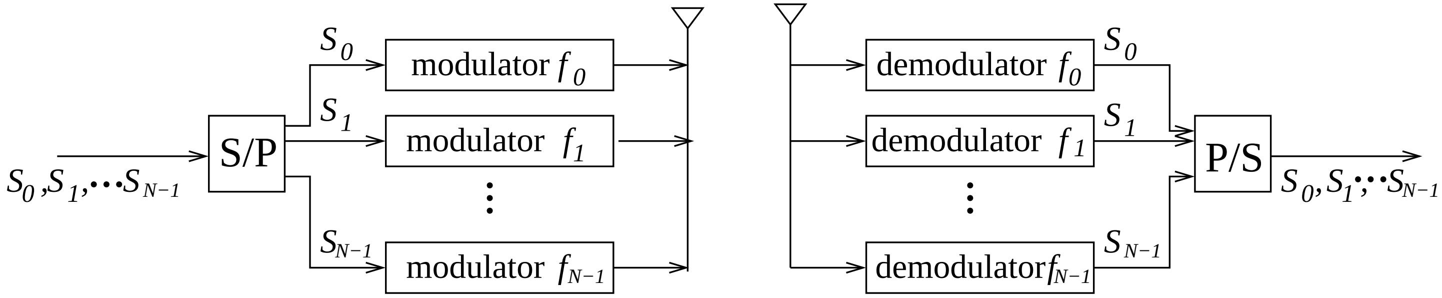
Revision of Lecture Twenty-Four

- Previous lecture focuses on high-layer protocol issues and emphasises with two different communication protocols:
  - Relying on end-to-end connections
  - Do not require end-to-end connections, relaying on store-carry-and-forward
- Next three lectures, we return to physical layer
- Transmission techniques we discussed so far are **single-carrier** systems
  - Information symbols are modulated with a single carrier to occupy the given channel bandwidth
- We now introduce **multi-carrier** systems:
  - A block of information symbols are modulated with multiple carriers to occupy the given channel bandwidth

OFDM: Motivations

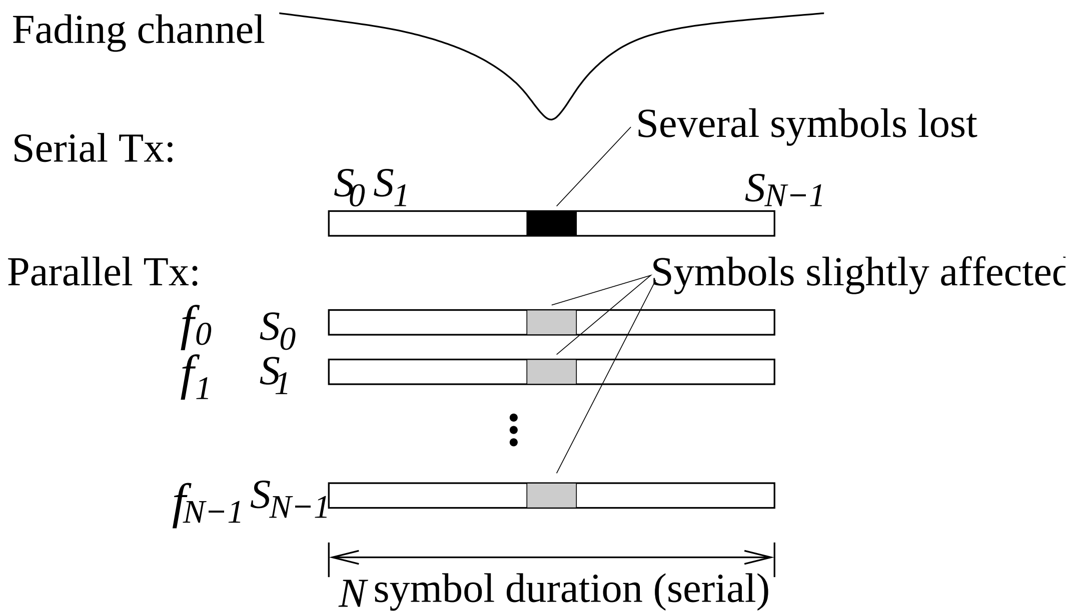
- Orthogonal frequency division multiplexing applies **multicarrier** modulation principle
  - Dividing the data stream into several bit streams, each of which has much lower bit rate, and using these substreams to modulate several carriers
- Basic **OFDM** system: Imaging this system "configuration"



What OFDM is good for?

