Chapter 5: Induction and Recursive Definitions

Kenneth Rosen, Discrete Mathematics, 8th Edition

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What is Mathematical Induction?

- A proof technique to establish that a statement $P(n)$ is true for all positive integers n.
- Based on the principle: if a statement holds for a base case and can be shown to hold for the next case, it holds for all cases.
- Two key steps:
  1. **Base Case**: Prove the statement for the smallest value of n (usually $n = 1$ or $n = 0$).
  2. **Inductive Step**: Assume the statement holds for $n = k$ (inductive hypothesis), and prove it for $n = k + 1$.

Principle of Mathematical Induction

Formal Statement

To prove $P(n)$ is true for all $n \geq n_0$:

- **Base Case**: Show $P(n_0)$ is true.
- **Inductive Step**: Assume $P(k)$ is true for some $k \geq n_0$ (inductive hypothesis). Prove $P(k+1)$ is true.

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- If both steps are proven, then $P(n)$ is true for all $n \geq n_0$.
- Analogy: Like climbing a ladder—prove you can start (base case) and move to the next rung (inductive step).

Example: Sum of First n Positive Integers

Statement

Prove: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

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- **Base Case** $(n = 1)$:

$$1 = \frac{1(1+1)}{2} = 1$$

Holds true.

- **Inductive Step**: Assume true for $n = k$:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Prove for $n = k + 1$:

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} =$$

Holds true.

Strong Induction

- A variation of mathematical induction.
- **Base Case**: Same as standard induction (prove for smallest n).
- **Inductive Step**: Assume $P(m)$ is true for all $m \leq k$, and prove $P(k+1)$.
- Useful when proving $P(k+1)$ requires more than just $P(k)$.

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Example Use Case

May sometimes make proofs easier.

Summary

- Mathematical induction proves statements for all positive integers.
- Requires a base case and an inductive step.
- Strong induction allows assuming all previous cases in the inductive step.
- Applications: Summations, inequalities, divisibility, and recursive definitions.

Key Takeaway

Induction is a powerful tool for proving statements about infinite sets by reducing them to finite steps.

Why Multiple Base Cases?

- Some proofs require more than one base case, especially in:
  - **Strong Induction**: Inductive step may depend on multiple previous cases.
  - **Recursive Definitions**: Statements defined using several prior terms (e.g., Fibonacci: $F(n) = F(n-1) + F(n-2)$).
  - Problems where the smallest n needs extra cases to establish the pattern.
- Multiple base cases ensure the inductive step can "reach back" to valid cases.

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Key Idea

The number of base cases depends on how many prior values the inductive step needs.

Example: Fibonacci Numbers Are Positive

Statement

Prove: Fibonacci numbers $F(n)$, defined by $F(1) = 1$, $F(2) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$, are positive for all $n \geq 1$.

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- **Base Cases**:
  - $n = 1$: $F(1) = 1 > 0$.
  - $n = 2$: $F(2) = 1 > 0$.
- **Inductive Step (Strong Induction)**:
  - Assume $F(m) > 0$ for all $m \leq k$, where $k \geq 2$.
  - Prove $F(k+1)$: Since $F(k+1) = F(k) + F(k-1)$, and $F(k) > 0$, $F(k-1) > 0$ by hypothesis, so $F(k+1) > 0$.
- Two base cases needed because $F(k+1)$ depends on $F(k)$ and $F(k-1)$.

Example: All Horses Are the Same Color

Claim (Flawed)

All horses in any set of n horses are the same color.

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- **Base Case**: For $n = 1$, one horse has its own color. True.
- **Inductive Step**: Assume true for $n = k$: any set of k horses is the same color.
- Prove for $n = k + 1$:
  - Take a set of $k + 1$ horses: $\{h_1, h_2, \ldots, h_{k+1}\}$.
  - Subset $\{h_1, \ldots, h_k\}$ has k horses, so all are the same color (by hypothesis).
  - Subset $\{h_2, \ldots, h_{k+1}\}$ has k horses, so all are the same color.
  - Since h_2 is in both subsets, all $k + 1$ horses are the same color.

Why the Horse Proof Fails

- The inductive step fails for $n = 2$ (base case is $n = 1$):
  - For $k = 1$, consider $\{h_1, h_2\}$.
  - Subset $\{h_1\}$ has one horse, $\{h_2\}$ has one horse—both trivially true.
  - But no overlap exists (no common horse), so we cannot conclude h_1 and h_2 are the same color.
- **Lesson**: The inductive step must hold for all cases, including the transition from base case to the next step.
- **Fix**: Adding a base case for $n = 2$ reveals the flaw, as two horses may differ in color.

