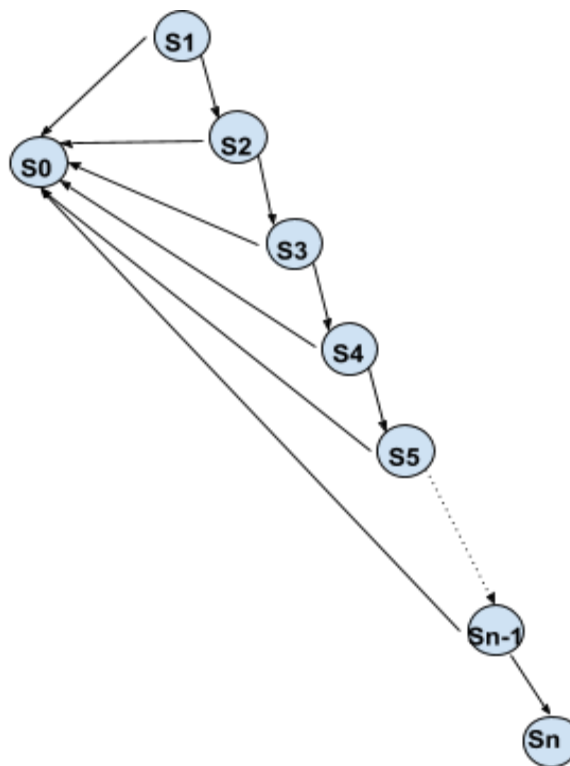


**CS 540: Introduction to Artificial Intelligence**  
**Homework Assignment # 11**  
**Assigned: 12/11 Due: 12/18 before class**

1. (Search) Consider  $n + 1$  states.  $S_1$  is the initial state,  $S_n$  is the goal state.  $S_0$  is a dead-end state with no successors. For each state  $S_i$ ,  $i = 1, \dots, n - 1$ , it has two successors:  $S_{i+1}$  and  $S_0$ .  $S_n$  also has no successors. There is no cycle check nor CLOSED list. How many goal-checks will be performed by depth first search? Assume everything being equal, state with small index is checked first. If a state is goal-checked multiple times, count it multiple times.



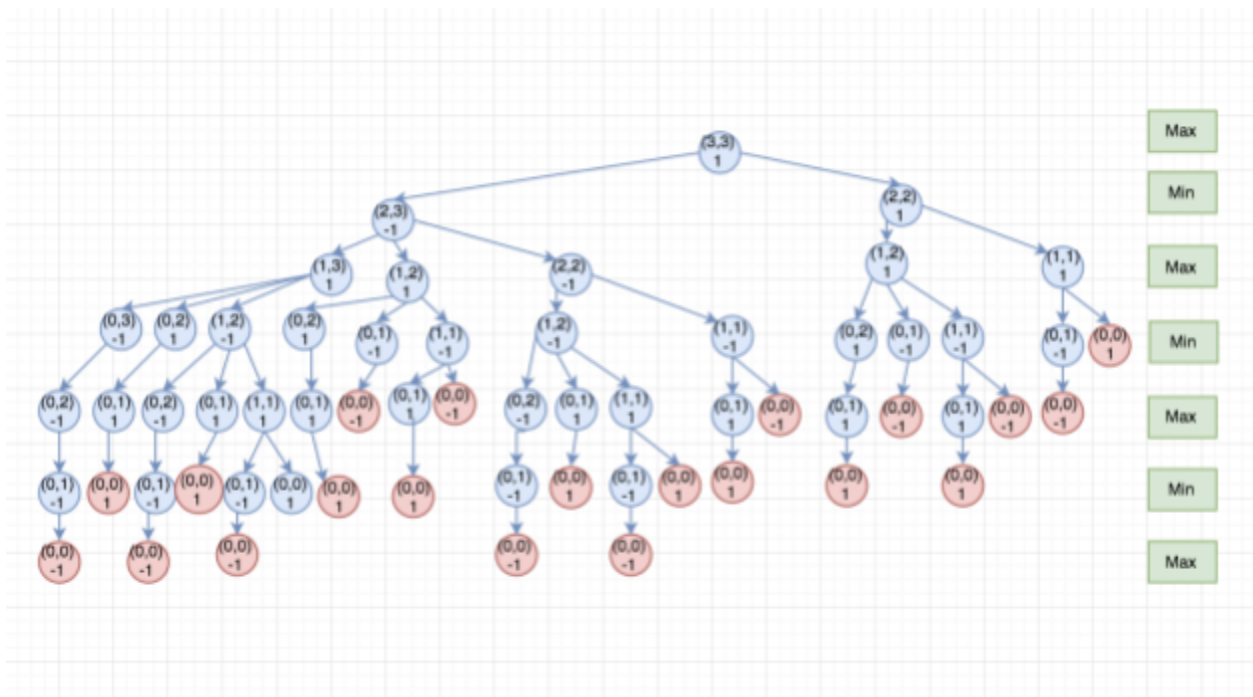
At depth 1: (Root at depth 1)  
States in CLOSED:  $S_1$   
Total No of Goal checks: 1