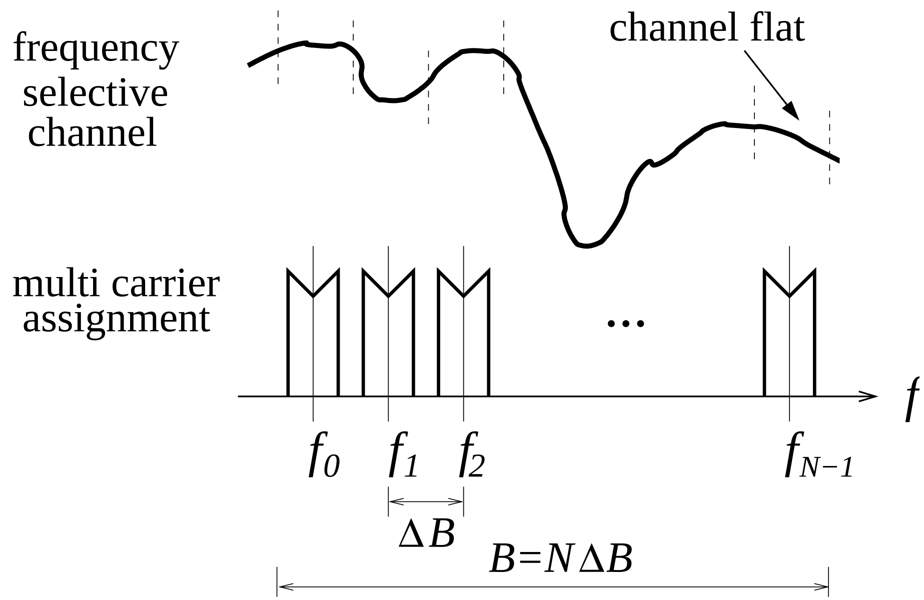
1. **Combating fading**: in a "parallel" transmission, each symbol in a sub-carrier has a much larger symbol duration, equal to N times of the symbol duration in "serial" transmission

   In a deep fade, several symbols in the single carrier system can be affected seriously and lost completely. However, in parallel transmission, each of the N symbols is only slightly affected and can still be recovered correctly



OFDM: Motivations (continue)

2. **Combating frequency selective**:
   - Channel can be severely frequency selective, but for each sub-carrier, the sub-channel is flat or at least only slightly frequency selective



What OFDM is bad for?

- **High peak to average power**
  - With N sinusoidal signals added together, the peak amplitude becomes very large, which will be clicked by amplifier and channel's nonlinear saturation, causing distortion

$$\sum_{i=1}^{N} A_i \cos(\omega_i t + \varphi_i) \Rightarrow \text{ Peak: } \sum_{i=1}^{N} A_i$$

- **High sensitivity** to carrier offset, phase noise, and timing error
- **High complexity**: to be effective, number of sub-carriers N should be large
  - If OFDM is implemented with N modulators/demodulators, the complexity will be enormous
  - Fortunately, it can be implemented alternatively using DFT/FFT to reduce this high complexity
- As DFT/FFT is vital to communication signal processing in general, let us discuss it

Fourier Transform Pair

- If a discrete-time aperiodic signal $x(k)$ satisfies
$$\sum_{k=-\infty}^{\infty} |x(k)| < \infty$$
then
$$\text{FT:} \quad X(\omega) = \sum_{k=-\infty}^{\infty} x(k) \exp(-j\omega k) \qquad \text{IFT:} \quad x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\exp(j\omega k)\, d\omega$$
Integration in IFT can also be over 0 to 2π
- Spectra: $X(\omega) = |X(\omega)| \exp(j\angle X(\omega))$, with $|X(\omega)|$ being the amplitude spectrum and $\angle X(\omega)$ the phase spectrum of $x(k)$
- Parseval's theorem:
$$\sum_{k=-\infty}^{\infty} |x(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$
where $|X(\omega)|^2$ is the energy spectral density, giving distribution of signal energy in frequency domain. In practice, the power spectral density is more often used
- **Differences**:
  - Continuous-time: f or $2\pi f$ has the unit of Hz or radian/s, and ranges in $(-\infty, \ \infty)$. FT is an integral
  - Discrete-time: ω has the unit of radian, and ranges in $[-\pi, \ \pi]$ or $[0, \ 2\pi]$. FT is a summation and $X(\omega)$ is periodic with period 2π