What is Strong Induction?

- A variation of mathematical induction for proving statements $P(n)$ for all positive integers n.
- Differs from standard induction in the inductive step:
  - **Base Case**: Prove $P(n)$ for the smallest value(s) (e.g., $n = 1$ or multiple base cases).
  - **Inductive Step**: Assume $P(m)$ is true for all $m \leq k$, and prove $P(k+1)$.
- Useful when $P(k+1)$ depends on multiple previous cases, not just $P(k)$.

Principle of Strong Induction

Formal Statement

To prove $P(n)$ is true for all $n \geq n_0$:

- **Base Case**: Show $P(n_0), P(n_0 + 1), \ldots$ for necessary starting values.
- **Inductive Step**: Assume $P(m)$ is true for all m where $n_0 \leq m \leq k$. Prove $P(k + 1)$.

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- If both steps hold, $P(n)$ is true for all $n \geq n_0$.
- Analogy: Like climbing a ladder, but you can use all previous rungs to reach the next one.

Example: Divisibility of Numbers

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- **Base Cases**:
  - $n = 12$: $12 = 3 \cdot 4 + 7 \cdot 0$.
  - $n = 13$: $13 = 3 \cdot 2 + 7 \cdot 1$.
  - $n = 14$: $14 = 3 \cdot 0 + 7 \cdot 2$.
- **Inductive Step**: Assume true for all m where $12 \leq m \leq k$. Prove for $k + 1$:
  - Since $k + 1 \geq 15$, consider $m = k + 1 - 3 = k - 2$. Since $k - 2 \geq 12$, $k - 2 = 3a + 7b$.
  - Then, $k + 1 = (k - 2) + 3 = 3a + 7b + 3 = 3(a + 1) + 7b$.

Well-Ordering Property

Definition

Every non-empty set of positive integers has a least element.

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- Key principle in proofs, especially for strong induction and contradiction.
- Example use: Prove a property by assuming a counterexample exists, then showing the smallest counterexample leads to a contradiction.
- Connection to induction: Well-ordering ensures the "smallest" case exists, supporting the base case and inductive reasoning.

Example: Fundamental Theorem of Arithmetic

Statement

Every integer $n > 1$ can be expressed as a product of primes (unique up to order).

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- Proof by well-ordering (sketch):
  - Assume the set S of integers $n > 1$ with no prime factorization is non-empty.
  - By well-ordering, S has a least element m.
  - If m is prime, it is its own factorization (contradiction).
  - If m is composite, $m = a \cdot b$ where $1 < a, b < m$. Since $a, b \notin S$, they have prime factorizations, so m does (contradiction).
  - Thus, S is empty, and every $n > 1$ has a prime factorization.

Summary

- **Strong Induction**: Assumes all cases $n_0 \leq m \leq k$ to prove $P(k+1)$.
- **Well-Ordering Property**: Every non-empty set of positive integers has a least element.
- Applications: Recursive sequences, divisibility, prime factorization, and algorithm correctness.
- Strong induction is more flexible than standard induction for complex dependencies.

Key Takeaway

Strong induction and well-ordering provide powerful tools for proving statements about integers, especially when multiple prior cases are needed.

Definitions

- **Weak Induction (Standard Induction)**:
  - Base Case: Prove $P(n_0)$ is true.
  - Inductive Step: Assume $P(k)$ is true, prove $P(k+1)$.
- **Strong Induction**:
  - Base Case: Prove $P(n_0), P(n_0+1), \ldots$ for necessary starting values.
  - Inductive Step: Assume $P(m)$ for all $m \leq k$, prove $P(k+1)$.
- **Well-Ordering Principle**:
  - Every non-empty set of positive integers has a least element.

Equivalence Overview

- Weak induction, strong induction, and well-ordering are logically equivalent.
- Proof strategy:
  1. Show weak induction implies strong induction.
  2. Show strong induction implies well-ordering.
  3. Show well-ordering implies weak induction.
- Equivalence means any one can be used as a foundation for proofs about integers.

Equivalence Overview

- Weak induction, strong induction, and well-ordering are logically equivalent.
- Proof strategy:
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  2. Show strong induction implies well-ordering.
  3. Show well-ordering implies weak induction.
- Equivalence means any one can be used as a foundation for proofs about integers.

