# Weekly Homework 3

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Brain Inspired Computing

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**Exercise 3.1.** Stability Conditions in 2D

a) Requirement for stability $r_\pm < 0$ wit $\lambda_\pm = r \pm \omega$; eigenvalues given by:

$$\lambda_\pm = \frac{1}{2}\left(F_u + G_w \pm \sqrt{(F_u + G_w)^2 - 4(F_u G_w - F_w G_u)}\right) \tag{1}$$

if $\sqrt{\ldots}$ gets imaginary $\Rightarrow r_\pm < 0$ since $F_u + G_w < 0$ if $\sqrt{\ldots}$ gets real:

$$\sqrt{(F_u + G_w)^2 - 4(F_u G_w - F_w G_u)} < \sqrt{(F_u + G_w)^2} = |F_u + G_w| \tag{2}$$

where we used condition number two; plugging this into eigenvalue expression leads us to:

$$\lambda_\pm \leq \frac{1}{2}(F_u + G_w \pm |F_u + G_w|) \leq 0 \tag{3}$$

in both cases the system is stable since $r_\pm < 0$

b) Using the following Mathematica code to obtain fixed points and decide whether they are stable:

```
eps = 0.1;
a = 15 / 8;
b = 3 / 2;
II = 0;
F[u_, w_] = u - u^3 / 3 - w + II;
G[u_, w_] = eps (a + b u - w);
sol = Solve[F[u, w] == 0 && G[u, w] == 0, {u, w}]
Fu = D[F[u, w], u] /. sol;
Fw = D[F[u, w], w] /. sol;
Gu = D[G[u, w], u] /. sol;
Gw = D[G[u, w], w] /. sol;
(Fu + Gw) < 0
(Fu Gw - Fw Gu) > 0
```

Solve::ratnz : Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. ≫

```
{{u → -1.5, w → -0.375}, {u → 0.75 - 1.78536 i, w → 3. - 2.67804 i},
 {u → 0.75 + 1.78536 i, w → 3. + 2.67804 i}}

{-1.35, 3.525 + 2.67804 i, 3.525 - 2.67804 i} < 0

{0.275, -0.2125 - 0.267804 i, -0.2125 + 0.267804 i} > 0
```

```
eps = 0.1;
a = 15 / 8;
b = 3 / 2;
II = 15 / 8;
F[u_, w_] = u - u^3 / 3 - w + II;
G[u_, w_] = eps (a + b u - w);
sol = Solve[F[u, w] == 0 && G[u, w] == 0, {u, w}]
Fu = D[F[u, w], u] /. sol;
Fw = D[F[u, w], w] /. sol;
Gu = D[G[u, w], u] /. sol;
Gw = D[G[u, w], w] /. sol;
(Fu + Gw) < 0
(Fu Gw - Fw Gu) > 0
```

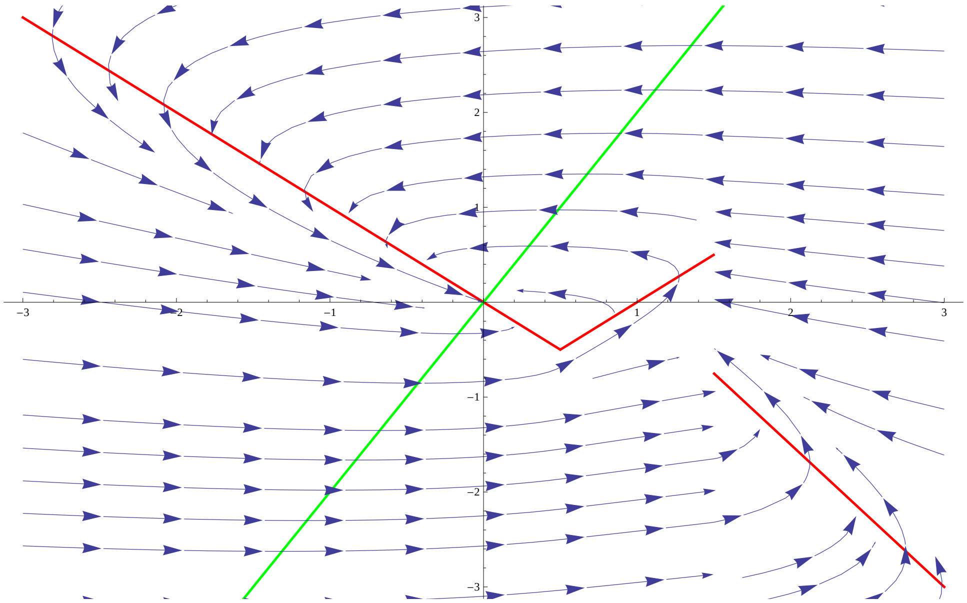
Solve::ratnz : Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. ≫

```
{{u → 0., w → 1.875}, {u → 0. - 1.22474 i, w → 1.875 - 1.83712 i},
 {u → 0. + 1.22474 i, w → 1.875 + 1.83712 i}}

{0.9, 2.4 + 0. i, 2.4 + 0. i} < 0

{0.05, -0.1 + 0. i, -0.1 + 0. i} > 0
```

