# A Performance Analysis of Coherently Demodulated PSK using a Digital Phase Estimate in Frequency Hopped Systems

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*Abstract*-Most existing frequency hopping AJ systems utilize a non-coherent waveform such as Differential Phase Shift Keying (DPSK) or Frequency Shift Keying (FSK). This choice is based on a trade-off between the efficiency of the waveform and the difficulties entailed in reestablishing a phase reference at the beginning of every hop.

An alternative technique that is being explored for several planned systems is the use of reference symbols combined with coherent demodulation. If several reference symbols of known phase are sent during each hop then the receiver can establish the absolute carrier phase and perform coherent demodulation. This paper explores the performance of frequency hopped systems utilizing reference symbols with a PSK waveform in both a benign and stressed environment. The results indicate that if as few as 10 reference symbols/hop are used that near coherent performance is obtained. Furthermore, the use of reference symbols yielded performance better than that obtained by differential encoding and non-coherent detection for all cases considered.

## 1 Introduction

In frequency hopping systems, the carrier phase changes from one hop to the next, making it particularly difficult for the receiver to establish a phase reference. This problem is usually overcome at the expense of energy efficiency by using differential encoding, and non-coherent detection. It is well known that the use of coherent demodulation results in an improvement in energy efficiency over non-coherent demodulation of the same waveform. This paper explores the use of digital phase estimation techniques which allow a phase reference to be established with little or no increase in demodulator complexity. The phase estimates are formed by observing periods of unmodulated carrier or alternatively periods of a-priori known symbols. This approach yields an energy efficient waveform which will provide improved performance in future SATCOM systems. A detailed analysis of the performance of frequency hopped systems that use digital phase estimation in both unstressed and jammed scenarios is provided.

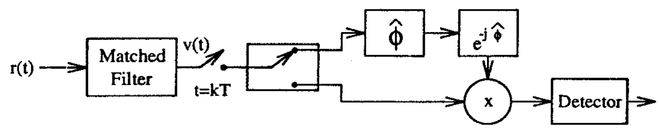


Figure 1: System with Digital Phase Estimation

The results indicate that the digital phase estimation approach gives better energy efficiency than non-coherent DPSK or DQPSK. In the unstressed case, almost coherent performance is achieved when 4 symbols/hop are used to form the phase estimate. In the case of jamming, 10 reference symbols/hop yield similar results.

## 2 Analysis

A block diagram of a system employing digital phase estimation appears in Figure 1. The system consists of two parts, a normal detector and a phase estimation/correction network. Note that all paths are complex and that it has been assumed that the phase reference has the same pulse shape (and hence uses the same matched filter) as the data. During normal operation, the phase estimator will only operate on the part of the hop that contains the phase reference, while the detector will work on the part of the hop that contains the data. The analysis will proceed from the following assumptions:

1. The reference symbols are a-priori known.
2. The carrier frequency is known.

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3. Bit synchronization has already been performed.
4. There is no inter-symbol interference.

Using complex baseband notation, the received waveform for BPSK or QPSK over one hop can be expressed as

$$r(t) = \sqrt{2E_b}\sum_{n=0}^{M}(I_n + jQ_n)u(t - nT_s)e^{j\phi} + n(t) \quad (1)$$

where

$$\begin{aligned} I_n, Q_n &= \text{I and Q data bits respectively} \\ u(t) &= \text{normalized modulating pulse,} \\ &\quad \int_{-\infty}^{\infty} u(t)u^*(t)\,dt = 1 \\ n(t) &= \text{complex Gaussian noise process} \\ &\quad \text{with single-sided PSD } N_0 \\ E_b &= \text{bit energy} \\ T_s &= \text{symbol duration} \\ \phi &= \text{carrier phase} \\ M &= \text{number of symbols/hop} \end{aligned}$$