At depth 2:  
States in CLOSED:  $S_1, S_0, S_2$   
Total No of Goal checks: 3

At depth 3:  
States in CLOSED:  $S_1, S_0, S_2, S_0, S_3$   
Total No of Goal checks: 5

**=  $2n-1$  goal checks will performed**

**2. (Game) Consider a variant of the II-nim game. There are two piles, each pile has three sticks. A player can take one stick from a single pile; or take two sticks: one from each pile (when available). The player who takes the last stick wins. Let the game value be 1 if the first player wins. Show the game tree and give the game theoretical value at all nodes.**



Game theoretical value is 1.

**3. (Probability)** There are two biased coins in my wallet: coin A has  $P(\text{Heads}) = a$ , coin B has  $P(\text{Heads}) = b$ . I took out one coin at random (with equal probability choosing A or B) and flipped it twice: the outcome was Head, Tail. What is the probability that the coin was A?

Given,

$$P(A) = P(B) = 0.5$$

Coin A,  $P(H) = a$ , so  $P(T) = 1-a$

Coin B,  $P(H) = b$ , so  $P(T) = 1-b$

Find  $P(A|H,T) = ?$

$$\begin{aligned}
 P(A|H,T) &= \frac{P(A,H,T)}{P(H,T)} \\
 &= \frac{P(H,T|A) * P(A)}{P(H,T|A)*P(A) + P(H,T|B)*P(B)} \quad // \text{ By chain rule and joint probability} \\
 &\text{Probability of getting head/tail from coin A/B are independent events} \\
 &= \frac{P(H|A)*P(T|A)*P(A)}{P(H|A)*P(T|A)*P(A) + P(H|B)*P(T|B)*P(B)} \\
 &= \frac{a * (1-a) * 0.5}{a * (1-a) * 0.5 + b * (1-b) * 0.5} = \frac{0.5 * a * (1-a)}{0.5 * [a * (1-a) + b * (1-b)]} \\
 &= \frac{a(1-a)}{a(1-a) + b(1-b)}
 \end{aligned}$$

**4. (PCA) You performed PCA in R2 . It turns out that the first principal component is  $u_1 = (\sqrt{1/2}, \sqrt{1/2})$ , and the second principal component is  $u_2 = (-\sqrt{1/2}, \sqrt{1/2})$ . One of your data points has its new representation as (1, 2). What was the original coordinates of the point?**

New data point = (1, 2)

first principal component,  $u_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$

second principal component,  $u_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$

Let's assume that the original point is  $\mathbf{X} = (a, b)$ , which we have to find here.

Projected data point (1, 2) is given by  $(u_1^T x, u_2^T x)$

So we have:

$$u_1^T x = 1$$

$$u_2^T x = 2$$

$$\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 1 \quad \dots\dots\dots (1)$$

$$\frac{-a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 2 \quad \dots\dots\dots (2)$$

By Adding (1) and (2):

$$\Rightarrow \frac{b}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 3$$

$$\Rightarrow \frac{2b}{\sqrt{2}} = 3$$

$$\Rightarrow b = \frac{3}{\sqrt{2}}$$

By subtracting (2) from (1):

$$\Rightarrow \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}} = -1$$

$$\Rightarrow \frac{2a}{\sqrt{2}} = -1$$

$$\Rightarrow a = \frac{-1}{\sqrt{2}}$$

Thus the original point is  $\mathbf{X(a,b) = ( \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}} )}$

**5. (Resolution) Given the knowledge base (a)  $A \vee B$  (b)  $C \Rightarrow A$  use resolution to prove the query  $A \vee B \vee C$ .**

Given KB:

(a)  $A \vee B$

(b)  $C \Rightarrow A$ , in CNF form =  $\sim C \vee A$

Query:

(a)  $A \vee B \vee C$

To prove by resolution we take  **$\text{KB} \wedge \sim \text{Query}$**

$$\sim \text{Query} = \sim(A \vee B \vee C) = \sim A \wedge \sim B \wedge \sim C$$

For resolution we have:

KB terms:

1.  $A \vee B$

2.  $\sim C \vee A$

$\sim \text{Query}$ :

3.  $\sim A$

4.  $\sim B$

5.  $\sim C$

Now, using (1), (3) and (4) we get empty result

$(A \vee B) \wedge (\sim A) \wedge (\sim B) = A \wedge \sim A \vee B \wedge \sim B = \text{Empty}$ , thus resolution is done with the given KB and query.

**6. (Clustering)** There are six points in two-dimensional space:  $a = (0, 0)$ ,  $b = (1, 0)$ ,  $c = (3, 1)$ ,  $d = (7, 7)$ ,  $e = (9, 9)$ ,  $f = (3, 6)$ . Perform Hierarchical Agglomerative Clustering with single linkage and Euclidean distance. Complete the resulting clustering tree diagram (i.e., the dendrogram).

Note: Used simple linkage Euclidean distance to find distance between clusters.

**Iteration 1:**

	a(0,0)	b(1,0)	c(3,1)	d(7,7)	e(9,9)	f(3,6)
a(0,0)		1	3.162278	9.899495	12.727922	6.708204
b(1,0)			2.236068	9.219544	12.041595	6.324555
c(3,1)				7.211103	10	5
d(7,7)					2.828427	4.123106
e(9,9)						6.708204
f(3,6)						

So,  $a(0,0)$  and  $b(1,0)$  are close to each other and they merge to form a cluster :

**(a,b), c, d, e, f**

**Iteration 2:**

	a(0,0)+b(1,0)	c(3,1)	d(7,7)	e(9,9)	f(3,6)
a(0,0)+b(1,0)		2.236068	9.219544	12.041595	6.324555
c(3,1)			7.211103	10	5
d(7,7)				2.828427	4.123106
e(9,9)					6.708204
f(3,6)					

So, cluster  $(a(0,0), b(1,0))$  and  $c(3,1)$  are close to each other and they merge to form a new

cluster : **((a,b), c), d, e, f**

**Iteration 3:**

	$a(0,0)+b(1,0)+c(3,1)$	$d(7,7)$	$e(9,9)$	$f(3,6)$
$a(0,0)+b(1,0)+c(3,1)$		7.211103	10	5
$d(7,7)$			2.828427	4.123106
$e(9,9)$				6.708204
$f(3,6)$				

So,  $d(7,7)$  and  $e(9,9)$  are close to each other and they merge to form a cluster :  
 **$((a,b), c), (d, e), f$**

