Logics of Truthmaker Semantics: Comparison, Compactness and Decidability

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Abstract

In recent years, there has been a growing interest in truthmaker semantics as a framework for understanding a range of phenomena in philosophy and linguistics. Despite this interest, there has been limited study of the various logics that arise from the semantics. This paper aims to address this gap by exploring numerous 'truthmaker logics' and proving their compactness and decidability. This is in continuation with the inquiry of Fine and Jago (2019), who proved compactness and decidability for a particular kind of truthmaker logic.

The key results going into this are (1) 'standard translations' into first-order logic; (2) a truthmaker analogue of the finite model property; and (3) a proof showing that truthmaker consequence on semilattices coincides with truthmaker consequence on complete lattices.

Finally, the connection with modal logic is examined. Specifically, it is illustrated how endowing truthmaker semantics with classical negation results in modal information logics.

Keywords: Truthmaker semantics, finite model property, decidability, compactness, translations, modal logic, modal information logic

Precede: In this paper, we use margin notes for informal comments, typically providing intuition or recalling notation.

Here's a margin note.

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1 Introduction

Truthmaker semantics has gained increasing attention in recent years as a framework for analyzing philosophical and linguistic phenomena such as metaphysical grounding, counterfactuals and implicatures (Fine 2017c). Its characteristic feature is its modally-flavored semantics for conjunction '$\wedge$', where a state makes a conjunction true iff it is the fusion of states making the conjuncts true. Apart from this tenet, much of the truthmaker framework is open for interpretation. Depending on the phenomenon of concern and one's philosophical dispositions, one can vary the design, for instance: Are the *semantics* for disjunction inclusive or non-inclusive? Is it adequate that *frames* have all binary fusions (resulting in semilattices), or must they have all fusions simpliciter (resulting in complete lattices)? Need *valuations* be non-vacuous, or may they go to the powerset?

'Fusion' is synonymous with 'supremum' or 'join'.

Since each such design choice gives rise to a truthmaker logic, there is a variety of logics to explore. Unlike their applications, these have only received little attention so far, with only a few exceptions (notably Fine and Jago 2019 and Korbmacher 2022). In this paper, we seek to make up for this by conducting a model-theory-based study of these logics.[1] Our impetus for doing so extends beyond mere logical curiosity: numerous philosophical concepts find expression as consequence or equivalence within a truthmaker logic. For instance, according to Jago (2017), both samesaying of sentences and identity of propositions amount to truthmaker equivalence; and as studied by Fine (2017a,b), notions of ground and of containment can be captured by truthmaker consequence.[2]

Our contribution consists of, roughly speaking, proving and combining three results. The first two are (1) 'standard translations' into first-order logic and (2) a truthmaker analogue of the finite model property (FMP). Through (1), truthmaker logics inherit compactness and recursive enumerability from first-order logic. And combined with (2), we get decidability. Although (2) is general enough to get the FMP for 'all' truthmaker logics, the proof of (1) only holds for the truthmaker logics defined on semilattices. However, compactness and decidability of the remaining logics are then implications of our last main result: (3) truthmaker consequence on semilattices coincides with truthmaker consequence on complete lattices.

[1] This complements Fine and Jago (2019)'s and, especially, Korbmacher (2022)'s more proof-theoretical approach.

[2] Among more, on Fine's account, for propositions P and Q (of a certain form): (i) P weakly grounds Q iff P truthmaker entails Q; (ii) P weakly partially grounds Q iff $(P \wedge Q) \vee Q$ is truthmaker equivalent to Q; and (iii) P contains Q iff $P \wedge Q$ is truthmaker equivalent to P.

These proof techniques are all reminiscent of modal logic. This is no wonder since, as we spell out in the final section, a corollary of van Benthem (2019)'s translation is that augmenting the language of truthmaker semantics with classical negation yields a modal logic, namely a so-called modal information logic.

Our work is structured as follows: Section 2 presents the truthmaker framework and defines the truthmaker logics that it induces; Section 3 achieves compactness and recursive enumerability through standard translations; Section 4 develops and proves the FMP and concludes decidability; Section 5 shows that truthmaker consequence on semilattices coincides with truthmaker consequence on complete lattices; and Section 6 investigates how truthmaker logics connect with modal information logics through translations.

2 The truthmaker framework

We begin by formally laying out the truthmaker framework, its typical implementations and the resulting notions of entailment, the truthmaker logics.

Definition 2.1 (Language)**.** The language $\mathcal{L}_T$ of truthmaker semantics is defined using a countable set of proposition letters $\mathbf{P}$. The formulas $\varphi \in \mathcal{L}_T$ are then given by the BNF-grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi,$$

where $p \in \mathbf{P}$. ⊣

Implementations of the truthmaker framework vary from taking the class of frames to be semilattices (in, e.g., Fine and Jago 2019) to complete lattices (in, e.g., Fine 2017c). For the time being, we only consider the former, and return to other classes of frames in Section 5.

Definition 2.2 (Frames and models)**.** A *(semilattice) frame* for $\mathcal{L}_T$ is a pair $\mathbb{F} = (S, \leqslant)$ where

- S is a set; and
- $\leqslant$ is a semilattice on S, i.e., a partial order with all binary fusions.

That is, $\leqslant$ is reflexive, transitive, anti-symmetric and for all $s, s' \in S$, there is some $s'' \in S$ s.t. $s'' = \sup\{s, s'\}$. I.e., our semilattices are join-semilattices.

A *model* for $\mathcal{L}_T$ is a quadruple $\mathbb{M} = (S, \leqslant, V^+, V^-)$ where

- $(S, \leqslant)$ is a frame; and
- V^+ and V^- are valuations on S, i.e., functions $V^+, V^- : \mathbf{P} \to \mathcal{P}(S)$. ⊣

Remark 2.3 (Restrictions on the valuations)**.** Sometimes valuations do not simply go to the powerset, but have additional requirements, including:

- Closure under binary joins: if $\{s,t\} \subseteq V^{\pm}(p)$, then $\sup\{s,t\} \in V^{\pm}(p)$.
- Non-vacuity: $V^{+}(p) \neq \varnothing$ for all $p \in \mathbf{P}$ and/or $V^{-}(p) \neq \varnothing$ for all $p \in \mathbf{P}$. ⊣

Besides differing on the valuations, truthmaker logics can differ on the actual semantics. One version is the following:

