Universal Quadratic Forms Over Totally Real Number Fields

A project report
submitted in partial fulfillment
of the requirement of degree
of
Master of Science
in
Mathematics

Submitted by:

Krishna Nand Singh
(Enrollment Number 21616011)

Under the Supervision of

Dr. Mahendra Verma

Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee - 247667, India
May 2023

Candidate's Declaration

I hereby declare that the project titled **Universal Quadratic Forms Over Totally Real Number Fields** in partial fulfillment of the requirements of the Degree of **M.Sc. (Mathematics)** and submitted in the **Department of Mathematics, Indian Institute of Technology Roorkee**.

This work has been carried out from January, 2021 to May, 2021 under the supervision of **Dr. Mahendra Verma**, Department of Mathematics, Indian Institute of Technology Roorkee.

This work has not been submitted by me for the award of any other degree of this or any other institute.

Date: 8 May 2023
Place: Roorkee

Krishna Nand Singh
Enrollment Number: 21616011
M.Sc.(Mathematics)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dr. Mahendra Verma
Assistant Professor
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee, India

Acknowledgements

I express my deepest gratitude to my supervisor Dr. Mahendra Verma, Assistant Professor, Department of Mathematics, Indian Institute of Technology, Roorkee, for giving me the opportunity to work under his guidance. It is only due to his constant support, valuable advice and encouragement throughout this period which made this work possible.

I would like to thank Dr. Surojit Ghosh, Assistant Professor, Department of Mathematics, Indian Institute of Technology, Roorkee, for his insightful suggestions and comments in this work.

I would finally like to thank Prof. P. Bera, Head of Department, Department of Mathematics, Indian Institute of Technology, Roorkee, for giving me the permission to carry out this work in IIT Roorkee.

Abstract

This report begins with introduction of universal quadratic forms and the statements of landmark results in the classification of universal quadratic forms over integers. Then, a general number field is introduced with emphasis on totally real number fields and the ring of integers therein. Additively indecomposable elements are used as a tool to explicitly construct universal quadratic forms over real quadratic fields. Multiquadratic fields and the notion of totally positive integers therein is explored with explicit description of automorphisms for fields of small degree (4 and 8).

Contents

1. Introduction

The theory of universal quadratic forms over totally real number fields is an extension of the theory over the field of rational numbers, which in turn was developed as answer to the problem of representaion of numbers as sum of squares.

We begin by stating three classic results in the direction of representation of natural numbers as sum of squares.

Theorem 1.1 (Fermat's two squares theorem)**.** *An odd prime p can be expressed as $x^2 + y^2$, with $x, y \in \mathbb{Z}$, if and only if $p \equiv 1 \pmod 4$.*

Theorem 1.2 (Legendre's three squares theorem)**.** *A natural number n can be expressed as $n = x^2 + y^2 + z^2$, with $x, y, z \in \mathbb{Z}$, if and only if $n \neq 4^a(8b + 7)$, $a, b \in \mathbb{Z}^+$.*

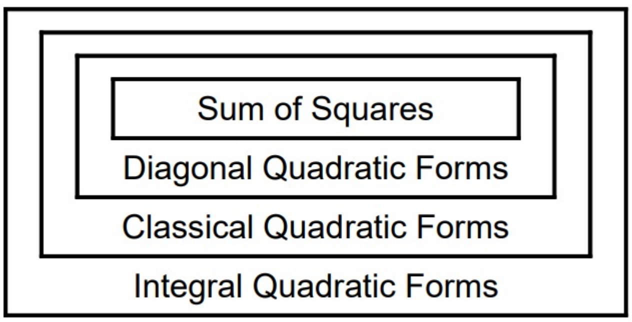
Theorem 1.3 (Lagrange's four squares theorem)**.** *Every natural number can be expressed as the sum of four squares.*

After three centuries' of developments and generalisations, the question of representation has changed to include "natural numbers" from other higher fields as well as "general" quadratic forms.

As an illustration, the study started from the bottom left set of positive integers ($\mathbb{N}$) and is now focussed on the top left set of totally positive integers ($\mathcal{O}_{\mathbb{K}}^{+}$). The notation will be clear by the end of this project.

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{K}}^{+} \arrow[r, hook] \arrow[d, dash, dashed] & \mathcal{O}_{\mathbb{K}} \arrow[r, hook] \arrow[d, dash, dashed] & \mathbb{K} \arrow[d, dash] \\
\mathbb{N} \arrow[r, hook] & \mathbb{Z} \arrow[r, hook] & \mathbb{Q}
\end{tikzcd}$$

And the simple "sum of squares" evolved to include "diagonal", followed by "classical" and "integral" quadratic forms. These concepts are explained in the next section.



2. Universal Quadratic Forms

We begin by introducing quadratic forms and their Gram matrix.

Definition 2.1. *A **quadratic form** of rank r over the ring of integers $\mathbb{Z}$ is a polynomial*

$$\mathcal{Q}(x_1, x_2, ..., x_r) = \sum_{1\le i\le j\le r} a_{ij}x_ix_j, \quad a_{ij} \in \mathbb{Z}.$$

The Gram matrix attached to the quadratic form $\mathcal{Q}$ is the symmetric matrix given by

$$M_{\mathcal{Q}} = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \dots & \frac{1}{2}a_{1r} \\ \frac{1}{2}a_{12} & a_{22} & \dots & \frac{1}{2}a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1r} & \frac{1}{2}a_{2r} & \dots & a_{rr} \end{pmatrix}.$$

If we represent the r variables as a column vector $\vec{x} = (x_1, x_2, ..., x_r)^t$, then the quadratic form can be expressed as

$$\mathcal{Q}(\vec{x}) = \vec{x}^t M_{\mathcal{Q}} \vec{x}.$$

We classify quadratic forms on the basis of their coefficients.

Definition 2.2. *A quadratic form $\mathcal{Q}(x_1, x_2, ..., x_r) = \sum_{1\le i\le j\le r} a_{ij}x_ix_j, \quad a_{ij} \in \mathbb{Z}$ is said to be:*

1. *diagonal, if $a_{ij} = 0$ for all $i \neq j$;*
2. *classical, if $\frac{1}{2}a_{ij} \in \mathbb{Z}$ for all $1 \le i \le j \le r$;*
3. *integral, if $a_{ij} \in \mathbb{Z}$ for all $1 \le i \le j \le r$.*

Thus, every diagonal quadratic form is classical, and every classical quadratic form is integral.

We denote the diagonal quadratic form $\mathcal{Q}(x_1, x_2, ..., x_r) = \sum_{j=1}^{r} a_jx_j^2, \ a_j \in \mathbb{Z}$ by $\langle a_1, a_2, ..., a_r\rangle$.

Example 2.3. (1) *Pythagorean triplets concern representation of 0 by $\langle 1, 1, -1\rangle$.*

(2) *The quadratic form under consideration in Lagrange's four square theorem is $\langle 1, 1, 1, 1\rangle$.*

(3) *Pell's equation is about the representation of 1 and other small integers by the binary diagonal quadratic form $\langle 1, -d\rangle$.*

We can also classify quadratic forms on the basis of their image in $\mathbb{Z}$.

