1 Flow variables and units (SI)

Flow variables and their units:

1. Density ρ is kg/m^3
2. Pressure p is Pa (is N/m^2 is J/m^3)
3. Flow velocity v is m/s
4. Total energy per unit volume $E = \rho e = \rho\varepsilon_{\text{int}} + \rho v^2/2$ is J/m^3
5. Total specific energy e is J/kg
6. Total specific internal energy ε is J/kg
7. Mass of the constituent species m is kg

2 Euler equations

For a two-dimensional flow of a single-species inviscid gas the compressible Euler equations governing such a flow are given by

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}_x + \frac{\partial}{\partial y}\mathbf{f}_y = 0. \tag{1}$$

The vector of conservative variables $\mathbf{u} \in \mathbb{R}^4$ is given by

$$\mathbf{u} = (\rho, \rho v_x, \rho v_y, E)^{\mathrm{T}}, \tag{2}$$

The inviscid fluxes are given by

$$\mathbf{f}_x = \left(\rho v_x, \rho v_x^2 + p, \rho v_x v_y, (E+p)v_x\right)^{\mathrm{T}}, \tag{3}$$

$$\mathbf{f}_y = \left(\rho v_y, \rho v_x v_y, \rho v_y^2 + p, (E+p)v_y\right)^{\mathrm{T}}. \tag{4}$$

Instead of E one can also write ρe.

Re-writing via scaled variables and reference quantities, we obtain

$$\frac{\partial}{t_{ref}\partial\hat{t}}\mathbf{u} + \frac{\partial}{L_{ref}\partial\hat{x}}\mathbf{f}_x + \frac{\partial}{L_{ref}\partial\hat{y}}\mathbf{f}_y = 0, \tag{5}$$

$$\mathbf{u} = \rho_{ref}\left(\hat{\rho}, v_{ref}\hat{\rho}\hat{v}_x, v_{ref}\hat{\rho}\hat{v}_y, p_{ref}\hat{\rho}\hat{e}\right), \tag{6}$$

$$\mathbf{f}_x = \left(\rho_{ref}v_{ref}\hat{\rho}\hat{v}_x, p_{ref}\hat{\rho}\hat{v}_x^2 + p_{ref}\hat{p}, p_{ref}\hat{\rho}\hat{v}_x\hat{v}_y, v_{ref}p_{ref}(\hat{\rho}\hat{e}+\hat{p})\hat{v}_x\right), \tag{7}$$

$$\mathbf{f}_y = \left(\rho_{ref}v_{ref}\hat{\rho}\hat{v}_y, p_{ref}\hat{\rho}\hat{v}_x\hat{v}_y, p_{ref}\hat{\rho}\hat{v}_y^2 + p_{ref}\hat{p}, v_{ref}p_{ref}(\hat{\rho}\hat{e}+\hat{p})\hat{v}_y\right). \tag{8}$$

Accounting for the fact that $t_{ref} = L_{ref}/v_{ref}$, we can write:

$$\frac{\partial}{\partial\hat{t}}\hat{\mathbf{u}} + \frac{\partial}{\partial\hat{x}}\hat{\mathbf{f}}_x + \frac{\partial}{\partial\hat{y}}\hat{\mathbf{f}}_y = 0, \tag{9}$$

$$\mathbf{u} = (\hat{\rho}, \hat{\rho}\hat{v}_x, \hat{\rho}\hat{v}_y, \hat{\rho}\hat{e},), \tag{10}$$

$$\mathbf{f}_x = \left(\hat{\rho}\hat{v}_x, \hat{\rho}\hat{v}_x^2 + \hat{p}, \hat{\rho}\hat{v}_x\hat{v}_y, (\hat{\rho}\hat{e}+\hat{p})\hat{v}_x\right), \tag{11}$$

$$\mathbf{f}_y = \left(\hat{\rho}\hat{v}_y, \hat{\rho}\hat{v}_x\hat{v}_y, \hat{\rho}\hat{v}_y^2 + \hat{p}, (\hat{\rho}\hat{e}+\hat{p})\hat{v}_y\right). \tag{12}$$

So the scaled Euler equations are identical to the non-scaled ones, as all reference quantities cancel out.

2.1 Scaling of specific heats

We have (for a single-component flow) $T = T_{ref}\hat{T} = m_{ref}p_{ref}/\rho_{ref}/k\hat{T}$, $\hat{T} = \hat{p}/\rho_{ref}$ (since at $T = T_{ref}$ and $n = n_{ref}$ the flow density is ρ_{ref} and the pressure is p_{ref}). Scaling of specific heats:

$$c_v(T) = \frac{\partial \varepsilon}{\partial T} = \frac{p_{ref}}{\rho_{ref}T_{ref}}\frac{\partial \hat{\varepsilon}}{\partial \hat{T}} = c_{v,ref}\hat{c}_v. \tag{13}$$

So $c_{v,ref} = p_{ref}/(\rho_{ref}T_{ref}) = k/m_{ref}$. Mayer's relation in scaled form then reads that $\hat{c}_p = \hat{c}_v + 1$.

3 Navier–Stokes equations

Let us consider the (multi-temperature) Navier–Stokes equations in general form:

$$\frac{d}{dt}\rho_s + \rho_s\nabla\cdot\mathbf{v} + \nabla\cdot(\rho_s\mathbf{V}_s) = 0, \quad s = 1,\ldots,N_s \tag{14}$$

$$\rho\frac{d}{dt}\mathbf{v} + \nabla\cdot\mathbf{P} = 0, \tag{15}$$

$$\rho\frac{d}{dt}\mathbf{U} + \nabla\cdot\mathbf{q} + \mathbf{P} : \nabla\mathbf{v} = 0, \tag{16}$$

$$\rho\frac{d}{dt}\mathbf{E}^{\mathbf{v}}_{\mathbf{s}} + \nabla\cdot\mathbf{q}^{\mathbf{v}}_{\mathbf{s}} = E_s\nabla\cdot(\rho_s\mathbf{V}_s). \tag{17}$$

With scaling:

$$\frac{\rho_{ref}}{t_{ref}}\frac{d}{d\hat{t}}\hat{\rho}_s + \frac{\rho_{ref}v_{ref}}{L_{ref}}\hat{\rho}_s\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{v}} + \frac{\rho_{ref}v_{ref}}{L_ref}\nabla_{\hat{\mathbf{x}}}\cdot(\rho_s\hat{\mathbf{V}}_s) = 0, \quad s = 1,\ldots,N_s \tag{18}$$

$$\frac{\rho_{ref}v_{ref}}{t_{ref}}\hat{\rho}\frac{d}{d\hat{t}}\hat{\mathbf{v}} + \frac{p_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{P}} = 0, \tag{19}$$

$$\frac{p_{ref}}{t_{ref}}\hat{\rho}\frac{d}{d\hat{t}}\hat{\mathbf{U}} + \frac{q_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{q}} + \frac{p_{ref}v_{ref}}{L_{ref}}\hat{\mathbf{P}} : \nabla_{\hat{\mathbf{x}}}\hat{\mathbf{v}} = 0, \tag{20}$$

$$\frac{p_{ref}}{t_{ref}}\rho\frac{d}{dt}\mathbf{E}^{\mathbf{v}}_{\mathbf{s}} + \frac{q_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\mathbf{q}^{\mathbf{v}}_{\mathbf{s}} = E_s\nabla\cdot(\rho_s\mathbf{V}_s). \tag{21}$$

So $q_{ref} = p_{ref}v_{ref}$.

4 Production terms