Definition 2.4 (Semantics)**.** Given a model $\mathbb{M} = (S, \leqslant, V^{+}, V^{-})$ and a state $s \in S$, *truthmaking* and *falsitymaking* of a formula $\varphi \in \mathcal{L}_T$ at s in $\mathbb{M}$ (written $\mathbb{M}, s \Vdash^{+} \varphi$ and $\mathbb{M}, s \Vdash^{-} \varphi$, respectively) are defined by the following recursive clauses:

$$
\begin{array}{lll}
\mathbb{M}, s \Vdash^{+} p & \textbf{iff} & s \in V^{+}(p). \\
\mathbb{M}, s \Vdash^{-} p & \textbf{iff} & s \in V^{-}(p). \\
\mathbb{M}, s \Vdash^{+} \neg\varphi & \textbf{iff} & \mathbb{M}, s \Vdash^{-} \varphi. \\
\mathbb{M}, s \Vdash^{-} \neg\varphi & \textbf{iff} & \mathbb{M}, s \Vdash^{+} \varphi. \\
\mathbb{M}, s \Vdash^{+} \varphi \wedge \psi & \textbf{iff} & \text{there exist } t, u \in S \text{ such that } \mathbb{M}, t \Vdash^{+} \varphi,\ \mathbb{M}, u \Vdash^{+} \psi, \\
 & & \hfill \text{and } s = \sup\{t, u\}. \\
\mathbb{M}, s \Vdash^{-} \varphi \wedge \psi & \textbf{iff} & \mathbb{M}, s \Vdash^{-} \varphi \ \textit{ or } \ \mathbb{M}, s \Vdash^{-} \psi. \\
\mathbb{M}, s \Vdash^{+} \varphi \vee \psi & \textbf{iff} & \mathbb{M}, s \Vdash^{+} \varphi \ \textit{ or } \ \mathbb{M}, s \Vdash^{+} \psi. \\
\mathbb{M}, s \Vdash^{-} \varphi \vee \psi & \textbf{iff} & \text{there exist } t, u \in S \text{ such that } \mathbb{M}, t \Vdash^{-} \varphi,\ \mathbb{M}, u \Vdash^{-} \psi, \\
 & & \hfill \text{and } s = \sup\{t, u\}. \quad \dashv
\end{array}
$$

Remark 2.5 (Alternative disjunction semantics)**.** These semantics are 'non-inclusive'. A common alteration is 'inclusive' semantics where (1) $\mathbb{M}, s \Vdash^{+} \varphi \wedge \psi$ also suffices for $\mathbb{M}, s \Vdash^{+} \varphi \vee \psi$, and, analogously, (2) $\mathbb{M}, s \Vdash^{-} \varphi \vee \psi$ also suffices for $\mathbb{M}, s \Vdash^{-} \varphi \wedge \psi$.[3] ⊣

Remark 2.6 (Convex truthmaking[4])**.** The presented semantics allow for *non-convex* truth- and falsitymaking: For any choice of admissible valuations and semantics, one

[3] Another option would be semantics where the disjunction is defined in terms of infimum—mirroring how conjunction is defined in terms of supremum. Under this definition, it might be natural to require the frame to not only be a join-semilattice but also a meet-semilattice, or in other words: a lattice.

[4] Special thanks to a reviewer for suggesting the inclusion of 'convex truthmaking'. Convex, inclusive semantics with non-vacuous valuations closed under binary joins – i.e., so-called 'replete' semantics – is discussed in, among more, Fine and Jago (2019) and Korbmacher (2022) and, on complete lattices, has been shown to be complete for Angell's logic for Analytic Containment (see Angell 1977, 1989) in Fine (2016).

can find models $\mathbb{M}$ with states $r \leqslant s \leqslant t$ and formulas φ such that $\mathbb{M}, r \Vdash^{\pm} \varphi$ and $\mathbb{M}, t \Vdash^{\pm} \varphi$, but $\mathbb{M}, s \nVdash^{\pm} \varphi$. To avoid this, one can work with the *convex closures* of the sets of truthmakers and falsitymakers instead: Given a model $\mathbb{M} = (S, \leqslant, V^{+}, V^{-})$ and a state $s \in S$, *convex truth-* and *falsitymaking* of a formula $\varphi \in \mathcal{L}_{T}$ at s in $\mathbb{M}$ (written $\mathbb{M}, s \Vdash^{+,c} \varphi$ and $\mathbb{M}, s \Vdash^{-,c} \varphi$, respectively) are defined as follows:

$$\mathbb{M}, s \Vdash^{\pm,c} \varphi \quad \textbf{iff} \quad \text{there exist } r, t \in S \text{ such that } \mathbb{M}, r \Vdash^{\pm} \varphi,\ \mathbb{M}, t \Vdash^{\pm} \varphi, \text{ and } r \leqslant s \leqslant t.$$ [5]

When considering convex truthmaking, we will typically continue with the notation '$\Vdash^{\pm}$' without the superscript '^{c}', employing the superscript only when needed for clarity. ⊣

Definition 2.7 (Truthmaker consequence)**.** *Entailment* $\Gamma \Vdash^{+} \varphi$ for $\Gamma \subseteq \mathcal{L}_{T} \ni \varphi$ is defined distributively. That is, $\Gamma \Vdash^{+} \varphi$ iff whenever $\mathbb{M}, s \Vdash^{+} \gamma$ for all $\gamma \in \Gamma$, it is also the case that $\mathbb{M}, s \Vdash^{+} \varphi$.[6] ⊣

While this paper isn't concerned with the laws of truthmaker logics (the paper's approach to the metalogic of truthmaking is model-theoretical rather than proof-theoretical), we believe it is valuable to briefly mention a few unusual characteristics of our logics in question:

- *No conjunction elimination:* Although $\varphi, \psi \Vdash^{+} \varphi \wedge \psi$, we do not have, for instance, $p \wedge q \Vdash^{+} p$ (nor $p \wedge q \Vdash^{+} q$) for any truthmaker logic (whether it be convex or not).
- *No absorption:* In all truthmaker logics: $p \vee (p \wedge q) \nVdash^{+} p$ and $p \wedge (p \vee q) \nVdash^{+} p$.
- *'Not' distributive:* For all but convex, inclusive, non-vacuous truthmaking: $(p \vee q) \wedge (p \vee r) \nVdash^{+} p \vee (q \wedge r)$.

This also highlights that the logics at hand are not intended as logics for valid reasoning. Instead, as mentioned in the introduction, they are more suited for

[5]Caution: Our formalization of 'convex truthmaking', as expressed by '$\Vdash^{+,c}$', differs from the definition of '$\Vdash_{cvx}$' found in Fine and Jago (2019). There are different methods for enforcing convexity while yielding the same consequence relation. We have opted for $\Vdash^{\pm,c}$, as it makes for a clearer presentation of our results.

[6]Note that decidability of the distributive entailment has as a special case decidability of the collective entailment, which is where $\Gamma \Vdash^{+} \varphi$ holds iff whenever $\mathbb{M}, s \Vdash^{+} \bigwedge_{\gamma \in \Gamma} \gamma$, $\mathbb{M}, s \Vdash^{+} \varphi$ (this is, of course, only defined for finite Γ [which suffices for decidability], but the definition can be extended to the infinite case).

expressing various philosophical concepts, such as samesaying of sentences, identity of propositions, and notions of ground.

3 Compactness and recursive enumerability

With the definitions of the previous section laid out, we are in a position to present the compactness and decidability results proved by Fine and Jago (2019).

