Modal Information Logics: Axiomatizations and Decidability

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Abstract

The present paper studies formal properties of so-called modal information logics (MILs)—modal logics first proposed in [15] as a way of using possible-worlds semantics to model a theory of information. They do so by extending the language of propositional logic with a binary modality defined in terms of being the supremum of two states.

First proposed in 1996, MILs have been around for some time, yet not much is known: [16, 17] pose two central open problems, namely (1) axiomatizing the two basic MILs of suprema on preorders and posets, respectively, and (2) proving (un)decidability.

The main results of the first part of this paper are solving these two problems: (1) by providing an axiomatization [with a completeness proof entailing the two logics to be the same], and (2) by proving decidability. In the proof of the latter, an emphasis is put on the method applied as a heuristic for proving decidability 'via completeness' for semantically introduced logics; the logics lack the FMP w.r.t. their classes of definition, but not w.r.t. a generalized class.

These results are build upon to axiomatize and prove decidable the MILs attained by endowing the language with an 'informational implication'—in doing so a link is also made to the work of [6] on the Lambek Calculus.*

Keywords: Modal information logic, modal logic, axiomatization, completeness, decidability

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Introduction

This introduction is divided into two parts. First, we give a more general introduction, forwarding the logics of concern and motivating their study. Second, we break down the paper section-by-section, outlining the mathematical issues at hand and how they are solved, ending with a list of the main results achieved.

Motivation and general introduction

Aiming to model a theory of information by using the possible-worlds semantics of modal logic, [15] introduces a modal logic of a single binary modality '$\langle\sup\rangle$' with semantics:

$$x \Vdash \langle\sup\rangle\varphi\psi \quad \textbf{iff} \quad \text{there exist } y, z \text{ such that } y \Vdash \varphi;\ z \Vdash \psi; \text{ and } x = \sup\{y, z\}.$$

This is motivated by construing the 'worlds' as information states; the relation as an ordering of the information states; and the supremum modality '$\langle\sup\rangle$' as providing language for speaking of 'merge' (or 'fusion') of information states. In accordance with this interpretation, modal logics with such a modality are called *modal information logics* (MILs).

The main focus of this paper is to study formal properties of MILs, primarily by providing axiomatizations and proving decidability results. Notably, this study also includes solutions to open axiomatization and decidability problems posed in [16–18]. However, it is worth emphasizing that there is a non-technical reason for studying these logics:[1] By describing basic patterns of information in- and decrease in qualitative yet logically precise manners, MILs are (or can be viewed as) attempting to solve a major problem in the foundations of information, namely that of unifying theories of information: ranging from 'quantitative' theories (such as Fisher information, Shannon entropy, and Kolmogorov complexity) to 'qualitative' ones (more akin to our everyday usage of the word 'information'), cf. [1].

Looking at modal information logics in this light, the paper is, foremost, concerned with the following two questions:

Axiomatization:	*What are, according to a MIL, the fundamental principles governing information?*
Decidability:	*Is there an algorithm that given any principle can tell whether it is a valid principle of information?*

Now, for these questions to be well-defined, we must get clear on a principal way in which MILs can differ, namely in their notion of 'fusion': on what class of structures do we want to interpret the '$\langle\sup\rangle$'-modality – what is our choice of frames? Rather general are preorders where the modality '$\langle\sup\rangle$' is defined in terms of quasi-least upper bounds; i.e., 'merges' are not unique but come in clusters. This defines the basic modal information logic, denoted MIL_{Pre}. The informational interpretation further

[1] In fact, there are many: see [11] for more reasons.

suggests examining the case where the relation is also anti-symmetric (resulting in posets).[2] We denote the corresponding logic as MIL_{Pos}.

After solving the problems of axiomatization and decidability for MIL_{Pre} and MIL_{Pos}, we show that our techniques for doing so extend to the MILs, $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, obtained by enlarging the language with the modality '$\backslash$' with semantics

$$y \Vdash \varphi\backslash\psi \qquad \textbf{iff} \qquad \text{for all } x, z, \text{ if } z \Vdash \varphi \text{ and } x = \sup\{y, z\}, \text{ then } x \Vdash \psi.$$

This extension was suggested in [18], and is motivated under the informational interpretation as an 'informational implication': an information state y 'satisfies' $\varphi\backslash\psi$ iff for all information states z and all merges $x = \sup\{y, z\}$ of information states y, z, if z satisfies φ (the antecedent), then the merge x satisfies ψ (the consequent).

It should be noted that connectives with this kind of semantics feature prominently in several logics: in fact, our informational interpretation is that of the relevance logic of [13, 14] where '$\backslash$' is relevant implication; and the symbol '$\backslash$' is the (left) residual of the Lambek Calculus [12] – a logic we will make a junction with. Moreover, '$\backslash$' compliments '$\langle\sup\rangle$' very naturally: if, say, $x = \sup\{y, z\}$, then '$\backslash$' accesses this from the perspective of y (or z) while '$\langle\sup\rangle$' accesses it from the perspective of x. It is thus not surprising that the 'intensional conjunction' of [13, 14] and the 'product connective' of [12] are analogues of '$\langle\sup\rangle$'.

As observed in [15–18], a final interesting aspect of MIL_{Pre} and MIL_{Pos} we want to mention is that, using '$\langle\sup\rangle$', the past-looking modality 'P' becomes definable, so by being modal logics of preorders and posets, they mildly extend **S4**. Moreover, using '$\backslash$', the future-looking modality 'F' becomes definable as well. Put in this light, MIL_{Pre} and MIL_{Pos} (and $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, respectively) are quite natural extensions of (temporal) **S4** obtained by adding vocabulary for describing further structure of preorders and posets. Thus, seen from a purely mathematical angle, these MILs can be motivated by an interest in seeking a modal perspective on rather ubiquitous mathematical structures, namely preorders and posets.

Guide to sections

Zooming in, in the order as they occur in this paper, we explain the mathematical problems we will be addressing. Starting off, we examine MIL_{Pre} and MIL_{Pos}, motivated by two central open problems posed by [16–18], namely (1) axiomatizing the logics and (2) proving (un)decidability. The first three sections of this paper are concerned with these two problems.

In Section 1, after having formally defined the logics, we, in particular, show that MIL_{Pre} lacks the finite model property (FMP) w.r.t. preorders. This proof extends to all above mentioned MILs on their respective classes of frames as well. Although this can be taken as a (clear) indication of undecidability, we end the section by explaining

[2] Moreover, this interpretation naturally suggests considering the case where any two worlds (or 'information states'), additionally, have a unique merge (resulting in join-semilattices). However, as to keep this paper within reasonable length, we concentrate on preorders and posets and postpone a study of modal information logics on join-semilattices (and other structures) for another paper – in line with us setting out to solve the open problems posed in [16–18].

why this need not be, forwarding a method for proving decidability 'via completeness' when dealing with semantically introduced logics (like MILs).[3]

In Section 2, we provide an axiomatization of MIL_{Pre} and prove it to be sound and strongly complete. We do so by, given a consistent set, constructing a model for it. As the constructed models are, in fact, posets, we get as a corollary that $MIL_{Pre} = MIL_{Pos}$; thus, solving problem (1) for both logics in one go.

Following the method laid out in Section 1, in Section 3, we, first, use this axiomatization to find another class of structures $\mathcal{C}$ for which the logic also is complete. Second, we show that on this class of structures we do, in fact, have the FMP—allowing us to deduce decidability.

Next, in Section 4, we explore the conservative extensions $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ obtained by adding the informational implication '$\backslash$'. Combining ideas from our study of $MIL_{Pre} = MIL_{Pos}$ with some new ones—among which some are ours and some, more interestingly, are due to work on the Lambek Calculus of [6]—we (i) axiomatize the logics, (ii) show that $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$, and (iii) prove them to be decidable. This crossing with the Lambek Calculus sheds one more illuminating light on modal information logics: $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$ is the Lambek Calculus (augmented with classical propositional logic) of suprema on preorders (or posets).

In summary, the main results achieved are:

- Axiomatizing MIL_{Pre} and deducing $MIL_{Pre} = MIL_{Pos}$.
- Proving MIL_{Pre} decidable.
- Axiomatizing $MIL_{\backslash\text{-}Pre}$ and deducing $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$.
- Proving $MIL_{\backslash\text{-}Pre}$ decidable.

1 Preliminaries

We start off this section by formally defining the basic modal information logics (subsection 1.1). Then, in subsection 1.2, we, first, show lacks of properties related to that of decidability, most notably proving that all of the logics of concern lack the finite model property w.r.t. their respective classes of definition; and then, second, sketch a general method for proving decidability in cases like ours – a method which we will employ in Section 2 and 3.

1.1 Defining the logics

Definition 1.1 (Language)**.** The basic language $\mathcal{L}_M$ of modal information logic is defined using a countable set of proposition letters $\mathbf{P}$ and a binary modality $\langle \text{sup} \rangle$. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \text{sup} \rangle \varphi\psi,$$

where $p \in \mathbf{P}$ and $\bot$ is the falsum constant. $\dashv$

[3] 'Semantically introduced' as contrasting logics introduced by a syntactic (or proof-theoretical) definition.

Modal information logics are defined by semantical means; i.e., as sets of $\mathcal{L}_M$-validities on classes of structures. The most general class of interest is that of preorders; formally, we define as follows:

Definition 1.2 (Frames and models)**.** A (Kripke) *preorder frame* for $\mathcal{L}_M$ is a pair $\mathbb{F} = (W, \leq)$ where

- W is a set; and
- $\leq$ is a preorder on W, i.e., reflexive and transitive.

A (Kripke) *preorder model* for $\mathcal{L}_M$ is a triple $\mathbb{M} = (W, \leq, V)$ where

- $(W, \leq)$ is a preorder frame; and
- V is a valuation on W, i.e., a function $V : \mathbf{P} \to \mathcal{P}(W)$. ⊣

For clarity, before defining the next class of structures we will be considering, we set out the basic modal information logic of preorders in full detail. Having defined the structures in which to interpret the $\mathcal{L}_M$-formulas, we are about to define the actual semantics. In order to do so, we provide the following definition generalizing the notion of supremum from partial orders to preorders:

Definition 1.3 (Supremum)**.** Given a preorder frame $(W, \leq)$ and worlds $u, v, w \in W$, we say that w is *a quasi-supremum* (or simply *supremum*) of $\{u, v\}$ and write $w \in \sup\{u, v\}$ iff

- w is an upper bound of $\{u, v\}$, i.e., $u \leq w$ and $v \leq w$; and
- $w \leq x$ for all upper bounds x of $\{u, v\}$.

In general, $\sup\{u, v\}$ denotes the set of quasi-suprema of $\{u, v\}$, and if this happens to be a singleton $\{w\}$, we may write $w = \sup\{u, v\}$.[4] ⊣

Definition 1.4 (Semantics)**.** Given a preorder model $\mathbb{M} = (W, \leq, V)$ and a world $w \in W$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at w in $\mathbb{M}$ (written '$\mathbb{M}, w \Vdash \varphi$' or '$w \Vdash \varphi$' for short) is defined using the following recursive clauses on φ:

$$
\begin{array}{lll}
\mathbb{M}, w \nVdash \bot, & & \\
\mathbb{M}, w \Vdash p & \textbf{iff} & w \in V(p), \\
\mathbb{M}, w \Vdash \neg\varphi & \textbf{iff} & \mathbb{M}, w \nVdash \varphi, \\
\mathbb{M}, w \Vdash \varphi \vee \psi & \textbf{iff} & \mathbb{M}, w \Vdash \varphi \ \ \textit{or} \ \ \mathbb{M}, w \Vdash \psi, \\
\mathbb{M}, w \Vdash \langle\sup\rangle\varphi\psi & \textbf{iff} & \text{there exist } u, v \in W \text{ such that } \mathbb{M}, u \Vdash \varphi; \\
 & & \mathbb{M}, v \Vdash \psi; \text{ and } w \in \sup\{u, v\}.
\end{array}
$$

Notions like *global truth*, *validity*, etc. are defined as usual in possible-worlds semantics (see, e.g., [4, ch. 1]). ⊣

With these notions laid out, we can define the logic as follows:

[4] Note how $w \in \sup_{\leq}\{u, v\}$ on a preorder $\leq$ iff $[w] = \sup_{\leq_\sim}\{[u], [v]\}$ on its 'skeletal' partial order $\leq_\sim$.

Definition 1.5. The basic modal information logic of suprema on preorders is denoted by MIL_{Pre}, and defined as the set of $\mathcal{L}_M$-validities on the class of all preorder frames; that is,

$$MIL_{Pre} := \{\varphi \in \mathcal{L}_M : (W, \leq) \Vdash \varphi \ \ \textit{for all preorder frames } (W, \leq)\}.$$

Analogously, we denote by MIL_{Pos} the basic modal information logic of suprema on *poset frames*, i.e., frames $(W, \leq)$ where '$\leq$' is a partial order (viz. an antisymmetric preorder). $\dashv$

1.2 Road to decidability

Having formally set out these logics and semantics, we continue with some preliminary remarks. Objective being to get a feel for how the semantics works by stating a few minor results, and, most notably, showing that the logics lack the FMP w.r.t. their respective frames of definition; viz., for instance, MIL_{Pre} does not have the FMP w.r.t. preorder frames. Foremost, we mention how to express the past-looking modality.

Remark 1.6. Besides the connectives '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', '[sup]', and '$\top$' being definable in the standard way, the past-looking unary modality 'P' is definable as

$$P\varphi := \langle\text{sup}\rangle\varphi\top.$$

This can be seen by recalling the definition

$$\mathbb{M}, w \Vdash P\varphi \qquad \textbf{:iff} \qquad \text{there exists } v \leq w \text{ such that } \mathbb{M}, v \Vdash \varphi,$$

and observing that also

$$\mathbb{M}, w \Vdash \langle\text{sup}\rangle\varphi\top \qquad \textbf{iff} \qquad \text{there exists } v \leq w \text{ such that } \mathbb{M}, v \Vdash \varphi.^5 \qquad \dashv$$

Using this observation, the first contribution of our paper is to show a lack of the FMP.

Proposition 1.7. *MIL_{Pre} does not have the FMP w.r.t. preorder frames.*

Proof. Consider the formula

$$\psi_N := HP\langle\text{sup}\rangle pp \wedge HP\neg\langle\text{sup}\rangle pp,$$

where $H := \neg P \neg$ is the dual of P. We claim that ψ_N only is satisfiable in infinite models.[6]

First, we show that ψ_N is, indeed, satisfiable on an infinite model. Accordingly, let $\mathbb{M} := (\mathbb{Z}_-, \leq, V)$ where

- $\mathbb{Z}_-$ is the set of negative integers;

[5] Thus, as promised in the introduction, MIL_{Pre} and MIL_{Pos} are (quite natural) extensions of **S4**.
[6] The subscript 'N' is short for 'negative', as ψ_N witnesses a negative property.

- $\leq$ is the less than or equal to relation on the negative integers; and
- $V(p)$ is the set of even negative integers.

Then $\mathbb{M}$, clearly, is a preorder model, and for all $z \in \mathbb{Z}_-$:

$$\mathbb{M}, z \Vdash \langle \sup \rangle pp \qquad \textbf{iff} \qquad z \text{ is even.}$$

Thus, for all $z \in \mathbb{Z}_-$:

$$\mathbb{M}, z \Vdash P\langle \sup \rangle pp \wedge P\neg\langle \sup \rangle pp.$$

But then ψ_N must be globally true in $\mathbb{M}$; in particular, ψ_N is satisfied in $\mathbb{M}$, proving the first part of the claim

Second, to see that ψ_N isn't satisfiable in any finite model, observe that for any preorder, if two points are situated in the same cluster,[7] then they are suprema of the exact same (sets of) points. It follows that for any preorder model, points in the same cluster satisfy the exact same '$\langle \sup \rangle$'-formulas (those are: formulas with '$\langle \sup \rangle$' as main connective).

With this in mind, it is easy to see that the satisfaction of ψ_N necessitates the existence of an infinite, strictly descending chain: if some $w \Vdash \psi_N$ and some $v_i \leq w$ satisfies, say, $\langle \sup \rangle pp$, then, in particular, there must be some $v_{i+1} \leq v_i$ s.t. $v_{i+1} \Vdash \neg\langle \sup \rangle pp$, whence v_{i+1} must be in a cluster strictly below v_i. Thus, ψ_N cannot be satisfied in any finite model. $\square$

It is worth noting how the proof made essential use of the additional expressive power of our language compared to that of **S4**. **S4** famously enjoys the FMP w.r.t. preorders—its language is, so to speak, too weak to distinguish clusters from chains.

Remark 1.8. The above proof applies to the class of posets as well since the frame $(\mathbb{Z}_-, \leq)$ was, in fact, a poset, hence neither does MIL_{Pos} enjoy the FMP w.r.t. its class of definition. $\dashv$

Beyond not having the FMP, there are even more indicators of undecidability. For the purpose of this paper, these are not central, so we mention them without elaborate proof.

