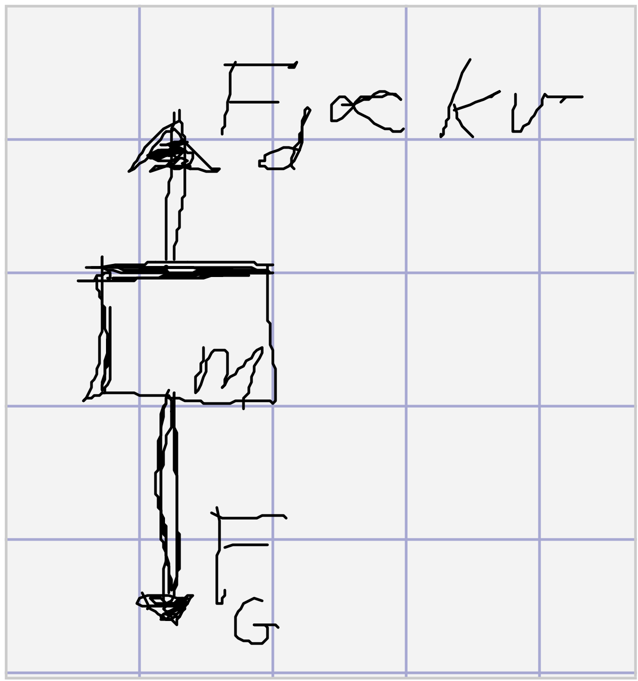
Friction can be a Drag!

This short ditty are the mathematical notes to a class on dealing with frictional drag. In particular we will be dealing with a form of drag common at low velocities called "Stokes Drag". The situation is simple: drop an object through air and allow "Air Resistance" "slow it down."

First off, Air Resistance doesn't slow down an object. This is another cognative disconnect in our language of motion. What is diminished by air drag is the acceleration not the velocity. Specifically what is happening is that the frictional force is dependant on the velocity:



Free Body Diagram with velocity dependant force.

An object is dropped. Initially there is no frictional drag. As the object accelerates the frictional drag increases until the frictional drag force equals the gravitational pull downward. When this happens the object stops accelerating ($F_{net} = 0$) and the partical reaches "terminal velocity" (v_{term}).

$F_{net} = F_G - F_d$ Note the implicit assumption that down is +

$m \frac{d}{dt} v(t) = m g - k v(t)$

$$m \left(\frac{d}{dt} v(t) \right) = m g - k v(t) \tag{1.1}$$

This is an example of the kind of equation that Newton originally had in mind when he set up his second law. It is the cause for motion. We can solve this without resorting to any magical mathematical tricks. You already know something of what must be done.

Note that when the force of gravity and the frictional drag are in balance the net effect of these forces is zero. At this point v(t) = v_{term}.

$0 = m \cdot g - k \cdot v_{term}$

$$0 = m g - k v_{term} \tag{1.2}$$

$\% + k \cdot v_{term}$

$$k v_{term} = m g \tag{1.3}$$

$$\frac{\%}{k}$$

$$v_{term} = \frac{m\,g}{k} \tag{1.4}$$

The first thing we must do is isolate the differential parts invoving one variable (v(t) and dv(t)) on one side of the equation and the other (dt) on the other side. Treat the derivative as if it were a fraction:

$$m \cdot dv = (m \cdot g - k \cdot v(t)) \cdot dt$$

$$m\,dv = (m\,g - k\,v(t))\,dt \tag{1.5}$$

$$\frac{\mathbf{(1.5)}}{(m \cdot g - k \cdot v(t))}$$

$$\frac{m\,dv}{m\,g - k\,v(t)} = dt \tag{1.6}$$

Lets do one small trick where we cast the denominator in terms of $v_{term} = \frac{m \cdot g}{k}$

$$\frac{\frac{m}{k} \cdot dv}{\frac{m \cdot g}{k} - v(t)} = dt$$

$$\frac{m\,dv}{k\left(\frac{m\,g}{k} - v(t)\right)} = dt \tag{1.7}$$

$$\frac{\% \cdot k}{m}$$

$$\frac{dv}{\frac{m\,g}{k} - v(t)} = \frac{k\,dt}{m} \tag{1.8}$$

Now we have the equation where we want it... Cornered! Lets use v_{term} instead of the fraction $\frac{m \cdot g}{k}$.

$$algsubs\left(\frac{m \cdot g}{k} = v_{term}, \mathbf{(1.8)}\right)$$

$$\frac{dv}{v_{term} - v(t)} = \frac{k\,dt}{m} \tag{1.9}$$

Now integrate each side. We need to see that each side relates the changes that happen to the particle as it accelerates downward. All you have to do is add up the changes. We know what the starting conditions are so we wont have any pesky constants left over.

$$\int_0^{v(t)} \frac{1}{\frac{m\,g}{k} - v}\,\mathrm{d}v \text{ assuming } \frac{m\,g}{k} > 0,\ v(t) > 0,\ \frac{m\,g}{k} > v(t)$$

$$\ln\left(\frac{m\,g}{k}\right) - \ln\left(\frac{m\,g - v(t)\,k}{k}\right) \tag{1.10}$$

$\int_0^t \frac{k}{m}\,\mathrm{d}t$

$$\frac{k\,t}{m} \tag{1.11}$$

(1.10) = **(1.11)**

$$\ln\left(\frac{m\,g}{k}\right)-\ln\left(\frac{m\,g-v(t)\,k}{k}\right)=\frac{k\,t}{m} \tag{1.12}$$

