Multimaterial Simulations using the Ghost Fluid Method

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Abstract

The unsteady, compressible Euler equations for multimaterial flow in one dimension have been solved numerically by employing a level set method and two versions of the Ghost Fluid Method.

1. Introduction

Leonhard Euler first presented the momentum and continuity equations in 1757[1], which were completed by the adiabatic condition presented by Laplace in 1816[2]. The energy balance equation, which is the last of what is now called the Euler equations, was not properly incorporated until the late nineteenth century[3], and although the much more general Navier-Stokes equations have been developed[4,5], the continued interest for and usefulness of the Euler equations is undisputable. They provide a robust framework for analyzing ideal fluids when viscous effects are negligible, but cannot generally be solved analytically. Applications of the Euler equations include aerodynamics[6–8], atmospheric modelling and weather forecasts[9,10], astrophysics[11] and detonations and explosives[12–14], to name a few. Accordingly, precise and efficient methods for solving these non-linear, hyperbolic partial differential equations on arbitrary domains has been, and still is, a major field in modern computational fluid dynamics.

The Euler equations govern adiabatic and inviscid flow of a fluid. In the Froude limit (no external body forces) in one dimension, with density ρ, velocity u, total energy E and pressure p, they are given by

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0\,, \tag{1}$$

where the vector of conserved quantities $\mathbf{U}$ and their fluxes $\mathbf{F}(\mathbf{U})$ are given by

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}.$$

It is sometimes convenient to work in terms of the primitive variables $\mathbf{W} = \begin{pmatrix} \rho, & u, & p \end{pmatrix}^T$. The total energy is the sum of the kinetic and potential energy of the system, i.e.

$$E = \frac{1}{2}\rho u^2 + \rho e\,, \tag{2}$$

where e is the internal energy, related to the other variables through the equation of state. For an ideal gas, the equation of state is

$$e = \frac{p}{(\gamma - 1)\rho}\,, \tag{3}$$

where γ denotes the ratio of specific heats for the gas.

Many important milestones have led us to the current state of the art of solving the Euler equations numerically. Riemann identified and worked on the initial value problem for Eq. (1) with discontinuous initial conditions as early as 1860[15], but the first exact solution did not arrive until Godunov proposed an iterative scheme a century later[16]. Today, efficient approximate solvers such as HLLC (Harten-Lax-van Leer-Contact)[17] and Rotated-hybrid Riemann solvers[18] are readily available. The development of numerical analysis methods for partial differential equations got its first proper boost after the famous paper by Courant, Friedrichs and Lewis[19], but local Riemann problems[16], conservative methods[20] and finite volume formulations[21] were necessary before the first proper, three-dimensional simulations of the Euler equations could be performed in the 1980s[22]. After the Monotonic Upstream-centered Scheme for Conservation Laws (MUSCL)[23] was introduced by van Leer as the first higher-order Total Variation Diminishing (TVD) scheme in 1979, several others have been developed, notably Toro's Weighted Average Flux (WAF)[24] and Flux/Slope LImiter Centred (SLIC/FLIC)[25] schemes.

considerations of multi-material cases and use of level set methods.

multimaterial flow: Differences in EoS means we need to think carefully about material interfaces.

The numerical methods which have been employed are explained in section 2, before several test cases and their results are discussed in section 3. Section 4 concludes the report.

2. Numerical methods

2.1. Numerical solutions for the Riemann Problem

Given a conservation equation and two sets of piecewise constant states separated by a single discontinuity, the initial value problem of evolving this system in time is called a Riemann problem. It is very useful in the study of the Euler equations for two reasons. Firstly, it allows for exact (up to an arbitrary accuracy) solutions for systems obeying Eq. (1) which have a single contact discontinuity. Secondly, the discretization of space which is inevitable in computational schemes for solving differential equations, allows for precise solvers of conservation equations based on the solutions of many local Riemann problems.

For the Euler equations and initial conditions

$$\mathbf{W}(x, t = 0) = \begin{cases} \mathbf{W}_L, & x \leq 0 \\ \mathbf{W}_R, & x > 0 \end{cases},$$

Figure 1 shows typical states the system can have, in addition to characteristics for waves propagating in spacetime. There are two types of resultant waves that propagate through space, in addition to the contact wave (dashed). The first is a shock wave, depicted as a thick line on the left, while the second is a rarefaction wave, shown as several gradually decaying (in strength) lines on the left. Any combination of shock and rarefaction waves can occur on the left and right sides of the contact discontinuity, and the result is only dependent on $\mathbf{W}(x, 0)$.

