Sequences, Recurrences and Series.

Lecture 10 CS20M - Theoretical Foundations of Computer Science.

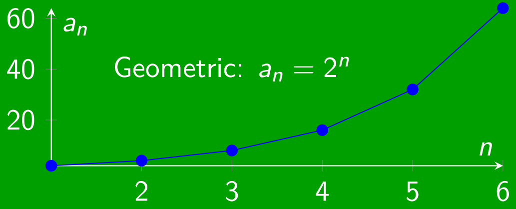
August 22, 2025

- Topics: Sequences, Limits, Cluster Points, Series, Linear Recurrences, Skolem Problem

Sequences: Definition and Basics

- **Definition**: A sequence over some set Σ, $\{a_n\}$, is a function from some subset of integers to elements of Σ, $f : \mathbb{M} \mapsto \Sigma$, where $a_n = f(n)$.
- In this chapter, we will only limit ourselves to real-valued sequences.
- A real-valued sequence, denoted by $\{a_n\}$, is a function $f : \mathbb{N} \rightarrow \mathbb{R}$, where $a_n = f(n)$.
- **Types**:
  - Arithmetic: $a_n = a_1 + (n-1)d$, e.g., 3, 5, 7, 9, $\ldots (d = 2)$
  - Geometric: $a_n = a_1 \cdot r^{n-1}$, e.g., 2, 4, 8, 16, $\ldots (r = 2)$
  - Harmonic: $a_n = \frac{1}{a_1 + (n-1)d}$, e.g., 1,1/2,1/3$\ldots (d = 1)$
  - ...
- **Properties**: Bounded if $\forall n \in \mathbb{N} : |a_n| \leq M$;
  Monotonic if $\forall n \in \mathbb{N} : a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$.

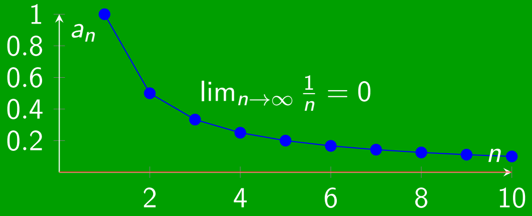
Sequence Visualization



Limits of Sequences

- For every definition from now, we will assume that our domain of discourse is Reals (unless specified otherwise).
- **Definition**: $\{a_n\}$ converges to L if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$.
- **Notation**: $\lim_{n\to\infty} a_n = L$.
- **Examples**:
  - $\{1/n\} \to 0$
  - $\{(-1)^n\}$: Does not converge
- **Properties**: Unique limit; convergent sequences are bounded.

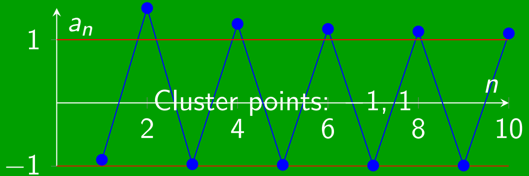
Limit Visualization



Cluster Points

- **Definition**: L is a cluster point if a subsequence $\{a_{n_k}\}$ converges to L.
- **Examples**:
  - $\{(-1)^n + \frac{1}{n}\}$: Cluster points at -1, 1.
  - $\{n\}$: No cluster points (unbounded).
- **Theorem**: Every bounded sequence in $\mathbb{R}$ has at least one cluster point (Bolzano-Weierstrass).

Cluster Points Visualization



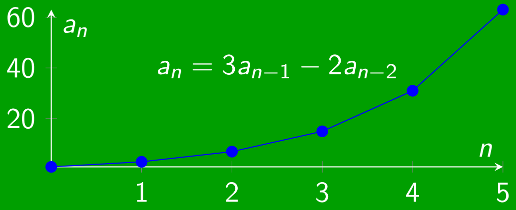
Series: Definition and Convergence

- **Definition**: A series is $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$. A series of a given sequence is a sequence of the partial sum of given sequence.
- **Convergence**: Converges if partial sums $s_n = \sum_{k=1}^{n} a_k \to S$.
- **Examples**:
  - Geometric series: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$.
  - Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$?.

Definition of Recurrences

- A **recurrence** is an equation that recursively defines a sequence, where each term is expressed as a function of its predecessors.
- Form: $a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k})$, with initial conditions for the first k terms.
- **Example**: $a_n = 2a_{n-1} + 1$, with $a_0 = 1$.
- Applied in algorithms, combinatorics, and dynamic systems to describe iterative processes.
- The recurrence is called linear if f is a linear function, otherwise it is called non-linear.

Recurrence Visualization



Closed-Form Solutions

- **Definition**: A closed-form solution is an explicit formula for the nth term, using a finite number of standard operations (e.g., +, -, *, /, exponents, logs) without recursion or infinite sums.
- **Characteristics**:
  - Direct computation for any n.
  - Useful for analysis, limits, and large n.
- **Examples**:
  - Arithmetic sequence: $a_n = a_1 + (n-1)d$.
  - Geometric sequence: $a_n = a_1 \cdot r^{n-1}$.
- **Pros/Cons**: Harder to derive; enables efficient evaluation and proofs.

The Skolem Problem

- **Definition**: Is it decidable to determine if a linear recurrence sequence $\{a_n\}$ with rational coefficients satisfies $a_n = 0$ for some n?
- **Formulation**: Given $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$, $c_i \in \mathbb{Q}$, does $\exists n$ such that $a_n = 0$?
- **Example**: Fibonacci $(a_n = a_{n-1} + a_{n-2})$ never reaches 0 for $n > 0$.
- **Is there an algorithm that terminates in finite time and give a yes/no answer to the above problem?**

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- **Is there an algorithm that terminates in finite time and give a yes/no answer to the above problem?**
- **Status**: Decidable for order ≤ 2; open for higher orders.

Skolem Problem: Challenges

- **Challenges (For interested ones)**:
  - Complex behavior (oscillatory, exponential).
  - Zeroes involve transcendental numbers.
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  - Zeroes involve transcendental numbers.
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- **Connections**: Diophantine equations, formal verification.
- **Example**: For $a_n = 3a_{n-1} - 2a_{n-2}$, $a_0 = 1$, $a_1 = 3$, $a_n = 1 + 2^{n+1} > 0$.

Summary

- **Sequences**: Ordered lists; study convergence, boundedness.
- **Limits**: Unique for convergent sequences.
- **Cluster Points**: Limits of subsequences.
- **Series**: Convergence via partial sums.
- **Recurrences**: Solved via characteristic equations.
- **Skolem Problem**: Open problem on decidability of zeroes.

Questions and Further Reading

- **Questions**:
  - Can you find a sequence with two cluster points?
  - Why is the Skolem Problem significant?
- **Extra Reading**:
  - Bartle and Sherbert, "Introduction to Real Analysis"
  - Graham, Knuth, Patashnik, "Concrete Mathematics".
  - Research on Skolem Problem (e.g., Tao's work).

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  - "It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for 'linear' automata!” - Terrance Tao.
- **Contact**: Your Email