Theorem 3.1 (Compactness (Fine and Jago 2019))**.** *The truthmaker logic of semilattices with valuations closed under binary joins and inclusive semantics is compact; that is,* ***if*** $\Gamma \Vdash^{+} \varphi$, ***then*** $\Gamma_F \Vdash^{+} \varphi$ *for some finite* $\Gamma_F \subseteq \Gamma$.

Theorem 3.2 (Decidability (Fine and Jago 2019))**.** *The truthmaker logic of semilattices with valuations closed under binary joins and inclusive semantics is decidable; that is, for a finite set of formulas* Γ_F*, it is decidable whether* $\Gamma_F \Vdash^{+} \varphi$.[7]

In what follows, we provide an alternative approach to obtaining these results by means of translations and a proof of the finite model property. This method generalizes to prove compactness and decidability of several truthmaker logics. We begin with compactness and restrict our attention to semilattices as our class of frames, deferring consideration of other classes to Section 5.

Essentially, the idea is the following: Since being a semilattice is first-order definable and the truth- and falsitymaking clauses of truthmaker semantics are as well, we get standard translations into first-order logic (FOL), and then compactness of FOL implies compactness of the truthmaker logics.

Spelt out a bit more, the key things are:

(a) The translations employ a double recursion trick [similar to the one van Benthem (2019) uses to translate truthmaker logics into modal information logics, cf. Section 6] to reduce two consequence relations (truth- and falsitymaking) to one consequence relation.

(b) Unlike truthmaker semantics, the target semantics can speak about 'not truthmaking' (hence, also 'not falsitymaking') through regular first-order negation.

(c) Everything is first-order definable.

[7]It is worth noting that the proofs of these two theorems given in Fine and Jago (2019) somewhat generalize to other truthmaker logics and are of independent interest.

We now present the translation into FOL, which can be thought of as the standard translation for truthmaker logics. Alternatively, it can be viewed as the composition of the translation given in van Benthem (2019) from truthmaker logics into modal information logics with a standard translation of modal information logics into FOL.

Definition 3.3. The target first-order language is with equality, contains a binary relation symbol '$\leqslant$', and two unary predicate symbols 'P^T', 'P^F' for each propositional letter $p \in \mathbf{P}$. The translation is then given by these double recursive clauses:

$$
\begin{array}{rcl}
ST_x^+(p) & = & P^T x \\
ST_x^-(p) & = & P^F x \\
ST_x^+(\neg\varphi) & = & ST_x^-(\varphi) \\
ST_x^-(\neg\varphi) & = & ST_x^+(\varphi) \\
ST_x^+(\varphi \wedge \psi) & = & \exists y, z\big(x = \sup\{y, z\} \wedge ST_y^+(\varphi) \wedge ST_z^+(\psi)\big) \\
ST_x^-(\varphi \wedge \psi) & = & ST_x^-(\varphi) \vee ST_x^-(\psi) \\
ST_x^+(\varphi \vee \psi) & = & ST_x^+(\varphi) \vee ST_x^+(\psi) \\
ST_x^-(\varphi \vee \psi) & = & \exists y, z\big(x = \sup\{y, z\} \wedge ST_y^-(\varphi) \wedge ST_z^-(\psi)\big)
\end{array}
$$

where $x = \sup\{y, z\}$ is short for $y \leqslant x \wedge z \leqslant x \wedge \forall u([y \leqslant u \wedge z \leqslant u] \rightarrow x \leqslant u)$. $\dashv$

Examining the translation, the succeeding proposition is almost self-explanatory (see Blackburn, Rijke, and Venema (2001, ch. 2) for similar results in the setting of modal logics).

Proposition 3.4 (Correspondence)**.** *For all models $\mathbb{M}$, all states $s \in \mathbb{M}$ and all $\varphi \in \mathcal{L}_T$:*

$$
\begin{array}{llll}
(i) & \mathbb{M}, s \Vdash^+ \varphi & \textbf{\textit{iff}} & \mathbb{M} \vDash ST_x^+(\varphi)[s];\ \textit{and} \\
(ii) & \mathbb{M}, s \Vdash^- \varphi & \textbf{\textit{iff}} & \mathbb{M} \vDash ST_x^-(\varphi)[s].
\end{array}
$$

To be clear, on the right-hand side of the 'iff's, strictly speaking, '$\mathbb{M}$' refers to the corresponding first-order definition of the truthmaker model $\mathbb{M}$.

Definition 3.5. Let J be the first-order formula defining being a (join-)semilattice; i.e., J is the conjunction of the formulas for reflexivity, transitivity, anti-symmetry, and having all binary joins. $\dashv$

Theorem 3.6 (Compactness)**.** *All truthmaker logics of Section 2 are compact.*

Proof. First, we give the compactness proof for the truthmaker logic of semilattices with valuations going to the powerset and the non-inclusive semantics of Definition 2.4. Second, we outline how the proof is modified to apply to other truthmaker logics.

Let $(\Gamma \cup \{\varphi\}) \subseteq \mathcal{L}_T$ be arbitrary, and set $ST_x^+(\Gamma) := \{ST_x^+(\gamma) \mid \gamma \in \Gamma\}$. Then

$$\begin{array}{lclcl} \Gamma \Vdash^+ \varphi & \overset{(i)}{\textbf{iff}} & ST_x^+(\Gamma) \cup \{J\} \vDash ST_x^+(\varphi) & & \\ & \overset{(c)}{\textbf{iff}} & ST_x^+(\Gamma_F) \cup \{J\} \vDash ST_x^+(\varphi) & \overset{(i)}{\textbf{iff}} & \Gamma_F \Vdash^+ \varphi \end{array}$$

where Γ_F is a finite subset of Γ obtained via compactness of FOL in the step *(c)*, and *(i)* follows from the first assertion of the above-stated proposition. This shows compactness.

The reasons this proof lifts to all of the truthmaker variants of Section 2 are:

(Sem) The various sorts of semantics for the connectives all admit a standard translation so that Proposition 3.4 holds. Additionally, in the context of convex truthmaking, we use a 'convex translation' instead:

$$C_x^\pm(\varphi) := \exists y, z \left(y \leqslant x \leqslant z \wedge ST_y^\pm(\varphi) \wedge ST_z^\pm(\varphi) \right).$$

(Val) For any propositional variable $p \in \mathbf{P}$, all of the listed potential conditions on its valuation (from Remark 2.3) can be defined by a first-order formula. Let V_p denote such a formula. Then, for the proof to go through, it is simply a matter of changing the first-order premise '$\{J\}$' to '$\{J\} \cup \{V_p \mid p \in \mathbf{P}\}$' in the just-proven sequence of 'iff's. □

Now for decidability of the truthmaker logics. First off, we observe that we achieve recursive enumerability (r.e.) through our standard translations connecting truthmaker logics to first-order logic.

