Propositional Equivalences

- Discrete Mathematics (Kenneth Rosen)
  - 8th edition – Section 1.3

Motivation

- How do we simplify complex logical statements?
- Can we replace a proposition with an equivalent one?
- Why it matters: circuits, automating proofs, solving puzzles, checking correctness of a program, checking feasibility etc.

Definition of Logical Equivalence

- $p \equiv q$ means $p \leftrightarrow q$ is a tautology.
- Example: $\neg(p \land q) \equiv \neg p \lor \neg q$

Tautology, Contradiction, Contingency

- Tautology: always true (e.g., $p \lor \neg p$)
- Contradiction: always false (e.g., $p \land \neg p$)
- Contingency: sometimes true, sometimes false (e.g., $p \land q$)

Common Logical Equivalences

- Identity: $p \lor F \equiv p$, $p \land T \equiv p$
- Domination: $p \lor T \equiv T$, $p \land F \equiv F$
- Idempotent: $p \lor p \equiv p$, $p \land p \equiv p$
- Double Negation: $\neg(\neg p) \equiv p$

More Logical Equivalences

- Commutative: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Associative: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

De Morgan’s Laws

- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $\neg(p \lor q) \equiv \neg p \land \neg q$

Implication Laws

- $p \rightarrow q \equiv \neg p \vee q$
- Contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- Note: Converse ($q \rightarrow p$) and inverse ($\neg p \rightarrow \neg q$) are not equivalent to $p \rightarrow q$. Are they equivalent to each other?

Biconditional Equivalences

- $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
- $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

Puzzle: Which Are Equivalent?

- A. $\neg(p \wedge q)$
- B. $\neg p \wedge \neg q$
- C. $\neg p \vee \neg q$
- D. $\neg(p \vee q)$

Puzzle: Simplify to p

- 1. $(p \vee q) \wedge (p \vee \neg q)$
- 2. $\neg(\neg p \wedge \neg q)$
- 3. $p \vee (p \wedge q)$
- 4. $p \wedge (p \vee q)$

Truth Table Challenge

- Simplify: $\neg(p \rightarrow q) \vee (\neg q \rightarrow \neg p)$
- Hint: Use implication identity and De Morgan's laws

Application: Simplifying Circuits

- Use equivalence to reduce logic gates.
- Example: $(p \wedge T) \vee (p \wedge F) \rightarrow p$

Satisfiability Problem (SAT)

- A formula is satisfiable if some assignment makes it true.
- Unsatisfiable if no assignment makes it true.
- SAT is the first known NP-complete problem.

SAT vs Tautology

- Satisfiable: at least one assignment is true.
- Tautology: all assignments are true.
- Example: $p \lor \neg p$ is both; $p \land \neg p$ is neither.

Solving SAT – Brute Force

- 1. List all assignments
- 2. Evaluate the expression
- 3. Check if any assignment makes it true

Solving SAT – Smart Techniques

- Use equivalence rules and simplification.
- Convert to CNF for SAT solvers.
- Apply DPLL, resolution, etc.

Satisfiability in Daily Life

- Scheduling constraints
- Smart home automation logic
- Games like Sudoku, Minesweeper

Knights and Knaves – Setup

- Island where:
- - Knights always tell the truth
- - Knaves always lie
- You meet A and B.
- A says: 'B is a knave'
- B says: 'A and I are opposites'

Knights and Knaves – Case 1: A is a knight

- If A is a knight → B is a knave (truth)
- B says 'A and I are opposites'
- But knaves lie → statement must be false
- So A and B are not opposites ⇒ contradiction
- → A cannot be a knight

Knights and Knaves – Case 2: A is a knave

- If A is a knave → 'B is a knave' is false ⇒ B is a knight
- B says 'A and I are opposites' → true (B is a knight)
- ☑ Consistent with all statements
- Conclusion: A is a knave, B is a knight

Truth-Teller, Liar & Random – Setup

- Three people:
  - One always tells the truth
  - One always lies
  - One answers randomly
- **Three Doors**
  - One leads to **freedom**
  - The other two lead to **doom** (or nothing)
- You can ask **one yes/no question** to one person
- Goal: Find the correct door to freedom

Truth-Teller, Liar & Random – Hint

- Can't rely on random person
- Need a question that works for both truth-teller and liar
- Use a self-referential question:
- Something similar to :
  'If I asked you whether door A is correct, would you say yes?'

Truth-Teller, Liar & Random – Logic

- Truth-teller answers truthfully about what they'd say
- Liar lies about what they'd say (double negation)
- → Both give consistent answer!
- ☑ Go to door A if answer is YES

Logic Grid Puzzle – Setup

- People: Alice, Bob, Charlie
- Drinks: Tea, Coffee, Juice
- Information Given as follows:
  - Alice didn’t order coffee
  - The person who ordered juice is not Bob
  - Charlie didn't order tea
- Who Ordered what?

Logic Grid Puzzle – Deduction

- From the information:
  - Alice → tea or juice
  - Bob → tea or coffee
  - Charlie → coffee or juice
- Try assigning:
- Alice → Tea, Bob → Coffee, Charlie → Juice
- ✔ All the constraints are satisfied.

Take Home Assignment:

Solve all this puzzle using propositional logic.

Summary

- Equivalence: Replace and simplify logic expressions
- SAT: Determine if some assignment satisfies a formula
- Logic applies to daily life, puzzles, AI, and computing.