**Exercise 3.2.** Piecewise linear nullclines

a) nullclines and flow in Phase plane of FitzHugh-Nagumo model using Mathematica and an input current of $I = 0$:

```
II = -2;

a = -1;
c1 = -1;
b = 2;
eps = 0.1;
c0 [u_] = -0.5 u - 1.5 c1;
NCw[u_] = b u;


NCu[u_] = II + a u HeavisideTheta [- (u - 0.5)] + a (1 - u) HeavisideTheta [u - 0.5]
    HeavisideTheta [- (u - 1.5)] + (c0[u] + c1 u ) HeavisideTheta [u - 1.5];
nullcline1 = Plot[NCu[u], {u, -3, 3}, PlotStyle → {Red, Thick}];
nullcline2 = Plot[NCw[u], {u, -3, 3}, PlotStyle → {Green, Thick}];

Du[u_, w_] = NCu[u] - w;(* II already in there*)
Dw[u_, w_] = eps ( b u - w);
flow = StreamPlot[{Du[u, w], Dw[u, w]}, {u, -3, 3}, {w, -6, 6}];
(*SetDirectory[ ]
Export["excercise_3.2_I0.jpg",*)
 Show[nullcline1, nullcline2, flow]
```



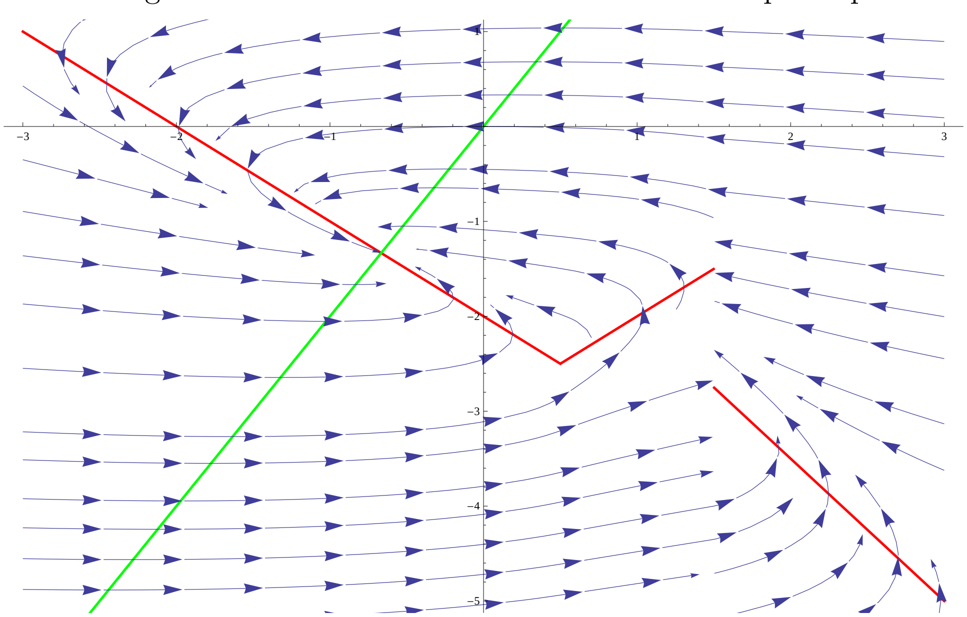
b) Same as in a but with an input current of $I = -2$. The fixed point is calculated by the intersection of the following equations ($u < 0.5$ compare figure):