Proposition 3.7 (Recursive enumerability)**.** *All truthmaker logics of Section 2 are recursively enumerable; that is, there is an effective procedure for enumerating the pairs* (Γ_F, φ) *s.t.* $\Gamma_F \Vdash^+ \varphi$ *for finite* Γ_F.

Proof. Once again, we begin by covering the case of our set out truthmaker logic, before explaining how the proof generalizes to the other truthmaker logics of Section 2.

For this, simply observe that for any (Γ_F, φ), we have that

$$\Gamma_F \Vdash^+ \varphi \quad \textbf{iff} \quad ST^+_x(\Gamma_F) \cup \{J\} \vDash ST^+_x(\varphi)$$
$$\overset{(ii)}{\textbf{iff}} \quad \vDash \bigwedge(ST^+_x(\Gamma_F) \cup \{J\}) \to ST^+_x(\varphi),$$

where (ii) follows by there being finitely many premises and the first-order semantics for conjunction and implication. In other words, this is the deduction theorem of first-order logic in a semantic disguise.

Since first-order logic is r.e. (and this procedure of constructing the formula $\bigwedge(ST^+_x(\Gamma_F) \cup \{J\}) \to ST^+_x(\varphi)$ from a pair (Γ_F, φ), evidently, is effective), we have attained r.e.

Albeit the arguments of (Sem) still go through to account for why this proof generalizes to other truthmaker logics, the argument given in (Val) pertaining to potential requirements on valuations does not generalize straight away. The problem is that the set $\{V_p \mid p \in \mathbf{P}\}$ is infinite. Fortunately, we can restrict this set to the propositional variables occurring in $\Gamma_F \cup \{\varphi\}$, thus obtaining a finite set of formulas instead. This is adequate for the proof to apply to truthmaker logics with restrictions on the admissible valuations. □

4 FMP and co-r.e.

To establish decidability, it remains to prove co-r.e.: viz., giving an effective procedure for enumerating the pairs (Γ_F, φ) s.t. $\Gamma_F \nVdash^+ \varphi$ for finite Γ_F. This will be our main concern in this section.

In many a logic, not least in modal logic, the most common way of establishing co-r.e. is by means of proving the FMP. Mirroring this, we develop and prove what, arguably, is the truthmaker analogue of the FMP.

Before doing so, notice that a direct analogue of the FMP, namely that whenever a formula is made true (resp. false) [or truth-refuted (resp. falsity-refuted)], it is made true (resp. false) [or truth-refuted (resp. falsity-refuted)] in a finite model, is trivial and unhelpful for the purpose at hand: the single-state model making true and false all propositional letters [or none at all], makes true and false all formulas [or none at all] in general. And, importantly, this does nothing for proving co-r.e. Instead, we must prove an 'FMP' that—just like the FMP of, e.g., modal logic—allows for a model-theoretical proof of co-r.e. via some sort of finite-model checking. To do so, we need two preparatory lemmas and a definition.

Lemma 4.1. *Suppose* $\mathbb{M}_0 = (S_0, \leqslant_0, V_0^+, V_0^-)$ *and* $\mathbb{M}_1 = (S_1, \leqslant_1, V_1^+, V_1^-)$ *are models s.t. (i)* $(S_1, \leqslant_1)$ *is a sub-semilattice of* $(S_0, \leqslant_0)$*, and (ii) for all* $p \in \mathbf{P}$*:*

$$V_1^+(p) = V_0^+(p) \cap S_1, \qquad V_1^-(p) = V_0^-(p) \cap S_1.$$

Then for all formulas $\varphi \in \mathcal{L}_T$ *and all states* $s_1 \in S_1$*, we have that*

$$\mathbb{M}_0, s_1 \not\Vdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s_1 \not\Vdash^+ \varphi$$

and

$$\mathbb{M}_0, s_1 \not\Vdash^- \varphi \quad \Rightarrow \quad \mathbb{M}_1, s_1 \not\Vdash^- \varphi.$$

Proof. By induction on $\varphi \in \mathcal{L}_T$. Base cases are by definition and the inductive steps follow by use of the IH and $(S_1, \leqslant_1)$ being a sub-semilattice of $(S_0, \leqslant_0)$. To illuminate, we cover the inductive case of not truthmaking $\varphi = \psi \wedge \chi$. So suppose

$$\mathbb{M}_0, s_1 \not\Vdash^+ \psi \wedge \chi,$$

and let $(t_1, u_1) \in S_1 \times S_1$ be arbitrary s.t. $s_1 = \sup_{\leqslant_1}\{t_1, u_1\}$. Since $(S_1, \leqslant_1)$ is a sub-semilattice of $(S_0, \leqslant_0)$, the inclusion mapping $\iota : S_1 \hookrightarrow S_0$ is a semilattice homomorphism, hence $s_1 = \sup_{\leqslant_0}\{t_1, u_1\}$. But then since $\mathbb{M}_0, s_1 \not\Vdash^+ \psi \wedge \chi$, we must have that

$$\mathbb{M}_0, t_1 \not\Vdash^+ \psi \qquad \textit{or} \qquad \mathbb{M}_0, u_1 \not\Vdash^+ \chi,$$

whence, by the IH,

$$\mathbb{M}_1, t_1 \not\Vdash^+ \psi \qquad \textit{or} \qquad \mathbb{M}_1, u_1 \not\Vdash^+ \chi,$$

which suffices for the claim since (t_1, u_1) was arbitrary. □

Observe that this proof goes through for all truthmaker variants of Section 2; the convex case $\Vdash^{\pm,c}$ is a corollary of the corresponding $\Vdash^{\pm}$, as semilattice homomorphisms, in particular, are order-preserving.[8]

Definition 4.2. For any model $\mathbb{M}$, state $s \in \mathbb{M}$ and formula $\gamma \in \mathcal{L}_T$ s.t. $\mathbb{M}, s \Vdash^+ \gamma$ (resp. $\mathbb{M}, s \Vdash^- \gamma$), we define a set $T(\gamma, s)$ (resp. $F(\gamma, s)$), which we denote a *T-selection*

[8] If one also deals with infima and, e.g., requires the underlying frames to be lattices, one shall assume $(S_1, \leqslant_1)$ to be a sub*lattice* of $(S_0, \leqslant_0)$.

w.r.t. (γ, s) (resp. *F-selection*), by the following recursive clauses:

The intuition for $T(\gamma, s)$ (resp. $F(\gamma, s)$) is that it is a set of states by virtue of which $s \Vdash^+ \gamma$ (resp. $s \Vdash^- \gamma$).