Remark 1.9. MIL_{Pre} does not have the tree model property (TMP) w.r.t. preorder frames. That is, there is a formula χ_N which is satisfiable in a preorder frame, but not in a preorder frame $(W, \leq)$ where $(W, \geq)$ is a reflexive and transitive tree.[8] $\dashv$

Proof. The following formula is satisfiable but not in a (converse) tree

$$\begin{aligned}\chi_N := \; & p \wedge q \wedge \langle \sup \rangle (p \wedge \neg q)(\neg p \wedge q) \\ & \wedge H\big([p \wedge \neg q] \to P(\neg p \wedge \neg q)\big) \wedge H\big([\neg p \wedge q] \to P(\neg p \wedge \neg q)\big)\end{aligned}$$

[7]For clarity, recall that given a preorder $\leq$, w, v are said to be in the same cluster :iff $w \leq v \leq w$.

[8]Consult, e.g., [4, ch. 1, def. 1.7] for the definition of a tree and, in particular, a reflexive and transitive one. Additionally, note how we define the TMP in terms of the converse relation '$\geq$'; this is motivated by the way in which '$\langle \sup \rangle$' is backward-looking. Otherwise, for the case of '$\leq$', a formula like '$p \wedge \langle \sup \rangle (q \wedge \neg p)(\neg q \wedge \neg p)$' already shows the lack of 'a TMP'.

$$\wedge\, H\big(\langle \sup \rangle (\neg p \wedge \neg q)^2 \to [\neg p \wedge \neg q]\big).$$

To see that it is satisfiable, consider the four-element Boolean algebra $(\mathcal{P}(\{a,b\}), \subseteq)$ with valuation V s.t. $\{a,b\} \Vdash p \wedge q$, $\{a\} \Vdash p \wedge \neg q$, $\{b\} \Vdash \neg p \wedge q$, $\varnothing \Vdash \neg p \wedge \neg q$. Then $\{a,b\} \Vdash \chi_N$.

To see that it is not satisfiable in a (converse) tree, suppose that some $\mathbb{M} = (W, \leq), x \Vdash \chi_N$. Since trees, in particular, are antisymmetric, we may assume that '$\leq$' is a partial order.[9] Then $x \Vdash p \wedge q$ and there are y, z s.t. $x = \sup\{y, z\}$, $y \Vdash p \wedge \neg q$ and $z \Vdash \neg p \wedge q$. I.e., $|\{x, y, z\}| = 3$, so $y \nleq z \nleq y$. Further, by the second line of χ_N, there must be y', z' s.t. $y \geq y' \Vdash \neg p \wedge \neg q$ and $z \geq z' \Vdash \neg p \wedge \neg q$. Now if the partial order were to be a tree, we would have that $x = \sup\{y', z'\}$, but then by the third line of χ_N, we would have that $x \Vdash \neg p \wedge \neg q$, which would be a contradiction. $\square$

Remark 1.10. Witnessed by the same proof, not having the TMP extends to MIL_{Pos} as well. $\dashv$

Observation 1.11. Our modal information logics are neither guarded nor packed (as, e.g., the guarded and packed fragments do enjoy the FMP). $\dashv$

At first glance, the results of this subsection might make decidability appear unlikely. But, as it turns out, there is an alternative way of proving decidability, circumventing these problems. We end this section by laying out our method for doing so. This will serve two purposes: by describing the method, we hope to (i), generally, elucidate how and when our method can work as a heuristic for proving decidability, and (ii), specifically, help the reader get a better grasp of the underlying ideas and structure of the ensuing two sections of this paper.

We (1) axiomatize the logics (and show that $MIL_{Pre} = MIL_{Pos}$), (2) use this to show the logic(s) to be complete with respect to another class of structures (where the ternary relation of $\langle \sup \rangle$ won't necessarily be the supremum relation of a preorder, but something more general), and then (3) prove that the logic(s) enjoy the FMP on this other class of structures, from which we can deduce decidability. So to make the salient point clear: when dealing with logics introduced by a semantic definition, not having (e.g.) the FMP w.r.t. the class of definition might not be very telling. The reason being that the resulting logic can very well be complete w.r.t. to another, bigger class of structures for which it does have the FMP.

2 Axiomatizing MIL_{Pre}

While [18] obtains an axiomatization of a variant of MIL_{Pre} extended with nominals and the global modality, the very same paper also inquires finding an axiomatization without hybrid extensions. In this section, we answer this inquiry, providing a purely modal axiomatization. In subsection 2.1, we give a proof-theoretic description of MIL_{Pre}, prove it to be sound, and lay some groundwork for the completeness proof of subsection 2.2, which also allows us to conclude that $MIL_{Pre} = MIL_{Pos}$.

[9] If we also allowed for general preorders '$\leq$' only satisfying that their skeletal partial orders '$\leq_\sim$' are converse trees, then the proof of no TMP would go through by changing the conjunct '$\langle \sup \rangle (p \wedge \neg q)(\neg p \wedge q)$' in χ_N to the more complicated '$\langle \sup \rangle (p \wedge \neg q \wedge \neg P(p \wedge q))(\neg p \wedge q \wedge \neg P(p \wedge q))$'.

2.1 Soundness and preparatory lemmas

We begin by syntactically defining a logic, suggestively called $\mathbf{MIL_{Pre}}$.[10] Through a soundness and completeness proof, we then show $\mathbf{MIL_{Pre}}$ exactly is an axiomatization of our semantically defined logic MIL_{Pre}.

Definition 2.1 (Axiomatization)**.** We define $\mathbf{MIL_{Pre}}$ to be the least normal modal logic (NML)[11] in the language of $\mathcal{L}_M$ containing the following axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp \quad (= \langle \text{sup} \rangle (\langle \text{sup} \rangle p\top)\top \to \langle \text{sup} \rangle p\top$, cf. Remark 1.6)

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$[12] $\dashv$

Having proof-theoretically defined the logic $\mathbf{MIL_{Pre}}$, we can promptly show it to be sound w.r.t. MIL_{Pre}.

Theorem 2.2 (Soundness)**.** $\mathbf{MIL_{Pre}} \subseteq MIL_{Pre}$.

Proof. Standard, tedious task checking that MIL_{Pre} is a normal modal logic and that (Re.), (4), (Co.), and (Dk.) all are valid on preorder frames. $\square$

As oftentimes is the case, while proving soundness is straightforward, proving completeness is much more intricate. Our proof will be a construction using maximal consistent sets (MCSs) for which some preparatory observations and lemmas are needed.

First hurdle is that the $\langle \text{sup} \rangle$-modality is in a general sense a 'logical modality': although accompanied by a *ternary* relation (namely the supremum relation) its interpretation is fixed given a *binary* relation (namely a preorder). For starters, this means that the standard construction of the canonical frame for $\mathbf{MIL_{Pre}}$ won't come equipped with a binary relation for interpreting the binary modality $\langle \text{sup} \rangle$—as is the case for the preorder frames of MIL_{Pre}—but with a ternary one. Fortunately, defining an underlying preorder from this ternary relation spells no trouble. This is summarized in the definition below.

Definition 2.3. We denote the set of all maximal consistent $\mathbf{MIL_{Pre}}$-sets by $W_{\mathbf{Pre}}$, and the ternary relation of the canonical $\mathbf{MIL_{Pre}}$-frame by $C_{\mathbf{Pre}}$.[13] That is $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ holds just in case

$$\forall \delta \in \Delta, \theta \in \Theta \left(\langle \text{sup} \rangle \delta\theta \in \Gamma \right).$$

[10] As a convention, we **boldface** when having 'syntactic' presentations of logics in mind and *italicize* when having 'semantic' presentations of logics in mind.

[11] Caution: As a reviewer has brought to my attention, the definition of a normal modal logic in a language with polyadic modalities given in [4, def. 4.13] is wrong. The necessitation rules are too weak: each occurring '$\bot$' should be replaced by arbitrary formulae ψ_i.

[12] (Re.) is short for 'Reflexivity'; (4) is the transitivity axiom; (Co.) is short for 'Commutativity'; and (Dk.) is short for 'Don't know what to call this axiom'.

[13] Consult [4, ch. 4] for basic definitions, results, and techniques regarding canonical models for modal logics; we have sought to align our notation and terminology with this.

From $C_{\mathbf{Pre}}$, we define the following binary relation on the canonical frame:

$$\leq_{\mathbf{Pre}} := \{(\Delta, \Gamma) \in W_{\mathbf{Pre}} \times W_{\mathbf{Pre}} : \exists\Theta\, (C_{\mathbf{Pre}}\Gamma\Delta\Theta)\}. \qquad \dashv$$

We want to show that $\leq_{\mathbf{Pre}}$ actually is a preorder. To do so, we begin by making two observations.

Observation 2.4. Since $\mathbf{MIL_{Pre}}$ is an NML, we have all the usual lemmas regarding its canonical model. $\dashv$

Observation 2.5. The formula

(T) $p \to Pp$

is derivable in $\mathbf{MIL_{Pre}}$.

In fact, $\{(\mathrm{T}), (4), (\mathrm{Co.}), (\mathrm{Dk.})\}$ is an alternative axiomatization of $\mathbf{MIL_{Pre}}$. $\dashv$

Proof. Some straightforward syntactical manipulations prove the claim; the key steps being

(Re.) $\Rightarrow$ (T): uniformly substitute q for $\top$ in (Re.); and

(T) $\Rightarrow$ (Re.): use (T) to get $p \wedge q \to p \wedge Pq$ and then use (Dk.). $\square$

Using these observations, in the ensuing lemma, we prove that not only is $\leq_{\mathbf{Pre}}$ a preorder, but more 'supremum-like' properties hold of the canonical relation $C_{\mathbf{Pre}}$.

Lemma 2.6. *The following hold:*

(a) $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ ***iff*** $C_{\mathbf{Pre}}\Gamma\Theta\Delta$

(b) $\Delta \leq_{\mathbf{Pre}} \Gamma \overset{(i)}{\textbf{\textit{iff}}} C_{\mathbf{Pre}}\Gamma\Gamma\Delta \overset{(ii)}{\textbf{\textit{iff}}} \forall\delta \in \Delta : P\delta \in \Gamma.$

(c) $\leq_{\mathbf{Pre}}$ *is a preorder.*

(d) $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ ***only if*** $\Delta \leq_{\mathbf{Pre}} \Gamma, \Theta \leq_{\mathbf{Pre}} \Gamma.$

Proof. Since (Re.), (4), (Co.), (Dk.) all are Sahlqvist, one can prove all but *(b)(ii)* via the Sahlqvist-van Benthem algorithm (cf. next section's Lemma 3.1). As often is the case, though, a direct argument is faster; we provide such here.

(a) Let $\{\theta, \delta\} \subseteq \mathcal{L}_M$ be arbitrary. Then – by (Co.), uniform substitution (US) of $\mathbf{MIL_{Pre}}$, and closure under modus ponens (MP) of MCSs – we have

$$\langle\mathrm{sup}\rangle\theta\delta \in \Gamma \Leftrightarrow \langle\mathrm{sup}\rangle\delta\theta \in \Gamma,$$

which suffices to prove the claim.

(b) Right-to-left of (i) is immediate (using (a)). For left-to-right, suppose that $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ for some $\Theta \in W_{\mathbf{Pre}}$ and that $\gamma \in \Gamma, \delta \in \Delta$. Since $\top \in \Theta$, we have that $\langle\mathrm{sup}\rangle\delta\top \in \Gamma$, hence $(\gamma \wedge \langle\mathrm{sup}\rangle\delta\top) \in \Gamma$ and so we get by (Dk.) (and US and MP of MCSs) that $\langle\mathrm{sup}\rangle\gamma\delta \in \Gamma$—as suffices. Regarding (ii), left-to-right follows by (a), while right-to-left is proven using (Dk.).

(c) *Reflexivity.* Let $\Gamma \in W_{\mathbf{Pre}}$ and $\gamma \in \Gamma$ be arbitrary. By *(b)*, it suffices to show that $P\gamma \in \Gamma$, but this follows by $\mathbf{MIL_{Pre}} \vdash p \to Pp$.
Transitivity. Suppose $\Gamma_1 \leq_{\mathbf{Pre}} \Gamma_2 \leq_{\mathbf{Pre}} \Gamma_3$ and $\gamma_1 \in \Gamma_1$. Then by applying *(b)* twice, we get that $PP\gamma_1 \in \Gamma_3$, hence since $\mathbf{MIL_{Pre}} \vdash PPp \to Pp$, we're done.

(d) Consequence of *(a)*. □

2.2 Completeness: constructing our model

Given the previous subsection's results – indicating that the canonical frame is well behaved – one might start wondering whether the canonical relation $C_{\mathbf{Pre}}$ is, in fact, the supremum relation on $\leq_{\mathbf{Pre}}$. If so, we would have completeness in our pocket. Unfortunately, this is far from being the case: not only is the canonical relation $C_{\mathbf{Pre}}$ not the supremum relation on $\leq_{\mathbf{Pre}}$, it is utterly wild.[14]

This forces us to make a rather complicated construction where we do not work with the canonical model per se. Instead, we construct our frame by recursively repairing so-called 'defects' and 'labeling' points of a *sub*set of our frame with MCSs for which we prove a truth lemma. This somewhat generalized approach is useful since it (a) allows for reuse of the same MCS – i.e., *different* points of the frame might get labeled with the *same* MCS – and (b) utilizes that, in the extreme, we only need a truth lemma for one MCS, namely the one extending a given consistent set; thus, we may and will include (non-labeled) points in our construction only to ensure that other (labeled) points satisfy formulas dictated by their MCS-label. That is, we do not care what formulas these points satisfy themselves—their role is entirely auxiliary.[15]

To be more concrete, when recursively constructing this frame, we make sure that at each stage, its corresponding 'approximating frame' is determined by a triple $(l, \leq, D)$ satisfying the definition (of $\mathbb{P}$) below. Specifically, in the recursive step from, say, n to $n+1$, we will make sure that if $(l_n, \leq_n, D_n) \in \mathbb{P}$ then also $(l_{n+1}, \leq_{n+1}, D_{n+1}) \in \mathbb{P}$.[16] This is needed for the colimit construction – i.e., the structure obtained after all finite stages in the recursive construction – to be of the right form.

Definition 2.7. Let W be countable set, and $\mathbb{P}$ the set of all triples $(l, \leq, D)$ such that

1. l is a partial function from W to the set of all MCSs, $W_{\mathbf{Pre}}$.
2. $|\mathrm{dom}(l)| < \aleph_0$, where 'dom$(l)$' refers to the domain of l.
3. $D \subseteq W, |D| < \aleph_0$.
4. $D \cap \mathrm{dom}(l) = \varnothing$.
5. $d \in D \wedge d \leq a \Rightarrow a = d$.
6. $\leq$ is a partial order on $\mathrm{dom}(l) \cup D$, and the diagonal on $W \setminus (\mathrm{dom}(l) \cup D)$.

[14] As to not interlude the completeness proof, observations regarding the wildness of the canonical frame have been put off to Appendix A.

[15] It is worth noting that it is not that we *cannot* make a construction in which all points are labeled (as, essentially, is done in our later Lemma 4.20 and Proposition 4.23), but doing so would obscure the central idea making the construction work.

[16] Our framework is loosely that of [5] with terminology borrowed from [4, sec. 4.6]. More generally, this is a 'step-by-step' construction for which an(other) excellent introduction is the exposition of the 'construction method C' in [9].

7. If $y \leq x$ then $l(y) \leq_{\mathbf{Pre}} l(x)$ (whenever $x, y \in \mathrm{dom}(l)$).[17] $\dashv$

As mentioned, the recursion is carried out by repeatedly repairing 'defects'. Since our goal will be to prove a truth lemma for labeled points, any defect is, in essence, either

(1) that a point x's MCS-label Γ dictates that x satisfy some formula $\langle\mathrm{sup}\rangle\varphi\psi$ which it doesn't, i.e., $\langle\mathrm{sup}\rangle\varphi\psi \in \Gamma = l(x)$ but $x \nVdash \langle\mathrm{sup}\rangle\varphi\psi$; or

(2) that a point x's MCS-label Γ dictates that x satisfy some formula $\neg\langle\mathrm{sup}\rangle\varphi\psi$ which it doesn't, i.e., $\neg\langle\mathrm{sup}\rangle\varphi\psi \in \Gamma = l(x)$ but $x \Vdash \langle\mathrm{sup}\rangle\varphi\psi$.

Although this captures the gist of what defects are, as it turns out, for the proof to work, the precise definitions must be more detailed than this. We proceed giving these.