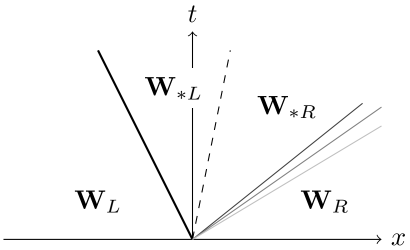

Figure 1: Possible wave configurations for the Riemann problem for Euler's equations in one dimension.

History of solving Riemann problems. Mention different approximate solvers.

For this project, the exact solver has been implemented.

Each of the four states separated by the waves are constant. Additionally, pressure and velocity is constant for the star states. How much more should be said about this?

2.2. Schemes for the Euler equations

Explain the order of schemes and the difference between centered and RP-based schemes.

2.2.1. Slope-Limiting Centered (SLIC)

2.2.2. Weighted-Average Flux (WAF)

2.3. Level-set method

2.4. Ghost Fluid Methods

2.4.1. Original Ghost Fluid Method

2.4.2. Riemann Ghost Fluid Method

3. Results

3.1. Moving contact discontinuity

3.2. Simple ghost fluid tests

3.3. Multimaterial shock tubes for gases

3.4. Water-gas shock tube test

Several other fluids can be approximated by a so-called stiffened ideal gas equation of state,

$$e = \frac{p + \gamma p_\infty}{(\gamma - 1)\rho}. \tag{4}$$

Here, a material-dependent stiffening parameter p_∞ has been introduced. Note that Eq. (4) reduces to Eq. (3) for materials with $p_\infty = 0$.

4. Conclusions

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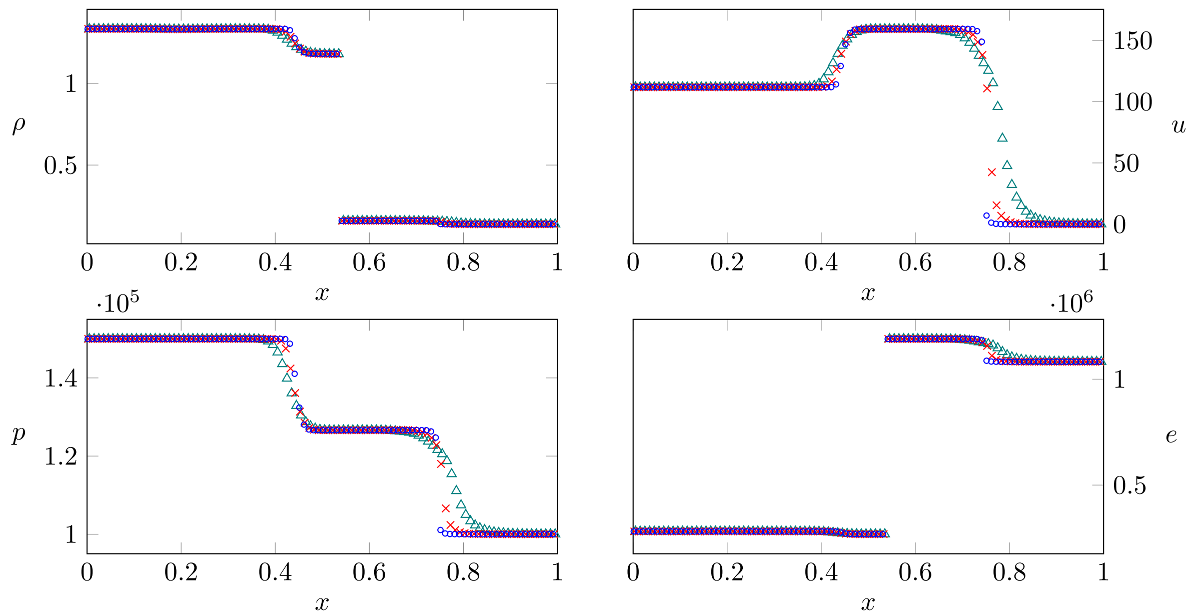