Definition 2.4. *A quadratic form $\mathcal{Q} : \mathbb{Z}^r \to Z$ is said to be*

(1) *positive definite, if* $\mathcal{Q}(x) > 0 \ \forall \ x \in \mathbb{Z}^r \setminus \{0\}$*;*
(2) *positive semidefinite, if* $\mathcal{Q}(x) \geq 0 \ \forall \ x \in \mathbb{Z}^r$*;*
(3) *negative definite, if* $\mathcal{Q}(x) < 0 \ \forall \ x \in \mathbb{Z}^r \setminus \{0\}$*;*
(4) *negative semidefinite, if* $\mathcal{Q}(x) \leq 0 \ \forall \ x \in \mathbb{Z}^r$*;*
(5) *indefinite, otherwise.*

We now define universality of quadratic forms.

Definition 2.5. *The quadratic form $\mathcal{Q} : \mathbb{Z}^r \to \mathbb{Z}$ is said to be universal if it represents all positive integers, that is, if $\mathcal{Q}(\mathbb{Z}^r) = \mathbb{N}$.*

We will study the universality of positive definite quadratic forms only.

Definition 2.6. *Two quadratic forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are said to be equivalent, if there is bijective linear map $T : \mathbb{Z}^r \to \mathbb{Z}^r$, such that $\mathcal{Q}_1(Tx) = \mathcal{Q}_2(x)$ for all $x \in \mathbb{Z}^r$.*

That is, if the Gram matrices $M_{\mathcal{Q}_1}$ and $M_{\mathcal{Q}_2}$ are similar.

Example: The quadratic forms $x^2 + 2xy + 2y^2$ and $x^2 - 2xy + 2y^2$ are equivalent to $x^2 + y^2$, since

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Equivalent quadratic forms have equal images. So, we need to study universality of quadratic forms only up to equivalence.

The problem of characterizing universal quadratic forms was solved in three steps:

(i) In 1917, Ramanujan compiled all universal diagonal quadratic forms of rank 4:

Theorem 2.7. *There are exactly 55 universal quaternary diagonal quadratic forms*[1].

(ii) In 1995, Conway-Schneeberger characterized all universal classical forms:

Theorem 2.8 (15-theorem). *If $\mathcal{Q}$ is classical and represents the integers* $1, 2, 3, 5, 6, 7, 10, 14,$ *and* 15*, then it is universal*[2].

(iii) In 2005, Bhargava-Hanke characterized all integral quadratic forms:

Theorem 2.9 (290-theorem). *If $\mathcal{Q}$ represents the integers* $1, 2, 3, 5, 6, 7, 10, 13, 14, 15,$ $17, 19, 21, 22, 23, 26, 29, 30, 31, 36, 35, 37, 42, 58, 93, 110, 145, 203,$ *and* 290*, then it is universal*[3].

3. Totally Real Number Fields

We begin by introducing number fields.

Definition 3.1. *An algebraic* ***number field*** $\mathbb{K}$ *is a field extension of the field* $\mathbb{Q}$ *having finite degree.*

We will denote the degree of the number field $\mathbb{K}$ over $\mathbb{Q}$ by d, that is, $d = [\mathbb{K} : \mathbb{Q}]$.

Example 3.2. *Some examples of number fields for degree* d *are given below:*

- $d = 2$, $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ *is called Quadratic number field, where* D *is a square-free integer.*
- $d = 3$, $\mathbb{K} = \mathbb{Q}[x]/(f(x))$ *is called a cubic number field, where* $f(x)$ *is an irreducible cubic polynomial with coefficients in* $\mathbb{Q}$.
- $d = 3$, $\mathbb{K} = \mathbb{Q}(\sqrt[3]{n})$ *for cube free integer* n *is called pure cubic field.*
- $d = 2^k$, $\mathbb{K} = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}, ..., \sqrt{q_k},)$ *is called a multi-quadratic field, where* $q_1, q_2, ..., q_k$ *are square-free positive integer, such that* $[\mathbb{K} : \mathbb{Q}] = 2^k$.

For a number field $\mathbb{K}$ of degree d, there are d-many distinct field embeddings $\sigma_i : \mathbb{K} \to \mathbb{C}$ where $i = 1, 2, ..., d$.

Definition 3.3. *For a number field* $\mathbb{K}$, *with* $[\mathbb{K} : \mathbb{Q}] = d$, *and* $\sigma_1, \sigma_2, ..., \sigma_d : \mathbb{K} \hookrightarrow \mathbb{C}$, *the* $d-$*many embeddings, we define norm and trace of an element* $\alpha \in \mathbb{K}$ *as:*

- $\mathrm{N}(\alpha) = \prod_{i=1}^{d} \sigma_i(\alpha)$,
- $\mathrm{Tr}(\alpha) = \sum_{i=1}^{d} \sigma_i(\alpha)$.

Definition 3.4. *A number field* $\mathbb{K}$ *is called* ***totally real number field****, if all the embeddings are real, that is,* $\sigma_i(\mathbb{K}) \subset \mathbb{R}$ *for all* $i \in \{1, 2, ..., d\}$.

Example 3.5. *Some examples of totally real number fields are:*

- *For square-free* $D > 0$, *the quadratic field* $\mathbb{Q}[\sqrt{-D}] = \{a + ib\sqrt{D} : a, b \in \mathbb{Q}\}$ *is* ***not*** *totally real, the embeddings* $(a+ib\sqrt{D}) \mapsto (a+ib\sqrt{D})$ *and* $(a+ib\sqrt{D}) \mapsto (a-ib\sqrt{D})$ *being complex.*
- *For square-free* $D > 0$, *the quadratic field* $\mathbb{Q}[\sqrt{D}] = \{a + b\sqrt{D} : a, b \in \mathbb{Q}\}$ *is totally real, the two embeddings being*

$$(a + b\sqrt{D}) \mapsto (a + b\sqrt{D}) \text{ and } (a + b\sqrt{D}) \mapsto (a - b\sqrt{D}).$$

- *More generally, the multiquadratic field* $\mathbb{K} = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}, ..., \sqrt{q_k})$ *is totally real field, where* $q_1, q_2, ..., q_k$ *are square-free positive integer, such that* $d = [\mathbb{K} : \mathbb{Q}] = 2^k$.
- *The cubic field* $\mathbb{K} = \mathbb{Q}[x]/(f(x))$ *is totally real, where* $f(x)$ *is an irreducible cubic polynomial with three real roots.*

Just like the field $\mathbb{Q}$ has a ring of integers $\mathbb{Z}$, which in turn has an additive semigroup of positive integers $\mathbb{N}$, we have a similar situation for a totally real number field.

Definition 3.6. *The algebraic integers of* $\mathbb{C}$ *that are elements in* $\mathbb{K}$, *form the ring of integers of* $\mathbb{K}$, *denoted by* $\mathcal{O}_{\mathbb{K}}$.