Definition 2.8 ($\langle\mathrm{sup}\rangle$-defect)**.** Let $(l, \leq, D) \in \mathbb{P}$. Then a pair $(\langle\mathrm{sup}\rangle\chi\chi', x)$ denotes a $\langle\mathrm{sup}\rangle$-*defect* (of $(l, \leq, D)$) :iff

$$\text{(i) } x \in \mathrm{dom}(l), \qquad \text{(ii) } \langle\mathrm{sup}\rangle\chi\chi' \in l(x),$$

and (iii) there are no $y, z \in \mathrm{dom}(l)$ s.t.

$$\begin{array}{lll} \chi \in l(y), & C_{\mathbf{Pre}}l(x)l(y)l(z), & {\uparrow}y = {\uparrow}x \cup \{y\} \cup ({\uparrow}y \cap \{w \mid {\uparrow}w \cap {\uparrow}x = \varnothing\}), \\ \chi' \in l(z), & x = \sup\{y, z\}, & {\uparrow}z = {\uparrow}x \cup \{z\} \cup ({\uparrow}z \cap \{w \mid {\uparrow}w \cap {\uparrow}x = \varnothing\}), \end{array}$$

where ${\uparrow}w := \{v \mid w \leq v\}$.[18] $\dashv$

Definition 2.9 ($\neg\langle\mathrm{sup}\rangle$-defect)**.** Let $(l, \leq, D) \in \mathbb{P}$. Then a quadruple $(\neg\langle\mathrm{sup}\rangle\psi\psi', x, y, z)$ is denoted a $\neg\langle\mathrm{sup}\rangle$-*defect* (of $(l, \leq, D)$) :iff

$$\begin{array}{lll} x \in \mathrm{dom}(l), & x = \sup\{y, z\},^{19} & \neg\langle\mathrm{sup}\rangle\psi\psi' \in l(x), \\ \psi \in l(y), & \psi' \in l(z). & \end{array}$$
$\dashv$

With these defects defined, next up is repairing them. Before providing the actual repair lemmas demonstrating how to coherently repair each of the defects (making sure that if $(l_n, \leq_n, D_n) \in \mathbb{P}$, then also $(l_{n+1}, \leq_{n+1}, D_{n+1}) \in \mathbb{P}$), we give an example to convey intuition for the repairs and the general construction.

Example 2.10. Suppose $(l, \leq, D) \in \mathbb{P}$ and $(\langle\mathrm{sup}\rangle\chi_0\chi_0', x)$ constitutes a $\langle\mathrm{sup}\rangle$-defect; that is, (i) $x \in \mathrm{dom}(l)$, (ii) $\langle\mathrm{sup}\rangle\chi_0\chi_0' \in l(x)$, and there are no y, z fulfilling (iii). Put crudely, the problem is that x's label $l(x)$ requires x to satisfy $\langle\mathrm{sup}\rangle\chi_0\chi_0'$, but x is not the supremum of any y, z s.t. $\chi_0 \in l(y), \chi_0' \in l(z)$. To solve this, we simply add two fresh points y, z immediately below x. Then using the existence lemma of the canonical model for the case $\langle\mathrm{sup}\rangle\chi_0\chi_0' \in l(x)$, we get two MCSs Γ_y, Γ_z s.t. $C_{\mathbf{Pre}}l(x)\Gamma_y\Gamma_z$.[20]

---

[17] l (short for 'label') labels worlds with MCSs, while D-worlds (short for 'dummy worlds') *sole* purpose is to ensure '$\neg\langle\mathrm{sup}\rangle$'-formulas are satisfied at $\mathrm{dom}(l)$-worlds.

[18] That is, a $\langle\mathrm{sup}\rangle$-defect is a failure of a rather strict requirement on a $\mathrm{dom}(l)$-world x when $\langle\mathrm{sup}\rangle\chi\chi' \in l(x)$. The 'upset requirements' on y, z, state that – besides from themselves – if they see a point that x does not, then that point is 'incompatible' with x.

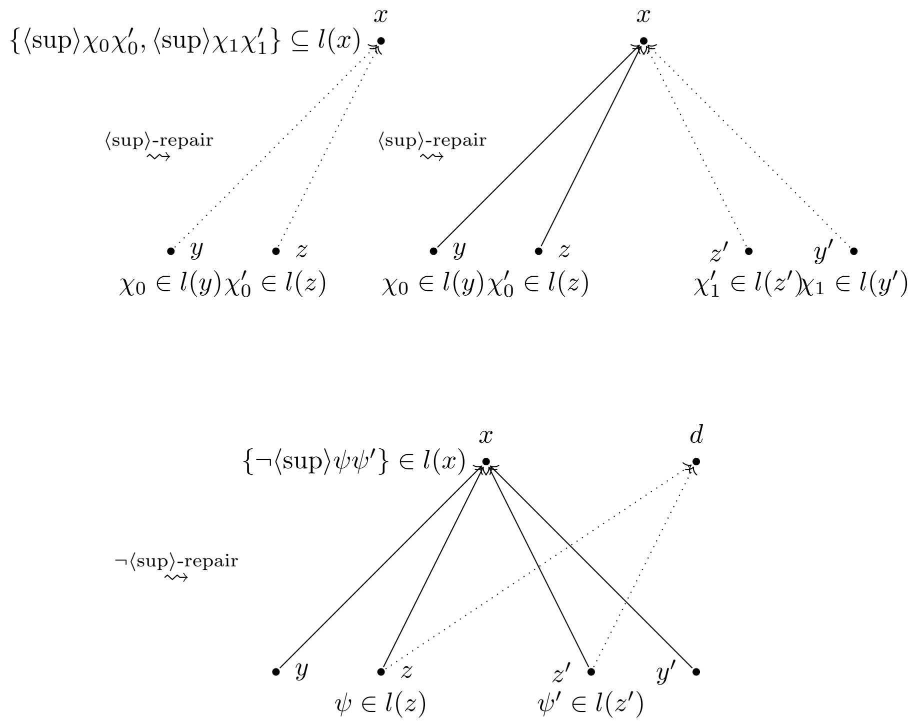
[19] Note that if $x \in \mathrm{dom}(l)$ and $x = \sup\{y, z\}$, then $y, z \in \mathrm{dom}(l)$. This follows by dummies being quasi-blind, cf. 5.

[20] Regarding the existence lemma, recall Observation 2.4.

Setting $l(y) := \Gamma_y$ and $l(z) := \Gamma_z$, the defect has been repaired. The idea is illustrated in the top left corner of the figure below.

Further, if, say, $(\langle\text{sup}\rangle\chi_1\chi_1', x)$ also constitutes a $\langle\text{sup}\rangle$-defect, we simply repeat the process as illustrated in the top right corner of the figure below.

While these two repairs did solve the problems they intended to, they might have created new ones. If, say, $\neg\langle\text{sup}\rangle\psi\psi' \in l(x), \psi \in l(z)$ and $\psi' \in l(z')$, in solving these problems they have made $(\neg\langle\text{sup}\rangle\psi\psi', x, z, z')$ constitute a $\neg\langle\text{sup}\rangle$-defect. This is where we need the 'dummies': to repair this defect, we add a quasi-blind point d as an incomparable upper bound of $\{z, z'\}$ so that x no more is the supremum of $\{z, z'\}$ (cf. the bottom part of the figure).[21] Since d is quasi-blind—and stays quasi-blind (viz. condition 5.)—whatever formula it satisfies is of absolutely no influence to the rest of the points: they cannot 'access' d. So, at bottom, adding dummies is a technique for altering the supremum relation without having to give second thought to what formulas the added points (the dummies) are to satisfy: they are entirely auxiliary (and, hence, do not get labeled, cf. condition 4.). And, most importantly, the alteration of a supremum relation caused by adding a dummy is sufficiently local to not mess up previously repaired defects; in this simplest of cases, we still have $x = \sup\{y, z\} = \sup\{y', z'\}$ after having added the dummy d.



[21] Note that x does stay a *minimal* upper bound of $\{z, z'\}$. This might suggest that the MIL defined by interpreting 'sup' in terms of *minimal* upper bounds instead (i.e., allowing for multiple, incomparable 'fusions' of info states) results in a different logic. Perhaps surprisingly, in other work (excluded from this paper for reasons of length), we have shown that this isn't the case.

We continue by making this basic intuition rigorous – starting with providing the repair lemmas.

Lemma 2.11 ($\langle\sup\rangle$-repair lemma)**.** *Suppose $(\langle\sup\rangle\chi\chi', x)$ is a $\langle\sup\rangle$-defect of some $(l, \leq, D) \in \mathbb{P}$. Then we can extend to $(l', \leq', D') \in \mathbb{P}$ by taking distinct $y, z \in W \setminus (\mathrm{dom}(l) \cup D)$ s.t.*

$$l' := l \cup \{(y, \Gamma), (z, \Delta)\}, \qquad \leq' := \leq \cup \{(y, u), (z, u) \mid x \leq u\}, \qquad D' := D,$$
$$\chi \in \Gamma, \chi' \in \Delta, \qquad C_{\boldsymbol{Pre}} l(x)\Gamma\Delta,$$

and y, z witness that $(\langle\sup\rangle\chi\chi', x)$ does not constitute a $\langle\sup\rangle$-defect of $(l', \leq', D')$.

Proof. Define as in the lemma by taking fresh $y \neq z$ and mapping them to Γ, Δ obtained via the existence lemma for $(\langle\sup\rangle\chi\chi', l(x))$. Then the last claim is easily checked to be satisfied, and $(l', \leq', D')$ also clearly satisfies 1.-6.; thus, it remains to show 7. Since $(l, \leq, D) \in \mathbb{P}$ – and having the definition of $\leq'$ in mind – it suffices to consider the subset

$$\{(y, u), (z, u) \mid x \leq u\} \;\subseteq\; \leq'$$

and the cases $y \leq' y, z \leq' z$. For these, we find:

- $(y \leq' y)$ $l(y) \leq_{\mathbf{Pre}} l(y)$ follows by $\leq_{\mathbf{Pre}}$ being a preorder, hence reflexive, cf. Lemma 2.6 *(c)*.
- $(y \leq' x)$ $l(y) \leq_{\mathbf{Pre}} l(x)$ follows by Lemma 2.6 *(d)*.
- $(y \leq' u)$ For $u > x$, $l(y) \leq_{\mathbf{Pre}} l(u)$ follows by transitivity of $\leq_{\mathbf{Pre}}$.
- $(z \leq' z, x, u)$ Same as for y.[22] □

Lemma 2.12 ($\neg\langle\sup\rangle$-repair lemma)**.** *Suppose $(\neg\langle\sup\rangle\psi\psi', x, y, z)$ is a $\neg\langle\sup\rangle$-defect of some $(l, \leq, D) \in \mathbb{P}$. Then we can extend to $(l', \leq', D') \in \mathbb{P}$ by taking $d \in W \setminus (\mathrm{dom}(l) \cup D)$, letting*

$$l' := l, \qquad \leq' := \leq \cup \{(u, d), (v, d) \mid u \leq y, v \leq z\}, \qquad D' := D \cup \{d\},$$

and getting $x \neq \sup_{\leq'}\{y, z\}$.

Proof. Extend to $(l', \leq', D')$ as described. It follows that $(l', \leq', D') \in \mathbb{P}$. To show

$$x \neq \sup{}_{\leq'}(y, z),$$

since $d \geq' y$ and $d \geq' z$, it suffices to show

$$d \not\geq' x.$$

[22]Observe how the axioms are being used via Lemma 2.6; each 'item' employs an axiom: first (Re.), then (Dk.), then (4), then (Co.). This elucidates their role, and why they are – even if rather weak – adequate: they need only 'encode' this lemma 2.6, which enables extending to $(l', \leq', D')$, and then the 'dummies' do the rest.

To see this, observe that if $x = y$, since $z \le x$, we would have by 7. that $l(z) \le_{\mathbf{Pre}} l(x)$ hence (cf. Lemma 2.6 *(b)*)

$$C_{\mathbf{Pre}} l(x) l(y) l(z),$$

but then $(\neg\langle \sup \rangle \psi\psi', x, y, z)$ couldn't have been a $\neg\langle \sup \rangle$-defect. Same for $x = z$. Thus,

$$y < x \quad \text{and} \quad z < x,$$

whence $d \not\ge' x$ by definition of $\le'$ and $\le$ being a *partial* order by assumption (cf. condition 6.). $\square$

With all of these preliminaries out of the way, we are finally in a position to construct the needed frame and prove completeness.

Theorem 2.13 (Completeness)**.** $\mathbf{MIL_{Pre}}$ *is strongly complete w.r.t.* MIL_{Pre}*. So, in particular,* $\mathbf{MIL_{Pre}} \supseteq MIL_{Pre}$*.*

Proof. Suppose Γ_0 is consistent. It suffices to show that Γ_0 is satisfiable. As previously mentioned, to show so, we will construct a model satisfying a truth lemma for labeled points by taking the colimit of a sequence of models getting ever closer to satisfying this truth lemma. We begin by extending Γ_0 to a maximal consistent set $\Gamma \supseteq \Gamma_0$, letting $\le_0$ be the diagonal on some (any) countable set W, and setting $D_0 := \varnothing$ and $l_0 := \{(x_0, \Gamma)\}$ for some $x_0 \in W$. Then 1.-7. are satisfied, where 7. follows by reflexivity of $\le_{\mathbf{Pre}}$. We continue by constructing a sequence

$$(l_0, \le_0, D_0), (l_1, \le_1, D_0), \dots, (l_n, \le_n, D_n), \dots$$

s.t. for all $i \in \omega$

$$l_i \subseteq l_{i+1}, \quad \le_i \subseteq \le_{i+1}, \quad D_i \subseteq D_{i+1},$$

using the repair lemmas repeatedly. We do so by enumerating the set of all pairs $(\langle \sup \rangle \chi\chi', x)$ and all quadruples $(\neg\langle \sup \rangle \psi\psi', x, y, z)$.[23] Then at each stage $n+1$ we pick the least tuple constituting a defect to $(l_n, \le_n, D_n)$, which we repair obtaining $(l_{n+1}, \le_{n+1}, D_{n+1})$. Letting

$$(l_\omega, \le_\omega, D_\omega) := \left(\bigcup_{n\in\omega} l_n, \bigcup_{n\in\omega} \le_n, \bigcup_{n\in\omega} D_n \right),$$

we get that (1) $(l_\omega, \le_\omega, D_\omega)$ satisfies 4.-7.; (2) l_ω is a (partial) function from W to the set of all MCSs; and (3) $(l_\omega, \le_\omega, D_\omega)$ neither has any $\langle \sup \rangle$- nor $\neg\langle \sup \rangle$-defects. Only (3) isn't straightforward. To show this, we prove two claims, and in order to do so, we need the following observation.

[23] Such an enumeration exists because (1) W is countable, and (2) there are countably many formulas.

Observation. Let $n \in \omega$ and $\{x, v\} \subseteq \mathrm{dom}(l_n)$ be arbitrary s.t.

$$\uparrow_n v = \uparrow_n x \cup \{v\} \cup (\uparrow_n v \cap \{w \mid \uparrow_n w \cap \uparrow_n x = \varnothing\}),$$

where $\uparrow_n y := \{z \mid y \leq_n z\}$. Then for all $m \geq n$:

$$\uparrow_m v = \uparrow_m x \cup \{v\} \cup (\uparrow_m v \cap \{w \mid \uparrow_m w \cap \uparrow_m x = \varnothing\}),$$

hence also

$$\uparrow_\omega v = \uparrow_\omega x \cup \{v\} \cup (\uparrow_\omega v \cap \{w \mid \uparrow_\omega w \cap \uparrow_\omega x = \varnothing\}).$$

This is easily seen by induction, using that each $(l_{m+1}, \leq_{m+1}, D_{m+1})$ is obtained from $(l_m, \leq_m, D_m)$ using either of the repair lemmas.

Claim (a). Suppose $(\langle \sup \rangle \chi\chi', x)$ does not constitute a defect for some $(l_n, \leq_n, D_n)$ at which (i) $x \in \mathrm{dom}(l_n)$ and (ii) $\langle \sup \rangle \chi\chi' \in l_n(x)$. Then this must be witnessed by some y, z (cf. Definition 2.8). We show that for all $m \geq n$:

$(\langle \sup \rangle \chi\chi', x)$ does not constitute a defect for $(l_m, \leq_m, D_m)$, witnessed by y, z.

A fortiori, neither does it for $(l_\omega, \leq_\omega, D_\omega)$.

By the observation and noting that $l_i \subseteq l_{i+1}$ for all $i \in \omega$, it suffices to show that for all $m \geq n$:

$$x = \sup_m\{y, z\},$$

where $\sup_m\{y, z\} := \sup_{\leq_m}\{y, z\}$ is the least upper bound of $\{y, z\}$ w.r.t. the relation $\leq_m$. We prove this by induction on $m \geq n$. By assumption, this holds for $m = n$. Accordingly, suppose it holds for an arbitrary $m \geq n$. We show it holds for $m + 1$. We have two cases, depending on the type of defect being repaired at stage $m + 1$.

First, suppose the defect repaired was a $\langle \sup \rangle$-defect for some world s. Since the corresponding introduced $\mathrm{dom}(l_{m+1})$-worlds y_s, z_s have no proper $\leq_{m+1}$-predecessors, the claim follows. Reason being that, cf. the IH and the definition

$$\leq_{m+1} := \leq_m \cup \{(y_s, u), (z_s, u) \mid s \leq_m u\},$$

y_s and z_s are the only possible counterexamples to the claim.