Discrete-Time Fourier Series

- If $x(k)$ is periodic with period K, i.e. $x(k) = x(k+K)$, $x(k)$ can be expressed by DFS:

$$x(k) = \sum_{n=0}^{K-1} c_n \exp(j\omega_n k), \quad \omega_n = \frac{2\pi n}{K}$$

  Note there are K frequency components $\exp(j\omega_n k)$ for $0 \le n \le K-1$ and $0 \le \omega_n < 2\pi$, and the Fourier coefficients

$$c_n = \frac{1}{K} \sum_{k=0}^{K-1} x(k) \exp(-j\omega_n k), \quad 0 \le n \le K-1$$

  provide the amplitudes and phases for frequency components $\exp(j\omega_n k)$

- **Differences** in periodic signal:
  - Continuous-time: has infinite frequency components, and Fourier coefficients are integrals
  - Discrete-time: has finite frequency components, and Fourier coefficients are summations
- In theory, $X(\omega)$ is all we need but let us consider some practical constraints
  - Computing $X(\omega)$ requires infinite summation, that is, infinite number of samples $\rightarrow$ one can only approximate it by a finite signal samples in a finite summation
  - Displaying $X(\omega)$ requires ω taking values continuously in $[0,\ 2\pi)$ $\rightarrow$ one can only approximate it at finite discrete points ω_n, that is, sample $X(\omega)$ and take only a finite spectrum samples.

  These considerations leads to discrete-time Fourier transform

Discrete-Time Fourier Transform

- Windowing data so that $x(k) = 0$ for $k < 0$ and $k \geq L$, i.e. a finite sequence $x(k)$ of length L $\longrightarrow$ the corresponding Fourier transform is

$$X(\omega) = \sum_{k=0}^{L-1} x(k) \exp(-j\omega k), \quad 0 \leq \omega < 2\pi$$

- Sample $X(\omega)$ at frequencies $\omega_n = 2\pi n/K$, $0 \leq n \leq K-1$, where $K \geq L$ $\longrightarrow$ the resulting spectrum samples or DFT of $\{x(k)\}$ is

$$X(n) = X(\omega_n) = \sum_{k=0}^{L-1} x(k) \exp(-j2\pi nk/K) = \sum_{k=0}^{K-1} x(k) \exp(-j2\pi nk/K)$$

- Inverse DFT (IDFT) is:

$$x(k) = \frac{1}{K} \sum_{n=0}^{K-1} X(n) \exp(j2\pi nk/K), \; 0 \leq k \leq K-1$$

- **DFT: time samples $\{x(k)\}$ of length $L \leq K$ $\Leftrightarrow$ frequency samples $\{X(n)\}$ of length K**
- For $K \geq L$, $\{x(k)\}_{k=0}^{L-1}$ can be exactly reconstructed from $\{X(n)\}_{n=0}^{K-1}$
  Otherwise, time folding or aliasing occurs $\longrightarrow$ This is dual to spectral folding or aliasing when sampling frequency is less than the Nyquist rate

Example

For 6-point sequence $x(k) = k + 1$, $0 \leq k \leq 5$, the spectrum $X(\omega)$:

$$X(\omega) = \sum_{k=0}^{5} x(k) \exp(-j\omega k) = \sum_{k=0}^{5} (k+1) \exp(-j\omega k), \quad 0 \leq \omega < 2\pi$$

Evaluate $X(\omega)$ at the 4 frequencies $\omega_n = 2\pi n/4$, $0 \leq n \leq 3$:

$$X(n) = \sum_{k=0}^{5} (k+1) \exp(-j2\pi nk/4), \quad 0 \leq n \leq 3$$

or

$$X(0) = 21, \; X(1) = 3 - 4j, \; X(2) = -3, \; X(3) = 3 + 4j$$

The IDFT for the resulting 4 samples $X(n)$, $0 \leq n \leq 3$:

$$\hat{x}(k) = \frac{1}{4} \sum_{n=0}^{3} X(n) \exp(j2\pi nk/4), \quad 0 \leq k \leq 3$$

or

$$\hat{x}(0) = 6, \; \hat{x}(1) = 8, \; \hat{x}(2) = 3, \; \hat{x}(3) = 4$$

This example illustrates time aliasing (note $x(0) = 1, x(1) = 2, x(2) = 3, x(3) = 4$)