$$
\begin{array}{lll}
T(\gamma, s) = \{s\} & \textbf{iff} & \gamma = p. \\
F(\gamma, s) = \{s\} & \textbf{iff} & \gamma = p. \\
T(\gamma, s) = \{s\} \cup F(\varphi, s) & \textbf{iff} & \gamma = \neg\varphi. \\
F(\gamma, s) = \{s\} \cup T(\varphi, s) & \textbf{iff} & \gamma = \neg\varphi. \\
T(\gamma, s) = \{s\} \cup T(\varphi, t) \cup T(\psi, u) & \textbf{iff} & \gamma = \varphi \wedge \psi \text{ and } \mathbb{M}, t \Vdash^+ \varphi, \\
 & & \quad \mathbb{M}, u \Vdash^+ \psi, s = \sup\{t, u\}. \\
F(\gamma, s) = \begin{cases} \{s\} \cup F(\varphi, s), & \text{if } \mathbb{M}, s \Vdash^- \varphi \\ \{s\} \cup F(\psi, s), & \text{otherwise} \end{cases} & \textbf{iff} & \gamma = \varphi \wedge \psi. \\
T(\gamma, s) = \begin{cases} \{s\} \cup T(\varphi, s), & \text{if } \mathbb{M}, s \Vdash^+ \varphi \\ \{s\} \cup T(\psi, s), & \text{otherwise} \end{cases} & \textbf{iff} & \gamma = \varphi \vee \psi. \\
F(\gamma, s) = \{s\} \cup F(\varphi, t) \cup F(\psi, u) & \textbf{iff} & \gamma = \varphi \vee \psi \text{ and } \mathbb{M}, t \Vdash^- \varphi, \\
 & & \quad \mathbb{M}, u \Vdash^- \psi, s = \sup\{t, u\}.
\end{array}
$$

Clearly, this need not define unique sets because, e.g., the truthmaking case of $\gamma = \varphi \wedge \psi$ might be satisfied by multiple choices of t, u; for the purpose of what we are to prove, this is irrelevant: any choice will do, so no reason to complicate the definition.[9] $\dashv$

Lemma 4.3. *For any model $\mathbb{M}$, state $s \in \mathbb{M}$ and formula $\gamma \in \mathcal{L}_T$ s.t. $\mathbb{M}, s \Vdash^+ \gamma$ (resp. $\mathbb{M}, s \Vdash^- \gamma$), the corresponding set $T(\gamma, s)$ (resp. $F(\gamma, s)$) contains $\{s\}$ and is finite.*

Proof. By induction on $\gamma \in \mathcal{L}_T$. □

With these results at hand, we can prove our truthmaker analogue of the FMP.

Proposition 4.4 (Truthmaker FMP). *For any model $\mathbb{M}_0 = (S_0, \leqslant_0, V_0^+, V_0^-)$, state $s \in S_0$, and finite set of formulas $\Gamma_F \subseteq \mathcal{L}_T$ s.t.*

$$\mathbb{M}_0, s \Vdash^+ \Gamma_F,$$

there is a finite submodel $\mathbb{M}_1$ s.t. (a)

$$\mathbb{M}_1, s \Vdash^+ \Gamma_F,$$

[9] For other semantics (e.g. inclusive), we modify this definition in the obvious way.

and (b) for all $\varphi \in \mathcal{L}_T$:

$$\mathbb{M}_0, s \nVdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s \nVdash^+ \varphi.$$

Notice that this proposition implies the analogous proposition stated in terms of falsitymaking, qua negating all formulas.

Proof. For each $\gamma \in \Gamma_F$, choose a set $T(\gamma, s)$ according to the previous definition, and let $(S_1, \leqslant_1)$ be the sub-semilattice generated by $\bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$. Since Γ_F is finite, the set of generators $\bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ is finite by the preceding lemma, hence S_1 is finite.

Further, as in Lemma 4.1, define V_1^+ and V_1^- as the restrictions of V_0^+ and V_0^-, respectively. Then $\mathbb{M}_1 := (S_1, \leqslant_1, V_1^+, V_1^-)$ is a model.[10] By Lemma 4.1, we have that (b) for all $\varphi \in \mathcal{L}_T$,

$$\mathbb{M}_0, s \nVdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s \nVdash^+ \varphi.$$

It remains to show (a)

$$\mathbb{M}_1, s \Vdash^+ \Gamma_F.$$

To do so, we prove that for all formulas $\varphi \in \mathcal{L}_T$ and all generator states $s' \in \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$: if there is some T-selection $T(\varphi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ (resp. F-selection $F(\varphi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$), then

$$\begin{array}{ccc} \mathbb{M}_0, s' \Vdash^+ \varphi & \Rightarrow & \mathbb{M}_1, s' \Vdash^+ \varphi \\ (\text{resp. } \mathbb{M}_0, s' \Vdash^- \varphi & \Rightarrow & \mathbb{M}_1, s' \Vdash^- \varphi). \end{array}$$

The proof is by structural induction on φ. The base cases follow by definition of V_1^+ and V_1^-. Among the inductive steps, we cover the case of truthmaking $\varphi = \psi \wedge \chi$.

Accordingly, suppose $s' \in \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ and there is some T-selection $T(\psi \wedge \chi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$. By definition of a T-selection, there must be some $\{t', u'\} \subseteq S_0$ s.t. (i) $\mathbb{M}_0, t' \Vdash^+ \psi$; (ii) $\mathbb{M}_0, u' \Vdash^+ \chi$; (iii) $s' = \sup_{\leqslant_0}\{t', u'\}$; and (iv)

$$T(\psi \wedge \chi, s') = \{s'\} \cup T(\psi, t') \cup T(\chi, u').$$

Cf. the preceding lemma and (iv), we have that

$$t' \in T(\psi, t') \subseteq T(\psi \wedge \chi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s) \supseteq T(\psi \wedge \chi, s') \supseteq T(\chi, u') \ni u',$$

hence from (i), (ii) and the IH, we get that

$$\mathbb{M}_1, t' \Vdash^+ \psi \qquad \textit{and} \qquad \mathbb{M}_1, u' \Vdash^+ \chi.$$

[10]In case we require, e.g., all $V^+(p) \neq \varnothing$, the proof goes through by simply adding a state $s_{p^+} \in V_0^+(p)$ to the set of generators for all propositional letters occurring in the formulas $\Gamma_F \cup \{\varphi\}$.

Moreover, because of (iii) and the fact that $(S_1, \leqslant_1)$ is a sub-semilattice of $(S_0, \leqslant_0)$, we have that $s' = \sup_{\leqslant_1}\{t', u'\}$. Consequently, $\mathbb{M}_1, s' \Vdash \psi \wedge \chi$ – exactly as we wanted. This completes the induction, which entails (a) as a special case and, therefore, finishes the proof. $\square$

Again, using the canonical modifications, this proof works for all truthmaker logics mentioned so far. For instance, in the convex case, the modification is simply to let $(S_1, \leqslant_1)$ be the sub-semilattice generated by

$$\{s\} \cup \bigcup_{\gamma \in \Gamma_F} \left[\mathsf{T}(\gamma, r_\gamma) \cup \mathsf{T}(\gamma, t_\gamma)\right],$$

where r_γ, t_γ are witnesses of $\mathbb{M}_0, s \Vdash^{+,c} \gamma$. In other words, $\mathbb{M}_0, r_\gamma \Vdash^{+} \gamma$; $\mathbb{M}_0, t_\gamma \Vdash^{+} \gamma$; and $r_\gamma \leqslant s \leqslant t_\gamma$.

