-9.02001884642815	$6.25311358248837 \cdot 10^{-10}$	$5.63 \cdot 10^{-6}$
250	-9.02001884568026	$1.22575727345975 \cdot 10^{-10}$	$1.12 \cdot 10^{-6}$

Table 3.5
Error report of Precorrected Trapezoidal rule approximation for Integral-(42)

Here $N = n - 1$, where n is the number of quadrature points used. $\mathcal{T}^n_{\alpha,\beta}(I)$ is Integral value calculated via n-point Pre-corrected Trapezoidal rule. Error is the absolute difference in values between analytic Integration and n-point Pre-corrected Trapezoidal rule approximation ($\mathcal{T}^n_{\alpha,\beta}$) of integrand . We get high order approximation of $I(f)$ as required by the Pre-corrected Trapezoidal rule.

3.3.2 Calculation of convolutions

Pre-corrected trapezoidal quadrature rules, $\mathcal{T}^n_{\alpha,\beta}$ can also be used to calculate convolutions of the type:

$$(Av)(t) = \int_{-\pi}^{\pi} v(x) log(w(1 - cos(t - x)))dx \quad , t \in [-\pi, \pi]$$

Where v is a smooth function of period 2π and w is a positive real number. One other way to approximate such convolution is via Colton-Kress Quadrature.

3.5. Colton-Kress Quadrature *let v be a smooth function of period 2π and $w \in \mathbb{R}_{>0}$. Then*

$$(Av)(t) = \int_{-\pi}^{\pi} v(x)log(w(1 - cos(t - x)))dx \quad , t \in [-\pi, \pi]$$

Can be approximated by

$$(Av)(t) \approx (a_n v)(t) = \sum_{j=0}^{2n-1} R_j^n(t)v(t_j) \tag{43}$$

where weights $R_j^n(t)$ are given by:

$$R_j^n(t) = -\frac{1}{n}\left\{ \sum_{m=1}^{n-1} \frac{1}{m} cos(\frac{mj\pi}{n}) + \frac{(-1)^j}{2n} \right\} \tag{44}$$

3.6 Example. Table shows the results of the accuracy of the Pre-corrected Trapezoidal rule to approximate the colvolution:

$$\int_{-\pi}^{\pi} e^{2cos(8x)+sin(9x)} log(2(1 - cos(x - t))) \quad , t \in [-\pi, \pi] \tag{45}$$

Using Pre-corrected Trapeziodal rule $\mathcal{T}_{\alpha,\beta}^n$ and Colton-Kress Quadrature Approximation respectively ,we get:

	N	t = 0	t = π
	100	-2.735266635099835	-2.735266635099867
Error:		$9.744514762 \cdot 10^{-6}$	$9.744498517 \cdot 10^{-6}$
	200	-2.735276379373059	-2.735276379373091
Error:		$2.41537 \cdot 10^{-10}$	$2.25293 \cdot 10^{-10}$
	280	-2.735276379489301	-2.735276379489333
Error:		$1.25296 \cdot 10^{-10}$	$1.09051 \cdot 10^{-10}$

Table 3.6
Error report of Precorrected Trapezoidal rule approximation for Convolution-(45)

	N	t = 0	t = π
	100	-2.735165105378464	-2.735165105378455
Error:		$1,11274219921 \cdot 10^{-4}$	$1.11274236142 \cdot 10^{-4}$
	200	-2.735276380239916	-2.735276380239910
Error:		$6.41532 \cdot 10^{-10}$	$6.25314 \cdot 10^{-10}$
	280	-2.735276379489354	-2.735276379489378
Error:		$1.09031 \cdot 10^{-10}$	$1.25219 \cdot 10^{-10}$

Table 3.7
Error report of Colton-Kress Quadrature Approximation for Convolution-(45)

For both tables $N = n-1$, where n is the number of quadrature points used. Error is the absolute difference in values between analytic Integration and n-point Quadrature approximation.

4. CONCLUDING REMARKS

We started this thesis by first reviewing the simple and composite Trapezoidal rule and illustrating its use via the approximation of Integral involving smooth functions in Chapter-1.