## 2.1 Unstressed Performance

In the unstressed case, the primary source of degradation will be errors in the phase estimate caused by the channel noise. Let $v(t)$ be the output of the matched filter, then at the sampling instances we have

$$\begin{aligned} v_n = v(nT_s) &= \int_{(n-1)T_s}^{nT_s} r(t)u^*(t)\,dt \\ &= \sqrt{2E_b}e^{j\phi}(I_n + jQ_n) + \tilde{n}, \end{aligned}$$

where

$$\tilde{n} \equiv \int_{-\infty}^{\infty} n(t)u^*(t - nT_s)\,dt$$

is WGN with zero mean, and variance

$$\sigma_{\tilde{n}}^2 = \frac{1}{2}\mathcal{E}\{\tilde{n}\tilde{n}^*\} = N_0. \quad (2)$$

In order to reduce the variance of the phase estimate $\hat{\phi}$ it will be assumed that the output of the matched filter will be digitally integrated over a time period of $kT_s$ (i.e., we will sum $k$ samples for each phase estimate). The phase estimate will therefore be formed from

$$v = \frac{1}{s}\sum_{i=1}^{k} v_{m_i}(I_{m_i} + jQ_{m_i})^*, \quad (3)$$

where $s = |I + jQ|$ is 1 for BPSK and $\sqrt{2}$ for QPSK and $m_i$ is the index of the $i^{th}$ reference symbol (i.e. we are simply de-rotating the phase of the reference chips so that they may be coherently combined). Therefore $v$ is a complex Gaussian random variable with mean

$$\mu_v = ks\sqrt{2E_b}e^{j\phi}$$

and variance

$$\sigma_v^2 = kN_0.$$

The phase estimate considered here is formed as

$$\hat{\phi} = \tan^{-1}\left(\frac{\Im\{v\}}{\Re\{v\}}\right) \quad (4)$$

which is both the Maximum likelihood [1] and the Maximum a-posteriori [2] estimate of a constant phase in Gaussian noise under the assumption of a uniform prior. Since $\hat{\phi}$ is only used to unwrap the carrier phase, it is not necessary to calculate it explicitly. We may simply compute the decision statistic for the $n^{th}$ symbol as

$$y_n = v_n v^*$$

and then perform detection on $y_n$ as if it were the output of the matched filter in a perfectly coherent system.

**BPSK:**
For BPSK, the detector will decide that a 1 was sent if $\Re\{y_n\} > 0$ and 0 otherwise. Using the fact that $\Re\{x\} = \frac{1}{2}(x + x^*)$, we can write the error probability as

$$p_e = \mathcal{P}\{y + y^* < 0 | 1 \text{ was sent}\}.$$

But this is simply a special case of a Hermitian form of Gaussian random variables which is detailed in Appendix 4B of [3]. The error rate is given as

$$p_e = Q(a,b) - \frac{1}{2}I_0(ab)e^{-\frac{a^2+b^2}{2}}, \quad (5)$$

where $Q(\cdot,\cdot)$ is Marcum's Q function, $I_0(\cdot)$ is the $0^{th}$ order modified Bessel function of the first kind, and the values of $a$ and $b$ may be calculated from the statistics of $v_n$ and $v$.

For the case of BPSK, we have

$$\begin{aligned} a &= \sqrt{\frac{E_b}{2N_0}}\left(\sqrt{k} - 1\right) \\ b &= \sqrt{\frac{E_b}{2N_0}}\left(\sqrt{k} + 1\right). \end{aligned}$$

The results are plotted in Figure 2. Note that for $k = 1$,

$$p_e = Q\left(0, 2\sqrt{\frac{E_b}{2N_0}}\right) - \frac{1}{2}I_0(0)e^{-\frac{E_b}{N_0}} = \frac{1}{2}e^{-\frac{E_b}{N_0}}$$

which is the well known result for the non-coherent detection of DBPSK.

**QPSK:**

For QPSK, we can make independent decisions on the I and the Q channels for each symbol. Using the same method outlined above, we obtain

$$\begin{aligned} a &= \sqrt{\frac{E_b}{N_0}}\sqrt{k - \sqrt{2k} + 1} \\ b &= \sqrt{\frac{E_b}{N_0}}\sqrt{k + \sqrt{2k} + 1}. \end{aligned}$$

The results appear in Figure 3. Once again setting $k = 1$ yields the result for the non-coherent detection of DQPSK, [3] p. 271.