Second, suppose $(l_{m+1}, \leq_{m+1}, D_{m+1})$ was obtained via $\neg\langle \sup \rangle$-repairing some s, y_s, z_s by introducing the dummy d_s. Notice that, by IH and the definition

$$\leq_{m+1} := \leq_m \cup \{(u, d_s), (v, d_s) \mid u \leq_m y_s, v \leq_m z_s\},$$

the only possible counterexample to the claim is d_s. Accordingly, suppose $d_s \geq_{m+1} y, z$. Going by cases, we prove that this implies $d_s \geq_{m+1} x$:

- If $y_s \geq_m y, z$, then, by IH, $y_s \geq_m x$ so $d_s \geq_{m+1} x$.
- If $z_s \geq_m y, z$, then as above.

- If $y_s \geq_m y$ and $z_s \geq_m z$, then, by the observation, either (a) $y_s = y$ or (b) $y_s \geq_m x$ or (c) $\uparrow_m y_s \cap \uparrow_m x = \varnothing$. If (b), then $d_s \geq_{m+1} x$. And if (c), then note that as s is a $\leq_m$-upper bound of $\{y_s, z_s\}$, it must also be a $\leq_m$-upper bound of $\{y, z\}$, hence, by IH, $x \leq_m s$ – contradicting $\uparrow_m y_s \cap \uparrow_m x = \varnothing$. Thus, we may assume (a) $y_s = y$; and, analogously, $z_s = z$. But then $s = \sup_{\leq_m}\{y_s, z_s\} = \sup_{\leq_m}\{y, z\} = x$, hence $(s, y_s, z_s) = (x, y, z)$ couldn't have constituted a $\neg\langle\text{sup}\rangle$-defect because $C_{\mathbf{Pre}} l_m(x) l_m(y) l_m(z)$.
- If $z_s \geq_m y$ and $y_s \geq_m z$, then as above.

This exhausts all cases, showing $d_s \geq_{m+1} x$, which completes the induction. $\square_{(a)}$

Claim (b). Suppose $n \in \omega$ and $a, b \in (\mathrm{dom}(l_n) \cup D_n)$ are s.t. $a \not\geq_n b$. Then for all $m \geq n$, we have that $a \not\geq_m b$. A fortiori, $a \not\geq_\omega b$.

Follows by induction on m, noting that if $(l_{m+1}, \leq_{m+1}, D_{m+1})$ was obtained by $\langle\text{sup}\rangle$-repairing some x by introducing some y, z, we would have

$$\leq_{m+1} := \leq_m \cup \{(y, u), (z, u) \mid x \leq u\};$$

that is, there is no change in successors of b.

Likewise, if $(l_{m+1}, \leq_{m+1}, D_{m+1})$ was obtained by $\neg\langle\text{sup}\rangle$-repairing some x, y, z by introducing a dummy d, there is no change in predecessors of a. This exhaust the cases, hence proves the claim. $\square_{(b)}$

Using *(a)* and *(b)*, it is straightforward to see *(c)*: If some tuple *did* constitute a defect at some stage n, but no longer at some later stage $m > n$, then it didn't for all $k \geq m$.

With these claims at hand, we can show (3) that $(l_\omega, \leq_\omega, D_\omega)$ neither has $\langle\text{sup}\rangle$- nor $\neg\langle\text{sup}\rangle$-defects. For $\langle\text{sup}\rangle$, let

$$(\langle\text{sup}\rangle\chi\chi', x)_i$$

be an arbitrary pair in our enumeration s.t. $x \in \mathrm{dom}(l_\omega)$ and $\langle\text{sup}\rangle\chi\chi' \in l_\omega(x)$. Then $x \in \mathrm{dom}(l_n)$ for some $n \in \omega$, hence

$$x \in \mathrm{dom}(l_m), \langle\text{sup}\rangle\chi\chi' \in l_m(x)$$

for all $m \geq n$. If, on one hand, $(\langle\text{sup}\rangle\chi\chi', x)_i$ didn't constitute a defect to $(l_n, \leq_n, D_n)$ – using *claim (a)* (and the observation) – we get that it wouldn't for $(l_\omega, \leq_\omega, D_\omega)$ either. On the other, in case it did, it would no more no later than at stage $n + i + 1$ (cf. *(c)*), and henceforth – by *claim (c)* – remain repaired. Thus, $(l_\omega, \leq_\omega, D_\omega)$ has no $\langle\text{sup}\rangle$-defects.

For $\neg\langle\text{sup}\rangle$, suppose towards contradiction that

$$(\neg\langle\text{sup}\rangle\psi\psi', x, y, z)_i$$

denotes a $\neg\langle\sup\rangle$-defect. Then $x \geq_\omega y$ and $x \geq_\omega z$, so there is some $n \in \omega$ s.t.

$$x \geq_n y, z.$$

If

$$x \neq \sup_{\leq_m}\{y, z\}$$

for some $m \geq n$, there must be some $a \in (\mathrm{dom}(l_m) \cup D_m)$ s.t.

$$y, z \leq_m a \not\geq_m x,$$

but then – cf. *claim (b)* –

$$y, z \leq_\omega a \not\geq_\omega x\ ,$$

which, in particular, shows $x \neq \sup_\omega\{y, z\}$—contradicting $(\neg\langle\sup\rangle\psi\psi', x, y, z)_i$ being a $\neg\langle\sup\rangle$-defect. Thus, we must have

$$x = \sup_{\leq_m}\{y, z\}$$

for all $m \geq n$, implying – and simultaneously contradicting – that the defect will be repaired no later than at stage $n + i + 1$ (cf. *(c)*). That is, there can be no $\neg\langle\sup\rangle$-defects either.

Finally, setting

$$V(p) := \{x \in \mathrm{dom}(l_\omega) : p \in l_\omega(x)\},$$

we show our truth lemma, namely that for all $x \in \mathrm{dom}(l_\omega)$ and all $\varphi \in \mathcal{L}_M$:

$$(W, \leq, V), x \Vdash \varphi \qquad \textbf{iff} \qquad \varphi \in l_\omega(x).$$

The proof goes by induction on the complexity of formulas. Base case is by definition and Boolean cases are straightforward. For the $\langle\sup\rangle$-case, we get

$$\begin{aligned}
x \Vdash \langle\sup\rangle\varphi_1\varphi_2 \quad &\overset{\textit{Def}}{\textbf{iff}} \quad \exists y, z\, [x = \sup_\omega\{y, z\}, y \Vdash \varphi_1, z \Vdash \varphi_2] \\
&\overset{\textit{(IH)}}{\textbf{iff}} \quad \exists y, z\, [x = \sup_\omega\{y, z\}, \varphi_1 \in l_\omega(y), \varphi_2 \in l_\omega(z)] \\
&\overset{\textit{(i)}}{\textbf{iff}} \quad \langle\sup\rangle\varphi_1\varphi_2 \in l_\omega(x),
\end{aligned}$$

where we in the left-to-right direction of *(IH)* use – apart from the induction hypothesis itself – that $(l_\omega, \leq_\omega, D_\omega)$ satisfies 5.-6.; i.e., in particular, neither of the witnessing y, z are dummies nor in $W \setminus (\mathrm{dom}(l_\omega) \cup D_\omega)$, and so they must be in $\mathrm{dom}(l_\omega)$. Further, left-to-right of (i) holds by there being no $\neg\langle\sup\rangle$-defects, while right-to-left follows from there being no $\langle\sup\rangle$-defects.

This completes the induction, from which it follows that

$$(W, \leq_\omega, V), x_0 \Vdash \Gamma_0,$$

showing that Γ_0 is satisfiable in a preorder model and, thus, at long last, finalizing our proof of completeness. $\square$

Corollary 2.14. $MIL_{Pre} = MIL_{Pos}$.

Proof. Clearly, $MIL_{Pre} \subseteq MIL_{Pos}$, and the other inclusion follows from the frame constructed in the completeness proof being a *partial* order. $\square$

3 Decidability of $\boldsymbol{MIL_{Pre}}$

This section consists of two parts. In subsection 3.1, we show that $\mathbf{MIL_{Pre}}$ is complete w.r.t. another class of structures $\mathcal{C}$. Then, in subsection 3.2, we show that $\mathbf{MIL_{Pre}}$ has the FMP *w.r.t.* $\mathcal{C}$*-frames* and conclude that MIL_{Pre} (and MIL_{Pos}) are, after all, decidable—solving a problem posed in [16–18].

3.1 Reinterpreting $\langle \text{sup} \rangle$ on generalized structures $\boldsymbol{\mathcal{C}}$

Following the method laid out in subsection 1.2, and with an axiomatization of MIL_{Pre} at hand, we continue our road to decidability by proving completeness relative to a different class of structures. These structures will be named $\mathcal{C}$*-frames*, alluding to our denoting this class of structures as $\mathcal{C}$.

Before we get that far, though, the first key observation to make is that there is nothing in the *syntactic* definition of $\mathbf{MIL_{Pre}}$ implying that the binary modality-symbol $\langle \text{sup} \rangle$ need be interpreted in terms of the supremum relation on a preorder. I.e., there is nothing a priori hindering us from reinterpreting $\mathbf{MIL_{Pre}}$ through reinterpreting the symbol $\langle \text{sup} \rangle$.

Further, $\mathbf{MIL_{Pre}}$ being an NML means that there might be a canonical reinterpretation, namely the one reached through frame correspondence of $\mathbf{MIL_{Pre}}$ on the class of all pairs (W, C) where W is a set and C is an *arbitrary* ternary relation on W. And, indeed, that is how we proceed.

Lemma 3.1. *Let* (W, C) *be a frame for the modal language with a single binary modality. Then we have the following frame correspondences:*

(i)	$(W, C) \Vdash (Re.)$	***iff***	$(W, C) \vDash \forall w \, (Cwww)$
(ii)	$(W, C) \Vdash (4)$	***iff***	$(W, C) \vDash \forall w, v, x \, (Cwvx \wedge Cvuy \to \exists z \, [Cwuz])$
(iii)	$(W, C) \Vdash (Co.)$	***iff***	$(W, C) \vDash \forall w, v, u \, (Cwvu \to Cwuv)$
(iv)	$(W, C) \Vdash (Dk.)$	***iff***	$(W, C) \vDash \forall w, v, u \, (Cwvu \to Cwwv)$

Proof. Standard frame correspondence proofs work, using arguments similar to the ones in the proof of Lemma 2.6*(a), (b)(i), (c)* and *(d)*. Alternatively, the Sahlqvist-van Benthem algorithm also applies because the formulas are Sahlqvist. $\square$

Definition 3.2. We denote the first-order correspondents of (Re.), (4), (Co.) and (Dk.) as (Re.f), (4f'), (Co.f) and (Dk.f), respectively. $\dashv$

While (Re.f), (Co.f), and (Dk.f) all match neatly with (Re.), (Co.), and (Dk.), respectively, (4f') is a slightly less elegant FO-correspondent of (4). However, as the following proposition shows, in the presence of the other axioms, the correspondence crystallizes.

Proposition 3.3. *Let (W, C) be a frame for the modal language with a single binary modality. Then $(W, C) \Vdash \mathbf{MIL_{Pre}}$ iff*

$$(W, C) \vDash (Re.f) \wedge (Co.f) \wedge (Dk.f) \wedge \forall w, v, u\, (Cwwv \wedge Cvvu \to Cwwu)$$

In other words, (4f') and (4f) are equivalent modulo (Re.f), (Co.f) and (Dk.f),[24] *where*

$$(4f) := \forall w, v, u\, (Cwwv \wedge Cvvu \to Cwwu)\,.$$

Proof. Straightforward consequence of Lemma 3.1. $\square$

It now follows that we have obtained a different class of structures, namely $\mathcal{C}$, which is complete w.r.t. $\mathbf{MIL_{Pre}}$. This is summarized in the ensuing corollary.

Corollary 3.4. $\mathbf{MIL_{Pre}}$ *is sound and (strongly) complete w.r.t.*

$$\mathcal{C} := \{(W, C) \vDash (Re.f) \wedge (Co.f) \wedge (Dk.f) \wedge (4f)\}\,.$$

In particular,

$$\mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}),$$

where $\mathrm{Log}(\mathcal{C}) := \{\varphi \in \mathcal{L}_M \mid (W, C) \Vdash \varphi, (W, C) \in \mathcal{C}\}$ *denotes the NML of* $\mathcal{C}$.

Proof. The preceding proposition implies soundness, and then our earlier completeness theorem (2.13) gives us (strong) completeness because preorder frames are particular instances of $\mathcal{C}$-frames, namely those where the ternary relation 'C' happens to be the supremum relation of an underlying preorder.[25] $\square$

This corollary proven, we have arrived at the final step described in subsection 1.2: showing the FMP of $\mathbf{MIL_{Pre}}$ when reinterpreted on $\mathcal{C}$. Before proving this in the next subsection, we find it instructive to revisit the formula ψ_N from Proposition 1.7 and show that, although not satisfiable on a finite *preorder frame*, it is satisfiable on a finite $\mathcal{C}$*-frame*. We do this right after observing the following:

Observation 3.5. It is not hard to prove that for any $(W, C) \in \mathcal{C}, x \in W$, valuation V on (W, C) and formula φ, we have that

$$(W, C, V), x \Vdash P\varphi \qquad \textbf{iff} \qquad \exists y \in W\, (Cxy y \wedge y \Vdash \varphi)\,,$$

[24] In fact, even modulo (Dk.f) and (Co.f).

[25] Alternatively, as noted by an anonymous referee, this follows by Sahlqvist canonicity.

and hence also

$$(W, C, V), x \Vdash H\varphi \qquad \textbf{iff} \qquad \forall y \in W\,(Cxxy \to y \Vdash \varphi)\,. \qquad \dashv$$

Remark 3.6. Although

$$\psi_N := HP\langle\text{sup}\rangle pp \wedge HP\neg\langle\text{sup}\rangle pp$$

only is satisfiable on infinite preorder models under the standard interpretation of $\langle\text{sup}\rangle$ (cf. Proposition 1.7), it is satisfiable on a finite $\mathcal{C}$-frame. $\dashv$

Proof. Set

$$\begin{aligned} &W := \{w, v\}\,, \qquad V(p) := \{w\}, \\ &C := \{(w,w,w),(v,v,v),(w,w,v),(w,v,w),(v,v,w),(v,w,v)\}. \end{aligned}$$

We claim that $(W, C) \in \mathcal{C}$ and $(W, C, V), w \Vdash \psi_N$.

The former can be seen by a quick (yet tedious) check that (W, C) models the given first-order conditions. The latter can be seen by first noting that

$$\text{(a) } w \Vdash \langle\text{sup}\rangle pp \qquad \textit{while} \qquad \text{(b) } v \nVdash \langle\text{sup}\rangle pp,$$

since, respectively, (a) $Cwww$ and $w \Vdash p$, and (b) $\neg Cvww$ and $v \nVdash p$.

Moreover, using that $(W, C) \in \mathcal{C}$, we get

$$\begin{aligned} w \Vdash HP\langle\text{sup}\rangle pp \qquad &\textbf{iff} \qquad \forall x \in W\,(Cwwx \to x \Vdash P\langle\text{sup}\rangle pp) \\ &\textbf{iff} \qquad \forall x \in W\,(Cwwx \to \exists y\,[Cxxy \wedge y \Vdash \langle\text{sup}\rangle pp])\,, \end{aligned}$$

hence also

$$w \Vdash HP\neg\langle\text{sup}\rangle pp \qquad \textbf{iff} \qquad \forall x \in W\,(Cwwx \to \exists y\,[Cxxy \wedge y \nVdash \langle\text{sup}\rangle pp])\,.$$

With this spelt out, we find that $w \Vdash \psi_N$ as we have $Cwww, Cwwv, Cvvv, Cvvw$; i.e., the existential consequents are always fulfilled. $\square$

3.2 The finite model property

As promised, we go on proving that $\mathbf{MIL_{Pre}}$ enjoys the FMP w.r.t. $\mathcal{C}$ and then use this to deduce decidability of MIL_{Pre}. The proof of the FMP is done by employing a filtration-style argument. To this end, we define a notion extending the standard notion of a set of formulas being subformula closed.

Definition 3.7. We say that a set Σ of $\mathcal{L}_M$-formulas is $\mathcal{C}$*-closed* :iff

(Sub) it is subformula closed;

(Com) $\langle\text{sup}\rangle\varphi\psi \in \Sigma$ implies $\langle\text{sup}\rangle\psi\varphi \in \Sigma$; and

(S-P) $\langle \sup \rangle \varphi\psi \in \Sigma$ implies $P\varphi \in \Sigma$.

Moreover, for any set of formulas Σ_0, we say that Σ is the $\mathcal{C}$*-closure* of Σ_0 :iff it is the least $\mathcal{C}$-closed set of formulas extending Σ_0.[26] $\dashv$

An immediate consequence of the definition is the following lemma:

Lemma 3.8. *Suppose Σ_0 is a finite set of $\mathcal{L}_M$-formulas. Then its $\mathcal{C}$-closure $\Sigma \supseteq \Sigma_0$ is finite as well.*

Less immediate is how to use this notion for a filtration-style argument of the FMP. This is the content of the following theorem, whose proof contains the actual definition of a filtration through a $\mathcal{C}$-closed set of formulas.