$$0 = au - w - 2 \tag{4}$$
$$0 = bu - w \tag{5}$$

which leads us to:

$$u = \frac{1}{a-b} = -\frac{2}{3} \tag{6}$$
$$w = au - 2 = -\frac{4}{3} \tag{7}$$

the next figure shows the nullclines and the flow in the phase plane for $I = -2$

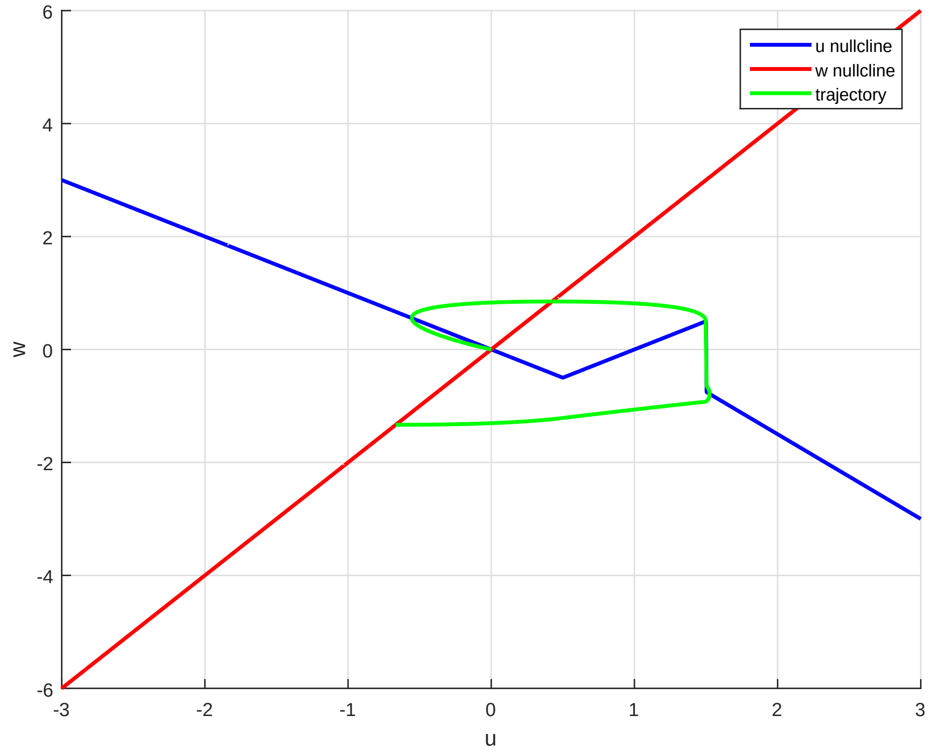


c) MATLAB code for simulation of the nullclines and trajectory in phase plane:

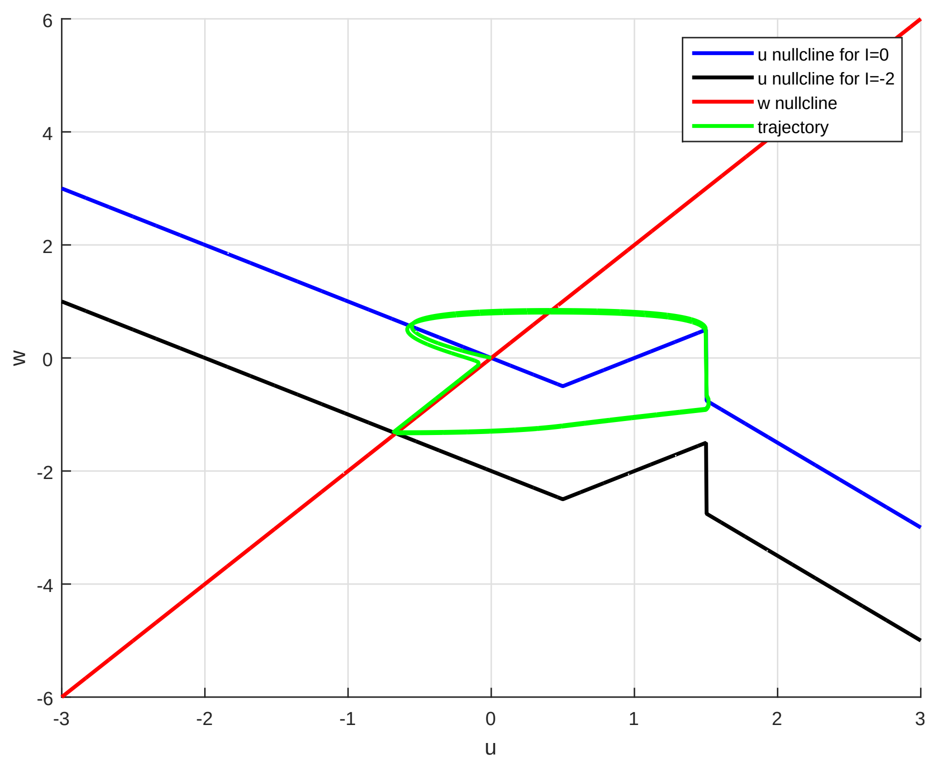
```
close all
clear all
clc

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% 0. PARAMETERS

% Euler integration
h = 10e-5;      % step size parameter
time = 500;     % simulation time

% Model
a = -1;
c1 = -1;
b = 2;
epsilon = 0.1;

t = 0:h:time; % generate time vector
I = [-1*(linspace(0,2,round(length(t)/2-1))) zeros(round(length(t)/2),1)']'; % create current vector

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% I. CALCULATE NULLCLINES

u = linspace(-3,3,length(t)); % preallocate voltage array
w = linspace(-3,3,length(t)); % preallocate w array
w = zeros(length(u),1); % preallocate aray for piecewise function

for i = 1:length(u)
    u1 = u(i);
    w(i) = fu(u1,a,c1);
end

nu = w + 0; % calculate voltage nullcline
nu_new = w - 2; % calculate voltage nullcline
nw = b*u; % calculate w nullcline
%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% II. PERFORM INTEGRATION

u_sol = zeros(size(t)); % Preallocate array for velocities
w_sol = zeros(size(t)); % Preallocate array for positions

u_sol(1) = -2/3;%1.5;               % Initial condition gives solution for position at t=0.
w_sol(1) = -4/3;%0.375;             % Initial condition gives solution for velocity at t=0.

for i=1:(length(t)-1) % loop over time
    u_sol(i+1) = u_sol(i) + (fu(u_sol(i),a,c1) - w_sol(i) + I(i))*h; % integrate voltage DE
    w_sol(i+1) = w_sol(i) + epsilon*(b*u_sol(i) - w_sol(i))*h; % integrate w DE
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% III. PLOT REULTS
figure
hold on
grid on
plot(u,nu,'b','linewidth',2)
plot(u,nu_new,'k','linewidth',2)
plot(u,nw,'r','linewidth',2)
plot(u_sol,w_sol,'g','linewidth',2)
legend('u nullcline for I=0','u nullcline for I=-2','w nullcline','trajectory')
xlabel('u')
ylabel('w')
print(gcf,'-depsc','exercise32c_full.eps')
```

```
function f = fu(u,a,c1)
c0 = -0.5*u -1.5*c1;
if u<0.5
    f = a*u;
elseif 0.5 <= u && u < 1.5
    f = a*(1-u);
elseif u >= 1.5
    f = c0 + c1*u;
end
end
```

The next figure shows the trajectory in phase plane after the switch-of of the hyperpolarization current. The initial values for the Euler integration correspond to the calculated fixed point (Equation 7)