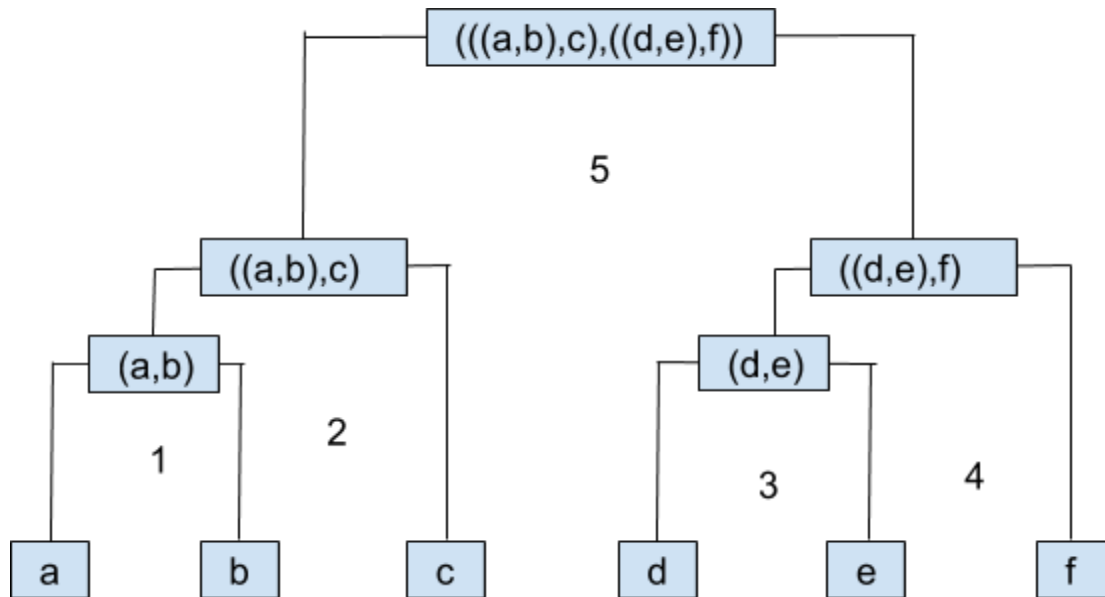
**Iteration 4:**

	$a(0,0)+b(1,0)+c(3,1)$	$d(7,7)+e(9,9)$	$f(3,6)$
$a(0,0)+b(1,0)+c(3,1)$		7.211103	5
$d(7,7)+e(9,9)$			4.123106
$f(3,6)$			

So, cluster  $(d(7,7), e(9,9))$  and  $f(3,6)$  are close to each other and they merge to form a new cluster :  **$((a,b), c), ((d, e), f)$**

**Iteration 5:**

So in this iteration the two clusters  **$((a,b), c)$  and  $((d, e), f)$**  merge to form one cluster  **$((a,b), c), ((d, e), f)$**



7. (Gradient descent) Let  $x = (x_1, x_2) \in \mathbb{R}^2$ . We want to minimize the objective function  $f(x) = \sin((x_1 + x_2)\pi) + e^{x_1 - x_2} + x_1 x_2$ . Let the stepsize  $\eta = 0.1$ . If we start at  $x^{(0)} = (1, 1)$ , what is the next vector  $x^{(1)}$  produced by gradient descent?

Let  $x = (x_1, x_2) \in \mathbb{R}^2$

Given function  $f(x) = \sin((x_1 + x_2)\pi) + e^{x_1 - x_2} + x_1 x_2$

stepsize  $\eta = 0.1$

$x^{(0)} = (1, 1)$

$x^{(1)} = ?$

From gradient descent, we know

$$x^{(1)} = x^{(0)} - \eta \nabla f(x^{(0)})$$

$$\begin{aligned} \nabla f(x) &= \nabla (\sin(x_1 + x_2)\pi) + e^{x_1 - x_2} + x_1 x_2 \\ &= (\partial(\sin((x_1 + x_2)\pi) + e^{x_1 - x_2} + x_1 x_2) / \partial x_1, \partial(\sin((x_1 + x_2)\pi) + e^{x_1 - x_2} + x_1 x_2) / \partial x_2) \\ &= (\pi \cos((x_1 + x_2)\pi) + e^{x_1 - x_2} + x_2, \pi \cos((x_1 + x_2)\pi) - e^{x_1 - x_2} + x_1) \end{aligned}$$

$$\begin{aligned} \text{now, } \nabla f(x^{(0)}) &= (\pi \cos((1 + 1)\pi) + e^{1-1} + 1, \pi \cos((1 + 1)\pi) - e^{1-1} + 1) \\ &= (\pi \cos(2\pi) + e^0 + 1, \pi \cos(2\pi) - e^0 + 1) \\ &= (\pi + 1 + 1, \pi - 1 + 1) \\ &= (\pi + 2, \pi) \end{aligned}$$

Now substituting values in gradient descent equation,  $x^{(1)} = x^{(0)} - \eta \nabla f(x^{(0)})$

$$\begin{aligned} x^{(1)} &= (1, 1) - 0.1 \nabla f(x^{(0)}) \\ &= (1, 1) - 0.1(\pi + 2, \pi) \\ &= (1, 1) - (0.1\pi + 0.2, 0.1\pi) \\ &= (0.8 - 0.1\pi, 1 - 0.1\pi) \end{aligned}$$

$$x^{(1)} = (0.8 - 0.1 * \pi, 1 - 0.1 * \pi)$$

**8. (Sigmoid) Derive the derivative of the sigmoid function  $\sigma(x) = 1 / 1 + \exp(-x)$  .**

$$\text{Sigmoid function } \sigma(x) = \frac{1}{1 + \exp(-x)}$$

$$\text{Derivative of the sigmoid function is given by, } \sigma'(x) = \frac{d}{dx} \frac{1}{1 + \exp(-x)}$$

Using the quotient and chain rule, we can differentiate above.