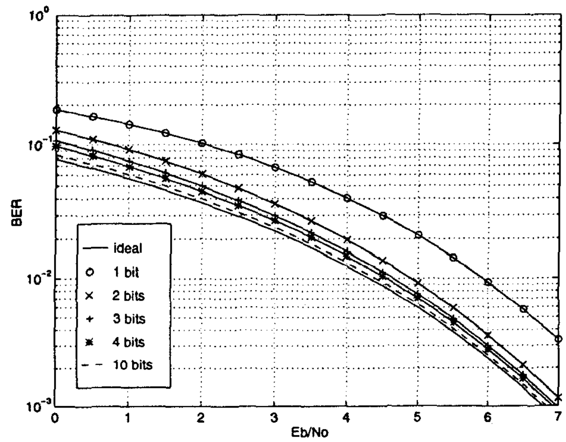


Figure 2: $p_e$ vs $E_b/N_0$ for BPSK

In comparing the performance of BPSK with QPSK, it is important to note that the SNR available to the phase estimator is $E_b/N_0$ for BPSK but is $2E_b/N_0$ for QPSK! This is due to the fact that our matched filter integrates over a complete symbol and therefore yields the energy of two bits for 4-ary modulation. The phase estimates give almost coherent performance for $k = 4$ with degradations of 0.2 dB (BPSK) and 0.75 dB (QPSK) from coherent at $p_e \approx 10^{-2}$, a typical operating point for coded systems. Therefore in an unstressed scenario, the performance of the phase estimate is quite good.

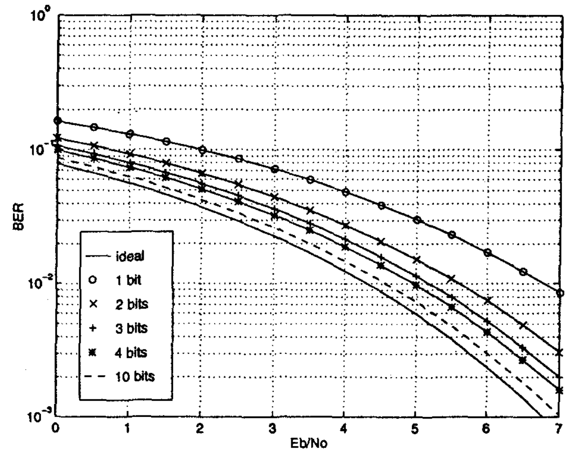


Figure 3: $p_e$ vs $E_b/N_0$ for QPSK

## 2.2 Stressed Performance

In this section, the results will be extended to include the case where the signal is jammed. A jammer will degrade performance in two ways; it will corrupt the phase reference increasing $p_e$ for all bits that depend on the jammed reference, and it will also interfere with the data bits directly. The BER for a single hop is determined here under the assumption that the jammer has landed on the center frequency of the signal. This can be expected for low resolution frequency hopped systems and is worst case for high resolution systems. It is also assumed that the jammer is transmitting at full power for the entire duration of the hop.

Before embarking on the analysis, it is worthwhile discussing some of the salient features of the system design. Since jamming the phase reference alone will prevent the data from being recovered, it is necessary to disguise the phase reference in the data. This can be done by pseudo-randomly permuting the reference symbols throughout the hop. If this is not done, then the jammer can simply turn on for the reference symbols, and achieve a large gain in effectiveness.

The actual reference symbols that are chosen also play a role in the susceptibility of the phase estimate to jamming. For an M-ary PSK system, we will show that it is desirable to use $lM$ reference symbols and send *exactly* $l$ of each phase reference symbol. If this is done, then the jammer's optimal strategy is to pretend to be a user, and send random symbols. Furthermore, under this criterion, the effects of a tone jammer (or a jammer that repeats the same symbol) will be totally canceled in the phase estimate. This can be seen in the phase diagrams for QPSK ($l = 1$).

