Universal Quadratic Forms over Totally Real Number Fields

Krishna Nand Singh

Indian Institute of Technology Roorkee

krishna_ns@ma.iitr.ac.in

MAN 612 FINAL PRESENTATION

Supervised by: Dr. Mahendra Kumar Verma

Presentation Overview

Historical Developments

Theorem (Fermat's two squares theorem)

An odd prime p can be expressed as $x^2 + y^2$, with $x, y \in \mathbb{Z}$, if and only if $p \equiv 1 \pmod 4$.

Theorem (Legendre's three squares theorem)

A natural number n can be expressed as $n = x^2 + y^2 + z^2$, with $x, y, z \in \mathbb{Z}$, if and only if $n \neq 4^a(8b + 7)$, $a, b \in \mathbb{Z}^+$.

Theorem (Lagrange's four squares theorem)

Every natural number can be expressed as the sum of four squares.

Quadratic Forms

Definition

A **quadratic form** of rank r over the ring of integers $\mathbb{Z}$ is a polynomial

$$\mathcal{Q}(x_1, x_2, ..., x_r) = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Z}.$$

The **Gram matrix** attached to the quadratic form $\mathcal{Q}$ is the symmetric matrix given by

$$M_{\mathcal{Q}} = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \dots & \frac{1}{2}a_{1r} \\ \frac{1}{2}a_{12} & a_{22} & \dots & \frac{1}{2}a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1r} & \frac{1}{2}a_{2r} & \dots & a_{rr} \end{pmatrix}$$

If $\vec{x} = (x_1, x_2, ..., x_r)^t$, then $\mathcal{Q}(\vec{x}) = \vec{x}^t M_{\mathcal{Q}} \vec{x}$.

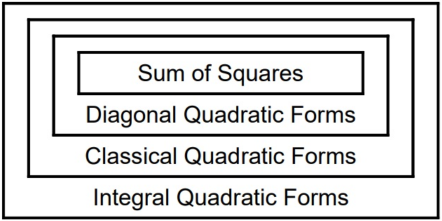
Quadratic Forms : Universality

Definition

A quadratic form $\mathcal{Q}(x_1, x_2, ..., x_r) = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j$, $a_{ij} \in \mathbb{Z}$ is said to be:

1. diagonal, if $a_{ij} = 0$ for all $i \neq j$;
2. classical, if $\frac{1}{2} a_{ij} \in \mathbb{Z}$ for all $1 \le i \le j \le r$;
3. integral, if $a_{ij} \in \mathbb{Z}$ for all $1 \le i \le j \le r$.

Notation: Diagonal form $\sum_{j=1}^{r} a_j x_j^2$ is denoted by $\langle a_1, a_2, ..., a_r \rangle$.



Definition

Universality refers to the representation of all positive integers by a given quadratic form.

Universal Quadratic Forms over $\mathbb{Z}$

Universal quadratic forms over $\mathbb{Z}$ were characterized in three big steps:

1. Ramanujan- all universal diagonal quadratic forms of rank 4- 1917:

Theorem

There are exactly 55 universal quaternary diagonal quadratic forms.

2. Conway-Schneeberger- all universal classical quadratic forms-1995:

Theorem (15-theorem)

If $\mathcal{Q}$ is classical and represents the integers $1, 2, 3, 5, 6, 7, 10, 14,$ and 15, then it is universal.

3. Bhargava-Hanke- all universal integral quadratic forms- 2005:

Theorem (290-theorem)

If $\mathcal{Q}$ represents the integers $1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 36, 35, 37, 42, 58, 93, 110, 145, 203,$ and 290, then it is universal.

Totally Real Number Fields

A number field $\mathbb{K}$ over $\mathbb{Q}$ of degree *d* has *d*-many embedding into $\mathbb{C}$.
If all the embeddings are real, then $\mathbb{K}$ is a **Totally Real** number field.

Ring of Integers: $\mathcal{O}_{\mathbb{K}}$

- For a totally real number field $\mathbb{K}$, the set of all numbers in $\mathbb{K}$ that are algebraic integers forms a ring. It is denoted by $\mathcal{O}_{\mathbb{K}}$.
- An integer $\alpha \in \mathcal{O}_{\mathbb{K}}$ is called **Totally Positive** if its image is positive under all embeddings.
- The set of all totally positive integers forms an additive semigroup $\mathcal{O}_{\mathbb{K}}^{+}$.
- $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ is **Indecomposable** if $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in \mathcal{O}_{\mathbb{K}}^{+}$.