The description of the ring of integers, $\mathcal{O}_{\mathbb{K}}$ for a general number field $\mathbb{K}$ difficult. But for some special fields, $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\alpha]$.

Example 3.7. *For some special number fields* $\mathbb{K}$, *the ring of integers* $\mathcal{O}_{\mathbb{K}}$ *is described below:*

- *For positive squarefree* D, *the quadratic field* $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$ *has the ring of integers* $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\alpha]$, *where* $\alpha = \begin{cases} \sqrt{D}, & \text{for } D = 2, 3 \ (mod\ 4), \\ \frac{\sqrt{D}+1}{2}, & \text{for } D = 1 \ (mod\ 4). \end{cases}$
- *For the simplest cubic field* $\mathbb{K} = \mathbb{Q}[\rho]$, *where* ρ *is a root of the polynomial*
$$f(x) = (x^3 - ax^2 - (a+3)x - 1), \ a \geq -1,$$
we have $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\rho]$.

Definition 3.8. *An element* $\alpha \in \mathcal{O}_{\mathbb{K}}$ *is called a* ***unit***, *if there exists* $\beta \in \mathcal{O}_{\mathbb{K}}$, *such that* $\alpha\beta = 1$.
In that case, such β *is unique, and is denoted by* α^{-1}.

Example 3.9.

(1) *The multiplicative identity* 1, *and its additive inverse* -1 *are units in any field* $\mathbb{K}$.

(2) *In* $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$, $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{2}]$ *and the units are:*
$$\pm 1, \ \pm(1 \pm \sqrt{2}), \ \pm(3 \pm 2\sqrt{2}), \ \pm(7 \pm 5\sqrt{2}), \ldots$$

(3) *In* $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$, $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{3}]$ *and the units are:*
$$\pm 1, \ \pm(2 \pm \sqrt{3}), \ \pm(7 \pm 4\sqrt{3}), \ \pm(26 \pm 15\sqrt{3}), \ldots$$

Definition 3.10. *The integer* $\alpha \in \mathcal{O}_{\mathbb{K}}$ *is called* ***totally positive integer***, *if* $\sigma_i(\alpha) > 0$. *for all the embeddings* σ_i *of* $\mathbb{K}$ *into* $\mathbb{C}$.

Notation: If $\alpha \in \mathcal{O}_{\mathbb{K}}$ is totally positive, we write $\alpha \succ 0$.
The set of all totally positive integers in $\mathcal{O}_{\mathbb{K}}$ form an additive semigroup denoted by $\mathcal{O}_{\mathbb{K}}^{+}$.

Example 3.11. *In totally real quadratic field:*

- $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$, $2+\sqrt{2} \succ 0$, $3+2\sqrt{2} \succ 0$ *and* $4+2\sqrt{2} \succ 0$, *but* $1+\sqrt{2} \not\succ 0$ *and* $2+2\sqrt{2} \not\succ 0$.
- $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$, $2+\sqrt{3} \succ 0$, $4+2\sqrt{3} \succ 0$ *and* $7+4\sqrt{3} \succ 0$, *but* $3+2\sqrt{3} \not\succ 0$ *and* $5+3\sqrt{3} \not\succ 0$.

In the study of quadratic forms and their universality on number fields, an important tool is the concept of indecomposable integers, which we define next.

Definition 3.12. $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ *is said to be additively indecomposable, if it cannot be decomposed as* $\alpha = \beta + \gamma$ *for some* $\beta, \gamma \in \mathcal{O}_{\mathbb{K}}^{+}$.

Example 3.13. *In the previous example,*

- *in* $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$, $2+\sqrt{2}$ *and* $3+2\sqrt{2}$ *are indecomposable, but* $4+2\sqrt{2} = (2+\sqrt{2}) + (2+\sqrt{2})$ *is not.*
- *in* $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$, $2+\sqrt{3}$ *and* $7+4\sqrt{3}$ *are indecomposable but* $4+2\sqrt{3} = (2+\sqrt{3}) + (2+\sqrt{3})$ *is not.*

Definition 3.14. *A quadratic form* $\mathcal{Q}$ *over* $\mathbb{K}$ *is said to be totally positive definite, if*

$$\mathcal{Q}(\sigma_i(\alpha)) > 0 \text{ for all embeddings } \sigma_i, \text{ and all } \alpha \in \mathcal{O}_{\mathbb{K}}^{+}.$$

Similar to the convention in $\mathbb{Q}$, we will discuss the universality of only totally positive definite quadratic forms over $\mathbb{K}$.

Definition 3.15. *A totally positive definite quadratic form* $\mathcal{Q} : (\mathcal{O}_{\mathbb{K}})^r \to \mathcal{O}_{\mathbb{K}}$ *is said to be* ***universal*** *if it represents all totally positive integers, that is, if for all* $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$, *there exists* $v \in (\mathcal{O}_{\mathbb{K}}^{+})^r$ *such that* $\mathcal{Q}(v) = \alpha$.

4. Real Quadratic Fields

In this section we consider real quadratic field and universal quadratic forms over them. Let $D > 1$ be a squarefree integer. Then, $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$ is a totally real field with degree $[\mathbb{K} : \mathbb{Q}] = 2$.

There are two embeddings $\mathbb{K} \hookrightarrow \mathbb{R}$,

$$(a + b\sqrt{D}) \mapsto (a + b\sqrt{D}) \text{ and } (a + b\sqrt{D}) \mapsto (a - b\sqrt{D}) \text{ for } a, b \in \mathbb{Q}.$$

The norm and trace are evaluated for $\alpha = a + b\sqrt{D} \in \mathbb{K}$:

- $\mathrm{N}(\alpha) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - b^2 D$,
- $\mathrm{Tr}(\alpha) = (a + b\sqrt{D}) + (a - b\sqrt{D}) = 2a$.

The ring of integers of $\mathbb{K}$ is given by $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\rho]$, where

$$\rho = \begin{cases} \sqrt{D}, & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{\sqrt{D}+1}{2}, & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

4.1. Diagonal Quadratic Forms.

Let us first consider diagonal quadratic forms, that is

$$\mathcal{Q}(x_1, x_2, \ldots, x_r) = a_1 x_1^2 + a_2 x_2^2 + \ldots + a_r x_r^2.$$

Suppose that $\mathcal{Q}$ is universal. Then any indecomposable $\alpha \in \mathcal{O}_{\mathbb{K}}^+$ can be expressed as

$$\alpha = a_1 v_1^2 + a_2 v_2^2 + \ldots + a_r v_r^2, \text{ for some } v_1, v_2, \ldots, v_r \in \mathcal{O}_{\mathbb{K}}^+,$$

and because α is indecomposable, so $\alpha = a_i v_i^2$ for some i.

Hence the rank of a universal diagonal quadratic form is bounded below by the number of square classes of indecomposables. The indecomposable elements in a real quadratic field are well understood in terms of the continued fraction of $\sqrt{D}$.

