# Projection Method for Incompressible Flow

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**Abstract**

## 1. Introduction

In physics, few equations span as wide a range of applications as the Navier-Stokes equations of fluid dynamics. They govern the motion of viscuous fluids in a continuum framework, and as such have applications in *e.g.* climate modelling [1,2], aerodynamics [3–5], medicinal research [6,7] and petroleum engineering [8–11], to name a few. Named after Claude Navier and George Stokes for their major contributions [12,13] to its formulation in the first half of the nineteenth century, the equations have constituted a major field of research in their own right since their formulation, and continue to do so today. Apart from their numerous applications, the equations are fundamentally interesting from a mathematical point of view. In fact, (dis-)proving the existence and uniqueness of their solutions is one of the seven Millenium prize problems [14] for which a prize of one million US dollars is associated.

The generalized Navier-Stokes equations are based on the conservation of mass through the continuity equation,

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0\,, \tag{1}$$

in addition to conservation of momentum as given by the Cauchy momentum equation

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \boldsymbol{f}\,. \tag{2}$$

Here, we have introduced the primitive variables density $\rho$, velocity $\boldsymbol{u}$ and pressure $p$, in addition to the deviatoric stress tensor $\boldsymbol{\tau}$. The vector $\boldsymbol{f}$ accounts for external body sources such as gravity acting on the fluid, and shall henceforth be disregarded. The Navier-Stokes equations are an extension of Eqns. (1) and (2) derived under the assumptions that the stress tensor $\boldsymbol{\tau}$ is a linear function of the strain tensor $\dot{\boldsymbol{\gamma}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$, that $\nabla \cdot \boldsymbol{\tau} = 0$ for fluids at rest, and that the fluid is isotropoc. Given these assumptions, Eq. (2) can be rewritten

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \nabla \cdot (\mu \dot{\boldsymbol{\gamma}}) + \nabla\left(\lambda \nabla \cdot \boldsymbol{u}\right) \tag{3}$$

where $\mu$ and $\lambda$ are the first and second coefficients of viscosity, respectively. Fluids which obey the Navier-Stokes equations as given by Eqns. (1) and (3) are labelled Newtonian fluids. The second coefficient of viscosity, $\lambda$, is related to bulk viscosity and disappears for incompressible flow, which we restrict ourselves to in this project. Consequently, we shall simply refer to $\mu$ as the viscosity of the fluid in the following.

When the density $\rho$ is constant within each control volume of the fluid, the flow is said to be isochoric or incompressible. A vast amount of cases in continuum mechanics relate to incompressible flow, and several simplifications arise in the description of the fluid. Firstly, Eq. (1) reduces to the incompressibility constraint

$$\nabla \cdot \boldsymbol{u} = 0\,. \tag{4}$$

Secondly, the viscosity $\mu$ is constant for incompressible flow. Consequently, Eq. (3) simplifies to

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u}\,, \tag{5}$$

where we have used the fact that $\nabla \cdot \dot{\boldsymbol{\gamma}} = \nabla^2 \boldsymbol{u}$ for incompressible flow. Eqns. (4) and (5) make up the incompressible Navier-Stokes equations for incompressible flow.

In order to nondimensionalize our system of equations, we move to dimensionless variables such that $\boldsymbol{u} \to \frac{1}{a}\boldsymbol{u}$, $\partial_t \to \frac{L}{a}\partial_t$ and $\nabla \to L\nabla$, where $L$ is the length of the system under consideration and $a$ is the maximum absolute value of the velocity. Applying these changes to Eq. (5) and multiplying through by $L/(\rho a^2)$ gives the dimensionless equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho a^2}\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}\,. \tag{6}$$

Here, $\mathrm{Re} = \rho a L/\mu$ is the Reynolds number of the flow, which equals the ratio of intertial forces to viscouous ones. Note that the pressure is still measured in physical units, as we wish to utilize different scalings depending on the Reynolds number. Low Reynolds numbers correspond to laminar flow, where the viscouous forces are dominant. In this case, we let $p \to \frac{\mathrm{Re}}{\rho a^2}p$, so that Eq. (6) becomes

$$\mathrm{Re}\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \nabla^2\boldsymbol{u}\,. \tag{7}$$

High Reynolds numbers, on the other hand, correspond to turbulent flow with the most important contributions arising from the intertial forces. We then let $p \to \frac{1}{\rho a^2}p$, yielding the alternative formulation

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}\,. \tag{8}$$

In our implementation of fractional step projection methods for incompresible flow of Newtonian fluids, Eq. (8) is the formulation of interest. When discussing creeping flow for Bingham plastic fluids, however, we return to (a variant of) Eq. (7).