Key Idea

Each principle allows us to prove statements $P(n)$ for all positive integers n, but they approach it differently.

Weak Induction $\implies$ Strong Induction

Proof Sketch

To prove $P(n)$ for all $n \geq n_0$ using weak induction:

- Define a new statement $Q(n)$: "$P(m)$ is true for all m where $n_0 \leq m \leq n$."
- **Base Case**: Prove $Q(n_0)$, i.e., $P(n_0)$ is true.
- **Inductive Step**: Assume $Q(k)$ (i.e., $P(m)$ for all $n_0 \leq m \leq k$). Prove $Q(k+1)$, i.e., $P(m)$ for all $n_0 \leq m \leq k+1$.
- Since $Q(k)$ implies $P(m)$ for $m \leq k$, use strong induction's hypothesis to prove $P(k+1)$, thus $Q(k+1)$.

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- Since $Q(k)$ implies $P(m)$ for $m \leq k$, use strong induction's hypothesis to prove $P(k+1)$, thus $Q(k+1)$.

- $Q(n)$ holds for all $n \geq n_0$, so $P(n)$ holds for all $n \geq n_0$.

Strong Induction $\implies$ Well-Ordering

Proof Sketch (by Contradiction)

Assume well-ordering is false: there exists a non-empty set S of positive integers with no least element.

- Define $P(n)$: "No integer $m \leq n$ is in S."
- **Base Case**: Prove $P(1)$ (1 is not in S, as S has no least element).
- **Inductive Step**: Assume $P(m)$ for all $m \leq k$ (no integer $\leq k$ is in S). Prove $P(k+1)$ (i.e., $k+1 \notin S$).
- If $k+1 \in S$, it would be the least element (since no $m \leq k$ is in S), a contradiction.

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- $P(n)$ true for all n implies S is empty, proving well-ordering.

Well-Ordering $\implies$ Weak Induction

Proof Sketch

To prove $P(n)$ for all $n \geq n_0$ using well-ordering:

- Assume $P(n)$ is false for some $n \geq n_0$. Let S be the set of all $n \geq n_0$ where $P(n)$ is false.
- By well-ordering, S has a least element m.
- **Base Case**: $m \neq n_0$, since $P(n_0)$ is true.
- Since m is the least element, $P(m-1)$ is true (for $m-1 \geq n_0$).
- **Inductive Step**: Use $P(m-1)$ to prove $P(m)$, contradicting $m \in S$.

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- **Inductive Step**: Use $P(m-1)$ to prove $P(m)$, contradicting $m \in S$.

- Thus, S is empty, so $P(n)$ is true for all $n \geq n_0$.

Summary

- **Weak Induction**: Uses $P(k)$ to prove $P(k+1)$.
- **Strong Induction**: Uses $P(m)$ for all $m \leq k$ to prove $P(k+1)$.
- **Well-Ordering**: Every non-empty set of positive integers has a least element.
- All three are equivalent: proving one implies the others.
- Practical use: Choose the principle that best fits the proof structure (e.g., strong induction for recursive cases, well-ordering for contradiction).

Key Takeaway

These principles form the foundation for proving statements about integers, offering flexible approaches to the same logical truth.

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- Well-Ordering Property - Every non-empty subset of S has a **least element**.
- Notice that we any set with the above proof of equivalence between strong induction, weak induction, and well-ordering principle will work for any set which satisfies the above properties.

Well-Ordered Sets.

- What is a Well-Ordered Set?
- Given some ordering $\prec$ on S, a set S along with a total order $\prec$ $(S; \prec)$ is **well-ordered** if every non-empty subset of S has a **least element**.
- **Total Order**: Given some ordering $\prec$ For any $a, b \in S$, exactly one of $a \prec b$, $a = b$, or $b \prec a$ holds.
- **Least Element**: An element $m \in S$ such that $m = x$ or $m \prec x$ for all x in a subset of S.

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Example

The positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$ with the usual order $\leq$ are well-ordered (every non-empty subset has a smallest element).

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Section 5.3: What are Recursive Definitions?

- A **recursive definition** defines an object in terms of itself.
- Two parts:
  - **Base Case**: Initial elements or values.
  - **Recursive Step**: Rules to construct new elements from existing ones.
- Used for:
  - Sequences (e.g., Fibonacci numbers).
  - Sets (e.g., well-formed formulas).
  - Functions (e.g., factorial).

