Projection Method for Incompressible Flow

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Abstract

1. Introduction

In physics, few equations span as wide a range of applications as the Navier-Stokes equations of fluid dynamics. They govern the motion of viscuous fluids in a continuum framework, and as such have applications in *e.g.* climate modelling [1,2], aerodynamics [3–5], medicinal research [6,7] and petroleum engineering [8–11], to name a few. Named after Claude Navier and George Stokes for their major contributions [12,13] to its formulation in the first half of the nineteenth century, the equations have constituted a major field of research in their own right since their formulation, and continue to do so today. Apart from their numerous applications, the equations are fundamentally interesting from a mathematical point of view. In fact, (dis-)proving the existence and uniqueness of their solutions is one of the seven Millenium prize problems [14] for which a prize of one million US dollars is associated.

The generalized Navier-Stokes equations are based on the conservation of mass through the continuity equation,

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0\,, \tag{1}$$

in addition to conservation of momentum as given by the Cauchy momentum equation

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \boldsymbol{f}\,. \tag{2}$$

Here, we have introduced the primitive variables density ρ, velocity $\boldsymbol{u}$ and pressure p, in addition to the deviatoric stress tensor $\boldsymbol{\tau}$. The vector $\boldsymbol{f}$ accounts for external body sources such as gravity acting on the fluid, and shall henceforth be disregarded. The Navier-Stokes equations are an extension of Eqns. (1) and (2) derived under the assumptions that the strain tensor $\boldsymbol{\tau}$ is a linear function of the strain tensor $\dot{\boldsymbol{\gamma}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$, that $\nabla \cdot \boldsymbol{\tau} = 0$ for fluids at rest, and that the fluid is isotropoc. Given these assumptions, Eq. (2) can be rewritten

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \nabla \cdot (\mu \dot{\boldsymbol{\gamma}}) + \nabla\left(\lambda \nabla \cdot \boldsymbol{u}\right) \tag{3}$$

where μ and λ are the first and second coefficients of viscosity, respectively.

When the fluid density ρ is constant, the flow is said to be incompressible. The second coefficient of viscosity, λ, is related to bulk viscosity and disappears for incompressible flow. Subsequently, we shall refer to μ as the viscosity of the fluid in the following.

- Show how they simplify under the assumption of incompressibility. Nondimensionalize the equations.
- Define stream function and vorticity.
- Talk about viscosity, and the two different representations we consider (Newtonian fluid with μ = const, Non-Newtonian fluid examplified by Bingham plastic).
- Introduce the lid-driven cavity test problem, explain what has been done in the literature.

2. Numerical methods

2.1. Newtonian fluid

- (Fractional step) projection methods
- Spatial discretization: the finite volume method
- Remember staggered grid!
- Boundary conditions

- Choice of time step for stability
- Solving the systems of equations with Eigen
- Checking if steady-state has been reached
- Transforming results to vorticity streamline formulation

2.2. Bingham plastic fluid

- Transient solution possible, cite appropriate paper and explain difficulty in discretizing the viscous term
- Reynolds number zero leads to removal of time-dependency because of alternative nondimensionalization of pressure
- Treatment of singularity in effective viscosity: regularization
- Finite volume method w/o need for staggered grid, discretization of viscuous term
- Solution of steady-state system: SIMPLE and its extensions

3. Results

3.1. Transient behaviour

- Impulsively started
- What happens as a function of time?
- Results for different Re?

3.2. Steady-state solution

All results for Re=100,400,1000,3200......

- 1D slices in the geometric center, including Ghia's results
- Stram lines and velocity vector fields
- Vorticity
- (Pressure field)!

3.3. Computational efficiency

- Computational complexity of the linear systems
- Runtime (and no of time steps) as a function of Re and N
- Plots of Δt vs. N for different Re

4. Discussion

- Everything works, results exactly as in literature
- Transient method is slow for high Re, SIMPLE could be better
- Other improvements include Hockney algorithm and multigrid methods
- Discuss stability and computational efficiency of

5. Conclusions

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