In the next chapter, we discussed singular integrands and utilised various integral methods such as "change in variables" and "Analytic treatment of Singularity" to deal with such integrals. In the change in variables method, we investigated how to integrate functions having Singularity at endpoints of singular functions numerically. In the Analytic treatment of the Singularity method, we learned how to integrate functions having a finite number of singular points numerically with high accuracy. After this, we examined the error in the approximation of Trapezoidal rule via *Euler–Maclaurin formula* and how it converges to high order for smooth and periodic function. Then we discussed the Endpoint corrections to the Trapezoidal rule and Investigated how Endpoint corrected Trapezoidal rule gives high order convergence for smooth and non-periodic functions. Then we further extended the "Endpoint corrected Trapezoidal rule" to "Endpoint corrected Trapezoidal rule for singular functions" to be able able to solve singular Integral of a certain class by adding another set of correction coefficients. As both types of corrections required precomputed or pre-corrected values. This corrected Trapezoidal rule is called the *"Pre-corrected trapezoidal rule"*.

In the Third Chapter, we constructed and demonstrated how, via the Nystrom method, how Fredholm integral equation of the second kind could be approximated via the Trapezoidal rule. After that, we listed the calculated correction coefficients for the Precorrected trapezoidal rule and then illustrated the "corrected trapezoidal rule" to solve Integral equations associated with Real-world problem like *Direct acoustic obstacle scattering problem in* $\mathbb{R}^2$.

We sincerely hope that the reader has developed an interest in the Simplicity of the Trapezoidal rule and how it can be modified to solve numerous practical, real-world problems.

Future work

Through this was just one application of the "Precorrected Trapezoidal rule". In the Literature, there are techniques to use the "Precorrected Trapezoidal rule" to solve several significant practical problems such as wave scattering, image processing, and medical imaging are shown in [11, 12, 13]. Also, Trapezoidal rule compatibility with Fast Fourier Transform adds another ton of other applications.

Also, recently, a new technique for the computation of pre-corrected weights is reported in [14]. Rules described in this paper can be seen as a Hybrid of both Colton-Kress Qradrature [10] and corrected trapezoidal quadrature of Kapur and Rokhlin [4]. The new method combines both methods' strengths and attains high-order convergence, numerical stability, ease of implementation, and compatibility with the "fast" algorithms (such as the Fast Multipole Method or Fast Direct Solvers).

Appendices

Appendix A

We shall now look at a proof of Theorem 2.8, also known as the Lagrange Interpolation for Equally-Spaced abscissa.

2.8 Theorem (Lagrange Interpolation for Equally-Spaced abscissa)**.** *Suppose a, b are a pair of real number such that $a < b$, $m \geq 3$ be an integer and $h = \frac{b-a}{m-1}$.*
let $f \in C^m[a-mh, b+mh]$ and equispaced points x_k be defined as $x_k = \frac{b-a}{2} + kh$. Then, for any real number p there exists a real number ξ, $-mh < \xi < mh$, such that

$$f(x_0 + ph) = \sum_k \mathcal{A}_k^m f(x_k) + \mathcal{R}_{m-1,p} \tag{11}$$

where k varies from

$$-\frac{1}{2}(m-2) \leq k \leq \frac{1}{2}m \quad \textit{for even } m$$
$$-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1) \quad \textit{for odd } m$$

with

$$\mathcal{A}_k^m(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \sum_t (p-t)$$

and

$$\mathcal{R}_{m-1,p} = \frac{h^m}{m!} f^{(m)}(\xi) \sum_n (p-n)$$

here both t and n varies same as k.

Proof. Assuming m to be odd. For a function $f \in C^m[a-mh, b+mh]$,its Lagrange interpolation polynomial constructed from equispaced points

$$x_{-\frac{1}{2}(m-1)}, x_{-\frac{1}{2}(m-1)+1}, \ldots\ldots, x_{\frac{1}{2}(m-1)-1}, x_{\frac{1}{2}(m-1)}$$

is given by

$$\mathcal{L}_{m-1}(x) = \sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} l_j(x_j) f(x_j) \tag{46}$$

and,

$$f(x) = \sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} l_j(x_j) f(x_j) + R_{m-1}(x) \tag{47}$$

with,

$$l_j(x) = \sum_{i=-\frac{1}{2}(m-1), i\neq j}^{\frac{1}{2}(m-1)} \frac{x - x_i}{x - x_j}$$

$$R_{m-1}(x) = \frac{f^{(m)}(\xi)}{m!}\Bigg[\sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} x - x_j\Bigg]$$

We can write x as $x = x_0 + ph$ for some real number p.