Illustration

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{K}}^{+} \arrow[r, hook] \arrow[d, dash, dashed] & \mathcal{O}_{\mathbb{K}} \arrow[r, hook] \arrow[d, dash, dashed] & \mathbb{K} \arrow[d, dash] \\
\mathbb{N} \arrow[r, hook] & \mathbb{Z} \arrow[r, hook] & \mathbb{Q}
\end{tikzcd}$$

- $\alpha \in \mathcal{O}_{\mathbb{K}}^{+} \subset \mathcal{O}_{\mathbb{K}} \subset \mathbb{K}$
- $1 \in \mathbb{Z}^{+} \subset \mathbb{Z} \subset \mathbb{Q}$
- $\alpha \in \mathbb{Z}[\sqrt{2}]^{+} \subset \mathbb{Z}[\sqrt{2}] \subset \mathbb{Q}[\sqrt{2}]$

  $\alpha = 1, (2+\sqrt{2}), (3+2\sqrt{2}), (10+7\sqrt{2}), \ldots$

Quadratic Fields: Diagonal Forms

$\mathcal{Q}(x_1, x_2, ..., x_r) = a_1x_1^2 + a_2x_2^2 + ... + a_rx_r^2$

If $\mathcal{Q}$ is universal, any indecomposable $\alpha \in \mathcal{O}_{\mathbb{K}}^+$ can be expressed as

$$\alpha = a_1v_1^2 + a_2v_2^2 + ... + a_rv_r^2, \text{ for some } v_1, v_2, ..., v_r \in \mathcal{O}_{\mathbb{K}}^+ \implies \alpha = a_iv_i^2.$$

Hence the rank of a universal diagonal quadratic form is bounded below by the number of square classes of indecomposables.

The **continued fraction** of quadratic irrationals is periodic.

$$\sqrt{D} = [u_0; \overline{u_1, u_2, ..., u_s}] = [u_0, u_1, ..., u_s, u_1, ..., u_s, u_1, ...] = u_0 + \cfrac{1}{u_1 + \frac{1}{u_2 + ...}}$$

It is known that $u_0 = \lfloor\sqrt{D}\rfloor$ and $u_s = 2\lfloor\sqrt{D}\rfloor$.
The convergents, $\frac{p_i}{q_i} = [u_0, ..., u_i]$, provide good approximations to $\sqrt{D}$.
We call the quadratic integers $\alpha_i = p_i + q_i\sqrt{D}$ convergents too.
Then $\alpha_0 = u_0 + \sqrt{D}$, and we define $\alpha_{-1} := 1$.

Lemma

The convergent α_i is totally positive iff i is odd.

Definition

The totally positive fundamental unit, denoted by ϵ, is the generator of the group of all totally positive units, $\mathcal{O}_{\mathbb{K}}^{\times,+}$. That is, $\mathcal{O}_{\mathbb{K}}^{\times,+} = \{\epsilon^l | l \in \mathbb{Z}\}$.

Then $\epsilon = \begin{cases} \alpha_{s-1} & \text{, if s is even} \\ \alpha_{2s-1} & \text{, if s is odd.} \end{cases}$

Definition

The semiconvergents are defined as $\alpha_{i,t} = \alpha_i + t\alpha_{i+1}$, where $i \geq -1$ is odd and $0 \leq t < u_{i+2}$.

Theorem

The indecomposables α are precisely the semiconvergents and their conjugates.

Lemma

Every $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ is a sum of indecomposables.

We consider the set S of representatives of indecomposables up to multiplication by $\mathcal{O}_{\mathbb{K}}^{\times,+}$. Then

$$S = \{\alpha_{i,t_i} | i = -1, 1, ..., k, \ \ 0 \leq t_i < u_{i+2}\}, \ \text{where } k = \begin{cases} s-3, & \text{if s is even} \\ 2s-3, & \text{if s is odd.} \end{cases}$$

So, $|S| = \begin{cases} u_1 + u_3 + ... + u_{s-3} + u_{s-1}, & \text{if s is even} \\ u_1 + u_2 + u_3 + ... + u_{s-1} + u_s, & \text{if s is odd.} \end{cases}$

From the lemma, we group the indecomposables according to their class in S to get the following result.