Figure 2: Original Ghost Fluid method for test C. $\triangle N = 100$ $\times N = 200$ $\circ N = 400$

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Acknowledgements

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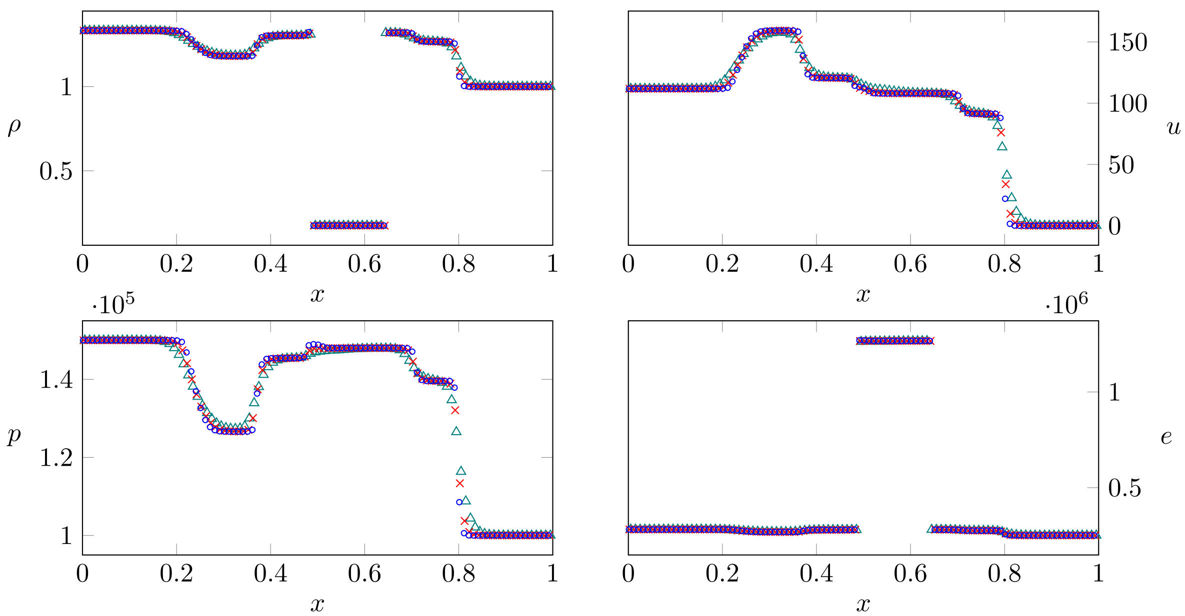