Theorem 3.9. $\mathbf{MIL_{Pre}}$ *admits filtration w.r.t. the class $\mathcal{C}$. Thus,*

$$\mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}_F),$$

where $\mathrm{Log}(\mathcal{C}_F)$ *denotes the NML of the class of finite $\mathcal{C}$-frames.*

Proof. Cf. Lemma 3.8 and the obvious inclusion $\mathrm{Log}(\mathcal{C}) \subseteq \mathrm{Log}(\mathcal{C}_F)$, it suffices to show that for any $\mathcal{C}$-model (W, C, V) and $\mathcal{C}$-closed set of formulas Σ, the following hold:

1. $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$, where our filtered universe is

$$W_\Sigma := \{|x|_\Sigma : x \in W\}$$

with relation

$$C^{\mathcal{C}}_\Sigma |x||y||z| \qquad \textbf{:iff} \qquad \forall \langle \sup \rangle \varphi\psi \in \Sigma \big([(y \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle \sup \rangle \varphi\psi] \ \textit{and} \ [(y \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi] \big),$$

where $|x|_\Sigma = |x|$ denotes the equivalence class on the set of worlds W defined as satisfying the same Σ-formulas as x.

2. For all $\varphi \in \Sigma, x \in W$:

$$(W_\Sigma, C^{\mathcal{C}}_\Sigma, V_\Sigma), |x| \Vdash \varphi \qquad \textbf{iff} \qquad (W, C, V), x \Vdash \varphi,$$

where $V_\Sigma(p) := \{|x| \in W_\Sigma : x \in V(p)\}$ for all $p \in \Sigma$.

We begin by proving 1.; i.e., showing $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$. This we do as follows:

- $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \vDash (Re.f)$ can be seen using $(W, C) \Vdash (Re.)$.[27]

---

[26] Note that the $\mathcal{C}$-closure of a set of formulas always exists.

[27] Alternatively, below we show that this filtration indeed satisfies the homomorphic filtration condition: $Cxyz \Rightarrow C^{\mathcal{C}}_\Sigma |x||y||z|$. From this and surjectivity of $x \mapsto |x|_\Sigma$, $(Re.f)$ follows.
On this node, it is worth (foot)noting that the culprit in hindering this inheritance argument for the three other FO-conditions are the implications in their respective definitions; e.g., for (Dk.f) we have $Cwvu \rightarrow Cwwv \equiv \neg Cwvu \vee Cwwv$, so when this implication holds by virtue of the first disjunct, namely '$\neg Cwvu$', we cannot likewise conclude $\neg C^{\mathcal{C}}_\Sigma |w||v||u|$.
This also explains that the filtration relation and the set of formulas we are filtering through have been defined to accommodate these three axioms. As for the transitivity axiom, we have drawn inspiration from the Lemmon filtration.

- $(W_\Sigma, C_\Sigma^\mathcal{C}) \vDash (Co.f)$ can be seen using $(W,C) \Vdash (Co.)$ and the (Com)-closure.
- Showing $(W_\Sigma, C_\Sigma^\mathcal{C}) \vDash (Dk.f)$ is a bit more tricky. Accordingly, suppose $C_\Sigma^\mathcal{C}|x||y||z|$ and let $\langle\sup\rangle\varphi\psi \in \Sigma$ be arbitrary. It then suffices to show

$$(x \Vdash \varphi, y \Vdash \psi) \Rightarrow x \Vdash \langle\sup\rangle\varphi\psi \quad \textit{and} \quad (x \Vdash P\varphi, y \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

  For the former, since $\langle\sup\rangle\psi\top = P\psi \in \Sigma$ by (Com)- and (S-P)-closure, we have that if $y \Vdash \psi$, then $x \Vdash \langle\sup\rangle\psi\top$ because $C_\Sigma^\mathcal{C}|x||y||z|$. So if also $x \Vdash \varphi$, then using $(W,C) \Vdash (Dk.)$, we get $x \Vdash \langle\sup\rangle\varphi\psi$.

  Further, for the latter, if $y \Vdash P\psi$, using $z \Vdash P\top$ because $(W,C) \Vdash (T)$ [and, again, $\langle\sup\rangle\psi\top = P\psi \in \Sigma$ and $C_\Sigma^\mathcal{C}|x||y||z|$], we get $x \Vdash P\psi$.
- Lastly, to prove $(W_\Sigma, C_\Sigma^\mathcal{C}) \vDash (4f)$, suppose $C_\Sigma^\mathcal{C}|x||x||y|, C_\Sigma^\mathcal{C}|y||y||z|$ and $\langle\sup\rangle\varphi\psi \in \Sigma$. We show that

$$(x \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle\sup\rangle\varphi\psi \quad \textit{and} \quad (x \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

  For the former, if $z \Vdash \psi$, then $C_\Sigma^\mathcal{C}|y||y||z|$ and $\langle\sup\rangle\top\psi \in \Sigma$ imply $y \Vdash \langle\sup\rangle\top\psi$, hence $y \Vdash P\psi$ by $(W,C) \Vdash (Co.)$. But then this along with $x \Vdash P\top$ and $C_\Sigma^\mathcal{C}|x||x||y|$ imply that $x \Vdash P\psi$. So if also $x \Vdash \varphi$, then $(W,C) \Vdash (Dk.)$ implies $x \Vdash \langle\sup\rangle\varphi\psi$.

  Further, if $z \Vdash P\psi$, using $z \Vdash P\top$, then $y \Vdash P\psi$, and in turn $x \Vdash P\psi$.

This completes our proof of 1. For proving 2., it suffices to show that $(W_\Sigma, C_\Sigma^\mathcal{C}, V_\Sigma)$ is a filtration of (W, C, V) through Σ. That is, we need to check two conditions, namely

(F1) $Cxyz \Rightarrow C_\Sigma^\mathcal{C}|x||y||z|$; and

(F2) $C_\Sigma^\mathcal{C}|x||y||z| \Rightarrow \forall\langle\sup\rangle\varphi\psi \in \Sigma\, [(y \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle\sup\rangle\varphi\psi]$.

(F2) follows by definition of our filtration relation. For (F1), suppose $Cxyz$ and $\langle\sup\rangle\varphi\psi \in \Sigma$. Then the only non-trivial part is to show that

$$(y \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

Since $(W,C) \vDash (Dk.f) \wedge (Co.f)$, we also have $Cxxy$ and $Cxxz$. Thus, if $y \Vdash P\varphi$ and $z \Vdash P\psi$, we get that

$$x \Vdash PP\varphi \wedge PP\psi,$$

hence from $(W,C) \Vdash (4)$, we get

$$x \Vdash P\varphi \wedge P\psi,$$

as desired. $\square$

Finally, we end this section by deducing decidability.[28]

[28] For the interested reader, in [11] it is also shown that the general heuristic regarding decidability and the FMP outlined in subsection 1.2 also applies to the TMP.

Corollary 3.10. *MIL_{Pre} is decidable (and so is MIL_{Pos}).*

Proof. Cf. Theorem 2.13 and Corollary 3.4, we know that

$$MIL_{Pre} = \mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}_F).$$

So since $\mathbf{MIL_{Pre}}$ is a finitely axiomatized NML admitting filtration w.r.t. $\mathcal{C}$, we get decidability.[29] $\square$

4 MIL with Informational Implication

With $MIL_{Pre} = MIL_{Pos}$ axiomatized and proven decidable, this section investigates their enrichments, $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, with the 'informational implication' '$\backslash$'. The main goals are to provide an axiomatization and a decidability proof.

In subsection 4.1, we formally set out the logics of concern and briefly comment on the increased expressibility. In subsection 4.2, we, first, put forward an axiomatization and point out on an interesting junction with the Lambek Calculus. Before, second, pausing our investigation of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ per se, to show that the proposed axiomatization is sound and complete w.r.t. the class $\mathcal{C}$. Using this result, in subsection 4.3, we obtain soundness and completeness w.r.t. our poset frames through combining two representation results: the first achieved via an adaptation of 'bulldozing', and the second via supplementing the framework of subsection 2.2 with an additional defect. We deduce that $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$. Lastly, in subsection 4.4, we modify the filtration technique of subsection 3.2 to attain decidability of $MIL_{\backslash\text{-}Pre}$.

4.1 Augmenting with '$\backslash$'

As noted, we seek to study the enrichment of the basic modal information logic(s), MIL_{Pre} and MIL_{Pos}, given by adding an informational implication as a binary modality. In this subsection we cover some preliminaries; specifically, some definitions followed by a few comments on expressivity. We start with supplying the following pertinent definitions:

Definition 4.1 (Language)**.** The language $\mathcal{L}_{\backslash\text{-}M}$ is given by extending the basic language of modal information logic $\mathcal{L}_M$ with a binary modality symbol '$\backslash$'.

As a convention we use infix notation for '$\backslash$' instead of prefix/Polish notation; that is, we write '$\varphi\backslash\psi$', rather than '$\backslash\varphi\psi$' (as we, e.g., would do with '$\langle\sup\rangle$' and '$[\sup]$').$\dashv$

Definition 4.2 (Semantics)**.** Given a preorder model $\mathbb{M} = (W, \leq, V)$, a world $v \in W$ and a formula $\varphi\backslash\psi \in \mathcal{L}_{\backslash\text{-}M}$ with main connective '$\backslash$', we let

$$\mathbb{M}, v \Vdash \varphi\backslash\psi \quad \textbf{iff} \quad \text{for all } u, w \in W, \text{ if } \mathbb{M}, u \Vdash \varphi \text{ and } w \in \sup\{u, v\}, \text{ then } \mathbb{M}, w \Vdash \psi.^{30} \qquad \dashv$$

[29] Similarly, using that our filtration argument establishes the strong finite model property, one can prove decidability.

[30] Notice how '$\backslash$' is the '$\Box$-ed' and not the '$\Diamond$-ed' half of a modality pair.

Definition 4.3 (Logic)**.** We denote the modal information logic on preorders in the enriched language of $\mathcal{L}_{\backslash\text{-}M}$ as $MIL_{\backslash\text{-}Pre}$, which – to be explicit – is defined as

$$MIL_{\backslash\text{-}Pre} := \{\varphi \in \mathcal{L}_{\backslash\text{-}M} : (W, \leq) \Vdash \varphi \ \ \textit{for all preorder frames } (W, \leq)\}.$$

$MIL_{\backslash\text{-}Pos}$ is defined analogously. ⊣

Remark 4.4. As a minor interlude, as mentioned in the introduction, the choice of symbol '\' concurs with standard notation in the Lambek Calculus. With the semantics given, the reason becomes evident: the interpretation is the same (given a supremum relation). It is also worth pointing out that the commutativity of suprema implies that the other Lambek residual – typically denoted by '/' – collapses into '\' in the sense that $\varphi/\psi \equiv \psi\backslash\varphi$. Lastly, the modality '$\langle\sup\rangle$' is interpreted (again, given a supremum relation) exactly as the binary product '$\cdot$' is in the Lambek Calculus. In the next subsection, we expound this connection even further. ⊣

Now, recall that the primary results we are after are (1) axiomatizing $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ and (2) showing them to be decidable. Once more, we will be following the heuristic laid out in subsection 1.2; however, this time our completeness theorem will not be proven via model constructions but via representation results. For this to work, we, needless to say, must (a) have another class of structures for which we can prove the representation results, and (b) also already have the logic of this other class axiomatized. Regarding (a), a natural candidate arises: the $\mathcal{C}$-frames of the previous section. Before being able to (b) axiomatize the logic of this class (as we will in the next subsection), we must clarify how '\' is to be interpreted on $\mathcal{C}$-models. This is the content of the following definition:

Definition 4.5. Given a frame $(W, C) \in \mathcal{C}$, a valuation V on (W, C), a world $v \in W$ and a formula $\varphi\backslash\psi \in \mathcal{L}_{\backslash\text{-}M}$ with main connective '\', we let

$$(W, C, V), v \Vdash \varphi\backslash\psi \qquad \textbf{iff} \qquad \text{for all } u, w \in W, \text{ if } (W, C, V), u \Vdash \varphi \text{ and } Cwvu, \text{ then } (W, C, V), w \Vdash \psi.$$

⊣

To be precise, we explicate how this generalizes our definition on preorder frames.

Definition 4.6. Let $\mathcal{S}_{\text{Pre}}$ (resp. $\mathcal{S}_{\text{Pos}}$) be the class of pairs $(W, S_{\leq})$ where W is a set and $S_{\leq} \subseteq W^3$ is a ternary relation for which there is some preorder (resp. partial order) $\leq$ on W s.t. for all $w, v, u \in W$:

$$S_{\leq}wvu \qquad \textbf{iff} \qquad w \in \sup_{\leq}\{u, v\}.^{31}$$

Then the semantics of '\' on a preorder model $(W, \leq, V)$ comes down to

$$(W, S_{\leq}, V), v \Vdash \varphi\backslash\psi \qquad \textbf{iff} \qquad \text{for all } u, w \in W, \text{ if } (W, S_{\leq}, V), u \Vdash \varphi \text{ and } S_{\leq}wvu, \text{ then } (W, S_{\leq}, V), w \Vdash \psi,$$

where $S_{\leq}$ is the supremum relation induced by $\leq$. ⊣

[31] I.e., $S_{\leq}$ is the supremum relation induced by a preorder (resp. poset).

As the last definition of this subsection, we set forth the logic of $\mathcal{C}$-frames in this extended language:

Definition 4.7. We write $\mathrm{Log}_\backslash(\mathcal{C})$ for the logic of $\mathcal{C}$-frames in the language $\mathcal{L}_{\backslash\text{-}M}$; i.e., $\mathrm{Log}_\backslash(\mathcal{C})$ denotes the set of $\mathcal{L}_{\backslash\text{-}M}$-validities on $\mathcal{C}$-frames. $\dashv$

With these definitions out of the way, we finish up this subsection with the promised comments on expressivity. First off, we show that with the additional vocabulary provided, we are not only able to express the past-looking unary modality 'P', but also the future-looking 'F'.

Remark 4.8. The future-looking unary modality 'F' (i.e., the standard '$\Diamond$') is definable as

$$F\varphi := \neg(\top\backslash\neg\varphi).^{32}$$

This can be seen by recalling the definition

$$\mathbb{M}, v \Vdash F\varphi \qquad \textbf{:iff} \qquad \exists w(v \leq w, w \Vdash \varphi),$$

and observing that also

$$\begin{aligned} \mathbb{M}, v \Vdash \neg(\top\backslash\neg\varphi) \qquad & \textbf{iff} \qquad \exists u, w(w \in \sup\{u, v\}, u \Vdash \top, w \nVdash \neg\varphi) \\ & \textbf{iff} \qquad \exists w(v \leq w, w \Vdash \varphi). \end{aligned}$$ $\dashv$

Finally, for good measure, observe that '$\backslash$' is not expressible in our simpler language $\mathcal{L}_M$. To see this, take, e.g., a two-chain $\{0, 1\}$ where $0 \leq 1$ and a one-chain $\{0'\}$; and let $0 \Vdash \neg p$, $1 \Vdash p$, and $0' \Vdash \neg p$. Then $0 \Vdash Fp$ while $0' \nVdash Fp$, but for all $\varphi \in \mathcal{L}_M$: $0 \Vdash \varphi$ iff $0' \Vdash \varphi$.

4.2 Axiomatizing $\mathrm{Log}_\backslash(\mathcal{C})$

Now for the promised axiomatization of $\mathrm{Log}_\backslash(\mathcal{C})$, which – via the representation results of the next subsection – entails that it even is an axiomatization of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$.

Definition 4.9 (Axiomatization)**.** We define $\mathbf{MIL_{\backslash\text{-}Pre}}$ to be the least set of $\mathcal{L}_{\backslash\text{-}M}$-formulas that (i) is closed under the axioms and rules of $\mathbf{MIL_{Pre}}$; (ii) contains the K-axioms for $\backslash$;[33] (iii) contains the axioms

(I1) $\langle\sup\rangle p(p\backslash q) \to q$, and

(I2) $p \to q\backslash(\langle\sup\rangle pq)$;[34]

and (iv) is closed under the rule

[32]Notice that this places us in an extension of *temporal* **S4**.

[33]For reference, the K-axioms for $\backslash$ are: $[(p \to q)\backslash r] \to [(p\backslash r) \to (q\backslash r)]$ and $[p\backslash(q \to r)] \to [(p\backslash q) \to (p\backslash r)]$.

[34]'(I1)' and '(I2)' are short for 'inverses': they capture how '$\langle\sup\rangle$' and '$\backslash$' relate.

$(N_\backslash)$ if $\vdash_{\backslash\text{-Pre}} \varphi$, then $\vdash_{\backslash\text{-Pre}} \psi\backslash\varphi$.[35] $\dashv$

Before showing that $\mathbf{MIL}_{\backslash\textbf{-Pre}}$ is sound and strongly complete w.r.t. $\mathcal{C}$-frames, some remarks are due.