Continued Fractions. The continued fraction of quadratic irrationals is periodic.

$$\sqrt{D} = [u_0; \overline{u_1, u_2, \ldots, u_s}] = [u_0, u_1, \ldots, u_s, u_1, \ldots, u_s, u_1, \ldots] = u_0 + \cfrac{1}{u_1 + \frac{1}{u_2 + \ldots}}.$$

It is known that $u_0 = \lfloor \sqrt{D} \rfloor$ and $u_s = 2\lfloor \sqrt{D} \rfloor$.

The convergents of the continued fraction, $\frac{p_i}{q_i} = [u_0, \ldots, u_i]$, provide good approximations to $\sqrt{D}$. We will call the quadratic integers $\alpha_i = p_i + q_i\sqrt{D}$ convergents too. It is clear that $\alpha_0 = u_0 + \sqrt{D}$. Also, we define $\alpha_{-1} := 1$.

Lemma 4.1. *The convergent α_i is totally positive iff i is odd.*

Definition 4.2. *The convergent α_{s-1} is called the fundamental unit as it is generator of the group of units, $\mathcal{O}_{\mathbb{K}}^{\times}$, that is,*

$$\mathcal{O}_{\mathbb{K}}^{\times} = \{\pm\alpha_{s-1}^{k} | k \in \mathbb{Z}\}.$$

Definition 4.3. *The totally positive fundamental unit, denoted by ϵ, is the generator of the group of all totally positive units, $\mathcal{O}_{\mathbb{K}}^{\times,+}$, that is,*

$$\mathcal{O}_{\mathbb{K}}^{\times,+} = \{\epsilon^{l} | l \in \mathbb{Z}\}.$$

Then, we have

$$\epsilon = \begin{cases} \alpha_{s-1}, & \text{if s is even,} \\ \alpha_{2s-1}, & \text{if s is odd.} \end{cases}$$

Definition 4.4. *The semiconvergents are defined as*

$$\alpha_{i,t} = \alpha_i + t\alpha_{i+1}, \text{ where } i \geq -1 \text{ is odd and } 0 \leq t < u_{i+2}.$$

Theorem 4.5. *The indecomposables α are precisely the semiconvergents and their conjugates.*

The construction of a universal diagonal quadratic form is done using the following results.

Lemma 4.6. *Every $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ is a sum of indecomposables.*

We consider the set S of representatives of indecomposables up to multiplication by $\mathcal{O}_{\mathbb{K}}^{\times,+}$. Then

$$S = \{\alpha_{i,t_i} | i = -1, 1, ..., k, \quad 0 \leq t_i < u_{i+2}\}, \quad \text{where } k = \begin{cases} s-3, & \text{if s is even,} \\ 2s-3, & \text{if s is odd.} \end{cases}$$

So, the number of elements in S is

$$|S| = \begin{cases} u_1 + u_3 + \ldots + u_{s-3} + u_{s-1}, & \text{if s is even,} \\ u_1 + u_2 + \ldots + u_{2s-3} + u_{2s-1} = u_1 + u_2 + u_3 + \ldots + u_{s-1} + u_s, & \text{if s is odd.} \end{cases}$$

From lemma 4.6, we group the indecomposables according to their class in S to get the following result.

Lemma 4.7. *Every $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ can be expressed as*

$$\alpha = \sum_{\sigma \in S} \sigma e_{\sigma},$$

where each e_{σ} is a linear combination of totally positive units with non-negative integral coefficients.

Next we can express the e_{σ} as linear combination of consecutive elements of $\mathcal{O}_{\mathbb{K}}^{\times,+}$.

Lemma 4.8. *For each e_{σ}, there exists $j \in \mathbb{Z}$ and integers $c, d \geq 0$ such that*

$$e_{\sigma} = c\epsilon^{j} + d\epsilon^{j+1}.$$

Finally, we construct a universal diagonal form using Lagrange's four square theorem and preceding results.

Theorem 4.9. *The diagonal quadratic form*

$$\bigoplus_{\sigma \in S} \langle \sigma, \sigma, \sigma, \sigma, \epsilon\sigma, \epsilon\sigma, \epsilon\sigma, \epsilon\sigma \rangle$$

is universal and has rank $r = 8 \cdot |S|$.

Example 4.10. *Consider the field $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$. Then $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{2}]$.*

$$\sqrt{3} = [1; 2, 2, 2, 2, \ldots] = [1; \overline{2}]$$

$\implies$ *$s = 1$; and $u_0 = 1$, $u_1 = 2$.*
The convergents are $\alpha_0 = 1 + \sqrt{2}$, $\alpha_1 = 3 + 2\sqrt{2}$, $\alpha_2 = 7 + 5\sqrt{2}$,...
and the totally positive fundamental unit, $\epsilon = \alpha_{2s-1} = \alpha_1 = 3 + 2\sqrt{2}$.
Thus, $|S| = u_1 = 2$ and $S = \{\alpha_{-1,0}, \alpha_{-1,1}\} = \{1, 2 + \sqrt{2}\}$.

Example 4.11. *Next, consider the field $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$. Then $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{3}]$.*

$$\sqrt{2} = [1; 1, 2, 1, 2, 1, 2, \ldots] = [1; \overline{1, 2}]$$

$\implies$ *$s = 2$; and $u_0 = 1$, $u_1 = 1$, $u_2 = 2$.*
The convergents are $\alpha_0 = 1 + \sqrt{3}$, $\alpha_1 = 2 + \sqrt{3}$, $\alpha_2 = 5 + 3\sqrt{3}$, $\alpha_3 = 7 + 4\sqrt{3}$,...
and the totally positive fundamental unit, $\epsilon = \alpha_{s-1} = \alpha_1 = 2 + \sqrt{3}$.
Thus, $|S| = u_1 = 1$ and $S = \{\alpha_{-1,0}\} = \{1\}$.

Example 4.12. *Next, consider the field* $\mathbb{K} = \mathbb{Q}[\sqrt{6}]$. *Then* $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{6}]$.

$$\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, \ldots] = [2; \overline{2, 4}]$$

$\implies s = 2$; *and* $u_0 = 2,\ u_1 = 2,\ u_2 = 4$.
The convergents are $\alpha_0 = 2 + \sqrt{6},\ \alpha_1 = 5 + 2\sqrt{6},\ \alpha_2 = 22 + 9\sqrt{6},\ \alpha_3 = 49 + 20\sqrt{6}, \ldots$
and the totally positive fundamental unit, $\epsilon = \alpha_{s-1} = \alpha_1 = 5 + 2\sqrt{6}$.
Thus, $|S| = u_1 = 2$ *and* $S = \{\alpha_{-1,0}, \alpha_{-1,1}\} = \{1, 3 + \sqrt{6}\}$.