In Figure 4 we show the signal and jammer vectors before de-rotation. Here, the jammer components add coherently while the reference symbols are distributed over the four phases. After de-rotation, the energy of the four reference symbols add coherently, while the jammer energy is out of phase and cancels completely, Figure 5. This forces the jammer to send random symbols, and hope that it's energy will add after de-rotation. The result is that the jammer's ability to corrupt the phase estimate is greatly reduced.

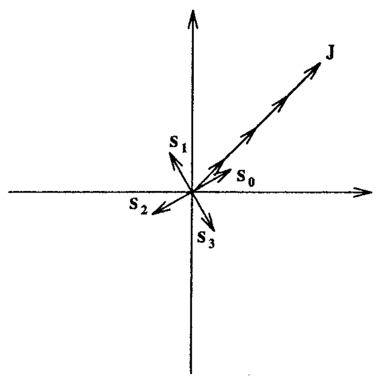


Figure 4: Phase Diagram for QPSK with Jamming

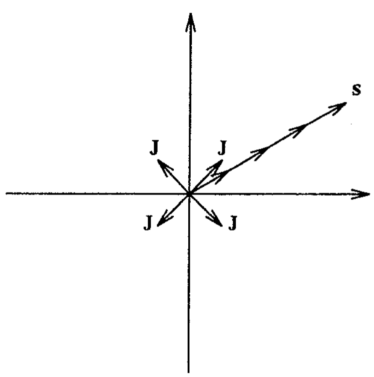


Figure 5: Phase Diagram for QPSK with Jamming after De-rotation

The results given here are for a hop that is jammed (i.e. the jammer lands on the center frequency of the signal). It is therefore necessary to adjust these results by the probability that a hop will be hit by the jammer to determine the actual performance of a given system, i.e.,

$$p_e = p_j \; p_{e|\,\text{jammed}} + (1 - p_j) \; p_{e|\,\text{not jammed}}, \quad (6)$$

where $p_j$ is the probability that the jammer will hit the signal.

The analysis proceeds as before but now equation (1) must be modified to take into account the fact that a jammer is present. The received waveform is given by

$$\begin{aligned} r(t) &= \sum_{n=0}^{M} \sqrt{2E_b}(I_n + jQ_n)u(t - nT_s)e^{j\phi} \\ &+ \sqrt{2E_j}(\alpha_n + j\beta_n)u_j(t - nT_s - \tau_j)e^{j\theta} + n(t), \end{aligned}$$

where $E_j$ is the jammer's energy, $u_j(t)$ is the normalized pulse shape of the jamming waveform, $\tau_j$ is the jammer's timing offset, $\alpha_n$ and $\beta_n$ are the I and Q jammer bits, and $\theta$ is the phase of the jammer's carrier. The output of the matched filter is now given by

$$\begin{aligned} v_n &= v(nT_s) \\ &= \sqrt{2E_b}e^{j\phi}(I_n + jQ_n) + \sqrt{2\delta E_b}e^{j\theta}\rho(\alpha_n + j\beta_n) \\ &\quad + \tilde{n}, \end{aligned}$$

where $E_j = \delta E_b$ and

$$\rho = \int_{-\infty}^{\infty} u_j(t - \tau_j)u^*(t)\, dt$$

is the cross-correlation between the user waveform and the jammer waveform. Note that because both $u_j(t)$ and $u(t)$ are normalized, we have $|\rho| \leq 1$ with equality when $u_j(t) = u(t)$ and $\tau_j = 0$. The mean of the matched filter output for the data symbols is given by

$$\mu_{v_n} = \sqrt{2E_b}e^{j\phi}(I_n + jQ_n) + \sqrt{2\delta E_b}e^{j\theta}\rho,$$

where the jammer's data symbol has been absorbed into the jammer phase $\theta$. The statistics for $v$ now become

$$\mu_v = k\sqrt{2E_b}e^{j\phi}(I_n + jQ_n) + (p + jq)\sqrt{2\delta E_b}e^{j\theta}\rho,$$

and

$$\sigma_v^2 = kN_0.$$

Note that $p$ and $q$ are the I and Q sum of the jammer symbols after the de-rotation in (3) is performed,

$$\begin{aligned} p &= \sum_{i=1}^{k} I_{m_i}\alpha_{m_i} \\ q &= \sum_{i=1}^{k} Q_{m_i}\beta_{m_i}. \end{aligned}$$