Figure 3: Riemann Ghost Fluid method for test C. △N = 100 ×N = 200 ○N = 400

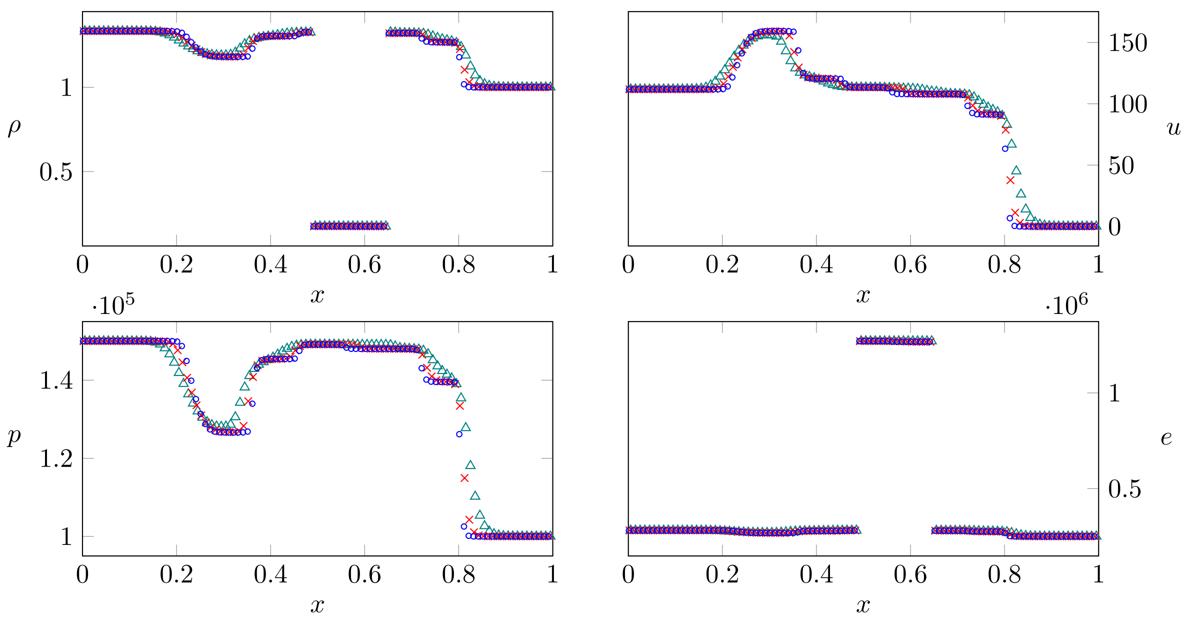

Figure 4: Original Ghost Fluid method for test D. △N = 100 ×N = 200 ○N = 400

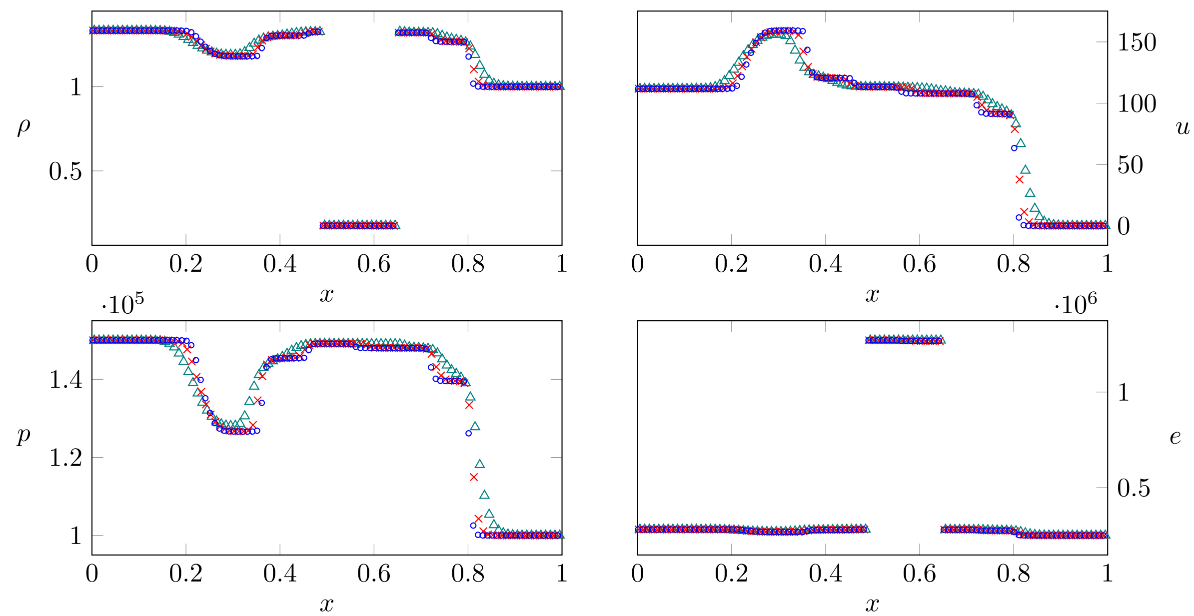

Figure 5: Riemann Ghost Fluid method for test D. △N = 100 ×N = 200 ○N = 400