The next figure shows the full trajectory in phase plane for a linearly increasing hyperpolarizing current, which is suddenly switched of. The initial values for the Euler integration are chosen to be $w = u = -2$. The $w$ nullcline is shown for $I = 0$ and $I = -2$

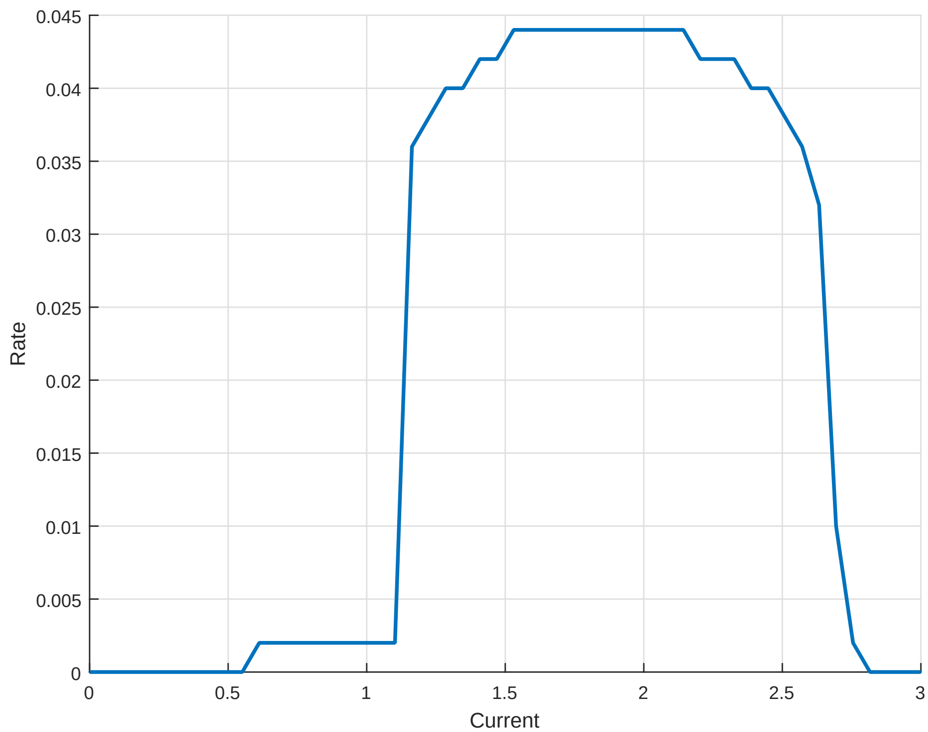


**Exercise 3.3.** Exploring the FitzHugh model

a) MATLAB code for Euler integration of FitzHugh-Nagumo model:

```
close all
clear all
clc

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% I. DEFINE PARAMETERs

% Euler
h = 10e-5; % step size parameter
time = 500; % simulation time
t = 0:h:time; % generate time vector

% model
epsilon = 0.1;
a = 15/8;
b = 3/2;
I = linspace(0,3,50); % create linear incresing current vector

rate = zeros(50,1); % preallocate rate array

for j = 1:length(I) % loop over currents
curr = I(j);
u = zeros(size(t)); % Preallocate array for velocities
w = zeros(size(t)); % Preallocate array for positions

u(1) = -1.5; % Initial condition gives solution for position at t=0.
w(1) = -0.375; % Initial condition gives solution for velocity at t=0.

numberOfPeaks = 0; % set counter
alreadyPeaked = 0; % set counter
threshold = 1; % set threshold

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% II. PERFORM INTEGRATION
for i=1:(length(t)-1)
    u(i+1) = u(i) + (u(i) -u(i)^3/3 -w(i) + curr)*h; % integrate u
    w(i+1) = w(i) + epsilon*(a + b*u(i) - w(i))*h; % inegrate w
    % detection algorithm
    if(u(i+1) >= threshold)
        alreadyPeaked = 1;
    else
        if(alreadyPeaked == 1)
            alreadyPeaked = 0;
            numberOfPeaks = numberOfPeaks + 1;
        end
    end
end
rate(j) = numberOfPeaks/time; % normalize rate
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% III. PLOT RESULTS

figure                      % prepare figure
hold on                     % plot in every loop cycle in same figure
grid on                     % plot mesh grid
xlabel('Current')
ylabel('Rate')
plot(I,rate,'Linewidth',2)
print(gcf,'-depsc','ecxercise3ac.eps')
```