$$\begin{aligned}
R_{m-1}(x) = R_{m-1}(x_0 + ph) &= \frac{f^{(m)}(\xi)}{m!}\Bigg[\sum_{j=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} x - x_j\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!}\Bigg[\Big(x - x_{-\frac{1}{2}(m-1)}\Big)\Big(x - x_{-\frac{1}{2}(m-1)+1}\Big).......\Big(x - x_{\frac{1}{2}(m-1)}\Big)\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!}\Bigg[\Big(x_0 + ph - x_0 + \frac{1}{2}(m-1)h\Big)....\Big(x_0 + ph - x_0 - \frac{1}{2}(m-1)h\Big)\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!}\Bigg[\Big(ph + \frac{1}{2}(m-1)h\Big)....\Big(ph - \frac{1}{2}(m-1)h\Big)\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!}\Bigg[h^m\Big(p + \frac{1}{2}(m-1)\Big)....\Big(p - \frac{1}{2}(m-1)\Big)\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!}\Bigg[h^m\Big(p + \frac{1}{2}(m-1)\Big)....\Big(p - \frac{1}{2}(m-1)\Big)\Bigg] \\
&= \frac{f^{(m)}(\xi)}{m!} h^m \sum_{n=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p - n) \\
&= \mathcal{R}_{m-1,p}
\end{aligned} \tag{48}$$

$$
\begin{aligned}
l_k(x) = l_k(x_0+ph) &= \sum_{i=-\frac{1}{2}(m-1), i\neq k}^{\frac{1}{2}(m-1)} \frac{x-x_i}{x_k-x_i} \\
&= \left[\frac{\big(x-x_{-\frac{1}{2}(m-1)}\big)......\big(x-x_{k-1}\big)\big(x-x_{k+1}\big).......\big(x-x_{\frac{1}{2}(m-1)}\big)}{\big(x_k-x_{-\frac{1}{2}(m-1)}\big)......\big(x_k-x_{k-1}\big)\big(x_k-x_{k+1}\big).......\big(x-x_{\frac{1}{2}(m-1)}\big)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{\Big(x_0+ph-x_0+\frac{1}{2}(m-1)h\Big)....\Big(x_0+ph-x_0-(k-1)h\Big)\Big(x_0+ph-x_0-(k+1)h\Big)....\Big(x_0+ph-x_0-\frac{1}{2}(m-1)h\Big)}{\Big(x_0+kh-x_0+\frac{1}{2}(m-1)h\Big)....\Big(x_0+kh-x_0-(k-1)h\Big)\Big(x_0+kh-x_0-(k+1)h\Big)....\Big(x_0+kh-x_0-\frac{1}{2}(m-1)h\Big)}\right] \\
&= \left[\frac{\Big(p+\frac{1}{2}(m-1)\Big)....\Big(p-(k-1)\Big)\Big(p-(k+1)\Big)....\Big(p-\frac{1}{2}(m-1)\Big)}{\Big(k+\frac{1}{2}(m-1)\Big)....\Big(k-(k-1)\Big)\Big(k-(k+1)\Big)....\Big(k-\frac{1}{2}(m-1)\Big)}\right]\frac{\Big(p-k\Big)}{\Big(p-k\Big)} \\
&= \left[\frac{\Big(p+\frac{1}{2}(m-1)\Big)....\Big(p-k+1\Big)\Big(p-k\Big)\Big(p-k-1\Big)....\Big(p-\frac{1}{2}(m-1)\Big)}{\Big(k+\frac{1}{2}(m-1)\Big)....\Big(k-k+1\Big)\Big(k-k-1\Big)....\Big(k-\frac{1}{2}(m-1)\Big)}\frac{1}{\Big(p-k\Big)}\right] \\
&= \frac{\alpha}{\beta}
\end{aligned}
\tag{49}
$$

where $\alpha = \big(p+\frac{1}{2}(m-1)\big)....\big(p-k+1\big)\big(p-k\big)\big(p-k-1\big)....\big(p-\frac{1}{2}(m-1)\big)$
, $\beta = \big(k+\frac{1}{2}(m-1)\big)....\big(k-k+1\big)\big(k-k-1\big)....\big(k-\frac{1}{2}(m-1)\big)\big(p-k\big)$
Now,