Because the data symbols are $\pm 1$, $p$ and $q$ are independent random walks of length $k$. The probability of a particular realization of a random walk of length $k$ is given by [4] as

$$\mathcal{P}\{x = 2i - k\} = 2^{-k} \binom{k}{k-i}, \quad 0 \leq i \leq k.$$

To determine the probability of bit error, it is now necessary to average over all realizations of $p$ and $q$ and all jammer phases,

$$p_e = \sum_{i=0}^{k} \sum_{j=0}^{k} 2^{-2k} \binom{k}{k-p} \binom{k}{k-q} \int_{-\pi}^{\pi} p_e(p, q, \theta)\, d\theta, \tag{7}$$

where $p = 2i - k$ and $q = 2j - k$. While the expression for $p_e(p, q, \theta)$ can be explicitly given in terms of the statistics of $v$ and $v_n$, it is not presented here as it is very cumbersome and lends no additional intuition into the problem.

The integral in (7) has been numerically evaluated using Romberg's method [5] for BPSK and QPSK. The results assume that $|\rho| = 1.0$, which is worst case. Figures 6 and 7 show the effect of varying the number of bits used for the phase estimate for the case where the jammer to signal power is $-10dB$ for BPSK and QPSK respectively. The curve labeled *coherent* shows the performance in the given jamming scenario when the signal phase is known perfectly,

$$p_e = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q\left(\sqrt{\frac{2E_b}{N_0}} \left[1 + \sqrt{\delta}\rho\cos\theta\right]\right)\, d\theta,$$

where $Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt$. Note that even for a $-10dB$ $J/S$ ratio, the loss at high SNR is quite dramatic as compared to the unstressed case in Figures 2 and 3. This degradation can be largely attributed the jammer's effect on the data bits as is evidenced by the fact that the performance is still very close to coherent. It is important to keep in mind that the figures give $p_e$ for the case where the jammer lands on the signal's center frequency, and that the composite system error rate must be obtained from (6).

The effects of jamming can also be viewed as a family of curves indexed by $J/S$ for a given phase estimate integration period. This is done for BPSK and QPSK in Figures 8 and 9 when $k = 10$. Note that the dashed lines give the coherent performance (i.e, when the phase is perfectly known). Once again, we see that the additional degradation due to the jamming of the reference symbols is quite small.

