Exam 2 Problem 1.

(a) $\hat{A} = \begin{pmatrix} \hat{A}_{00} & \hat{a}_{01} \\ \hat{a}_{10}^T & \hat{\alpha}_{11} \end{pmatrix}$ and $L = \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix}$. (^ denotes original values)

$$A = LL^T = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right) = \begin{pmatrix} L_{00}L_{00}^T & L_{00}l_{10} \\ l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{pmatrix}$$

• Assume $A_{00} = L_{00}L_{00}^T$, and L_{00} has been computed ($\because$ bordered algorithm)

1. $\hat{a}_{10}^T = l_{10}^T L_{00}^T$. L_{00}^T is known, $\hat{a}_{10}^T$ is given. $\Rightarrow$ $a_{10}^T := l_{10}^T = \hat{a}_{10}^T L_{00}^{T-1}$
2. $\hat{\alpha}_{11} = l_{10}^T l_{10} + \lambda_{11}^2$. Above, a_{10}^T ($:= l_{10}$) has been computed.
   Thus, $\lambda_{11}^2 = \hat{\alpha}_{11} - l_{10}^T l_{10}$ can be calculated.
3. Finally, $\alpha_{11} := \lambda_{11}$ and A becomes $\hat{A}_{00}$. Repeat the process.
   $= \sqrt{\hat{\alpha}_{11} - l_{10}^T l_{10}}$

(b) 1. base case. $n=1$ $A = \alpha_{11} = LL^T \Rightarrow \lambda_{11} = \sqrt{\alpha_{11}}$ Since α_{11} (A is SPD) is positive, unique λ_{11} exists.

2. Inductive case Assume $A = LL^T$ holds for $n = k$.
That suggests, $A \in \mathbb{R}^{k\times k}$. $A = LL^T$ and $\underset{k\times k}{L} = \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix}$ exists.

If $n = k+1$, $A' \in \mathbb{R}^{k+1 \times k+1}$.

$A' = \begin{pmatrix} \underset{k\times k}{A_{00}(:=A)} & a_{01} \\ \underbrace{a_{10}^T}_{\mathbb{R}^k} & \underbrace{\alpha_{11}}_{\mathbb{R}^1} \end{pmatrix}$ If $A = LL^T$, $L = \begin{pmatrix} L_{00}(:= \underset{k\times k}{L}) & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix}$.

① L_{00} can be calculated because we assume the hypothesis hold for $A \in \mathbb{R}^{k\times k}$ (by I.H.)

② $a_{10}^T = l_{10}^T L_{00}^T$. $a_{10}^T \in \mathbb{R}^{1\times k}$ And $l_{10}^T \in \mathbb{R}^{1\times k}$, $L_{00}^T \in \mathbb{R}^{k\times k}$ so l_{10}^T is well-defined
Also, L_{00}^T is nonsingular, so l_{10}^T is uniquely defined.

③ $\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$. The dot product ($= \|l_{10}\|_2^2$) is positive, so is the square. $\therefore \alpha_{11}$ must be positive. The diagonal elements of SPD are positive in line with the result.
That means α_{11} is greater than $l_{10}^T l_{10}$, which makes $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$ well-defined and uniquely exists.

Since all elements (L_{00}, l_{10}^T, λ_{11}) are unique, L exists, and this proves the theorem