To avoid time aliasing, frequency samples K must be no less than time samples L

Fast Fourier Transform

- Recall that DFT: $\{x(k)\}_{k=0}^{K-1} \Longleftrightarrow \{X(n)\}_{n=0}^{K-1}$. By introducing $W_K = \exp(-j2\pi/K)$,

$$\text{DFT:} \quad X(n) = \sum_{k=0}^{K-1} x(k) W_K^{kn}, \ 0 \le n \le K-1$$

$$\text{IDFT:} \quad x(k) = \frac{1}{K} \sum_{n=0}^{K-1} X(n) W_K^{-kn}, \ 0 \le k \le K-1$$

- Direct computation of DFT can be costly for large K: $2K^2$ trigonometric functions, K^2 multiplications, and $K(K-1)$ additions
- Let $K = LM$. Data can either be stored in one-dimensional array: $\{x(k)\}$ with $0 \le k \le K-1$ or in two-dimensional array: $x(l,m)$ indexed by l and m with $0 \le l \le L-1$ and $0 \le m \le M-1$
- Row wise:

$k = Ml + m$

$$\left|\begin{array}{ccc} x(0,0) & \cdots & x(0,M-1) \\ x(1,0) & \cdots & x(1,M-1) \\ & \vdots & \\ x(L-1,0) & \cdots & x(L-1,M-1) \end{array}\right\|\begin{array}{ccc} x(0) & \cdots & x(M-1) \\ x(M) & \cdots & x(2M-1) \\ & \vdots & \\ x((L-1)M) & \cdots & x(LM-1) \end{array}\right|$$

- Column wise:

$k = l + mL$

$$\left|\begin{array}{ccc} x(0,0) & & x(0,M-1) \\ x(1,0) & \cdots & x(1,M-1) \\ \vdots & & \vdots \\ x(L-1,0) & & x(L-1,M-1) \end{array}\right\|\begin{array}{ccc} x(0) & & x((M-1)L) \\ x(1) & \cdots & x((M-1)L+1) \\ \vdots & & \vdots \\ x(L-1) & & x(LM-1) \end{array}\right|$$

FFT Algorithms

- Similarly, $X(n), 0 \le n \le K-1 \Longleftrightarrow X(p,q), 0 \le p \le L-1, 0 \le q \le M-1$ with row wise: $n = Mp + q$ or column wise: $n = p + qL$
- Assuming column wise for $x(k)$ and row wise for $X(n)$, then

$$X(p,q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m) W_K^{(l+mL)(Mp+q)}$$

where $W_K^{(l+mL)(Mp+q)} = W_K^{Mlp} W_K^{Kmp} W_K^{lq} W_K^{Lmq}$. But $W_K^{Mlp} = W_{K/M}^{lp} = W_L^{lp}$, $W_K^{Kmp} = 1$, and $W_K^{Lmq} = W_{K/L}^{mq} = W_M^{mq}$. Thus:

$$X(p,q) = \underbrace{\sum_{l=0}^{L-1} \underbrace{\left(W_K^{lq} \underbrace{\left[\sum_{m=0}^{M-1} x(l,m) W_M^{mq} \right]}_{\text{step 1}} \right)}_{\text{step 2}} W_L^{lp}}_{\text{step 3}}$$

- The computation of DFT can be divided into three steps as shown in the next slide

FFT Algorithms (continue)

- Algorithm one:
  1. For $0 \le l \le L-1$, compute the M-point DFTs:

$$F(l,q) = \sum_{m=0}^{M-1} x(l,m) W_M^{mq}, \quad 0 \le q \le M-1$$

  2. For $0 \le l \le L-1$ and $0 \le q \le M-1$, compute the array $G(l,q) = W_K^{lq} F(l,q)$
  3. For $0 \le q \le M-1$, compute the L-point DFTs

$$X(p,q) = \sum_{l=0}^{L-1} G(l,q) W_L^{lp}, \quad 0 \le p \le L-1$$

- Rearrange the double summation in the same DFT expression $\Rightarrow$ another similar algorithm
- Choosing row wise for $x(k)$ and column wise for $X(n)$ $\Rightarrow$ two more similar algorithms
- Complexity of these 4 algorithms resulting from a two-stage decomposition is: $2(L^2 + M^2 + K)$ trigonometric functions, $K(M + L + 1)$ multiplications, $K(M + L - 2)$ additions
- With $L = 2$ and $M = \frac{K}{2}$, for example, complexity reduction factor is approximately 2
- Factoring $K = r_1 r_2 \cdots r_v$, with v the stage decomposition, leads to the computation of many small DFTs and, the more stage v, the more significant in complexity reduction