Several models exist to describe different types of Non-Newtonian fluids, each characterized by the way the stress $\tau = \sqrt{\frac{1}{2}\sum \boldsymbol{\tau}_{i,j}^2}$ depends on the strain rate $\dot\gamma = \sqrt{\frac{1}{2}\sum \dot{\boldsymbol{\gamma}}_{i,j}^2}$. The interesting variable is the apparent viscosity $\eta = \tau/\dot\gamma$. For Newtonian fluids, as we have already seen, the relationship is linear, with slope $\eta = \mu$. Non-Newtonian fluids include dilatants (shear-thickening, $\partial\eta/\partial\dot\gamma > 0$) and pseudoplastics (shear-thinning, $\partial\eta/\partial\dot\gamma < 0$), but in this project we restrict ourselves to the relatively simple case of Bingham plastics. Figure 1 illustrates the different types of fluids.

Bingham plastic fluids, named after Eugene Cook Bingham for his investigation into them in 1916 [15], have a threshold stress $\tau_0$, below which they do not yield to applied forces. For the nondimensionalized stress tensor, the yield stress corresponds to the dimensionless Bingham number $B = \tau_0 L/(\mu a)$. In other words, the strain rate is zero unless a stress higher than that characteristic stress is applied. Physically, this means that they behave as solids

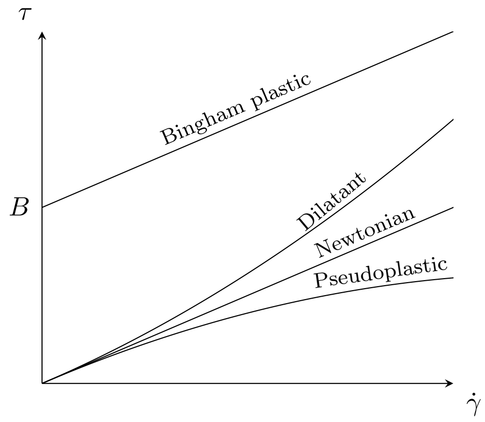


Figure 1: Classification of fluids based on apparent viscosity.

for small stresses, something which leads to interesting behaviour such as non-flat surfaces at rest. Regions where the flow is such that $\tau < B$ are also known as unyielded regions. The relationship between stress and shear rate for Bingham plastics is therefore

$$\dot\gamma = \begin{cases} 0\,, & \tau \le B \\ \tau - B\,, & \tau > B \end{cases}\,, \tag{9}$$

from which it is evident that the apparent viscosity

$$\eta = \frac{B}{\dot\gamma} + 1\,, \tag{10}$$

and as such has a singularity for $\dot\gamma = 0$.

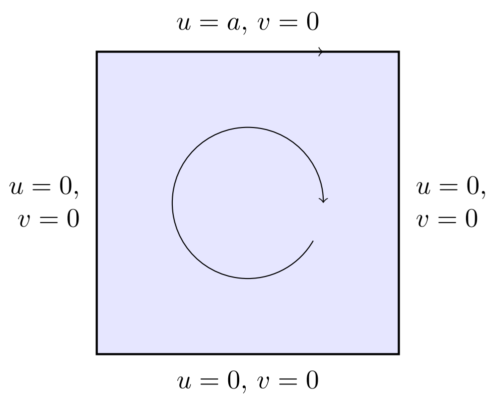


Figure 2: Lid-driven cavity test case.

- Introduce the lid-driven cavity test problem, explain what has been done in the literature.

## 2. Numerical methods

### 2.1. Newtonian fluid

- (Fractional step) projection methods
- Spatial discretization: the finite volume method
- Remember staggered grid!
- Boundary conditions
- Choice of time step for stability
- Solving the systems of equations with Eigen
- Checking if steady-state has been reached
- Define stream function and vorticity.
- Transforming results to vorticity streamline formulation

### 2.2. Bingham plastic fluid

- Transient solution possible, cite appropriate paper and explain difficulty in discretizing the viscous term
- Reynolds number zero leads to removal of time-dependency because of alternative nondimensionalization of pressure
- Treatment of singularity in effective viscosity: regularization
- Finite volume method w/o need for staggered grid, discretization of viscuous term
- Solution of steady-state system: SIMPLE and its extensions

## 3. Results

### 3.1. Transient behaviour

- Impulsively started
- What happens as a function of time?
- Results for different Re?

### 3.2. Steady-state solution

All results for Re=100,400,1000,3200……

- 1D slices in the geometric center, including Ghia's results
- Stram lines and velocity vector fields
- Vorticity
- (Pressure field)!

### 3.3. Computational efficiency

- Computational complexity of the linear systems
- Runtime (and no of time steps) as a function of Re and $N$
- Plots of $\Delta t$ vs. $N$ for different Re

## 4. Discussion

- Everything works, results exactly as in literature
- Transient method is slow for high Re, SIMPLE could be better
- Other improvements include Hockney algorithm and multigrid methods
- Discuss stability and computational efficiency of

## 5. Conclusions

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