Example 4.13. *For the field* $\mathbb{K} = \mathbb{Q}[\sqrt{7}]$, $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{7}]$.

$$\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, ...] = [2; \overline{1, 1, 1, 4}]$$

$\implies s = 4$; *and* $u_0 = 2,\ u_1 = 1,\ u_2 = 1,\ u_2 = 1,\ u_3 = 1,\ u_4 = 4$.
$\alpha_0 = 2 + \sqrt{7},\ \alpha_1 = 3 + \sqrt{7},\ \alpha_2 = 5 + 2\sqrt{7},\ \alpha_3 = 8 + 3\sqrt{7},\ \alpha_4 = 37 + 14\sqrt{7}, ...$
and the totally positive fundamental unit, $\epsilon = \alpha_{s-1} = \alpha_3 = 8 + 3\sqrt{7}$.
Thus, $|S| = u_1 + u_3 = 2$ *and* $S = \{\alpha_{-1,0}, \alpha_{1,0}\} = \{1, 3 + \sqrt{7}\}$.

Example 4.14. *Finally, consider the field* $\mathbb{K} = \mathbb{Q}[\sqrt{10}]$. *Then* $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{10}]$.

$$\sqrt{10} = [3; 6, 6, 6, 6, ...] = [3; \overline{6}]$$

$\implies s = 1$; *and* $u_0 = 3,\ u_1 = 6$.
$\alpha_0 = 3 + \sqrt{10},\ \alpha_1 = 3 + 10\sqrt{10},\ \alpha_2 = 19 + 6\sqrt{10},\ \alpha_3 = 117 + 37\sqrt{10}, ...$
and the totally positive fundamental unit, $\epsilon = \alpha_{2s-1} = \alpha_1 = 3 + \sqrt{10}$.
Thus, $|S| = u_1 = 6$ *and*
$S = \{\alpha_{-1,0}, \alpha_{-1,1}, \alpha_{-1,2}, \alpha_{-1,3}, \alpha_{-1,4}, \alpha_{-1,5}\}$
$= \{1, 4 + \sqrt{10}, 7 + 2\sqrt{10}, 10 + 3\sqrt{10}, 13 + 4\sqrt{10}, 16 + 5\sqrt{10}\}$.

From the table of examples, we notice that for $D = n^2 - 1$, $\sqrt{D} = [n - 1, \overline{1, 2(n - 1)}]$, so $|S| = 1$, leading to the following result.

Corollary 4.15. *There is a universal diagonal quadratic form of rank 8 on* $\mathbb{Q}[\sqrt{D}]$, *where* $D = n^2 - 1$, *squarefree.*

TABLE 1. For a squarefree D, $\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \mathbb{Z}[\alpha]$ where α is as

D	α	cont. frac. of α	s	# S
2	$\sqrt{2}$	$[1;\overline{2}]$	1	2
3	$\sqrt{3}$	$[1;\overline{1,2}]$	2	1
5	$\frac{\sqrt{5}+1}{2}$	$[\overline{1}]$	0	–
6	$\sqrt{6}$	$[2;\overline{2,4}]$	2	2
7	$\sqrt{7}$	$[2;\overline{1,1,1,4}]$	4	2
10	$\sqrt{10}$	$[3;\overline{6}]$	1	6
11	$\sqrt{11}$	$[3;\overline{3,6}]$	2	3
13	$\frac{\sqrt{13}+1}{2}$	$[2;\overline{3}]$	1	3
14	$\sqrt{14}$	$[3;\overline{1,2,1,6}]$	4	2
15	$\sqrt{15}$	$[3;\overline{1,6}]$	2	1
17	$\frac{\sqrt{17}+1}{2}$	$[2;\overline{1,1,3}]$	3	4
19	$\sqrt{19}$	$[4;\overline{2,1,3,1,2,8}]$	6	7
21	$\sqrt{21}$	$[4;\overline{1,1,2,1,1,8}]$	6	4
22	$\sqrt{22}$	$[4;\overline{1,2,4,2,1,8}]$	6	6
23	$\sqrt{23}$	$[4;\overline{1,3,1,8}]$	4	2
26	$\sqrt{26}$	$[5;\overline{10}]$	1	10
29	$\frac{\sqrt{29}+1}{2}$	$[3;\overline{5}]$	1	5
30	$\sqrt{30}$	$[5;\overline{2,10}]$	1	2
31	$\sqrt{31}$	$[5;\overline{5,1,1,3,5,3,1,1,10}]$	8	12
33	$\frac{\sqrt{33}+1}{2}$	$[3;\overline{2,1,2,5}]$	4	4
34	$\sqrt{34}$	$[5;\overline{1,4,1,10}]$	4	2
35	$\sqrt{35}$	$[5;\overline{1,10}]$	2	1
37	$\frac{\sqrt{37}+1}{2}$	$[3;\overline{1,1,5}]$	3	7
38	$\sqrt{38}$	$[6;\overline{6,12}]$	2	6
39	$\sqrt{39}$	$[6;\overline{4,12}]$	2	4
41	$\frac{\sqrt{41}+1}{2}$	$[3;\overline{1,2,2,1,5}]$	5	11
42	$\sqrt{42}$	$[6;\overline{2,12}]$	2	2
43	$\sqrt{43}$	$[6;\overline{1,1,3,1,5,1,3,1,1,12}]$	10	13
46	$\sqrt{46}$	$[6;\overline{1,3,1,1,2,6,2,1,1,3,1,12}]$	12	8
47	$\sqrt{47}$	$[6;\overline{1,5,1,12}]$	4	2

Table 2. For a squarefree D, $\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \mathbb{Z}[\alpha]$ where α is as

D	α	cont. frac. of α	s	# S
51	$\sqrt{51}$	$[7;\overline{7,14}]$	2	7
53	$\frac{\sqrt{53}+1}{2}$	$[4;\overline{7}]$	1	7
55	$\sqrt{55}$	$[7;\overline{2,2,2,14}]$	4	4
57	$\frac{\sqrt{57}+1}{2}$	$[4;\overline{3,1,1,1,3,7}]$	6	7
58	$\sqrt{58}$	$[7;\overline{1,1,1,1,1,1,14}]$	7	20
59	$\sqrt{59}$	$[7;\overline{1,2,7,2,1,14}]$	6	8
61	$\frac{\sqrt{61}+1}{2}$	$[4;\overline{2,2,7}]$	3	11
62	$\sqrt{62}$	$[7;\overline{1,6,1,14}]$	4	2
65	$\frac{\sqrt{65}+1}{2}$	$[4;\overline{1,1,7}]$	3	9
66	$\sqrt{66}$	$[8;\overline{8,16}]$	2	8
67	$\sqrt{67}$	$[8;\overline{5,2,1,1,7,1,1,2,5,16}]$	10	19
69	$\frac{\sqrt{69}+1}{2}$	$[4;\overline{1,1,1,7}]$	4	2
70	$\sqrt{70}$	$[8;\overline{2,1,2,1,2,16}]$	6	6
71	$\sqrt{71}$	$[8;\overline{2,2,1,7,1,2,2,16}]$	8	6
73	$\frac{\sqrt{73}+1}{2}$	$[4;\overline{1,3,2,1,1,2,3,1,7}]$	9	21
74	$\sqrt{74}$	$[8;\overline{1,1,1,1,16}]$	5	20
77	$\frac{\sqrt{77}+1}{2}$	$[4;\overline{1,7}]$	2	1
78	$\sqrt{78}$	$[8;\overline{1,4,1,16}]$	4	2
79	$\sqrt{79}$	$[8;\overline{1,7,1,16}]$	4	2
82	$\sqrt{82}$	$[9;\overline{18}]$	1	18
83	$\sqrt{83}$	$[9;\overline{9,18}]$	2	9
85	$\frac{\sqrt{85}+1}{2}$	$[5;\overline{9}]$	1	9
86	$\sqrt{86}$	$[9;\overline{3,1,1,1,8,1,1,1,3,18}]$	10	16
87	$\sqrt{87}$	$[9;\overline{3,18}]$	1	3
89	$\frac{\sqrt{89}+1}{2}$	$[5;\overline{4,1,1,1,1,9}]$	6	6
91	$\sqrt{91}$	$[9;\overline{1,1,5,1,5,1,1,18}]$	8	12
93	$\frac{\sqrt{93}+1}{2}$	$[5;\overline{3,9}]$	2	3
94	$\sqrt{94}$	$[9;\overline{1,2,3,1,1,5,1,8,1,5,1,1,3,2,1,18}]$	16	12
95	$\sqrt{95}$	$[9;\overline{1,2,1,18}]$	4	2
97	$\frac{\sqrt{97}+1}{2}$	$[5;\overline{2,2,1,4,4,1,2,2,9}]$	9	27

