Exam 2 Problem 2.

(a) $n=1$. $A=\alpha_{00} \Rightarrow L=\lambda_{00}=1$, $U=u_{00}=\alpha_{00}$

$$\check{A}=A+\Delta A=\check{L}\check{U}=\lambda_{00}\cdot\alpha_{00}(1+\epsilon_{mach}^{*})$$
$$=\alpha_{00}+\delta\alpha_{00} \quad \text{where } |\delta\alpha_{00}|=\epsilon_{mach} \text{ hence } |\delta\alpha_{00}|=|\epsilon||\alpha_{00}|$$
$$\leq \gamma_1|\alpha_{00}|$$
$\therefore \Delta A=\delta\alpha_{00}$

(b) Inductive step Assume $n=k$ holds

Then, $\underset{(k+1)\times(k+1)}{A}+\Delta A=\check{L}\check{U}$ with $|\Delta A|\leq\gamma_k|\check{L}||\check{U}|$

$$\check{A}=\begin{pmatrix}\underset{k\times k}{\check{A}_{00}} & \underset{k\times 1}{\check{a}_{01}}\\ \underset{1\times k}{\check{a}_{10}^T} & \underset{1\times 1}{\check{\alpha}_{11}}\end{pmatrix}=A+\Delta A$$
$$=\begin{pmatrix}\underset{k\times k}{\check{L}_{00}} & 0\\ \underset{1\times k}{\check{l}_{10}^T} & 1\end{pmatrix}\begin{pmatrix}\underset{k\times k}{\check{U}_{00}} & \underset{k\times 1}{\check{u}_{01}}\\ 0 & \upsilon_{11}\end{pmatrix}=\begin{pmatrix}L_{00}U_{00} & L_{00}u_{01}\\ \check{l}_{10}^T\check{U}_{00} & l_{10}^T\check{u}_{01}+\check{\upsilon}_{11}\end{pmatrix}$$

① $A_{00}+\Delta A_{00}=\check{L}_{00}\check{U}_{00}$ where $|\Delta A_{00}|\leq\gamma_k|\check{L}_{00}||\check{U}_{00}|$ (by I.H.).

② $\check{a}_{01}=\check{L}_{00}\check{u}_{01}=(L_{00}+\Delta L_{00})u_{01}$ where $|\Delta L_{00}|\leq\max(\gamma_2,\gamma_{k-1})|L_{00}|$

Also,
$\check{a}_{01}=a_{01}+\delta a_{01}=L_{00}u_{01}+\delta a_{01}$ where $|\delta\alpha_{i}|\leq\gamma_k|\tilde{l}_i|^T|u_{01}|$ hence $|\delta a_{01}|\leq\gamma_k|L_{00}||u_{01}|$

③ $\check{a}_{10}^T=a_{10}^T+\delta a_{10}^T=l_{10}^TU_{00}+\delta a_{10}^T$ (by extrapolation) where $|\delta\alpha_i|^T\leq\gamma_k|l_{10}|^T|u_i|$
hence $|\delta a_{10}|^T\leq\gamma_k|l_{10}|^T|U_{00}|$

Also, $\check{a}_{10}^T=l_{10}^T(U_{00}+\Delta U_{00})$ (By extrapolation), where $|\Delta U_{00}|\leq\max(\gamma_2,\gamma_{k-1})|U_{00}|$

④ $\check{\alpha}_{11}=\check{l}_{10}^T\check{u}_{01}+\check{\upsilon}_{11}=[K(=l_{10}^Tu_{01})+\delta K+\upsilon_{11}]$ where $|\delta K|=(1+\theta_k)|l_{10}^T||u_{01}|\leq\gamma_k|l_{10}|^T|u_{01}|$
$$=(K+\delta K)(1+\epsilon^{+})+(1+\epsilon^{+})\upsilon_{11}$$
$$=K(1+\theta_{k+1})+(1+\epsilon^{+})\upsilon_{11} \quad \text{where } |\delta K|=(1+\theta_{k+1})|l_{10}^T||u_{01}|$$
hence $|\delta K|\leq\gamma_{k+1}|l_{10}^T||u_{01}|$

$$\therefore \left|\begin{pmatrix}\delta A_{00} & \delta a_{01}\\ \delta a_{10}^T & \delta\alpha_{11}\end{pmatrix}\right|\leq\left|\begin{pmatrix}\gamma_k|L_{00}||U_{00}| & \gamma_k|L_{00}||u_{01}|\\ \gamma_k|l_{10}|^T|U_{00}| & \gamma_{k+1}|l_{10}|^T|u_{01}|\end{pmatrix}\right|\leq\gamma_{k+1}\left|\begin{pmatrix}\check{L}_{00} & 0\\ \check{l}_{10}^T & 1\end{pmatrix}\right|\left|\begin{pmatrix}\check{U}_{00} & \check{u}_{01}\\ 0 & \check{\upsilon}_{11}\end{pmatrix}\right|$$