Radix-2 FFT Algorithms

- When $K = r^v$, DFTs are of size r and computation has regular pattern, where r is called the radix of FFT algorithm. In particular, with $K = 2^v$, we have radix-2 FFT algorithms
- **Decimation-in-frequency** FFT: in the decomposition stage one, choose $L = K/2$ and $M = 2$:

$$X(n) = \sum_{k=0}^{K/2-1} x(k) W_K^{kn} + W_K^{nK/2} \sum_{k=0}^{K/2-1} x(k + K/2) W_K^{kn}$$

- Since $W_K^{nK/2} = (-1)^n$

$$X(n) = \sum_{k=0}^{K/2-1} \left(x(k) + (-1)^n x(k + K/2)\right) W_K^{kn}, \quad 0 \le n \le K - 1$$

- Next decimate $X(n)$ into even and odd samples and use $W_K^2 = W_{K/2}$:

$$X(2n) = \sum_{k=0}^{K/2-1} \left(x(k) + x(k + K/2)\right) W_{K/2}^{kn} \quad n = 0, 1, \cdots, \frac{K}{2} - 1$$

$$X(2n+1) = \sum_{k=0}^{K/2-1} \left[\left(x(k) - x(k + K/2)\right) W_K^k\right] W_{K/2}^{kn} \quad n = 0, 1, \cdots, \frac{K}{2} - 1$$

Radix-2 FFT Algorithms (continue)

- Define two $K/2$-point sequences

$$\left.\begin{aligned} g_1(k) &= x(k) + x(k+K/2) \\ g_2(k) &= \left[x(k) - x(k+K/2)\right] W_K^k \end{aligned}\right\} k = 0, 1, \cdots, \frac{K}{2} - 1$$

- Then

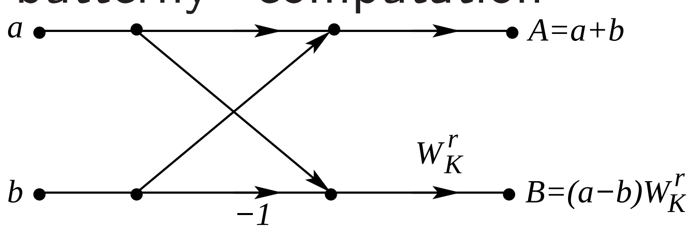
$$X(2n) = \sum_{k=0}^{K/2-1} g_1(k) W_{K/2}^{kn}, \quad X(2n+1) = \sum_{k=0}^{K/2-1} g_2(k) W_{K/2}^{kn}$$