4.2. **Classical Universal Quadratic Forms.** Although universal quadratic forms exist for every number field, the existence of universal quadratic forms of small rank is very rare once the degree of field gets large[4].

Definition 4.16. *Define the lower bounds on ranks of universal forms by*

- $m_{class}(\mathbb{K}) :=$ *the minimal rank of a classical universal quadratic form over* $\mathcal{O}_{\mathbb{K}}$.
- $m(\mathbb{K}) :=$ *the minimal rank of universal quadratic form over* $\mathcal{O}_{\mathbb{K}}$.

The simplest quadratic form, that is the **sum of n squares** is universal only for

- $\mathbb{K} = \mathbb{Q}$ $(n = 4)$,
- $\mathbb{K} = \mathbb{Q}[\sqrt{5}]$ $(n = 3)$.

It is also observed that there is never a universal quadratic form of rank $r = 1$ *or* 2. So, the next small rank is $r = 3$.

Theorem 4.17. *If* $\mathbb{Q}[\sqrt{D}]$ *has a ternary classical universal form, then* $D = 2, 3,$ *or* 5.

There are 11 such universal forms, their Gram matrices are given below:

- for $D = 2$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2\pm\sqrt{2} \end{pmatrix},$
- for $D = 3$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2+\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2+\sqrt{3} \end{pmatrix},$
- for $D = 5$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5+\sqrt{5}}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & \frac{5\pm\sqrt{5}}{2} \end{pmatrix}.$

Theorem 4.18. *If* $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$, *with D squarefree, has a universal quadratic form of rank* ≤ 7 *then* $D < (5762838677310720000000005)^2$.

Since every diagonal form is classical so we get a trivial upper bound for $m(\mathbb{Q}[\sqrt{D}])$ and $m_{class}(\mathbb{Q}[\sqrt{D}])$. The next theorem provides a lower bound too.

Theorem 4.19. *For $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$ with $\sqrt{D} = [u_0; \overline{u_1, u_2, \ldots, u_s}]$, we have:*

- *If s is even,*

  $$\frac{1}{2}\max\{u_1, u_3, \ldots, u_{s-1}\} \leq m_{class}(\mathbb{K}) \leq 8 \cdot |S| = 8(u_1 + u_3 + \ldots + u_{s-1})$$

  and

  $$\frac{1}{2}\sqrt{\max\{u_1, u_3, \ldots, u_{s-1}\}} \leq m(\mathbb{K}) \leq 8 \cdot |S| = 8(u_1 + u_3 + \ldots + u_{s-1}).$$

- *If s is odd,*

  $$\frac{1}{2}\sqrt{D} \leq m_{class}(\mathbb{K}) \leq 8 \cdot |S| = 8(u_1 + u_2 + \ldots + u_s)$$

  and

  $$\frac{1}{2}\sqrt[4]{D} \leq m(\mathbb{K}) \leq 8 \cdot |S| = 8(u_1 + u_2 + \ldots + u_s).$$

Example 4.20. *We compute the bounds explicitly for some large D:*

- $D = 67$.

  $$\sqrt{67} = [8; \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}] \implies s = 10, \text{ even}.$$

  So $\frac{7}{2} < 4 \leq m_{class}(\mathbb{K}) \leq 152$ *and* $\frac{\sqrt{7}}{2} < 2 \leq m(\mathbb{K}) \leq 152$.

- $D = 74$.

  $$\sqrt{74} = [8; \overline{1, 1, 1, 1, 16}] \implies s = 5, \text{ odd}.$$

  So $\frac{\sqrt{74}}{2} < 5 \leq m_{class}(\mathbb{K}) \leq 160$ *and* $\frac{\sqrt[4]{74}}{2} < 2 \leq m(\mathbb{K}) \leq 160$.

- $D = 86$.

  $$\sqrt{86} = [9; \overline{3, 1, 1, 1, 8, 1, 1, 1, 3, 8}] \implies s = 10, \text{ even}.$$

  So $\frac{8}{2} = 4 \leq m_{class}(\mathbb{K}) \leq 128$, *and* $\frac{\sqrt{8}}{2} < 2 \leq m(\mathbb{K}) \leq 128$.

From the above result and computations we observe that the lower bound on rank of universal quadratic forms grows without bound, this is the motivation for the next result. The following result describes the fact that universal quadratic forms can require arbitrary large ranks.

Theorem 4.21. *For any positive integer r, there are infinitely many quadratic fields $\mathbb{Q}[\sqrt{D}]$, such that $m(\mathbb{Q}[\sqrt{D}]) \geq r$ [5].*

5. Multiquadratic Fields

The previous result can be extended to multiquadratic fields. We will start with the definitions.

Definition 5.1. *A number field $\mathbb{K} = \mathbb{Q}[\sqrt{q_1}, \sqrt{q_1}, \ldots, \sqrt{q_k}]$, where $q_1, q_2, \ldots, q_k$ are square free positive integers, such that $[\mathbb{K} : \mathbb{Q}] = 2^k$, is a totally real number field and is called a multiquadratic real field.*

We now describe the ring of integers $\mathcal{O}_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{Q}[\sqrt{q_1}, \sqrt{q_1}, \ldots, \sqrt{q_k}]$.

Let $I \subset \{1, 2, \ldots, k\}$.