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\sigma'(x) = \frac{(1 + \exp(-x)) \left( \frac{d}{dx} 1 \right) - 1 \left( \frac{d}{dx} (1 + \exp(-x)) \right)}{(1 + \exp(-x))^2}$$

$$\sigma'(x) = \frac{0 - 1(0 + \exp(-x)(-1))}{(1 + \exp(-x))^2}$$

$$\sigma'(x) = \frac{\exp(-x)}{(1 + \exp(-x))^2} = \frac{1 + \exp(-x) - 1}{(1 + \exp(-x))^2}$$

$$\sigma'(x) = \frac{1 + \exp(-x)}{(1 + \exp(-x))^2} - \frac{1}{(1 + \exp(-x))^2}$$

$$\sigma'(x) = \frac{1}{1 + \exp(-x)} - \left( \frac{1}{1 + \exp(-x)} \right)^2$$

$$\sigma'(x) = \sigma(x) - \sigma(x)^2$$

$$\sigma'(x) = \sigma(x)(1 - \sigma(x))$$

**9. (MDP) Consider state space  $S = \{s_1, \dots, s_n\}$  and action space  $A = \{\text{left}, \text{right}\}$ :  
 $s_1 \dots s_n$**

**The actions move the agent one step in the corresponding direction, except when it is at an end: attempting to move beyond the end makes the agent stay in the current state.**



When the agent is in state  $s_{n-1}$ , taking the “right” action also gives it reward  $r = 1$ . All other state-action pairs have zero reward. Let  $\gamma$  be the discounting factor. What is the value  $v(s_1)$  under the optimal policy?

The optimal policy for our case is

$$\pi(s_1) = \text{right}$$

$$\pi(s_2) = \text{right}$$

$$\pi(s_3) = \text{right}$$

...

...

...

...

$$\pi(s_{n-1}) = \text{right}$$

$$\pi(s_n) = \text{left}$$

Now,

$$v(s_1) = \gamma^0.R_0 + \gamma^1.R_1 + \gamma^2.R_2 + \gamma^3.R_3 + \dots$$

$$v(s_1) = \gamma^0.0 + \gamma^1.0 + \gamma^2.0 + \gamma^3.0 + \dots \gamma^{n-3}.0 + \gamma^{n-2}.1 + \gamma^{n-1}.0 + \gamma^n.1 + \gamma^{n+1}.0 + \gamma^{n+2}.1 + \dots$$

$$v(s_1) = \gamma^{n-2}.1 + \gamma^n.1 + \gamma^{n+2}.1 + \dots$$

$$v(s_1) = \frac{\gamma^{n-2}}{1 - \gamma^2}$$

**10. (Q-learning)** A robot initializes Q-learning by setting  $q(s, a) = 1$  for all state  $s$  and action  $a$ . It has a learning rate  $\alpha$ , and discounting factor  $\gamma$ . The robot senses that it is in state  $s_1$  and decides to perform action  $a_1$ . For this action, the robot receives reward 100 and arrives at state  $s_2$ . After this one step of Q-learning, for all  $s, a$  pairs show their value  $q(s, a)$ .

Given that:  $q(s, a) = 1 \quad \forall s, a$

Learning rate =  $\alpha$

Discounting factor =  $\gamma$

$s_1, a_1 \rightarrow s_2$

$R(s_1, a_1) = 100$

From Bellman's equation,

$$q'(s_1, a_1) = \alpha[R + \gamma \cdot \max(q(s', a'))] + (1 - \alpha) \cdot q(s_1, a_1)$$

$$q'(s_1, a_1) = \alpha[100 + \gamma \cdot \max(q(s_2, a'))] + (1 - \alpha) \cdot 1$$

now,  $q(s_2, a') = 1 \quad \forall a'$

$$\Rightarrow \max(q(s_2, a')) = 1$$

$$\Rightarrow q'(s1, a1) = \alpha[100 + \gamma \cdot 1] + (1 - \alpha) \cdot 1$$

$$q'(s1, a1) = \alpha[100 + \gamma] + (1 - \alpha)$$

$$q'(s1, a1) = \alpha[99 + \gamma] + 1$$

Rest of the Q-table remains intact.

Thus, the updated values will be

$$q(s, a) = 1 \quad \forall s, a - \{s1, A1\}$$

$$q(s1, a1) = \alpha[99 + \gamma] + 1$$