Lemma

Every $\alpha \in \mathcal{O}_{\mathbb{K}}^{+}$ can be expressed as $\alpha = \sum_{\sigma \in S} \sigma e_\sigma$, where each e_σ is a linear combination of totally positive units with non-negative integral coefficients.

We express e_σ as linear combination of consecutive elements of $\mathcal{O}_{\mathbb{K}}^{\times,+}$.

Lemma

For each e_σ, there exists $j \in \mathbb{Z}$ and integers $c, d \geq 0$ such that $e_\sigma = c\epsilon^j + d\epsilon^{j+1}$

Lagrange's four square theorem with preceding results gives a universal diagonal form.

Theorem

The diagonal quadratic form $\bigoplus_{\sigma \in S} \langle \sigma, \sigma, \sigma, \sigma, \epsilon\sigma, \epsilon\sigma, \epsilon\sigma, \epsilon\sigma \rangle$ is universal and has rank $r = 8 \cdot |S|$.

Since, $\sqrt{n^2 - 1} = [n - 1, \overline{1, 2(n - 1)}]$, so $|S| = 1$, hence the following result.

Corollary

There is a universal diagonal quadratic form of rank 8 on $\mathbb{Q}[\sqrt{D}]$, where $D = n^2 - 1$, squarefree.

Quadratic Fields: Classical and Integral forms

Theorem

If $\mathbb{Q}[\sqrt{D}]$ has a ternary classical universal form, then $D = 2, 3,$ or 5.

Kitaoka's Conjecture : There are only finitely many totally real number fields $\mathbb{K}$ having a ternary universal form.

Notation

- $m_{class}(\mathbb{K}) :=$ the minimal rank of a classical universal form over $\mathcal{O}_{\mathbb{K}}$.
- $m(\mathbb{K}) :=$ the minimal rank of integral universal form over $\mathcal{O}_{\mathbb{K}}$.

Theorem

For $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$ with $\sqrt{D} = [u_0; \overline{u_1, u_2, ..., u_s}]$, we have

- *if s is even, $\frac{1}{2}\max\{u_1, u_3, ..., u_{s-1}\} \leq m_{class}(\mathbb{K}) \leq 8 \cdot |S|$ and $\frac{1}{2}\sqrt{\max\{u_1, u_3, ..., u_{s-1}\}} \leq m(\mathbb{K}) \leq 8 \cdot |S|$.*
- *if s is odd, $\frac{1}{2}\sqrt{D} \leq m_{class}(\mathbb{K}) \leq 8 \cdot |S|$ and $\frac{1}{2}\sqrt[4]{D} \leq m(\mathbb{K}) \leq 8 \cdot |S|$.*

Universal quadratic forms with small ranks become rare as the degree of the number field increases.
For quadratic fields, we have[1]

Theorem

For any positive integer r, there are infinitely many quadratic fields $\mathbb{Q}[\sqrt{D}]$*, such that* $m(\mathbb{K}) > r$*.*

For multiquadratic fields, we have[2]

Theorem

For all pairs of positive integers k, N *there are infinitely many totally real multiquadratic fields* $\mathbb{K}$ *such that* $[\mathbb{K} : \mathbb{Q}] = 2^k$ *and* $m(\mathbb{K}) \geq N$*.*

Higher number fields

Solving the problem of universality of quadratic forms over general real fields of higher degree is a formidable task[3]. For a particular field, a general approach may be followed.

General methods for particular fields

1. Find a totally real number field $\mathbb{K}$.
2. Describe the ring of integers $\mathcal{O}_{\mathbb{K}}$.
3. Characterize the structure of totally positive integers $\mathcal{O}_{\mathbb{K}}^{+}$.
4. Estimate the norm and trace of indecomposable integers.
5. Calculate the lower bound on rank of universal quadratic form.
6. Construct a few universal quadratic forms.
7. Classify all universal (classical) quadratic forms.

References

V. Blomer, V. Kala (2015)
Number fields without n-ary quadratic forms
Math. Proc. Cambridge Philos. Soc. **159**, 239 – 252.

V. Kala, J. Svoboda (2019)
Universal quadratic forms over multiquadratic fields
Ramanujan J. **48**, 151-157.

V. Kala (2023)
Universal quadratic forms and indecomposables in number fields: A survey
Preprint **arXiv**, 2301.13222.