There are 2^k such subsets. For each of these, consider

$$q_I = \frac{1}{l^2} \prod_{i \in I} q_i, \text{ where } l \text{ is the largest integer such that } q_I \text{ is an integer.}$$

Then, each q_I is square free, and we get the description of $\mathcal{O}_{\mathbb{K}}$ as

$$\mathbb{Z}[\sqrt{q_I} | I \subset \{1, 2, \ldots, k\}] \subset \mathcal{O}_{\mathbb{K}} \subset \frac{1}{2^k}\mathbb{Z}[\sqrt{q_I} | I \subset \{1, 2, \ldots, k\}].$$

This description for $k = 2$ and $k = 3$ is given below:

- $k = 2$:

  We have $\mathbb{K} = \mathbb{Q}[\sqrt{q}_1, \sqrt{q}_2]$, where q_1, q_2 are square free. Take the subsets of {1,2} as

  $$I_\phi = \phi, \ I_1 = \{1\}, \ I_2 = \{2\}, \ I_{12} = \{1, 2\}.$$

  Then the expression $q_I = \frac{1}{l^2} \prod_{i \in I} q_i$ gives

  $$q_{I_\phi} = 1; \ q_{I_1} = q_1; \ q_{I_2} = q_2; \ q_{I_{12}} = \frac{q_1 q_2}{gcd(q_1, q_2)^2}.$$

  So, the ring of integers $\mathcal{O}_{\mathbb{K}}$ satisfies

  $$\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{I_{12}}}] \subset \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{I_{12}}}].$$

  Example 5.2.

  (1) For $\mathbb{K} = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$, $\quad \mathbb{Z}[1, \sqrt{2}, \sqrt{3}, \sqrt{6}] \subset \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{2}, \sqrt{3}, \sqrt{6}]$.

  (2) For $\mathbb{K} = \mathbb{Q}[\sqrt{15}, \sqrt{21}]$, $\quad \mathbb{Z}[1, \sqrt{15}, \sqrt{21}, \sqrt{35}] \subset \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{15}, \sqrt{21}, \sqrt{35}]$.

  (3) For $\mathbb{K} = \mathbb{Q}[\sqrt{35}, \sqrt{42}]$, $\quad \mathbb{Z}[1, \sqrt{35}, \sqrt{42}, \sqrt{30}] \subset \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{35}, \sqrt{42}, \sqrt{30}]$.

- $k = 3$:

  We have $\mathbb{K} = \mathbb{Q}[\sqrt{q}_1, \sqrt{q}_2, \sqrt{q}_3]$, where q_1, q_2, q_3 are square free. Take the subsets of {1,2,3} as

$$I_\phi = \phi,\ I_1 = \{1\},\ I_2 = \{2\},\ I_{12} = \{1,2\},$$

$$I_3 = \{3\},\ I_{13} = \{1,3\},\ I_{23} = \{2,3\},\ I_{123} = \{1,2,3\}.$$

Then the expression $q_I = \frac{1}{l^2}\prod_{i\in I} q_i$ gives

(for simplified notation, we drop I and use (a,b) for $gcd(a,b)$)

$$q_\phi = 1;\ q_1 = q_1;\ q_2 = q_2;\ q_{12} = \frac{q_1q_2}{(q_1,q_2)^2};$$

$$q_3 = q_3;\ q_{13} = \frac{q_1q_3}{(q_1,q_3)^2};\ q_{13} = \frac{q_1q_3}{(q_1,q_3)^2};\ q_{123} = \frac{q_1q_2q_3}{l^2}$$

where l is the largest integer such that q_{123} is an integer.

So, the ring of integers $\mathcal{O}_\mathbb{K}$ satisfies

$$\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{12}}, \sqrt{q_3}, \sqrt{q_{13}}, \sqrt{q_{23}}, \sqrt{q_{123}}]$$

$$\subset \mathcal{O}_\mathbb{K} \subset \frac{1}{8}\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{12}}, \sqrt{q_3}, \sqrt{q_{13}}, \sqrt{q_{23}}, \sqrt{q_{123}}].$$

Example 5.3. (1) *For* $\mathbb{K} = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$,

$$\mathbb{Z}[1, \sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{5}, \sqrt{10}, \sqrt{15}, \sqrt{30}]$$

$$\subset \mathcal{O}_\mathbb{K} \subset \frac{1}{8}\mathbb{Z}[1, \sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{5}, \sqrt{10}, \sqrt{15}, \sqrt{30}].$$

(2) *For* $\mathbb{K} = \mathbb{Q}[\sqrt{6}, \sqrt{15}, \sqrt{21}]$,

$$\mathbb{Z}[1, \sqrt{6}, \sqrt{15}, \sqrt{10}, \sqrt{21}, \sqrt{14}, \sqrt{35}, \sqrt{210}]$$

$$\subset \mathcal{O}_\mathbb{K} \subset \frac{1}{8}\mathbb{Z}[1, \sqrt{6}, \sqrt{15}, \sqrt{10}, \sqrt{21}, \sqrt{14}, \sqrt{35}, \sqrt{210}].$$

Now we need to characterize totally positive integers inside $\mathcal{O}_\mathbb{K}$. For that, we describe the conjugates of $\alpha \in \mathcal{O}_\mathbb{K}$.

For $i \in \{1,2,...,k\}$, define $\sigma_i : \mathbb{K} \to \mathbb{K}$ by $\sigma_i(q_j) = (-1)^{\delta_{ij}} q_j$.

And for $I \subset \{1,2,...,k\}$, define $\sigma_I := \prod_{i\in I}\sigma_i$.

Then the conjugates of $\alpha \in \mathcal{O}_\mathbb{K}$ are given by

$$\{\sigma_I(\alpha) | I \subset \{1,2,...,k\}\}.$$

The cumbersome notation will become clear after some examples.

Example 5.4. *We find the set of conjugates explicitly for $k = 2$ and 3:*

- $k = 2, \quad \mathbb{K} = \mathbb{Q}[\sqrt{q_1}, \sqrt{q_2}] \implies \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{12}}]$.

  So, the automorphisms are:

$$\sigma_\phi : (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}}) \mapsto (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}})$$

$$\sigma_1 : (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}}) \mapsto (a - b\sqrt{q_1} + c\sqrt{q_2} - d\sqrt{q_{12}})$$

$$\sigma_2 : (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}}) \mapsto (a + b\sqrt{q_1} - c\sqrt{q_2} - d\sqrt{q_{12}})$$

$$\sigma_{12} : (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}}) \mapsto (a - b\sqrt{q_1} - c\sqrt{q_2} + d\sqrt{q_{12}})$$

  for $a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}} \in \mathcal{O}_{\mathbb{K}}$, where $a, b, c, d \in \frac{1}{4}\mathbb{Z}$.

- $k = 3, \quad \mathbb{K} = \mathbb{Q}[\sqrt{q_1}, \sqrt{q_2}, \sqrt{q_3}]$

  $\implies \mathcal{O}_{\mathbb{K}} \subset \frac{1}{8}\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_3}, \sqrt{q_{12}}, \sqrt{q_{13}}, \sqrt{q_{23}}, \sqrt{q_{123}}]$.