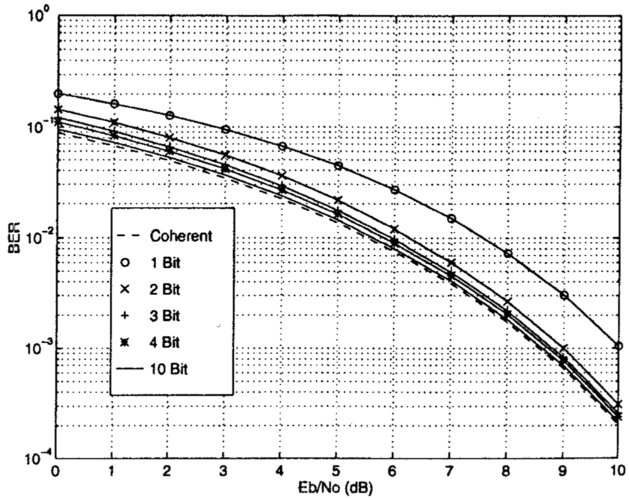


Figure 6: $p_e$ vs $E_b/N_0$ and $k$ for BPSK with $J/S = -10$ dB

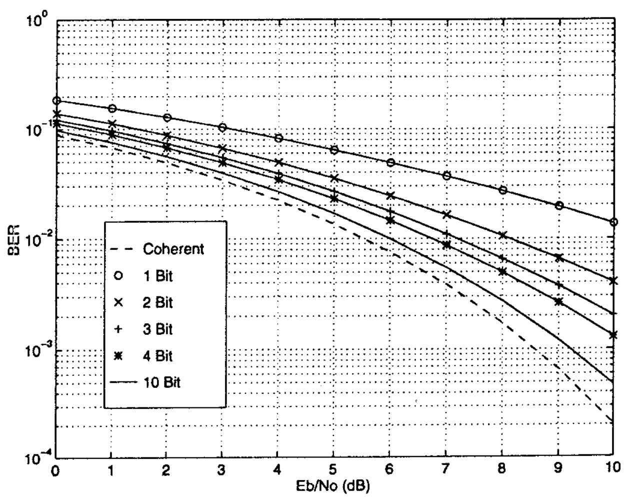


Figure 7: $p_e$ vs $E_b/N_0$ and $k$ for QPSK with $J/S = -10$ dB

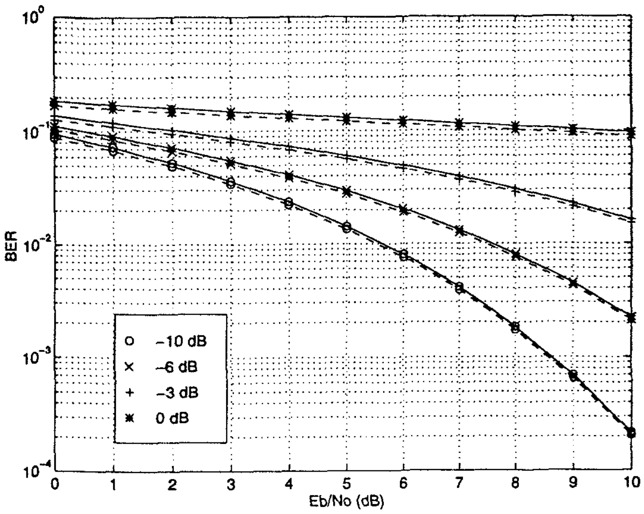


Figure 8: $p_e$ vs $E_b/N_0$ and $J/S$ for BPSK with $k = 10$

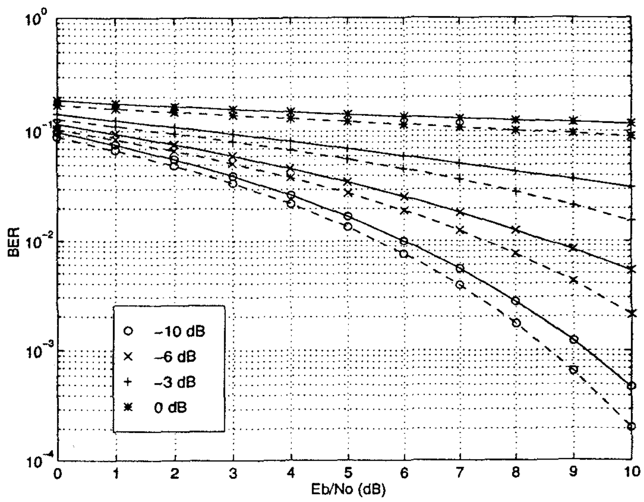


Figure 9: $p_e$ vs $E_b/N_0$ and $J/S$ for QPSK with $k = 10$

# 3 Conclusions

This paper has investigated the use of digital phase estimation in the detection of several PSK modulated waveforms. It has been shown that in an unstressed scenario, that as few as 4 reference symbols provide a reasonable phase estimate. In an environment with jamming, the phase estimate is degraded but remains reasonably close to coherent performance when 10 reference symbols are used. The results indicate that the use of reference symbols improves the energy efficiency of the waveform with respect to the non-coherent detection of differentially encoded PSK in both an unstressed and stressed environment.

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