$$
\begin{aligned}
\beta &= \big(k+\frac{1}{2}(m-1)\big)....\big(k-k+1\big)\big(k-k-1\big)....\big(k-\frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \big(k+\frac{1}{2}(m-1)\big)....\big(2\big)\big(1\big)\big(-1\big)\big(-2\big)....\big(k-\frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \Big(k+\frac{1}{2}(m-1)\Big)!\big(-1\big)\big(-2\big)....\big(k-\frac{1}{2}(m-1)\big)\big(p-k\big) \\
&= \Big(k+\frac{1}{2}(m-1)\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\big(1\big)\big(2\big)....\big(\frac{1}{2}(m-1)-k\big)\big(p-k\big) \\
&= \Big(\frac{1}{2}(m-1)+k\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\Big(\frac{1}{2}(m-1)-k\Big)!\Big(p-k\Big)
\end{aligned}
$$

putting α, β back in equation 49

$$\begin{aligned}
l_k(x) &= \frac{\alpha}{\beta} \\
&= \frac{\prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)}(p-t)}{\Big(\frac{1}{2}(m-1)+k\Big)!\Big(-1\Big)^{\frac{m-1}{2}-k}\Big(\frac{1}{2}(m-1)-k\Big)!\Big(p-k\Big)} \\
&= \frac{\Big(-1\Big)^{m-1}}{\Big(-1\Big)^{m-1}} \frac{\Big(-1\Big)^{k-\frac{m-1}{2}} \prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)}(p-t)}{\Big(\frac{1}{2}(m-1)+k\Big)!\Big(\frac{1}{2}(m-1)-k\Big)!\Big(p-k\Big)} \\
&= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \prod_{t=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p-t) \\
&= \mathcal{A}_k^m(p)
\end{aligned} \tag{50}$$

By using equations 50 , 47 , 48

$$f(x) = f(x_0 + ph) = \sum_k \mathcal{A}_k^m f(x_k) + \mathcal{R}_{m-1,p}$$

where k varies from $-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1) \quad , \quad m \textit{ is odd.}$

Similarly, for even m, we can get the result by constructing Lagrange interpolating polynomial from equispaced points.

$$x_{-\frac{1}{2}(m-2)}, x_{-\frac{1}{2}(m-2)+1}, \ldots\ldots, x_{\frac{1}{2}m-1}, x_{\frac{1}{2}m}$$

$\square$

We shall now look at a proof of Theorems 2.10 and 2.12 which have been taken from [4].

2.10 Theorem. *Suppose m, l, k are integers and coefficients $a^m_{k,l}$ are defined by recussive relations*

$$a^3_{1,1} = 1 \tag{13a}$$
$$a^3_{1,2} = 1 \tag{13b}$$
$$a^{2k+1}_{k,l} = (k - k^2)a^{2k-1}_{k-1,l} + a^{2k-1}_{k-1,l} + a^{2k-1}_{k-1,l-2} \tag{13c}$$
$$a^{m+2}_{k,l} = a^m_{k,l-2} - \big(\frac{m+1}{2}\big)^2 a^m_{k,l} \tag{13d}$$

with $a^m_{k,l} = 0 \ \forall \ k \leq 0$ *or* $l \leq 0$ *or* $m \leq 1$.
then

$$\mathcal{A}^m_k(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{\big(\frac{m-1}{2}+1\big)!\big(\frac{m-1}{2}-1\big)!} \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l$$

Proof. From (2.8),

$$\begin{aligned}\mathcal{A}^m_k(p) &= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \sum_{k=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p-t) \\ &= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} C^m_k(p)\end{aligned}$$

where

$$C^m_k(p) = \frac{1}{(p-k)} \sum_{k=-\frac{1}{2}(m-1)}^{\frac{1}{2}(m-1)} (p-t) \tag{52}$$

To prove all four recursive conditions, it is sufficient to show that.

$$C^m_k(p) = \frac{1}{(p-k)} \sum_{l=1}^{m-1} a^m_{k,l} p^l \tag{53}$$

This can be shown by induction. Indeed, if $m = 3$, then, due to (52),

$$C^3_1(p) = p^2 + p, \tag{54}$$

which is equivalent to (13a) , (13b). Assume now that for some m, k such that $-\frac{1}{2}(m-1) \leq k \leq \frac{1}{2}(m-1)$