With this proven, we got co-r.e., hence also decidability, in our pocket.

Theorem 4.5 (Decidability)**.** *All truthmaker logics of Section 2 are decidable.*

Proof. Since we have proven r.e. in Proposition 3.7, it suffices to provide an effective procedure for the case of $\Gamma_F \nVdash^{+} \varphi$ for finite Γ_F. This is done as follows:

1. Enumerate all finite semilattices.
2. For each such finite semilattice, check the finitely many valuations ($\Gamma_F \cup \{\varphi\}$ is finite, so only finitely many proposition letters occur in $\Gamma_F \cup \{\varphi\}$) and states for whether we witness $\Gamma_F \nVdash^{+} \varphi$.

This suffices because, by the preceding proposition, we have that *if* $\mathbb{M}_0, s \Vdash^{+} \Gamma_F$ and $\mathbb{M}_0, s \nVdash^{+} \varphi$, *then* there is a finite model $\mathbb{M}_1$ s.t. $\mathbb{M}_1, s \Vdash^{+} \Gamma_F$ and $\mathbb{M}_1, s \nVdash^{+} \varphi$. $\square$

5 Second-order frames

As promised, we now consider implementations of the truthmaker framework that do not take the class of frames to be semilattices. Most common perhaps is to assume the existence of all, or all non-empty, fusions. As a slogan, we show that truthmaker consequence is invariant for choice of frames.

In fact, it is already a corollary of the Truthmaker FMP that for finite Γ_F, we have that $\Gamma_F \Vdash^{+} \varphi$ on semilattices iff $\Gamma_F \Vdash^{+} \varphi$ on posets with all non-empty joins. Had we known that truthmaker consequence on posets with all non-empty joins was

Left-to-right is immediate, while right-to-left is by contraposition using the Truthmaker FMP and the fact that finite semilattices have all non-empty joins.

compact, this would generalize to arbitrary Γ_F. This brings us to a limitation of the standard-translation proof method: it only applies when conditions are first-order definable. In particular, we do not obtain compactness of truthmaker logics where the frames are taken to be posets with all (non-empty) joins, since this is not first-order definable.

Instead, we go about this by 'completing' any semilattice into a complete lattice in a satisfaction-preserving (and -reflecting) way. We begin by fixing some notation and formally stating the theorem warranting the abovementioned slogan.

Definition 5.1. We write:

$$\mathcal{S}_1 := \{(S, \leqslant) \mid (S, \leqslant) \textit{ is a semilattice}\},$$
$$\mathcal{S}_2 := \{(S, \leqslant) \mid (S, \leqslant) \textit{ is a semilattice with a bottom element}\},$$
$$\mathcal{C}_1 := \{(S, \leqslant) \mid (S, \leqslant) \textit{ is a poset with all non-empty joins}\},$$
$$\mathcal{C}_2 := \{(S, \leqslant) \mid (S, \leqslant) \textit{ is a poset with all joins}\}.$$

A poset $(S, \leqslant)$ has a bottom element :iff there is some $s \in S$ s.t. $s \leqslant t$ for all $t \in S$. So $\mathcal{C}_2$ is the class of complete lattices, or alternatively, $\mathcal{C}_2$ is the restriction of $\mathcal{C}_1$ to all and only its members with a bottom element.

Furthermore, given a choice of semantics and valuations (from Section 2), we write $\Vdash^+_X$ for the induced consequence relation on $X \in \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{C}_1, \mathcal{C}_2\}$. ⊣

Theorem 5.2 (Entailment Invariance for Choice of Frames)**.** *Given any choice of semantics and valuations (from Section 2), any $\Gamma \subseteq \mathcal{L}_T \ni \varphi$, and any $X, Y \in \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{C}_1, \mathcal{C}_2\}$,*

$$\Gamma \Vdash^+_X \varphi \qquad \textbf{\textit{iff}} \qquad \Gamma \Vdash^+_Y \varphi.$$ [11]

Proof. The following chains of implications follow by containment of the classes of frames:

$$\Gamma \Vdash^+_{\mathcal{S}_1} \varphi \quad \Rightarrow \quad \Gamma \Vdash^+_{\mathcal{S}_2} \varphi \quad \Rightarrow \quad \Gamma \Vdash^+_{\mathcal{C}_2} \varphi$$

and

$$\Gamma \Vdash^+_{\mathcal{S}_1} \varphi \quad \Rightarrow \quad \Gamma \Vdash^+_{\mathcal{C}_1} \varphi \quad \Rightarrow \quad \Gamma \Vdash^+_{\mathcal{C}_2} \varphi.$$

Therefore, it suffices to show that

$$\Gamma \Vdash^+_{\mathcal{S}_1} \varphi \quad \Leftarrow \quad \Gamma \Vdash^+_{\mathcal{C}_2} \varphi.$$

[11] Once more, while our notation is $\Vdash^+_X$, it will become apparent that this theorem likewise applies to the convex consequence relations $\Vdash^{+,c}_X$.

This is a consequence of the Completion Lemma below (5.9), which shows how to complete a semilattice into a complete lattice in a satisfaction-preserving and -reflecting way. $\square$

Corollary 5.3. *For any choice of semantics and valuations (from Section 2) and any class of frames* $X \in \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{C}_1, \mathcal{C}_2\}$*, the corresponding truthmaker logic is compact and decidable.*

In order to complete the proof of the entailment invariance under frame choice, we need to establish the Completion Lemma. This requires us to set up a few things first. We begin by defining the notion of an 'upset'.

Definition 5.4. Given a poset $(S, \leqslant)$, a subset $T \subseteq S$ is called an *upset* :iff

$$\forall s, t \in S[(t \in T \wedge t \leqslant s) \Rightarrow s \in T].$$

Moreover, for a point $s \in S$, we denote its upset as $\uparrow s := \{s' \in S \mid s \leqslant s'\}$. $\dashv$

The ensuing is then easily seen to hold.

Lemma 5.5. *Let* $(S, \leqslant)$ *be a semilattice and* $\mathcal{U}(S) \subseteq \mathcal{P}(S)$ *its collection of upsets. Then (i)* $(\mathcal{U}(S), \supseteq)$ *forms a complete lattice, and (ii) for all* $s, t, u \in S$*:*

I.e., we have dualized and taken '$\cap$' as our join and '$\cup$' as our meet.

$$s = \sup_{\leqslant}\{t, u\} \qquad \textbf{\textit{iff}} \qquad \uparrow s = \uparrow t \cap \uparrow u.$$

To facilitate the succeeding proof of Lemma 5.8, we review the notion of 'negation normal form'.