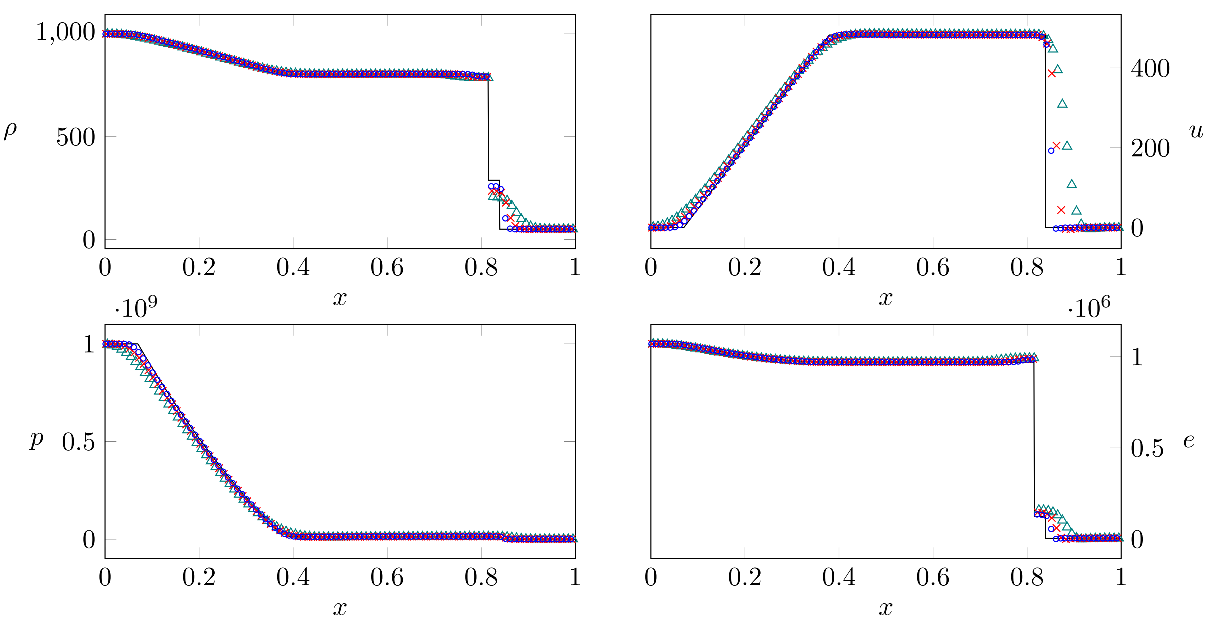

Figure 6: Riemann Ghost Fluid method for test E. △N = 100 ×N = 200 ○N = 400

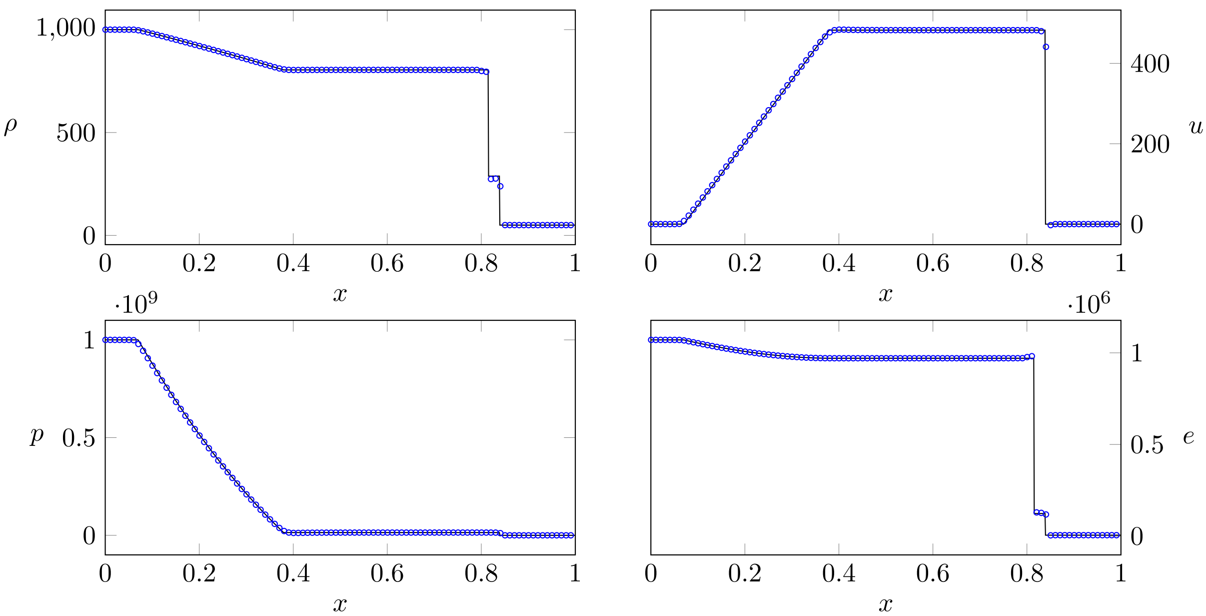

Figure 7: Riemann Ghost Fluid method for test E with higher accuracy (N = 1000).