$$C_k^m(p) = \frac{1}{(p-k)} \sum_{l=1}^{m-1} a_{k,l}^m p^l \tag{55}$$

Combining (52) and (55), we have

$$\begin{aligned} C_k^{m+2}(p) &= \Big(p + \frac{m+1}{2}\Big)\Big(p - \frac{m+1}{2}\Big) \sum_{l=1}^{\frac{m-1}{2}} a_{k,l}^m p^l \\ &= \left(p^2 - \Big(\frac{m+1}{2}\Big)^2\right) \sum_{l=1}^{\frac{m-1}{2}} a_{k,l}^m p^l \\ &= \sum_{l=1}^{\frac{m-1}{2}} a_{k,l}^m p^{l+2} - \Big(\frac{m+1}{2}\Big)^2 \sum_{l=1}^{\frac{m-1}{2}} a_{k,l}^m p^l \end{aligned} \tag{56}$$

which is equivalent to (13d). Now, assume that for some k

$$C_k^{m+2}(p) = \sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \tag{57}$$

Combining (57) and (52), we have

$$\begin{aligned} C_k^{2k+3}(p) &= \big(p-k\big)\big(p-(k+1)\big) \sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \\ &= \big(p^2 + p - (k^2+k)\big) \sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \\ &= \sum_{l=1}^{k} a_{k,l}^{2k+1} p^{l+1} - \big(k^2+k\big) \sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \end{aligned} \tag{58}$$

which is equivalent to (13c). □

Before we get to the theorem-2.12. and its proof, we need to see the following lemmas.

A.1 Lemma. *If $k \geq 2$ is an integer and $a_{k,l}^m$ is defined in Lemma-*(2.10)*, then*

$$|(l)! \cdot a_{k,l}^{2k+1}| < |(l+2)! \cdot a_{k,l+2}^{2k+1}| \tag{59}$$

for all $l = 1, 2, \cdots, 2k-3$.

A.2 Lemma. *If $k \geq 2$ is an integer, and $a^m_{k,l}$ is defined in Lemma-(2.10), then*

$$|(l)! \cdot a^m_{k,l}| < |(l+2)! \cdot a^m_{k,l+2}| \tag{60}$$

for all $m \geq 2k+1$ and $l = 1, 2, \cdots, 2k-3$.

A.3 Lemma. *If m, k are integers such that $m \geq 3$ is odd , and $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$ then*

$$|(1)! \cdot a^m_{k,1}| < |(3)! \cdot a^m_{k,3}| < |(5)! \cdot a^m_{k,5}| < \cdots |(m-2)! \cdot a^m_{k,m-2}| \tag{61}$$

A.4 Lemma. *If $m \geq 3$ is odd then,*

$$\frac{\big(m-1\big)\big(m-2\big)!}{2\big((\frac{m-1}{2})!\big)} < \frac{(2\pi)^{m-1}}{4} \tag{62}$$

A.5 Lemma. *If $m \geq 3$ is odd, then*

$$|\mathcal{D}^m_{i,k}| < \frac{(2\pi)^{m-1}}{4} \tag{63}$$

for any k, i such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, and $1 \leq i \leq \frac{m-1}{2}$

A.6 Lemma. *For any $l \geq 1$, the Bernoulli numbers $\mathcal{B}_{2l}$ satisfies the inequality*

$$\left|\frac{\mathcal{B}_{2l}}{(2l)!}\right| < \frac{4}{(2\pi)^{2l}} \tag{64}$$

Now to proof of Theorem-(2.12)

Proof. By Combing equation (15) and (14) , we obtain

$$\begin{aligned}\mathcal{T}_{\alpha,\beta}(f) &= \mathcal{T}_{\alpha^m}(f) + h \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \big(f(b+kh) - f(b-kh)\big) \sum_{l=1}^{\frac{m-1}{2}} \frac{\mathcal{D}^m_{i,k}\mathcal{B}_{2l}}{(2l)!} \\ &= \mathcal{T}_{\alpha^m}(f) + \sum_{l=1}^{\frac{m-1}{2}} \frac{h^{2l}\mathcal{B}_{2l}}{(2l)!} \left(\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{\mathcal{D}^m_{i,k}\big(f(b+kh) - f(a+kh)\big)}{h^{2i-1}} \right)\end{aligned} \tag{65}$$

Finally, combining (12), (**??**), and Euler–Maclaurin formula , we have

$$\mathcal{T}_{\alpha^m}(f) = \mathcal{T}_n(f) + \sum_{l=1}^{\frac{m-1}{2}} \frac{h^{2l}\mathcal{B}_{2l}}{(2l)!} \big(f^{(2i-1)}(b) f^{(2i-1)}(a)\big) + \mathcal{O}(h^{m+1}) \tag{66}$$

$\square$

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