The next plot shows the firing rate as a function of input current

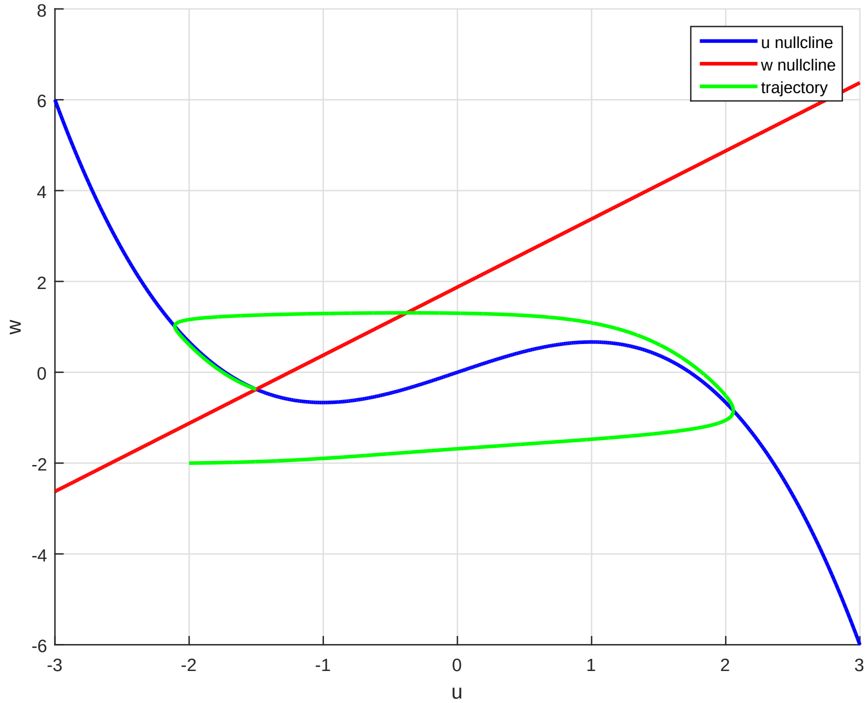



b) MATLAB code for calculating nullclines and an example trajectory

```
close all
clear all
clc

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% I. DEFINE PARAMETERS

% Euler integration
h = 10e-5; % step size parameter
time = 500; % simulation time
t = 0:h:time; % generate time vector

% model
a = 15/8;
b = 3/2;
epsilon = 0.1;
I = 2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% II. CALCULATE NULLCLINES

u0 = linspace(-3,3,20000);

w1 = u0 - u0.^3/3 + I; % calculate u nullcline
w2 = a + b*u0; % calculate w nullcline

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% III. SOLVE DEs

u = zeros(size(t)); % Preallocate array for velocities
w = zeros(size(t)); % Preallocate array for positions

u(1) = -2; % Initial condition gives solution for position at t=0.
w(1) = -2; % Initial condition gives solution for velocity at t=0.

for i=1:(length(t)-1)
    u(i+1) = u(i) + (u(i) -u(i)^3/3 -w(i) + I)*h;
    w(i+1) = w(i) + epsilon*(a + b*u(i) - w(i))*h;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%
%% IIII. PLOT RESULTS

figure
hold on
grid on
plot(u0,w1,'b','linewidth',2)
plot(u0,w2,'r','linewidth',2)
plot(u,w,'g','linewidth',2)
legend('u nullcline','w nullcline','trajectory')
```

```
xlabel('u')
ylabel('w')
print(gcf,'excercise33c_I2.pdf')
```

Plot of nullclines and trajectory at $I = 0$ where $\nu = 0$




Plot of nullclines and trajectory at $I = 2$ where $\nu \neq 0$