Remark 4.10 (Lambek Calculus of suprema on preorders)**.** In its basic version, the Lambek Calculus only contains the three binary connectives '$\cdot$', '$\backslash$' and '$/$', of which the first matches our '$\langle\text{sup}\rangle$' and the last two, modulo (Co.), both match our '$\backslash$'. It is defined proof-theoretically with the constitutive rules of the connectives (when given in our language) being

(L1) if $\vdash \langle\text{sup}\rangle\varphi\psi \to \chi$, then $\vdash \psi \to \varphi\backslash\chi$; and its converse

(L1) if $\vdash \psi \to \varphi\backslash\chi$, then $\vdash \langle\text{sup}\rangle\varphi\psi \to \chi$.

Unsurprisingly, both of these rules are derivable in our Hilbert system for $\mathbf{MIL}_{\backslash\textbf{-Pre}}$. We refer the reader to [6] for a proof; in this paper, Buszkowski considers the extensions of both the associative and non-associative Lambek Calculus—which he denotes **L** and **NL**, respectively—with the classical propositional calculus, resulting in the logical systems **L-CL** and **NL-CL**, respectively. It is his proof of derivability of (L1) and (L2) in his Hilbert system for **NL-CL** that readily applies to our $\mathbf{MIL}_{\backslash\textbf{-Pre}}$. Reason being that $\mathbf{MIL}_{\backslash\textbf{-Pre}}$ turns out to be nothing but an extension of **NL-CL** with the axioms (Re.), (4), (Co.), and (Dk.)—shedding another interesting light on modal information logics and, especially, $MIL_{\backslash\text{-}Pre}$ (and $MIL_{\backslash\text{-}Pos}$) when having in mind that we end up proving that $MIL_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre} = \mathbf{MIL}_{\backslash\textbf{-Pre}}$. In other words, $MIL_{\backslash\text{-}Pre}$ *is* the Lambek Calculus (augmented with CL) of suprema on preorders (or on posets). $\dashv$

Remark 4.11. Besides from [6] being a recent gem in the literature on the Lambek Calculus extended with CL (i.e., essentially, studying it as a classical modal logic with three binary modalities), it has received some newborn attention: in [7] **NL-CL** is denoted **BFNL**, and in [10] **L-CL** and **NL-CL** are denoted **PL** and **PNL**, respectively. $\dashv$

We continue with the pledged completeness proof.

Theorem 4.12. $\mathbf{MIL}_{\backslash\textbf{-Pre}}$ *is sound and strongly complete w.r.t. the class $\mathcal{C}$. Thus, in particular,* $\mathbf{MIL}_{\backslash\textbf{-Pre}} = \mathrm{Log}_\backslash(\mathcal{C})$.

Proof. Soundness $\mathbf{MIL}_{\backslash\textbf{-Pre}} \subseteq \mathrm{Log}_\backslash(\mathcal{C})$ is routine.[36]

For strong completeness, we define the canonical frame as we did in Definition 2.3, but now defined w.r.t. the language $\mathcal{L}_{\backslash\text{-}M}$ instead; i.e., we let $W_{\backslash\textbf{-Pre}}$ denote the set of $\mathbf{MIL}_{\backslash\textbf{-Pre}}$-MCSs, and set $C_{\backslash\textbf{-Pre}}\Gamma\Delta\Theta$:iff

$$\forall\delta \in \Delta, \theta \in \Theta\, (\langle\text{sup}\rangle\delta\theta \in \Gamma)\,.$$

[35] '$(N_\backslash)$' is short for 'necessitation'. Observe that the other necessitation rule is not validity preserving. We, e.g., have $\mathrm{Log}_\backslash(\mathcal{C}) \Vdash \top$ but we do not have $\mathrm{Log}_\backslash(\mathcal{C}) \Vdash \top\backslash\bot$.

[36] Nevertheless, for soundness, to understand how '$\backslash$' and '$\langle\text{sup}\rangle$' capture different aspects of the same relation, it might be instructive for the reader to check that (I1) and (I2) are valid on $\mathcal{C}$-frames.

Note that Lindenbaum's Lemma and standard properties of MCSs hold, since our logic contains all classical propositional tautologies and is closed under MP and US. As in Lemma 2.6, we then get that $(W_{\backslash\textbf{-Pre}}, C_{\backslash\textbf{-Pre}}) \in \mathcal{C}$.

Thus, it suffices to show the standard truth lemma. The base and Boolean cases are straightforward by standard properties of MCSs, and since '[sup]' is a normal modality and $C_{\backslash\textbf{-Pre}}$ is defined in terms of it(s dual), the corresponding inductive step of the truth lemma goes through. Therefore, it only remains to cover the inductive step for '$\backslash$'.[37] To this end, the following two claims will suffice:

- *Claim:* If $\varphi\backslash\psi \in \Delta, \varphi \in \Theta$ and $C_{\backslash\textbf{-Pre}}\Gamma\Delta\Theta$, then $\psi \in \Gamma$.
  Proof. Assume $\varphi\backslash\psi \in \Delta, \varphi \in \Theta$ and $C_{\backslash\textbf{-Pre}}\Gamma\Delta\Theta$. By definition of $C_{\backslash\textbf{-Pre}}$, we would have that $\langle\text{sup}\rangle(\varphi\backslash\psi)\varphi \in \Gamma$. $\psi \in \Gamma$ then follows by (I1), (Co.), US, and MP of MCSs.

- *Claim (existence lemma for '$\backslash$'):* If $\neg(\varphi\backslash\psi) \in \Delta$, then there are some Θ, Γ s.t. $\varphi \in \Theta, \neg\psi \in \Gamma$ and $C_{\backslash\textbf{-Pre}}\Gamma\Delta\Theta$.
  Proof. Assume $\neg(\varphi\backslash\psi) \in \Delta$. Then

  $$\Gamma_0 := \{\langle\text{sup}\rangle\delta\varphi \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

  is consistent because if not, then

  $$\vdash_{\backslash\text{-Pre}} \bigwedge_{i \leq k} \langle\text{sup}\rangle\delta_i\varphi \to \psi$$

  for some finite $\{\delta_0, \dots, \delta_k\} \subseteq \Delta$, hence (a)

  $$\vdash_{\backslash\text{-Pre}} \langle\text{sup}\rangle\widehat{\delta}\varphi \to \psi$$

  where $\widehat{\delta} := \bigwedge_{i \leq k} \delta_i$. Moreover, since $\widehat{\delta} \in \Delta$, we get by (I2), US, and MP of MCSs that (b) $\varphi\backslash(\langle\text{sup}\rangle\widehat{\delta}\varphi) \in \Delta$. Thus, since all MCSs extend $\textbf{MIL}_{\backslash\textbf{-Pre}}$ and the monotonicity rule

  $$\text{if } \vdash_{\backslash\text{-Pre}} \alpha_0 \to \alpha_1, \text{ then } \vdash_{\backslash\text{-Pre}} \beta\backslash\alpha_0 \to \beta\backslash\alpha_1$$

  is easily derived, we get by (a), (b), US and MP of MCSs that $\varphi\backslash\psi \in \Delta$ – contradiction. Consequently, Γ_0 must be consistent.

  Now, let $\chi_0, \chi_1, \dots$ be an enumeration of all $\mathcal{L}_{\backslash\text{-}M}$-formulas, and define

  $$\Theta_0 := \{\varphi\},$$

[37] For another, more elaborate proof of a truth lemma which resembles ours, see the one given for the canonical model of **NL-CL** (their **PNL**) in [10]. We provide our own proof and keep it brief, assuming familiarity with the techniques involved. This will be done in the terminology of [4, ch. 4], which also is an excellent resource for an explication of arguments and details sufficiently similar to the ones we will omit.

and

$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\chi_n\}, & \textit{if } \{\langle \sup \rangle \delta(\widehat{\Theta_n} \wedge \chi_n) \mid \delta \in \Delta\} \cup \{\neg\psi\} \textit{ is consistent} \\ \Theta_n \cup \{\neg\chi_n\}, & \textit{otherwise.} \end{cases}$$

We claim that the set

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

is consistent for all $n \in \omega$. For the base case, notice that Γ_0 being consistent precisely means that

$$\{\langle \sup \rangle \delta \widehat{\Theta_0} \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

is consistent.

So assume

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

is consistent for some $n \in \omega$. If

$$\{\langle \sup \rangle \delta(\widehat{\Theta_n} \wedge \chi_n) \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

is consistent, we are done, so suppose not. Enumerating the formulas of Δ as $\delta_0, \delta_1, \ldots$ and setting $\delta'_i := \{\delta_j : j \leq i\}$, there must then be some $k \in \omega$ s.t. for all $m \geq k$:

$$\vdash_{\backslash\text{-Pre}} \langle \sup \rangle \widehat{\delta'_m}(\widehat{\Theta_n} \wedge \chi_n) \wedge \neg\psi \to \bot. \tag{$*$}$$

Furthermore, since by the IH

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

is consistent, using Lindenbaum, we can extend it to an MCS Λ_n. For this MCS Λ_n, we must then have for all $i \in \omega$:

$$\langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge [\chi_n \vee \neg\chi_n]) \in \Lambda_n.$$

So for all $i \in \omega$:

$$\langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge \chi_n) \in \Lambda_n \quad \textit{or} \quad \langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge \neg\chi_n) \in \Lambda_n.$$

Thus, combining this with $(*)$ [and having in mind that $\neg\psi \in \Lambda_n$], we get that for all $m \geq k$:

$$\langle \sup \rangle \widehat{\delta'_m}(\widehat{\Theta_n} \wedge \neg\chi_n) \in \Lambda_n.$$

But this entails that

$$\Big(\{\langle\text{sup}\rangle\delta(\widehat{\Theta_n}\wedge\neg\chi_n)\mid\delta\in\Delta\}\cup\{\neg\psi\}\Big)\subseteq\Lambda_n,$$

wherefore $\{\langle\text{sup}\rangle\delta\widehat{\Theta_{n+1}}\mid\delta\in\Delta\}\cup\{\neg\psi\}$ is consistent, as required for the induction proof.

From this [and having in mind that $\Theta_i\subseteq\Theta_j$ for $i\leq j$], one easily sees that (1)

$$\Gamma_\omega:=\bigcup_{n\in\omega}\{\langle\text{sup}\rangle\delta\widehat{\Theta_n}\mid\delta\in\Delta\}\cup\{\neg\psi\}$$

is consistent, and (2)

$$\Theta:=\bigcup_{n\in\omega}\Theta_n$$

is an MCS. Extending $\Gamma_\omega\subseteq\Gamma$ to an MCS, we get that $\varphi\in\Theta_0\subseteq\Theta,\neg\psi\in\Gamma$ and $C_{\backslash\textbf{-Pre}}\Gamma\Delta\Theta$, which precisely shows the claim.

With these claims at our disposal, the inductive step regarding '$\backslash$' in a proof of the truth lemma is immediate (the two claims cover one direction each). Since this was the last obstacle for proving the truth lemma, and we have already noted that $(W_{\backslash\textbf{-Pre}},C_{\backslash\textbf{-Pre}})\in\mathcal{C}$, we can deduce strong completeness—finishing not only our proof, but also this subsection. $\square$

4.3 Bulldozing and completeness-via-representation

With $\text{Log}_\backslash(\mathcal{C})$ axiomatized, next up is showing $\text{Log}_\backslash(\mathcal{C})=\mathit{MIL}_{\backslash\text{-}\mathit{Pre}}=\mathit{MIL}_{\backslash\text{-}\mathit{Pos}}$ via representation; i.e., via onto 'p-morphisms'.[38]

Importantly, to find the technique of onto p-morphisms in our arsenal of validity-preserving techniques, when dealing with preorder frames, we have to define the 'back'- and 'forth'-conditions in terms of the accompanying ternary (and not binary) relations. For ease of reference, let us spell this out:

Definition 4.13. Given any two frames $\{(W,C),(W',C')\}\subseteq\mathcal{C}$, a function

$$f:W'\to W$$

is denoted a *p-morphism* if it satisfies the following conditions:

(forth) if $C'x'y'z'$, then $Cf(x')f(y')f(z')$; and

($\langle$sup$\rangle$-back) if $Cf(x')yz$, then there exist $\{y',z'\}\subseteq W'$ s.t. $f(y')=y,f(z')=z$ and $C'x'y'z'$.

If f additionally satisfies

[38] Another commonly used term for 'p-morphism' is 'bounded morphism'.

(\-back) if $Cxf(y')z$, then there exist $\{x', z'\} \subseteq W'$ s.t. $f(x') = x, f(z') = z$ and $C'x'y'z'$,

we denote it a \-*p-morphism*.[39]

When dealing with preorder frames $(W, \leq)$, [\-]p-morphisms are defined in terms of the induced $(W, S_{\leq}) \in \mathcal{S}_{\text{Pre}} \subseteq \mathcal{C}$.[40] $\dashv$

Now to be clear, onto p-morphisms preserve validity (and, generally, consequences) of $\mathcal{L}_M$-formulas, while onto \-p-morphisms even preserve validity (and consequences) of $\mathcal{L}_{\backslash\text{-}M}$-formulas. This means we have a formal framework for developing representation results. In this subsection, this is a substantial part of what we will be doing.[41]

First up is our plighted proof that any $\mathcal{C}$-frame (W, C) is the \-p-morphic image of a poset frame $(W', S'_{\leq}) \in \mathcal{S}_{\text{Pos}}$, entailing that with $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ we have achieved an axiomatization of both $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$. This representation is obtained by composing two other representations; the first of which generalizes 'bulldozing' from the usual unary-modality setting to our binary-modality setting. To explain how this works, we briefly observe the following:

Observation 4.14. For any $(W, C) \in \mathcal{C}$, let $\leq_C$ and $\leq'_C$ be given as follows:

$$\leq_C := \{(y, x) : Cxxy\}, \qquad \leq'_C := \{(y, x) : \exists z(Cxyz \vee Cxzy)\}.$$

Then, by definition of the class $\mathcal{C}$, it is not too hard to see that (a) $\leq_C = \leq'_C$, and (b) $\leq_C$ is a preorder on W.

Moreover, if C happened to be the supremum relation of some preorder $\leq$, i.e., if $Cxyz$ iff $x \in \sup_{\leq}\{y, z\}$, then $\leq_C = \leq$. $\dashv$

With this observed, we are ready for the first representation result, mending $\mathcal{C}$-frames (W, C) so that $\leq_C$ becomes a *partial* order.

Proposition 4.15 (Bulldozing)**.** *Let $(W, C) \in \mathcal{C}$. Then (W, C) is the \-p-morphic image of some $(W', C') \in \mathcal{C}$ for which $\leq_{C'}$ is a partial order.*

Proof. Let $(W, C) \in \mathcal{C}$ be arbitrary. We construct (W', C') by adapting the well-known bulldozing technique from the binary-relation setting to our ternary-relation setting. More precisely, let $\mathcal{K}$ denote the set of maximal non-degenerate clusters of (W, C)

[39] Notice the symmetry in the two back clauses: this is caused by '\' and '$\langle \text{sup} \rangle$' referring to the same relation, but from different perspectives.

[40] These definitions extend to the notion of p-morphisms between models as well. Moreover, the notion of bisimulation for modal information logics is also defined in terms of the induced $(W, S_{\leq}) \in \mathcal{S}_{\text{Pre}}$; i.e., the back and forth conditions are defined in terms of the supremum relation and not in terms of the preorder '$\leq$'.

[41] Regarding \-p-morphisms, it is important to have in mind that they are also required to meet ($\langle \text{sup} \rangle$-back). A notion for simply meeting (forth) and (\-back) would appear appropriate, but we will not be needing such since we do not deal with modal logics having only the modality '\'. In general, of course, the results of this section have these modal logics as special cases; e.g., our decidability proof in the next subsection.

w.r.t. the preorder $\leq_C$.[42] We then define the underlying set as

$$W' := \left(W \setminus \bigcup_{K \in \mathcal{K}} K \right) \cup \bigcup_{K \in \mathcal{K}} (K \times \mathbb{Z}),$$

and let the function

$$f : W' \to W$$

be given by

$$f(x) = \begin{cases} x, & x \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right) \\ k, & x = (k, z) \in K \times \mathbb{Z}, K \in \mathcal{K} \end{cases}.$$

To define the relation C', fix some linear order $\leq^K$ for each $K \in \mathcal{K}$, and for all $x, a, b \in W'$, let $C'xab$:iff $Cf(x)f(a)f(b)$ and

$$\left(\begin{array}{ll} \textit{(i)} & x \in W \setminus \bigcup_{K \in \mathcal{K}} K, \ \textbf{or} \\ \textit{(ii)} & x = (k, z) \in K \times \mathbb{Z}; (K \times \mathbb{Z}) \cap \{a, b\} = \varnothing, \ \textbf{or} \\ \textit{(iii)} & x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_a, z_a) = a; b \notin K \times \mathbb{Z}; [z_x > z_a \ \textit{or} \ (z_x = z_a \ \textit{and} \ k_x \geq^K k_a)], \ \textbf{or} \\ \textit{(iv)} & x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_b, z_b) = b; a \notin K \times \mathbb{Z}; [z_x > z_b \ \textit{or} \ (z_x = z_b \ \textit{and} \ k_x \geq^K k_b)], \ \textbf{or} \\ \textit{(v)} & \{x, a, b\} \subseteq K \times \mathbb{Z}; x = (k_x, z_x), a = (k_a, z_a), b = (k_b, z_b); \\ & [z_x > z_a \ \textit{or} \ (z_x = z_a \ \textit{and} \ k_x \geq^K k_a)]; [z_x > z_b \ \textit{or} \ (z_x = z_b \ \textit{and} \ k_x \geq^K k_b)] \end{array} \right).$$

We claim that (1) $(W', C') \in \mathcal{C}$; (2) (W, C) is a \-p-morphic image of (W', C') witnessed by f; and (3) $\leq_{C'}$ is a partial order.