  So, for $\alpha = (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_3} + e\sqrt{q_{12}} + f\sqrt{q_{23}} + g\sqrt{q_{13}} + h\sqrt{q_{123}})$, the images under automorphisms are:

$$\sigma_\phi(\alpha) : (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_3} + e\sqrt{q_{12}} + f\sqrt{q_{23}} + g\sqrt{q_{13}} + h\sqrt{q_{123}})$$

$$\sigma_1(\alpha) : (a - b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_3} - e\sqrt{q_{12}} + f\sqrt{q_{23}} - g\sqrt{q_{13}} - h\sqrt{q_{123}})$$

$$\sigma_2(\alpha) : (a + b\sqrt{q_1} - c\sqrt{q_2} + d\sqrt{q_3} - e\sqrt{q_{12}} - f\sqrt{q_{23}} + g\sqrt{q_{13}} - h\sqrt{q_{123}})$$

$$\sigma_3(\alpha) : (a + b\sqrt{q_1} + c\sqrt{q_2} - d\sqrt{q_3} + e\sqrt{q_{12}} - f\sqrt{q_{23}} - g\sqrt{q_{13}} - h\sqrt{q_{123}})$$

$$\sigma_{12}(\alpha) : (a - b\sqrt{q_1} - c\sqrt{q_2} + d\sqrt{q_3} + e\sqrt{q_{12}} - f\sqrt{q_{23}} - g\sqrt{q_{13}} + h\sqrt{q_{123}})$$

$$\sigma_{23}(\alpha) : (a + b\sqrt{q_1} - c\sqrt{q_2} - d\sqrt{q_3} - e\sqrt{q_{12}} + f\sqrt{q_{23}} - g\sqrt{q_{13}} + h\sqrt{q_{123}})$$

$$\sigma_{13}(\alpha) : (a - b\sqrt{q_1} + c\sqrt{q_2} - d\sqrt{q_3} + e\sqrt{q_{12}} - f\sqrt{q_{23}} - g\sqrt{q_{13}} + h\sqrt{q_{123}})$$

$$\sigma_{123}(\alpha) : (a - b\sqrt{q_1} - c\sqrt{q_2} - d\sqrt{q_3} + e\sqrt{q_{12}} + f\sqrt{q_{23}} + g\sqrt{q_{13}} - h\sqrt{q_{123}})$$

Now we can describe the trace and norm of any element $\alpha \in \mathbb{K}$.
From the definition of trace and norm we have,

$$N(\alpha) = \prod_{i=1}^{d} \sigma_i(\alpha) \text{ and } Tr(\alpha) = \sum_{i=1}^{d} \sigma_i(\alpha).$$

So, for $\alpha = (a_0 + a_1\sqrt{q_1} + a_2\sqrt{q_2} + \ldots + a_k\sqrt{q_k}) \in \mathbb{Q}[\sqrt{q_1}, \sqrt{q_2}, \ldots, \sqrt{q_k}]$,
we have $Tr(\alpha) = 2^k a_0$.

Lemma 5.5. *Suppose that $\alpha = \sum_{I \subset 1,\ldots,k} a_I\sqrt{q_I} \in \mathcal{O}_{\mathbb{K}}^{+}$(with $\mathrm{a}_I \in \frac{1}{2^k}\mathbb{Z}$). If $a_I \neq 0$, then $Tr(\alpha) > \sqrt{q_I}$.*

Example 5.6. *The following example motivate the proof of the above lemma.*

- $k = 2, \quad \mathbb{K} = \mathbb{Q}[\sqrt{q_1}, \sqrt{q_2}] \implies \mathcal{O}_{\mathbb{K}} \subset \frac{1}{4}\mathbb{Z}[1, \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{12}}]$.

  Let $\alpha = (a + b\sqrt{q_1} + c\sqrt{q_2} + d\sqrt{q_{12}}) \in \mathcal{O}_{\mathbb{K}}^{+}$.

  Without loss of generality, let $d \neq 0$.

  Then,

  $0 < \sigma_{\phi}(\alpha) + \sigma_{12}(\alpha) = 2(a + d\sqrt{q_{12}})$, *and*

  $0 < \sigma_1(\alpha) + \sigma_2(\alpha) = 2(a - d\sqrt{q_{12}})$.

  Combining the two inequalties we get $a > |d\sqrt{q_{12}}| \implies 4a > \sqrt{q_{12}}$.

  Hence, $Tr(\alpha) = 4a > \sqrt{q_{12}}$.

  Similarly we get for other cases, hence

$$Tr(\alpha) > \min\{\sqrt{q_1}, \sqrt{q_2}, \sqrt{q_{12}}\}, \text{ when } \alpha \notin \mathbb{Z}.$$

This description of totally positive integers in the ring of integers with combination of the principle of induction on theorem (4.21) gives the following generalisation:

Theorem 5.7. *For all pairs of positive integers* k, N *there are infinitely many totally real multiquadratic fields* $\mathbb{K}$ *of degree* 2^k *over* $\mathbb{Q}$ *such that* $m(\mathbb{K}) \geq N$[6].

6. Conclusion

Answering the questions regarding universality of quadratic forms over real number fields of higher degree is a formidable task. For a particular family of fields, the general approach which is followed is:

General methods for particular family of fields:

(1) Find a totally real number field $\mathbb{K}$.
(2) Describe the ring of integers $\mathcal{O}_{\mathbb{K}}$.
    Currently the focus is on the fields $\mathbb{K}$, where $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\rho]$, for some $\rho \in \mathbb{R}$.
(3) Characterize the structure of totally positive integers $\mathcal{O}_{\mathbb{K}}^{+}$.
    The structure of $\mathcal{O}_{\mathbb{K}}^{+}$ in terms of indecomposable integers and their square classes decides the universality of small rank quadratic forms.
(4) Estimate the norm and trace of indecomposable integers.
(5) Calculate the lower bound on rank of universal quadratic form.
(6) Construct a few universal quadratic forms.
(7) Classify all universal (classical) quadratic forms.

This approach was covered in this report for the family of quadratic real fields(up to step 6), and the family of multiquadratic fields(up to step 4).
The current status of research in this area is very promising with high hopes of getting results analogous to 15-theorem for atleast some particular fields.

References

[1] S. Ramanujan, *On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$*, Proc. Amer. Math. Soc. **19** (1917), 11–21.

[2] M. Bhargava, *On the Conway-Schneeberger Fifteen Theorem,* Contemp. Math. **272** (1999), 27–37.

[3] M. Bhargava, J. Hanke, *Universal quadratic forms and the 290-theorem,* Invent. Math., to appear.

[4] V. Blomer, V. Kala, *On the rank of universal quadratic forms over real quadratic fields,* Doc. Math. **23** (2018), 15–34.

[5] V. Blomer, V. Kala, *NUmber fields without universal n-ary quadratic forms,* Math. Proc. Cambridge Philos. Soc. **159** (2015), 239–252.

[6] V. Kala, J. Svoboda, *Universal quadratic forms over multiquadratic fields,* Ramanujan J. **48** (2019), 151–157.