Conditional Probability Homework Solutions

1. Because of air pollution, in some cities on days when the air pollution index is very high, people should not exercise outdoors. Sam lives in a city where the probability of the being to exercise outdoors on any randomly chosen day is .99. Henry lives in a city where the probability of being able to exercise outdoors is .9999 on any randomly chosen day. What is the probability that Sam can exercise outdoors every day of the year? What is the probability that Henry can exercise outdoors every day of the year?

   Answer: The probability of being able to exercise outdoors every day of the year where Sam now lives is
   $$(.99)^{365} \approx .0255 \approx 3\%$$
   so there is about a 97% probability of not being able to exercise outdoors one or more days of the year .

   The probability of being able to exercise outdoors every day of the year where Henry now lives is
   $$(.9999)^{365} \approx .9641 \approx 96\%$$
   so there is only about a 4% probability of ever having to miss his outdoor exercise during the year.

2. In a certain city 30% of the families have no bicycle, 50% have one bicycle, 15% have two bicycles, and 5% have at more than two bicycles. Assume that all people store their bicycles in the garage.

   a) What is the probability that a family has at least two bicycles?

   Answer: Let A be the event that a family has two or more bicycles.
   $$p(A) = .15 + .05 = .2$$

   b) It is observed that there is a bicycle belonging to a family in its garage. Now what is the probability that the family has at least two bicycles?

   Answer: Let B be the event that a family has at least one bicycle.
   $$p(A|B) = \frac{p(A \text{ and } B)}{p(B)} = \frac{p(A)}{p(B)} = \frac{.2}{.7} = \frac{2}{7}$$

3. Two dice are tossed. What is the probability that the sum is 7, given that the first die is 3?

   Answer:

   The second die must land only one way (as four,) so $p = 1/6$.

4. Two dice are tossed. What is the probability that the first die is 3, given that the sum is 7?

   Answer:

   First way of answering the question: Since we know the sum is 7, all possibilities for the first die are possible. It could be 1,2,3,4,5, or 6. The probability of getting a 3 is $p = 1/6$.

   Alternate way of answering the question:

   Let B be the event that the first die is a 3. Let A be the event that the sum is 7.

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)} = \frac{(1/6)(1/6)}{(6/36)} = \frac{1}{6}$$

5. Two dice are tossed. What is the probability that the first die is 3, given that the sum is 5?

   Answer:

   First way of answering the question: Since we know the sum is 5, only four possibilities for the first die are possible. It could be 1,2,3,4 The probability of getting a 3 is $p = 1/4$.

   Alternate way of answering the question:

   Let B be the event that the first die is a 3. Let A be the event that the sum is 5.

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)} = \frac{(1/6)(1/6)}{(4/36)} = \frac{1}{4}$$

Four cards are drawn from a standard deck.

6. What is the probability that they are all hearts?

   Answer:

$$\frac{{}_{13}C_4}{{}_{52}C_4} \approx 0.00264106$$

7. What is the probability that the fourth card is a heart, given that the first three cards are hearts?

   Answer: After drawing the first three cards, which we know are hearts, there are 49 cards left. There are 10 hearts left. So the probability is

$$\frac{10}{49} \approx 0.204082$$

8. In 1988, the state of Illinois required HIV testing for a couple to obtain a marriage license. The HIV testing at the time consisted of two separate tests, the ELISA test and the Western Blot test. The Elisa test was significantly less expensive. A person who is HIV positive would test positive under the ELISA test 95% of the time. A person who is HIV negative would test positive under the ELISA test 99% of the time. In 1988, it was estimated that the percentage of people applying for a marriage license that were actually HIV positive was 1%. Calculate the conditional probability under that someone who tested positive under the ELISA test was actually HIV positive. (Hint: To do this, you may assume a population of 10000 people, calculate how many of them are HIV positive and HIV negative. For each of these, you will then want to calculate how many test positive and test negative.)

   Answer: Under the conditions of the problem, $.99 \times 10000 = 9900$ people out of the 10000 are HIV negative. Of these, $.99 \times 9900 = 9801$ will test negative, while $.01 \times 990 = 99$ will test positive. On the other hand, 100 people are HIV positive, of whom 95 will test positive and 5 will test negative. All together, 194 people will test positive of whom 95 are actually positive. Thus, the conditional probability of being HIV positive, given that you test positive (and are from the overall population) is 95/194, or about 49%.

9. If someone test positive on the ELISHA test, then that person is given the Western Blot test. A person who is HIV positive will test positive on the Western Blot test 99% of the time, while a person who is HIV negative will test positive on the Western Blot test 5% of the time. Suppose we are in the problem before. What is the conditional probability that someone is HIV positive given that the person tests positive on both the Western Blot and the ELISHA test?

   Answer: Using the above numbers, of the 99 HIV negative people that test positive on the ELISHA test, 95% of them, or $.95 \times 99 = 94.05$ of them, will test negative and $.05 \times 99 = 4.95$ of them test positive on the Western Blot test. (This works better if you start with 1000000 people). Meanwhile, of the 99 people that were HIV positive that tested HIV positive on the ELISHA test, $.99 \times 99 = 98.01$ test positive on the Western Blot test. Thus, all together, $98.01 + 4.95 = 102.96$ test positive on both tests, of which 4.95 are actually HIV negative. Thus the conditional probability is 98.01/102.96 or about 95% that someone is HIV positive given that the person tested HIV positive.

10. In the 2000 Summer Olympics, the head of the Australian drug testing said that the odds of the lab giving a positive result on a negative sample for an athlete was about 1 in 1000, or about .1%. Suppose that the tests are 100% accurate on positive samples (so they will all test positive), and that 5% of the athletes are actually using drugs. What is the conditional probability that an athlete who tested positive for drug use, actually had a positive sample? What would be the conditional probability if 1% of the athletes had positive samples?

    Answer: We can suppose there were 1000000 samples and work through this. Doing so, we get the following chart for the 5% case:

		$Sample$	
		$Positive$(50000)	$Negative$(950000)
$TestResult$	$Positive$	50000	950
	$Negative$	0	949050

Consequently 50950 athletes would have positive samples of which 50000 of them would actually be positive. Thus the conditional probability would be 98%.

Alternatively, if the percentage of positive samples is 1%, we have the chart

		Sample	
		Positive(10000)	*Negative*(990000)
TestResult	*Positive*	10000	990
	Negative	0	989010

So in this case the number of positive tests is 10990 while only 10000 samples are positive. Thus the conditional probability is 10000/10990 or approximately 91%.

11. For Olympic drug testing, the best tests are used by the most careful labs. Suppose the rate of false positive tests given by a lab is about 5%, while about 0.5% of the population at large uses a given drug. Again assuming that all positive samples test positive, in this case, what is the conditional probability of a positive sample given a positive test?

    Answer: In this case, out of a population of 1000000, about 5000 samples are actually positive, while 995000 samples are negative. Of these $.05 \times 995000 = 24875$ will test positive. On the other hand, if all 5000 positive samples test positive, then there are 29875 samples that test positive, giving a conditional probability of 5000/29875 that a sample testing positive is actually positive, or about 17%.