$simplify(expand(solve(\mathbf{(1.12)}, v(t))))$

$$-\frac{m\,g\left(\mathrm{e}^{-\frac{k\,t}{m}}-1\right)}{k} \tag{1.13}$$

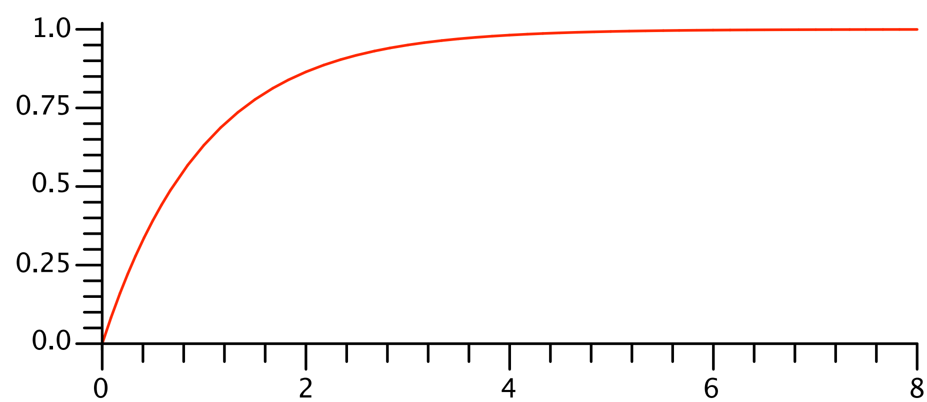
The equation for the velocity should look something like:

$algsubs\left(\frac{m\cdot g}{k}=v_{term}, v(t)=\mathbf{(1.13)}\right)$

$$v(t)=-\left(\mathrm{e}^{-\frac{k\,t}{m}}-1\right)v_{term} \tag{1.14}$$

The key piece of information is the funny expression in front of the terminal velocity

$plot(\ 1-\mathrm{e}^{-t}, t=0\,..8)$



As you can see the curve starts with a slope that looks alot like g and plateaus at v_{term}! This is what we guessed it should look like.

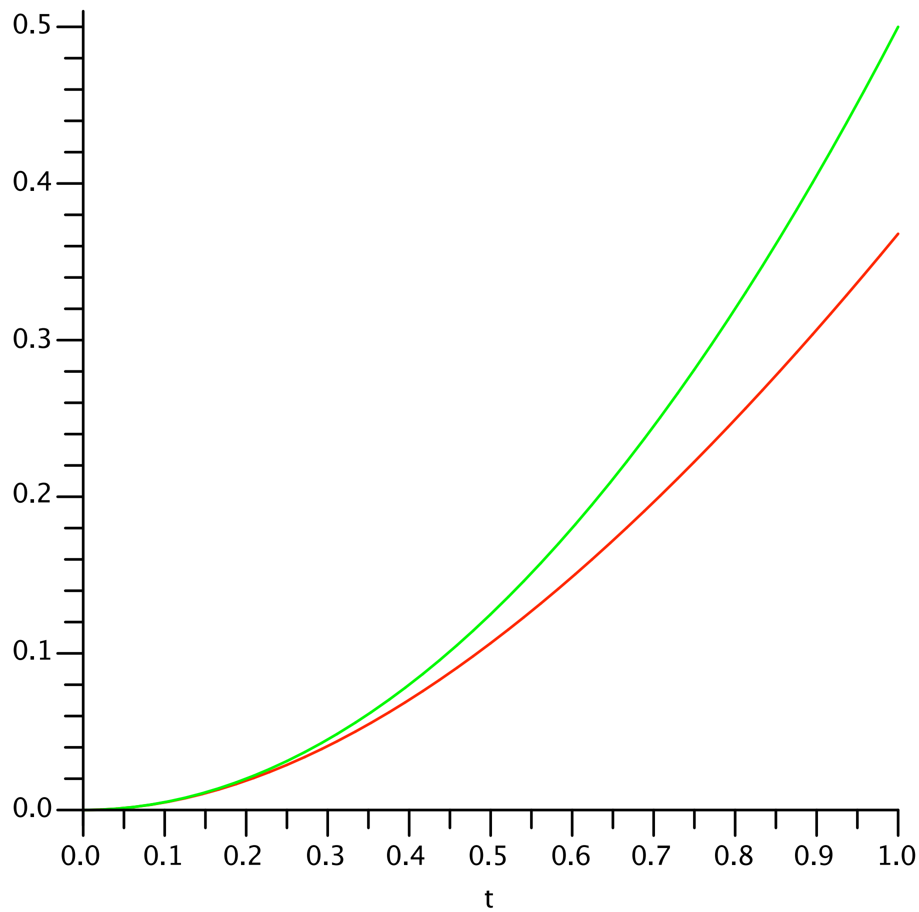
We can integate this again and see where it takes us. Let $v(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\,y(t)$ and let $y(0)=0$. Keeping with our implicit convention the distance increases downward.

$y(t)=\int_0^t -\left(\mathrm{e}^{-\frac{k\,t}{m}}-1\right)v_{term}\,\mathrm{d}t$

$$y(t)=\frac{v_{term}\left(-m+m\,\mathrm{e}^{-\frac{k\,t}{m}}+k\,t\right)}{k} \tag{1.15}$$

What does this look like (letting k/m=1) we can plot this function

$$plot\left(\left[e^{-t}-1+t,\ \frac{1}{2}t^2\right],\ t=0..1\right)$$



So you can see from my choice that at small time the speed is nearly indistinguishable from a freefall example. Once the speed starts going, the curve flattens out and eventually become a straight line caracteristic of uniform speed at v_{term}.