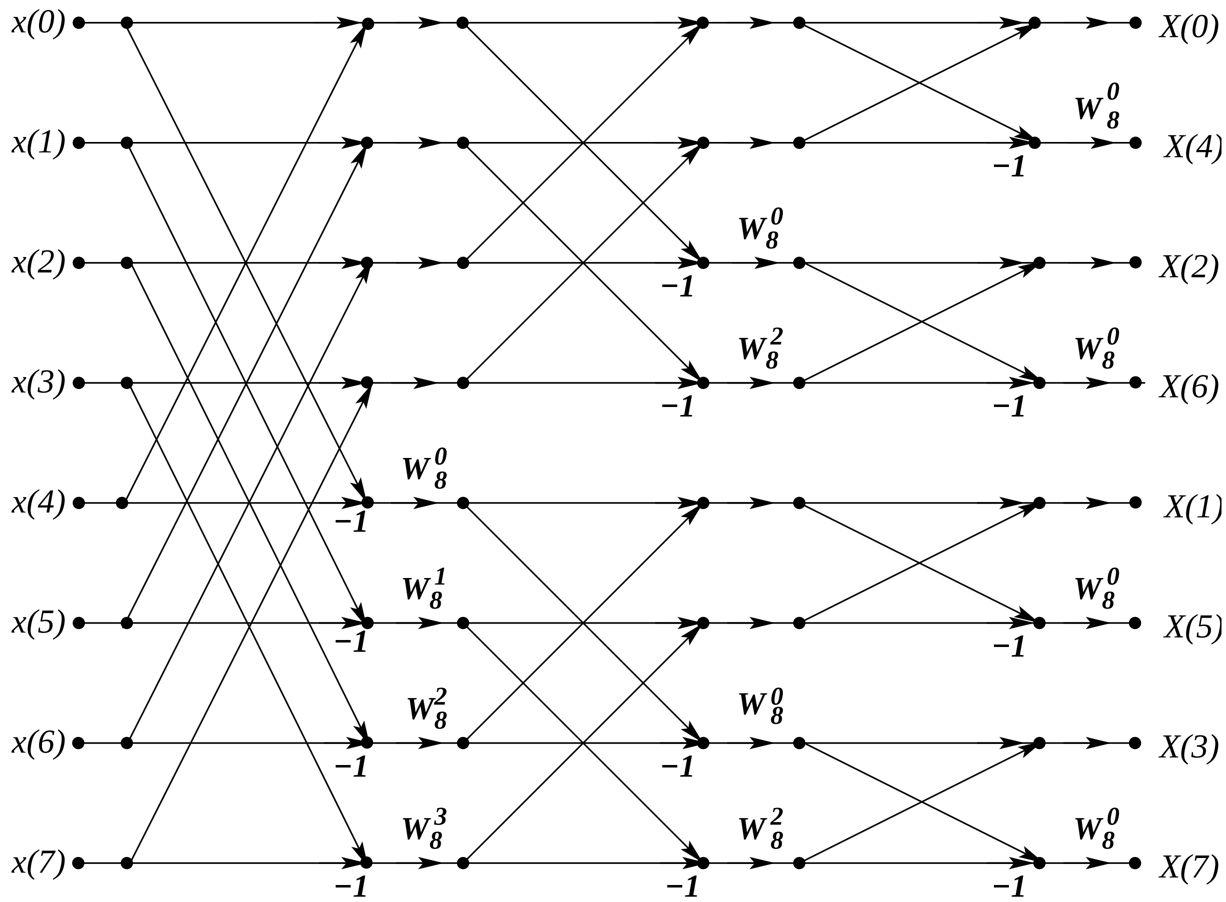
- $K/2$-point DFTs $X(2n)$ and $X(2n+1)$ can each be decimated into two $K/4$-point DFTs
- Procedure is repeated and entire procedure involves $v = \log_2(K)$ stages of decimation
- **Decimation-in-time** FFT: decimate $\{x(k)\}$ into even and odd samples and repeat the procedure
- Radix-2 FFT algorithm complexity: $(K/2)\log_2(K)$ complex multiplications, $K\log_2(K)$ complex additions
- **Example**. 1024-point DFT with $K = 2^{10}$:
  - Direct computing involves 1048576 multiplications and 1047552 additions
  - But radix-2 FFT only involves 5120 multiplications and 10240 additions $\longrightarrow$ speed improvement factor is approximately 100

8-Point Decimation-in-Frequency FFT

Algorithm:

Basic operation – "butterfly" computation





Bit Reversal Rule

8-point decimation-in-frequency

	n	bits	$\xrightarrow{\text{FFT}}$	bits	n	
$x(0)$	0	000		000	0	$X[0]$
$x(1)$	1	001		100	4	$X[4]$
$x(2)$	2	010		010	2	$X[2]$
$x(3)$	3	011		110	6	$X[6]$
$x(4)$	4	100		001	1	$X[1]$
$x(5)$	5	101		101	5	$X[5]$
$x(6)$	6	110		011	3	$X[3]$
$x(7)$	7	111		111	7	$X[7]$

8-point decimation-in-time

	$\xrightarrow{\text{FFT}}$	
$x(0)$		$X[0]$
$x(4)$		$X[1]$
$x(2)$		$X[2]$
$x(6)$		$X[3]$
$x(1)$		$X[4]$
$x(5)$		$X[5]$
$x(3)$		$X[6]$
$x(7)$		$X[7]$

Summary

- OFDM: basic concepts of multi carrier
  - Effective in combating channel fading and frequency selective, i.e. effective means of overcoming two big killers of mobile channels
  - Disadvantage of high peak to average power ratio, and sensitive to carrier phase noise and timing error

- Frequency analysis of discrete-time signals: differences with continuous-time case

- DFT: $\{x(k)\}_{k=0}^{K} \Longleftrightarrow \{X(n)\}_{n=0}^{K}$, practical considerations, time aliasing

- FFT: basic concepts, Radix-2, DFT implemented efficiently by FFT is widely used in communication systems