Definition 5.6 (Negation normal form)**.** The set $\mathcal{N} \subseteq \mathcal{L}_T$ of formulas in *negation normal form* is defined by:

- Literals are in negation normal form; i.e., $\mathbf{P} \subseteq \mathcal{N}$ and $\{\neg p \mid p \in \mathbf{P}\} \subseteq \mathcal{N}$.
- Disjunctions of formulas in negation normal form are in negation normal form; i.e., if $\{\varphi, \psi\} \subseteq \mathcal{N}$, then $\varphi \vee \psi \in \mathcal{N}$.
- Conjunctions of formulas in negation normal form are in negation normal form; i.e., if $\{\varphi, \psi\} \subseteq \mathcal{N}$, then $\varphi \wedge \psi \in \mathcal{N}$.
- Nothing else is in negation normal form. $\dashv$

Lemma 5.7. *For all formulas* $\varphi \in \mathcal{L}_T$*, there is some formula* $\varphi' \in \mathcal{N}$ *in negation normal form which is equivalent to* φ*.*

I.e., $M, s \Vdash^{\pm} \varphi$ *iff* $M, s \Vdash^{\pm} \varphi'$.

Proof. Follows from de Morgan's and double negation elimination (which holds for both inclusive and non-inclusive semantics). □

Lemma 5.8. *For all formulas $\varphi \in \mathcal{L}_T$ and $\mathbb{M}, s$ s.t. $\mathbb{M}, s \Vdash^+ \varphi$, there are literals $l_1, \dots l_n$ s.t.*

1. $(l_1 \wedge \cdots \wedge l_n) \Vdash^+_{\mathcal{S}_1} \varphi$,
2. $\mathbb{M}, s \Vdash^+ (l_1 \wedge \cdots \wedge l_n)$.[12]

Proof. Cf. the preceding lemma, we may assume that φ is in negation normal form. The proof then goes through via an easy induction (both for inclusive and non-inclusive semantics), noting that if $\chi \Vdash^+_{\mathcal{S}_1} \psi$ and $\chi' \Vdash^+_{\mathcal{S}_1} \psi'$, then $(\chi \wedge \chi') \Vdash^+_{\mathcal{S}_1} (\psi \wedge \psi')$. □

Having established these preliminary results, we can now proceed finalizing the proof of Theorem 5.2 via the following key lemma:

Lemma 5.9 (Completion Lemma)**.** *Let $\mathbb{M} = (S, \leqslant, V^+, V^-)$ be a semilattice model. Then for all $\varphi \in \mathcal{L}_T$ and all $s \in S$,*

$$\left(\mathcal{U}(S), \supseteq, V'^+, V'^-\right), \uparrow s \Vdash^+ \varphi \qquad \textit{\textbf{iff}} \qquad \mathbb{M}, s \Vdash^+ \varphi,$$

where $V'^{\pm}$ are defined by setting

$$V'^{\pm}(p) := \{\uparrow s \mid s \in V^{\pm}(p)\}.$$

Proof. First note that $\mathbb{M}' := (\mathcal{U}(S), \supseteq, V'^+, V'^-)$ is a complete-lattice model for the same choice of semantics and valuations for which $\mathbb{M}$ was a semilattice model.

Cf. Lemma 5.5(ii), $s \mapsto \uparrow s$ defines a semilattice embedding, hence Lemma 4.1 gives us the direction from right to left.

For the direction from left to right, suppose that

$$\mathbb{M}', \uparrow s \Vdash^+ \varphi.$$

By the preceding lemma, there are literals $l_1, \dots, l_n$ s.t.

1. $l_1 \wedge \cdots \wedge l_n \Vdash^+_{\mathcal{S}_1} \varphi$

[12]This lemma corresponds to a weaker version of Lemma 3.5 in Fine and Jago (2019), which describes all such conjunctions of literals corresponding to a formula φ.

2. $\mathbb{M}', {\uparrow}s \Vdash^+ l_1 \wedge \cdots \wedge l_n$.

From 2., we then get that there are upsets X_i s.t. $X_i \Vdash^+ l_i$ for all $i \in \{1, \ldots, n\}$ and

$$\bigcap_{i=1}^{n} X_i = {\uparrow}s.$$

But by the definition of the valuations $V'^{\pm}$, we have that $X_i = {\uparrow}x_i$ for some $x_i \in S$ s.t. $\mathbb{M}, x_i \Vdash^+ l_i$. Thus,

$$\bigcap_{i=1}^{n} {\uparrow}x_i = {\uparrow}s,$$

so, using Lemma 5.5(ii), we get that

$$\mathbb{M}, s \Vdash^+ l_1 \wedge \cdots \wedge l_n.$$

By 1., this implies that

$$\mathbb{M}, s \Vdash^+ \varphi,$$

which completes the proof. $\square$

To see that the Completion Lemma holds for convex truthmaking $\Vdash^{+,c}$ as well, observe that the right-to-left direction of

$$\left(\mathcal{U}(S), \supseteq, V'^{+}, V'^{-}\right), {\uparrow}s \Vdash^{+,c} \varphi \qquad \textbf{iff} \qquad \mathbb{M}, s \Vdash^{+,c} \varphi$$

is a consequence of the Completion Lemma for the corresponding $\Vdash^+$. For the left-to-right direction, we also use Lemma 5.8 and the fact that if $X = \bigcap_{i=1}^{n} {\uparrow}x_i$ then $X = {\uparrow}x$ for some $x \in S$ (namely $x = \sup_{\leqslant}\{x_1, \ldots, x_n\}$).

6 A modal perspective on truthmaker semantics

The techniques used in this paper bear a resemblance to modal logic. In this final section, we seek to explain why by elucidating the connection between truthmaker logics and so-called modal information logics. Specifically, we show how truthmaker logics can be viewed as $\{\vee, \langle \sup \rangle\}$-fragments of modal information logics, or vice versa, how modal information logics can be seen as augmenting truthmaker logics with classical negation.

The modal information framework, introduced by van Benthem (1996), is a normal modal framework. Analogous to the truthmaker framework, its characteristic feature is a binary modality '$\langle \text{sup} \rangle$' with the same semantics as the conjunction '$\wedge$' of truthmaker semantics. And as with implementations of the truthmaker framework, one can vary the choice of frames; while preorders and posets have been the main objects of interest, the modal information logics induced by semilattices and other classes have been considered as well.

We continue by presenting the translation of van Benthem (2019) from truthmaker logics to modal information logics.