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Example

Factorial: $n! = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$

Examples of Recursive Definitions

- **Sequence**: Fibonacci numbers
  - Base: $F(0) = 0$, $F(1) = 1$.
  - Recursive: $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$.

Examples of Recursive Definitions

- **Sequence**: Fibonacci numbers
  - Base: $F(0) = 0$, $F(1) = 1$.
  - Recursive: $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$.
- **Set**: Strings over alphabet $\{0, 1\}$
  - Base: Empty string ϵ is a string.
  - Recursive: If w is a string, then $w0$ and $w1$ are strings.

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- **Set**: Strings over alphabet $\{0, 1\}$
  - Base: Empty string ϵ is a string.
  - Recursive: If w is a string, then $w0$ and $w1$ are strings.
- **Function**: Ackermann's function
  - $A(m, n) = \begin{cases} n + 1 & \text{if } m = 0, \\ A(m-1, 1) & \text{if } m > 0, n = 0, \\ A(m-1, A(m, n-1)) & \text{if } m > 0, n > 0. \end{cases}$

What is Structural Induction?

- A proof technique for recursively defined objects (sets, sequences, trees, etc.).
- Similar to strong induction, but tailored to the structure of the definition.
- Steps:
  - **Base Case**: Prove the property holds for base elements.
  - **Inductive Step**: Assume the property holds for all elements used in the recursive step, then prove it for the new element.
- Useful for proving properties of recursive sets or structures.

Key Idea

Follow the recursive definition to ensure the property holds for all elements.

Example: Full Binary Trees

- A **full binary tree** is a tree where every node is either a leaf or has exactly two children (left and right).
- **Recursive Definition**:
  - **Base Case**: A single node (leaf) is a full binary tree.
  - **Recursive Step**: If T_1 and T_2 are full binary trees, a new tree T can be formed with a root node and T_1 as left child, T_2 as right child.
- Notation:
  - $L(T)$: Number of leaves in tree T.
  - $N(T)$: Total number of nodes in tree T.

Example: Theorem - Number of Nodes in a Full Binary Tree

Statement

For any full binary tree T, the number of nodes $N(T)$ satisfies:

$$N(T) = 2L(T) - 1,$$

where $L(T)$ is the number of leaves in T.

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- **Why Interesting?**
  - Relates leaves to total nodes in a recursive structure.
  - Useful in computer science (e.g., analyzing binary tree algorithms).

Structural Induction for the Proof

- We use **structural induction** to prove the theorem.
- Follows the recursive definition of full binary trees:
  - **Base Case**: Prove for the simplest tree (single node).
  - **Inductive Step**: Assume the property holds for subtrees T_1 and T_2, prove for a tree T with T_1 and T_2 as children.
- Goal: Show $N(T) = 2L(T) - 1$ for all full binary trees.

Base Case

Single Node (Leaf)

Consider a full binary tree T with a single node.

Base Case

Single Node (Leaf)

Consider a full binary tree T with a single node.

- Number of leaves: $L(T) = 1$ (the node is a leaf).
- Number of nodes: $N(T) = 1$ (only one node).
- Check:
$$2L(T) - 1 = 2 \cdot 1 - 1 = 1 = N(T).$$
- The property holds for the base case.

Inductive Step

- **Inductive Hypothesis**: Assume for full binary trees T_1 and T_2:
$$N(T_1) = 2L(T_1) - 1, \quad N(T_2) = 2L(T_2) - 1.$$
- Consider a tree T with root and children T_1 (left) and T_2 (right).
- Compute:
  - Leaves: $L(T) = L(T_1) + L(T_2)$.
  - Nodes: $N(T) = 1 + N(T_1) + N(T_2)$ (1 for the root).

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- Substitute hypothesis:
$$N(T) = 1 + (2L(T_1) - 1) + (2L(T_2) - 1).$$
$$= 1+2L(T_1)-1+2L(T_2)-1 = 2(L(T_1)+L(T_2))-1 = 2L(T)-1.$$
- The property holds for T.

Conclusion and Applications

- **Conclusion**: By structural induction, $N(T) = 2L(T) - 1$ holds for all full binary trees.
- **Applications**:
  - Analyzing binary tree structures in computer science (e.g., binary search trees, expression trees).
  - Understanding node-leaf relationships in algorithms and data structures.
- **Why Interesting?**:
  - Demonstrates structural induction on a recursive, hierarchical structure.
  - Connects mathematical proof to practical applications in computing.