We begin by proving **(1)** $(W', C') \in \mathcal{C}$. We have that

(Re.f) is satisfied because (a) $(W, C) \vDash (Re.f)$ by assumption and (b) for all $K \in \mathcal{K}$: $\leq^K$ is, as a (weak) linear order, in particular, reflexive;

(4f) can be seen to be satisfied by a straightforward, but tedious check using $(W, C) \vDash (4f)$. Only non-trivial case is when $C'xxa$ by virtue of *(iii)*: there one must observe that if $C'aab$ then $f(b)$ cannot be in the same cluster as $f(x)$ by maximality of clusters $K \in \mathcal{K}$;

(Co.f) is satisfied because (a) $(W, C) \vDash (Co.f)$ and (b) the definition of C' is symmetrical in the two last arguments; and

(Dk.f) is satisfied because (a) $(W, C) \vDash (Dk.f)$ and (b) if $C'xab$ holds by virtue of *(i)*, then $C'xxa$ holds by virtue of *(i)*; if $C'xab$ holds by virtue of *(ii)* or *(iv)*, then $C'xxa$ holds by virtue of *(iii)*; if $C'xab$ holds by virtue of *(iii)*, then $C'xxa$ holds by virtue of *(v)*; and if $C'xab$ holds by virtue of *(v)*, then $C'xxa$ holds by virtue of *(v)*.

[42] Recall that a cluster is non-degenerate :iff it contains at least two elements. It is maximal :iff no proper superset is a cluster.

Having proven (1), we continue by proving **(2)**. f is clearly (a) surjective and (b) a homomorphism. Therefore, it remains to show that (c) the back conditions are satisfied. Beginning with ($\langle$sup$\rangle$-back), suppose $Cf(x)a'b'$ for arbitrary $x \in W'$, $\{a', b'\} \subseteq W$. We then have to find $a, b \in W'$ s.t. $C'xab$, $f(a) = a'$, and $f(b) = b'$. We go by cases:

(i) If $x \in \left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$, pick any $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$ using surjectivity of f.

(ii) If $x = (k, z) \in K \times \mathbb{Z}$ and $\{a', b'\} \cap K = \varnothing$, pick any $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$.

(iii) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $a' \in K \not\ni b'$, set $a := (a', z_x - 1)$ and pick any $b \in f^{-1}(b')$.

(iv) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $b' \in K \not\ni a'$, set $b := (b', z_x - 1)$ and pick any $a \in f^{-1}(a')$.

(v) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $a' \in K \ni b'$, set $a := (a', z_x - 1)$ and $b := (b', z_x - 1)$.

This exhausts all cases, hence f satisfies the ($\langle$sup$\rangle$-back) condition, thus is a p-morphism. Continuing with ($\setminus$-back), suppose $Cxf(a')b$ for some $a' \in W'$ and $\{x, b\} \subseteq W$. Again, we go by cases:

(i) If $a' \in W \setminus \bigcup_{K\in\mathcal{K}} K$ or $[a' = (k_a, z_a) \in K \times \mathbb{Z}$ and $x \notin K]$, then begin by picking any $x' \in f^{-1}(\{x\})$. Then $Cf(x')f(x')b$ by (Dk.f) and (Co.f) of $(W, C) \in \mathcal{C}$ and because $Cxf(a')b$, so by the just proved $\langle$sup$\rangle$-back condition and the definition of C', we can find a $b' \in W'$ s.t. $C'x'x'b'$ and $f(b') = b$. We claim that $C'x'a'b'$. To see this, first recall that $Cxf(a')b$, $f(x') = x$ and $f(b') = b$. Second, notice that $C'x'x'b'$ must hold by virtue of *(i)*, *(iii)* or *(v)*. If by virtue of *(i)*, then so does $C'x'a'b'$; if by virtue of *(iii)*, then $C'x'a'b'$ holds by virtue of *(ii)* (since, by assumption, either $a' \in W \setminus \bigcup_{K\in\mathcal{K}} K$ or $[a' = (k_a, z_a) \in K \times \mathbb{Z}$ and $x \notin K]$); and if by virtue of *(v)*, then $C'x'a'b'$ holds by virtue of *(iv)*.

(ii) And if $a' = (k_a, z_a) \in K \times \mathbb{Z}$ and $x \in K$, then setting $x' := (x, z_a + 1)$, we, again by (Dk.f) and (Co.f), get that $Cf(x')f(x')b$, hence by the $\langle$sup$\rangle$-back condition we can find a $b' \in W'$ s.t. $C'x'x'b'$ and $f(b') = b$. Now because (1) $Cxf(a')b$ and (2) $C'x'x'b'$ must hold by virtue of *(iii)* or *(v)*, we get that $C'x'a'b'$ likewise holds by virtue of either *(iii)* or *(v)* since $z_a + 1 > z_a$.

This covers all cases—completing our proof of f being an onto $\setminus$-p-morphism.

Lastly, we show that **(3)** $\leq_{C'}$ is a partial order. Reflexivity and transitivity are consequences of $(W', C') \in \mathcal{C}$. To show anti-symmetry, let $x, y \in W'$ be arbitrary s.t. $C'xxy$ and $C'yyx$. We have to show that $x = y$. Going by cases we find that:

- If $\{x, y\} \subseteq \left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$, then $Cxxy$ and $Cyyx$ by definition of f and C', so since $\left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$ contains no non-degenerate clusters by definition, we must have $x = y$.
- If $x \in \left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$ and $y = (k, z) \in K \times \mathbb{Z}$, then $Cxxk$ and $Ckkx$ so $x \in K$—contradicting $x \in \left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$.

- If $y \in \left(W \setminus \bigcup_{K\in\mathcal{K}} K\right)$ and $x = (k, z) \in K \times \mathbb{Z}$, then as above.
- If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $y = (k_y, z_y) \in K' \times \mathbb{Z}$ for $K \neq K'$, then $Ck_xk_xk_y$ and $Ck_yk_yk_x$ so $k_x \in K'$—contradicting maximality of the clusters (which implies that whenever $K \neq K'$, we even have $K \cap K' = \varnothing$).
- If $x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_y, z_y) = y$, then $x = y$ follows by anti-symmetry of our lexicographical ordering (since the ordering of the integers is linear and so is $\leq^K$).

Thus, we've shown $\leq_{C'}$ to be anti-symmetric, which completes our proof of (3) $\leq_{C'}$ being a partial order, thus finalizing our bulldozing proof. $\square$

Using this representation, we continue further mending $\mathcal{C}$-frames (W, C) into real poset frames (i.e., frames whose ternary relation is the supremum relation of a partial order). We do so through another representation, which is obtained by adopting the framework of the completeness proof of Section 2 (2.13). In brief, in the proof to come, we will also be constructing a poset frame recursively by repairing defects. However, this time, the defects will be determined by an onto function, which we iteratively extend seeking to make it an onto \-p-morphism. And, although the ⟨sup⟩- and ¬⟨sup⟩-defects only need minor revision, we do need to include a third kind of defect corresponding to (\-back).

Many of the arguments will be almost identical to the ones of the completeness proof of Section 2, and so will be omitted or only hinted at. But – although the general set-up is very similar – since there are some differences, it is worth spelling out. We proceed doing so.

Definition 4.16. Given any $(W, C) \in \mathcal{C}$, we let E be some set disjoint from W of cardinality $\max\{|W|, \aleph_0\}$, and $\mathbb{P}_{(W,C)}$ be the set of all quadruples $(f, D, X, \leq)$ such that

1.' f is an onto function from $(W \cup D \cup X)$ to W;

2.' $|D \cup X| < |E|$;

3.' $(D \cup X) \subseteq E$;

4.' $D \cap X = \varnothing$;

6.' $\leq$ is a partial order on $(W \cup D \cup X)$; and

7.' if $y \leq x$ then $f(y) \leq_C f(x)$.[43] $\dashv$

Next, we define the revised versions of the ⟨sup⟩- and ¬⟨sup⟩-defects and their complementary revised repair lemmas, before subsequently stating and proving the last defect/repair pair.

Definition 4.17 ((⟨sup⟩-back)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then a triple $(x', y, z) \in (W \cup E) \times W \times W$ denotes a *(⟨sup⟩-back)-defect* (of $(f, D, X, \leq$ $))$:iff

$$\text{(i) } x' \in (W \cup D \cup X), \qquad \text{(ii) } Cf(x')yz,$$

[43] The only significant change is the present deletion of what was condition 5. of Definition 2.7.

and (iii) there are no $y', z' \in (W \cup D \cup X)$ s.t. $x' = \sup\{y', z'\}$ and

$$\begin{aligned} f(y') &= y, & \uparrow y' &= \uparrow x' \cup \{y'\} \cup (\uparrow y' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}), \\ f(z') &= z, & \uparrow z' &= \uparrow x' \cup \{y'\} \cup (\uparrow z' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}).^{44} \end{aligned} \qquad \dashv$$

Definition 4.18 ((forth)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then a triple $(x', y', z') \in (W \cup E) \times (W \cup E) \times (W \cup E)$ is denoted a *(forth)-defect* (of $(f, D, X, \leq)$) :iff

$$\{x', y', z'\} \subseteq (W \cup D \cup X), \qquad x' = \sup\{y', z'\}, \qquad \neg C f(x') f(y') f(z').^{45} \qquad \dashv$$

Lemma 4.19 (($\langle\sup\rangle$-back)-repair lemma)**.** *Suppose (x', y, z) is a ($\langle\sup\rangle$-back)-defect of some $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D, X', \leq') \in \mathbb{P}_{(W,C)}$ by taking distinct $y', z' \in E \setminus (D \cup X)$ and setting*

$$\begin{aligned} &f' := f \cup \{(y', y), (z', z)\}, \qquad X' := X \cup \{y', z'\}, \\ &\leq' := \leq \cup \{(y', u), (z', u) \mid x' \leq u\} \cup \{(y', y'), (z', z')\}. \end{aligned}$$

Then, witnessed by y' and z', (x', y, z) does not constitute a ($\langle\sup\rangle$-back)-defect of $(f', D, X', \leq')$.

Proof. Defining as described, the proof of $(f', D, X', \leq') \in \mathbb{P}_{(W,C)}$ resembles the one of Lemma 2.11: 1.'-6.' are obvious, and 7.' is shown using $C f'(x') f'(y') f'(z')$ and $(W, C) \in \mathcal{C}$.

Moreover, the latter claim is immediate. $\square$

Lemma 4.20 ((forth)-repair lemma)**.** *Suppose (x', y', z') is a (forth)-defect of some $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D', X, \leq') \in \mathbb{P}_{(W,C)}$ by (a) taking $d' \in E \setminus (D \cup X)$, (b) letting*

$$\begin{aligned} &f' := f \cup \{(d', f(x'))\}, \qquad D' := D \cup \{d'\}, \\ &\leq' := \leq \cup \{(u, d'), (v, d') \mid u \leq y', v \leq z'\} \cup \{(d', d')\}, \end{aligned}$$

and (c) getting $x' \neq \sup_{\leq'}\{y', z'\}$.[46]

Proof. Extending to $(f', D', X, \leq')$ as described, it follows similarly to the proof of Lemma 2.12 that $(f', D', X, \leq')$ satisfies 1.'-7.' and $x' \neq \sup_{\leq'}\{y', z'\}$. Only two things are worth mentioning: (1) for proving 7.', we use that if $u <' d'$ then $u \leq x'$, hence $f'(u) = f(u) \leq_C f(x') = f'(d')$, and (2) for proving $x' \neq \sup_{\leq'}\{y', z'\}$, we need that $\leq$ is a *partial* order (this is where we use bulldozing). $\square$

Our third and last defect, naturally, bears much resemblance to the ($\langle\sup\rangle$-back)-defect. It is defined as follows:

[44] Notice the similarity between ($\langle\sup\rangle$-back)-defects and $\langle\sup\rangle$-defects (2.8).

[45] And between (forth)-defects and $\neg\langle\sup\rangle$-defects (2.9).

[46] Now 'd'' is no longer short for 'dummy', but for 'duplicate' (of x'): $f'(d') = f(x')$. We stress: this is key. (However, this is only a good intuition for the D-worlds introduced in this repair lemma—not for those in the next.)

Definition 4.21 (($\backslash$-back)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then a triple $(x, y', z) \in W \times (W \cup E) \times W$ denotes a *($\backslash$-back)-defect* (of $(f, D, X, \leq)$) :iff

$$\text{(i) } y' \in (W \cup D \cup X), \qquad \text{(ii) } Cxf(y')z,$$

and (iii) there are no $x', z' \in (W \cup D \cup X)$ s.t. $x' = \sup\{y', z'\}$ and

$$\begin{aligned} f(x') &= x, \qquad & {\uparrow}y' &= {\uparrow}x' \cup \{y'\} \cup ({\uparrow}y' \cap \{w' \mid {\uparrow}w' \cap {\uparrow}x' = \varnothing\}), \\ f(z') &= z, \qquad & {\uparrow}z' &= {\uparrow}x' \cup \{z'\} \cup ({\uparrow}z' \cap \{w' \mid {\uparrow}w' \cap {\uparrow}x' = \varnothing\}). \end{aligned} \quad \dashv$$

This new defect is repaired in this fashion:

Lemma 4.22 (($\backslash$-back)-repair lemma)**.** *Suppose (x, y', z) is a ($\backslash$-back)-defect of some $(f, D, X, \leq) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D', X', \leq') \in \mathbb{P}_{(W,C)}$ by taking distinct $x', z' \in E \setminus (D \cup X)$ and setting*

$$\begin{aligned} &f' := f \cup \{(x', x), (z', z)\}, \qquad D' := D \cup \{x'\}, \qquad X' := X \cup \{z'\}, \\ &\leq' := \leq \cup \{(u, x') \mid u \leq y'\} \cup \{(x', x'), (z', z'), (z', x')\}. \end{aligned}$$

Then, witnessed by x' and z', (x, y', z) does not constitute a $\backslash$-back defect of $(f', D', X', \leq')$.

Proof. A matter of going over the definition. $\square$

Employing these repairs, we are ready to prove the desired representation result.

Proposition 4.23. *Every $(W, C) \in \mathcal{C}$ for which $\leq_C$ is a partial order, is a $\backslash$-p-morphic image of a poset frame.*

Proof. Let $(W, C) \in \mathcal{C}$ be arbitrary s.t. $\leq_C$ is a partial order. For the sake of simplicity, assume W is countable: as oftentimes is the case, the adjustments of the ensuing proof needed for the case where $|W| > \aleph_0$ are conceptually insignificant but notationally taxing.[47] Besides, by a 'standard translation' and the Löwenheim-Skolem Theorem, $\mathcal{C}$ has the countable model property w.r.t. $\mathcal{L}_{\backslash\text{-}M}$-formulas, so, for instance, starting with a countable frame, we can bulldoze it into a countable $\mathcal{C}$-frame whose underlying preorder is a partial order.

As in the completeness proof of section 2, using the repair lemmas repeatedly, we will be constructing a sequence

$$(f_0, D_0, X_0, \leq_0), (f_1, D_1, X_1, \leq_1), \ldots$$

such that for all $i \in \omega$

$$(f_i, D_i, X_i, \leq_i) \in \mathbb{P}_{(W,C)}, \quad f_i \subseteq f_{i+1}, \quad D_i \subseteq D_{i+1}, \quad X_i \subseteq X_{i+1}, \quad \leq_i \subseteq \leq_{i+1}.$$

[47]The adjustments in case $|W| > \aleph_0$ are doing transfinite recursion and induction instead.

We begin the sequence by setting

$$f_0 := Id : W \to W, \quad D_0 := X_0 := \varnothing, \quad \leq_0 := \leq_C.$$

Then $(f_0, D_0, X_0, \leq_0) \in \mathbb{P}_{(W,C)}$.