Definition 6.1. The target language, $\mathcal{L}_M$, is the language of modal information logic (namely a classical propositional language with a binary modality '$\langle \text{sup} \rangle$') but where we have two propositional variables for each propositional variable $p \in \mathcal{L}_T$, namely p^T and p^F.[13] The translation is then given by these double recursive clauses:

$$
\begin{array}{rclrcl}
(p)^+ & = & p^T, & (p)^- & = & p^F, \\
(\neg\varphi)^+ & = & \varphi^-, & (\neg\varphi)^- & = & \varphi^+, \\
(\varphi \wedge \psi)^+ & = & \langle \text{sup} \rangle \varphi^+ \psi^+, & (\varphi \wedge \psi)^- & = & \varphi^- \vee \psi^-, \\
(\varphi \vee \psi)^+ & = & \varphi^+ \vee \psi^+, & (\varphi \vee \psi)^- & = & \langle \text{sup} \rangle \varphi^- \psi^-. \quad \dashv
\end{array}
$$

Inspecting the translation,[14] as van Benthem (2019) notes, we see that

Proposition 6.1 (Correspondence)**.** *For all models $\mathbb{M}$, all states $s \in \mathbb{M}$ and all $\varphi \in \mathcal{L}_T$:*

$$
\begin{array}{llcl}
(i) & \mathbb{M}, s \Vdash^+ \varphi & \textbf{\textit{iff}} & \mathbb{M}, s \Vdash (\varphi)^+; \textit{ and} \\
(ii) & \mathbb{M}, s \Vdash^- \varphi & \textbf{\textit{iff}} & \mathbb{M}, s \Vdash (\varphi)^-.
\end{array}
$$

To be perfectly clear, on the right-hand side of the 'iff's, '$\mathbb{M}$' refers to the corresponding modal definition of the truthmaker model $\mathbb{M}$, and '$\Vdash$' to modal satisfaction.

It, thus, becomes clear that for 'complementary' truthmaker logics and modal information logics,[15] we get the following proposition (as stated in van Benthem (2019)):

[13] From a mathematical perspective, we can also think of these as corresponding to whether p^T [resp. p^F] was even [odd] in an enumeration of the propositional variables of $\mathcal{L}_M$.

[14] For inclusive semantics, the translation modifies canonically.

[15] 'Complementary' as in the logics being defined on the same class of structures with the same admissible valuations. For instance, the modal information logic on semilattices is 'complementary' to the truthmaker logic first specified in Section 2, as well as to the truthmaker logic that uses inclusive semantics instead.

Proposition 6.2. *For all* $(\Gamma \cup \{\varphi\}) \subseteq \mathcal{L}_\mathsf{T}$*:*

$$\Gamma \Vdash^{\pm} \varphi \qquad \textbf{\textit{iff}} \qquad (\Gamma)^{\pm} \Vdash (\varphi)^{\pm},$$

where $(\Gamma)^{\pm} := \{(\gamma)^{\pm} \mid \gamma \in \Gamma\}$ *and the left-hand-side refers to truthmaker entailment while the right-hand-side refers to modal entailment.*

With these results re-capped, we explore this translation a bit more. As stated, most glaring is that it, in a way, licenses us to characterize truthmaker logics as the $\{\vee, \langle\mathrm{sup}\rangle\}$-fragments of modal information logics—or modal information logics as endowing truthmaker logics with classical negation. To explicate this a bit further, consider the following translation:

Definition 6.3. Let $\mathcal{L}_\mathsf{M}^{\{p^\mathsf{T}, p^\mathsf{F}, \vee, \langle\mathrm{sup}\rangle\}} \subseteq \mathcal{L}_\mathsf{M}$ be the fragment of the language of modal information logic restricted to the propositional letters, connective '$\vee$' and modality '$\langle\mathrm{sup}\rangle$'. Then for all $\varphi \in \mathcal{L}_\mathsf{M}^{\{p^\mathsf{T}, p^\mathsf{F}, \vee, \langle\mathrm{sup}\rangle\}}$, we recursively define its translation $(\varphi)^\bullet$ into $\mathcal{L}_\mathsf{T}$ as follows:[16]

$$\begin{array}{rclcrcl} (p^\mathsf{T})^\bullet & = & p, & \qquad & (p^\mathsf{F})^\bullet & = & \neg p, \\ (\langle\mathrm{sup}\rangle\varphi\psi)^\bullet & = & \varphi^\bullet \wedge \psi^\bullet, & & (\varphi \vee \psi)^\bullet & = & \varphi^\bullet \vee \psi^\bullet. \quad \dashv \end{array}$$

Once again, we deduce a correspondence proposition.

Proposition 6.4 (Correspondence)**.** *For all models* $\mathbb{M}$*, all states* $s \in \mathbb{M}$ *and all* $\varphi \in \mathcal{L}_\mathsf{M}^{\{p^\mathsf{T}, p^\mathsf{F}, \vee, \langle\mathrm{sup}\rangle\}}$*:*

$$(i) \qquad \mathbb{M}, s \Vdash \varphi \qquad \textbf{\textit{iff}} \qquad \mathbb{M}, s \Vdash^{+} (\varphi)^\bullet.$$

Symmetric results for falsitymaking are achieved by a symmetric translation.

Thus, the translations $(\cdot)^+$ *and* $(\cdot)^\bullet$ *are, essentially, each other's 'inverses':*

$$\begin{array}{lccc} \textit{For all } \varphi \in \mathcal{L}_\mathsf{T} \textit{ and all } \mathbb{M}, s\textit{:} & \mathbb{M}, s \Vdash^{+} \varphi & \textbf{\textit{iff}} & \mathbb{M}, s \Vdash^{+} \left(\varphi^{+}\right)^\bullet. \\ \textit{For all } \varphi \in \mathcal{L}_\mathsf{M}^{\{p^\mathsf{T}, p^\mathsf{F}, \vee, \langle\mathrm{sup}\rangle\}} \textit{ and all } \mathbb{M}, s\textit{:} & \mathbb{M}, s \Vdash \varphi & \textbf{\textit{iff}} & \mathbb{M}, s \Vdash (\varphi^\bullet)^{+}. \end{array}$$

Corollary 6.5 (Characterization)**.** *Truthmaker logics are (in a precise mathematical sense) the* $\{\vee, \langle\mathrm{sup}\rangle\}$*-fragments of modal information logics, or alternatively, modal information logics arise from augmenting truthmaker logics with classical negation.*

[16] For truthmaker logics with inclusive semantics, we restrict the fragment of $\mathcal{L}_\mathsf{M}$ so that '$\vee$' only occurs as '$\varphi \vee \psi \vee \langle\mathrm{sup}\rangle\varphi\psi$' and modify the translation accordingly.

Besides providing a technically precise and, hopefully, perspicuous modal view on truthmaker semantics, these translations are also mathematically conducive. For instance, in other work (Knudstorp Forthcoming), the present author proves that the modal information logics on preorders and posets, respectively, coincide and are decidable. Using these translations, we then obtain similar results for the complementary truthmaker logics defined on preorders and posets, respectively. However, of perhaps greater interest are the differences between truthmaker and complementary modal information logics, which suggest directions for further research.

For example, while truthmaker consequence on semilattices coincides with truthmaker consequence on complete lattices, this is not the case upon adding classical negation to the language. As observed in the author's Master's thesis (Knudstorp 2022), modal information consequence on lattices already differs from modal information consequence on semilattices. Moreover, while all of the truthmaker logics considered in this paper are decidable, it is an open problem whether the addition of classical negation makes for undecidability; in particular, it remains wide open whether modal information logic on semilattices is decidable.

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