At each stage $n+1$, we then pick the least tuple constituting a defect to $(f_n, D_n, X_n, \leq_n)$—according to a fixed enumeration of the set of all triples $(x', y, z) \in (W \cup E) \times W \times W$ and all triples $(x', y', z') \in (W \cup E)^3$ and all triples[48] $(x, y', z) \in W \times (W \cup E) \times W$—and repair it to obtain $(f_{n+1}, D_{n+1}, X_{n+1}, l_{n+1})$. Letting

$$(f_\omega, D_\omega, X_\omega, \leq_\omega) := \left(\bigcup_{n\in\omega} f_n, \bigcup_{n\in\omega} D_n, \bigcup_{n\in\omega} X_n, \bigcup_{n\in\omega} \leq_n \right),$$

we get that (1) $(f_\omega, D_\omega, X_\omega, \leq_\omega)$ satisfies 1.' and 3.'-7.', and (2) $(f_\omega, D_\omega, X_\omega, \leq_\omega)$ has no defects whatsoever. Again, only (2) is not straightforward, and, again, for proving (2) two claims and an observation are helpful.

Observation'. Let $n \in \omega$ and $\{x, v\} \subseteq (W \cup D_n \cup X_n)$ be arbitrary s.t.

$$\uparrow_n v' = \uparrow_n x' \cup \{v'\} \cup (\uparrow_n v' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}).$$

Then for all $m \geq n$:

$$\uparrow_m v' = \uparrow_m x' \cup \{v'\} \cup (\uparrow_m v' \cap \{w' \mid \uparrow_m w' \cap \uparrow_m x' = \varnothing\}),$$

hence also

$$\uparrow_\omega v' = \uparrow_\omega x' \cup \{v'\} \cup (\uparrow_\omega v' \cap \{w' \mid \uparrow_\omega w' \cap \uparrow_\omega x' = \varnothing\}).$$

This follows by an easy induction, using that each $(f_{m+1}, D_{m+1}, X_{m+1}, \leq_{m+1})$ is obtained from $(f_m, D_m, X_m, \leq_m)$ using one of the repair lemmas.

Claim (a'). Let $n \in \omega$ and $\{x', y', z'\} \subseteq (W \cup D_n \cup X_n)$ be arbitrary s.t.

$$\begin{aligned} &x' = \sup_n\{y', z'\}, \qquad Cf_n(x')f_n(y')f_n(z'), \\ &\uparrow_n y' = \uparrow_n x' \cup \{y'\} \cup (\uparrow_n y' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}), \\ &\uparrow_n z' = \uparrow_n x' \cup \{z'\} \cup (\uparrow_n z' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}). \end{aligned}$$

Then for all $m \geq n$:

$$x' = \sup_m\{y', z'\};$$

a fortiori, $x' = \sup_\omega\{y', z'\}$.

We prove the claim by induction. By assumption, it holds for $m = n$, so assume it holds for some $m \geq n$. We show it holds for $m+1$. This time we have three

[48] For simplicity of argument, we assume all $(x_0, y_0, x_0) \in (W \cup E) \times W \times W$ to be distinct from all $(x_1, y_1, z_1) \in (W \cup E)^3$ – and so forth.

cases, depending on the type of defect being repaired at stage $m+1$. The cases of a $(\langle \sup \rangle$-back)-repair and (forth)-repair are the exact same as in Theorem 2.13.

Consequently, suppose stage $m+1$ was obtained by $(\backslash$-back)-repairing some (s, y'_s, z_s) through introducing the worlds s', z'_s. Then s' is the only possible counterexample to $x' = \sup_{m+1}\{y', z'\}$, so assume $y' \leq_{m+1} s' \geq_{m+1} z'$. Then we must have $y' \leq_m y'_s \geq_m z'$, so by the IH $x' \leq_m y'_s$, hence $x' \leq_{m+1} s'$. $\square_{Claim\ (a')}$

Claim (b'). Let $n \in \omega$ and suppose that $a, b \in (W \cup D_n \cup X_n)$ are s.t. $a \not\geq_n b$. Then for all $m \geq n$, we have that $a \not\geq_m b$. A fortiori, $a \not\geq_\omega b$.

Once again by induction on $m \geq n$ with no change concerning the cases of $(\langle \sup \rangle$-back)-repairs and (forth)-repairs. Therefore, assume $(l_{m+1}, \leq_{m+1}, D_{m+1})$ was obtained by $(\backslash$-back)-repairing some (x, y', z) by introducing x', z'. Then there is no change in predecessors of a, which suffices for the claim. $\square_{Claim\ (b')}$

Finally, from these claims we likewise get *(c)*: If some tuple *did* constitute a defect at some stage n, but no longer at some later stage $m > n$, then it didn't for all $k \geq m$.

Noteworthy is the overlap between our definitions of $(\langle \sup \rangle$-back)-defects and $(\backslash$-back)-defects, which assures that *claim (a')* applies to both types of defects. And using *(c)* along with *claim (a')* and *(b')* in an analogous manner to what we did in the completeness proof, we get that $(f_\omega, D_\omega, X_\omega, \leq_\omega)$ neither has (forth)-, $(\langle \sup \rangle$-back)- nor $(\backslash$-back)-defects.

Lastly, the fact that there are no defects, entails that f_ω is a $\backslash$-p-morphism from $(W \cup D_\omega \cup X_\omega, \leq_\omega)$ to (W, C), so since f_ω also is onto, we've shown the desired. $\square$

At long last, combining the two representations, we can deduce that we have achieved the axiomatization we were seeking.

Theorem 4.24. *Every $(W, C) \in \mathcal{C}$ is a $\backslash$-p-morphic image of a poset frame.*

Thus, $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ is sound and strongly complete w.r.t. preorder frames, and, in particular,

$$MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}.$$

Additionally, as a special case, we get another proof of $\mathbf{MIL}_{\mathbf{Pre}}$ being sound and strongly complete w.r.t. preorder frames, and, particularly

$$MIL_{Pre} = MIL_{Pos} = \mathbf{MIL}_{\mathbf{Pre}}.$$

Proof. The first assertion follows from propositions 4.15 and 4.23 because onto $\backslash$-p-morphisms are closed under composition.

Soundness and strong completeness is the upshot of onto $\backslash$-p-morphisms preserving the consequence relation of a frame and the fact that $\mathcal{S}_{\mathrm{Pos}} \subseteq \mathcal{S}_{\mathrm{Pre}} \subseteq \mathcal{C}$; so also, in particular

$$MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}.$$

Lastly, since $\backslash$-p-morphisms are p-morphisms and $\mathcal{L}_M \subseteq \mathcal{L}_{\backslash\text{-}M}$, this also restricts to the special case of the basic modal information language. $\square$

4.4 Decidability

The problem of axiomatizing our conservative extension(s), $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$, of the basic modal information logic(s), $MIL_{Pre} = MIL_{Pos}$, solved, the biggest remaining problem is, arguably, that of decidability. As already mentioned, we continue being guided by the procedure outlined in subsection 1.2, thus showing decidability qua a proof of the FMP w.r.t. another class of frames, which, of course, is $\mathcal{C}$ anew. Albeit the $\mathcal{L}_M$-filtration through a $\mathcal{C}$-closed set of formulas (cf. subsection 3.2) is *not* an $\mathcal{L}_{\backslash\text{-}M}$-filtration—that is, through a $\mathcal{C}$-closed set of formulas it does not preserve satisfaction of $\mathcal{L}_{\backslash\text{-}M}$-formulas, but only of $\mathcal{L}_M$-formulas—we are not at a loss: only some minor modifications are needed.

Borrowing the idea of a *suitable* set of formulas from [6], we define a notion extending our notion of a $\mathcal{C}$-closed set of formulas.

Definition 4.25. We say that a set Σ of $\mathcal{L}_{\backslash\text{-}M}$-formulas is *$\mathcal{C}$-suitably closed* :iff

($\mathcal{C}$) it is $\mathcal{C}$-closed; and

(Suit) $\varphi\backslash\psi \in \Sigma$ implies $\langle\sup\rangle\varphi(\varphi\backslash\psi) \in \Sigma$.

Moreover, for any set of $\mathcal{L}_{\backslash\text{-}M}$-formulas Σ_0, we say that Σ is the *$\mathcal{C}$-suitable closure* of Σ_0 :iff it is the least $\mathcal{C}$-suitably closed set of formulas extending Σ_0.[49] ⊣

Afresh, an immediate consequence is:

Lemma 4.26. *For any finite set of $\mathcal{L}_{\backslash\text{-}M}$-formulas Σ_0, its $\mathcal{C}$-suitable closure $\Sigma \supseteq \Sigma_0$, too, is finite.*

As the last ingredient for achieving decidability, we show that when filtrating through $\mathcal{C}$-*suitably* closed sets of formulas, the $\mathcal{L}_M$-filtration of Theorem 3.9 lifts to an $\mathcal{L}_{\backslash\text{-}M}$-filtration:

Theorem 4.27. $\mathbf{MIL}_{\backslash\textbf{-Pre}}$ *admits filtration w.r.t. the class $\mathcal{C}$. Consequently,*

$$\mathbf{MIL}_{\backslash\textbf{-Pre}} = \mathrm{Log}_{\backslash}(\mathcal{C}_F),$$

where $\mathrm{Log}_{\backslash}(\mathcal{C}_F)$ denotes the logic of the class of finite $\mathcal{C}$-frames in the language of $\mathcal{L}_{\backslash\text{-}M}$.

Proof. Let (W, C, V) be an arbitrary $\mathcal{C}$-model; Σ an arbitrary $\mathcal{C}$-suitably closed set of formulas; and $(W_\Sigma, C^{\mathcal{C}}_\Sigma, V_\Sigma)$ be the filtration of (W, C, V) through Σ defined in Theorem 3.9. Then, as shown in the proof of said theorem, $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$ and the filtration conditions (F1) and (F2) hold for the modality '$\langle\sup\rangle$'. Thus, because of Lemma 4.26 and the inclusion $\mathrm{Log}_{\backslash}(\mathcal{C}) \subseteq \mathrm{Log}_{\backslash}(\mathcal{C}_F)$, we need only show that the synonymous filtration conditions for the modality '$\backslash$' likewise are met.

The former, homomorphism condition is evidently the same, while the latter becomes

(F2') $C^{\mathcal{C}}_\Sigma|x||y||z| \Rightarrow \forall\varphi\backslash\psi \in \Sigma\,[(y \Vdash \varphi\backslash\psi, z \Vdash \varphi) \Rightarrow x \Vdash \psi]$.[50]

[49] Note that the $\mathcal{C}$-suitable closure of a set of formulas always exists.

[50] Recall that '$\backslash$' is a '$\Box$-ed' modality; therefore, this presentation of the second filtration clause.

Consequently, all that remains to be proven is (F2').[51] So assume $C^{\mathcal{C}}_{\Sigma}|x||y||z|$, and let $\varphi\backslash\psi \in \Sigma$ be arbitary s.t. $y \Vdash \varphi\backslash\psi$ and $z \Vdash \varphi$. By (Suit), $\langle\sup\rangle\varphi(\varphi\backslash\psi) \in \Sigma$ so by (Com) we have that $\langle\sup\rangle(\varphi\backslash\psi)\varphi \in \Sigma$. But then (F2) entails that $x \Vdash \langle\sup\rangle(\varphi\backslash\psi)\varphi$, whence $x \Vdash \langle\sup\rangle\varphi(\varphi\backslash\psi)$ by $(W, C) \vDash (Co.f)$, so finally since $(W, C) \Vdash (I1)$, we have $x \Vdash \psi$ as required. □

Using this, we can conclude that the basic modal information logic of preorders (or posets) endowed with the informational implication is decidable.

Corollary 4.28. *$MIL_{\backslash\text{-}Pre}$ is decidable (and so is $MIL_{\backslash\text{-}Pos}$).*

Proof. We have shown that

$$MIL_{\backslash\text{-}Pre} = \mathbf{MIL_{\backslash\text{-}Pre}} = \mathrm{Log}_{\backslash}(\mathcal{C}_F),$$

so since $\mathbf{MIL_{\backslash\text{-}Pre}}$ is finitely axiomatizeable and complete w.r.t. a recursively enumerable (r.e.) class of finite frames [simply check for satisfaction of the first-order formulas (Re.f), (4f), (Co.f), and (Dk.f)], we obtain decidability of $MIL_{\backslash\text{-}Pre}$. □

Closing off this section, we state the following corollary:

Corollary 4.29. *Let $\mathcal{L}_{\Diamond\text{-}M}$ be the extension of the basic language $\mathcal{L}_M$ with the unary modality '$\Diamond$', and let the semantics for '$\Diamond$' be the usual one, namely those of the forward-looking modality 'F' given in Remark 4.8. Then letting $MIL_{\Diamond\text{-}Pre}$ and $MIL_{\Diamond\text{-}Pos}$ be the MILs of this language on preorders and posets, respectively, we get that both are decidable.*

Proof. A decision procedure is given as follows: For any $\mathcal{L}_{\Diamond\text{-}M}$-formula φ, translate it into a formula $t(\varphi) \in \mathcal{L}_{\backslash\text{-}M}$ in accordance with Remark 4.8, and then use the decision procedure of the preceding corollary. □

Conclusion and Future Work

This paper's exploration of modal information logics has come to an end. We summarize this inquiry, clarify where it leaves us, and point to future lines of research.

First, we examined the basic modal information logics of suprema on preorders and posets, namely MIL_{Pre} and MIL_{Pos}. We showed that – even if they do not enjoy the FMP w.r.t. their frames of definition – they are decidable. This was shown 'via completeness' by (1) axiomatizing them; (2) deducing that they are one and the same logic; and (3) obtaining another class of frames $\mathcal{C}$ complete w.r.t. the logic(s), which, importantly, did enjoy the FMP.

Second and last, we tackled these same problems but for the enriched logics $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ and achieved analogous results. However now, already having a clear candidate for a generalized class of frames, namely $\mathcal{C}$, we could axiomatize this logic first, and subsequently solve the problem of axiomatization of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ through representation.

[51] The proof of Lemma 2 in [6] pertains to showing the satisfaction of (F2') in our present setting, so the ensuing argument is only given for the sake of completeness of the current proof—we claim no originality whatsoever.

This brings us to directions for further research. We find two such to be of great interest:

- Further examining how MILs relate to other logics, not only those mentioned in the introduction, but also more: analogues of '$\langle \text{sup} \rangle$' occur in an array of logical systems: 'intensional conjunction' or simply 'fusion' in relevance logics ([3]); regular conjunction in semantics for exact truthmaking ([8, 19]); 'tensor disjunction' in the team semantics of [21, 22]; and 'split disjunction' in the state-based semantics of [2], to name some.

  This could shed new perspicuous lights on not only MILs but also on the logics of comparison (cf. 4.10).

- Expanding the inquiry from MILs on preorders and posets to MILs on different, more concrete structures.[52] This would be in line with the work of [20], where the authors axiomatize the MIL on lattices in the language extended with an '$\langle \text{inf} \rangle$'-modality and nominals. Additionally, it could enable linking up with work on other information-oriented logics, such as relevance logics and domain theory.

[52] As mentioned in the introduction, we have already initiated this study by finding an axiomatization of the basic MIL on join-semilattices, but for reasons of length, we have decided not to include it in this paper.

Appendix A Wildness of the canonical frame

As an informal addendum to Section 2, we briefly remark that the canonical relation $C_{\mathbf{Pre}}$ of the canonical frame for $\mathbf{MIL_{Pre}}$ is not the supremum relation of $\leq_{\mathbf{Pre}}$.

Remark A.0.1. The following hold:

1. There are MCSs Γ, Δ s.t. $C_{\mathbf{Pre}}\Gamma\Delta\Delta$ even if $\Gamma \not\leq_{\mathbf{Pre}} \Delta$. In other words, although Γ and Δ aren't in the same cluster ($\Gamma \not\leq_{\mathbf{Pre}} \Delta$), Γ 'claims' to be the 'supremum' of Δ.

2. In fact, there are continuum many such MCSs Γ_i all claiming to be the supremum of Δ. $\dashv$

Proof. Consider the model depicted below where the worlds satisfy all and only the proposition letters shown.

$$\begin{tikzcd}
 & w \Vdash p & \\
v_1 \Vdash \neg p \arrow[ur] & & v_2 \Vdash \neg p \arrow[ul]
\end{tikzcd}$$

Then $||v_1||, ||v_2||, ||w||$ are MCSs where $||x|| := \{\varphi \in \mathcal{L}_M \mid x \Vdash \varphi\}$. Moreover, $Pp \notin ||v_1|| = ||v_2||$, so (a) since $p \in ||w||$ we have that $\Gamma := ||w|| \not\leq_{\mathbf{Pre}} ||v_1|| =: \Delta$, and (b) since $w = \sup\{v_1, v_2\}$ we also have $C_{\mathbf{Pre}}\Gamma\Delta\Delta$, which proves the first claim.

For the second, simply change the valuation of w for proposition letters $q \neq p$ to get the same results for different MCSs Γ_i. Since there are countably many proposition letters (so continuum many subsets of proposition letters), we get continuum many MCSs claiming to be supremum of $\Delta = ||v_1|| = ||v_2||$.[53] $\square$

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[53] Evidently, this technique can be adapted to quickly show many more 'absurdities'; for instance, an infinite strictly ascending chain of MCSs all claiming to be the supremum of two MCSs below. Even if amusing, by now, this becomes too much of a sidetrack (even for an appendix), so we leave this as an activity for the reader.

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