

# Relativity I: Kinematics

Chapter 11 of Morin is a clear, comprehensive, highly recommended introduction to relativistic kinematics. Also read sections 13.1–13.3 for four-vectors, and see appendices F, G, H, and I for enrichment. Alternatively, see chapter 12 and sections 14.1–14.3 of Kleppner and Kolenkow, chapter 11 of Wang and Ricardo, volume 2, or Morin’s newer book, *Special Relativity: For the Enthusiastic Beginner*, which covers similar ground with slightly more detail. An entertaining introduction is also given in chapters I-15 through I-17 of the Feynman lectures. To learn about tests of special relativity, see *The Special Theory of Relativity* by Christodoulides. There is a total of **84** points.

## 1 Lorentz Transformations

Special relativity is uniquely subtle among introductory physics topics, and requires a solid, detailed introduction. This problem set assumes you’ve already done that, by reading chapter 11 of Morin or the equivalent in another book. (The short chapter in Halliday, Resnick, and Krane is not sufficient.)

### Idea 1: Lorentz Transformation

Let  $S'$  be the frame of an observer moving to the right with velocity  $v\hat{x}$  with respect to the frame  $S$ . Then the coordinates in  $S$  and  $S'$  are related by the Lorentz transformation

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

This implies that the lengths of moving objects are contracted by  $\gamma$ , moving clocks run slow by a factor of  $\gamma$ , and that if two clocks are synchronized in the frame  $S'$  and separated by a distance  $L$ , then in the frame  $S$  the rear clock is ahead by  $Lv/c^2$ .

- [2] **Problem 1** (Morin 11.2). Two planets,  $A$  and  $B$ , are at rest with respect to each other, a distance  $L$  apart, with synchronized clocks. A spaceship flies at speed  $v$  past planet  $A$  toward planet  $B$  and synchronizes its clock with  $A$ ’s right when it passes  $A$  (they both set their clocks to zero). The spaceship eventually flies past planet  $B$  and compares its clock with  $B$ ’s. We know, from working in the planets’ frame, that when the spaceship reaches  $B$ ,  $B$ ’s clock reads  $L/v$ . And the spaceship’s clock reads  $L/\gamma v$ , because it runs slow by a factor of  $\gamma$  when viewed in the planets’ frame.

How would someone on the spaceship quantitatively explain to you why  $B$ ’s clock reads  $L/v$  (which is *more* than its own  $L/\gamma v$ ), considering that the spaceship sees  $B$ ’s clock running *slow*?

- [2] **Problem 2** (Morin 11.4). A stick of (proper) length  $L$  moves past you at speed  $v$ . There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in:
- your frame? (Calculate this by working in your frame.)
  - the stick’s frame? (Work in the stick’s frame.)
  - your frame? (Work in the stick’s frame.)
  - the stick’s frame? (Work in your frame. This is the tricky one.)

- [2] **Problem 3** (Morin 11.9). Two balls move with speed  $v$  along a line toward two people standing along the same line. The proper distance between the balls is  $\gamma L$ , and the proper distance between the people is  $L$ . Due to length contraction, the people measure the distance between the balls to be  $L$ , so the balls pass the people simultaneously (as measured by the people), as shown.



Assume that the people's clocks both read zero at this time. If the people catch the balls, then the resulting proper distance between the balls becomes  $L$ , which is shorter than the initial proper distance of  $\gamma L$ . Your task is to explain how the proper distance between the balls decreases from  $\gamma L$  to  $L$ , by working in the frame where the balls are initially at rest.

- Draw the beginning and ending pictures for the process. Indicate the readings on both clocks in the two pictures, and label all relevant lengths.
  - Explain in words how the proper distance between the balls decreases from  $\gamma L$  to  $L$ .
- [3] **Problem 4.** ⌚ USAPhO 2016, problem A3. Print out the custom answer sheet before starting.
- [5] **Problem 5.** ⌚ IPhO 2006, problem 2. A nice problem about relativistic visual effects.

## 2 Velocity Addition

### Idea 2: Velocity Addition

Again, let frame  $S'$  moves with velocity  $v\hat{x}$  with respect to frame  $S$ . If an object has velocity  $(u'_x, u'_y)$  in frame  $S'$ , then the velocity in  $S$  is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$$

where  $\gamma = 1/\sqrt{1 - v^2/c^2}$  as usual.

### Idea 3: Relativistic Doppler Shift

If a light source with (proper) frequency  $f'$  is moving directly towards you at speed  $v$ , then in nonrelativistic physics, we would measure a frequency

$$f_{\text{nr}} = \frac{f'}{1 - v/c}.$$

In relativity, we also need to account for the source being time dilated, so

$$f = \frac{f_{\text{nr}}}{\gamma} = \sqrt{\frac{1 + v/c}{1 - v/c}} f'.$$

This additional, second-order correction was first measured by Ives and Stilwell, in the late 1930s. (The transverse Doppler effect is more subtle, and we'll come back to it in problem 22.)

### Example 1

A reference frame is a formal object made of rulers and synchronized clocks. The length of an object in a given reference frame isn't necessarily the same thing as how long the object *looks*, to somebody at rest in the frame using their own eyes. That is different, because one has to account for the time the light needs to travel to the eyes.

Consider a train of rest length  $L$  moving with speed  $v$  to the right in the ground frame. How long does the train look to somebody standing on the ground directly to the right of it?

### Solution

Both ends of the train continually emit light. Suppose two flashes of light, one from each end, hit an observer's eyes simultaneously. Then the apparent length of the train  $L_{\text{app}}$  is the distance between the points where the light flashes were originally launched.

For somebody to the right of the train, the pulse from the left end of the train had to travel an extra distance  $L_{\text{app}}$ , so it must have been emitted a time  $L_{\text{app}}/c$  earlier. At this time, the left end of the train was  $vL_{\text{app}}/c$  behind where it is when the pulse from the right end of the train hits the observer. So the apparent length is

$$L_{\text{app}} = \frac{L}{\gamma} + \frac{v}{c}L_{\text{app}}.$$

Solving this for  $L_{\text{app}}$  gives

$$L_{\text{app}} = L\sqrt{\frac{1+v/c}{1-v/c}}.$$

Unlike the abstract lab frame length  $L/\gamma$ , this directly observable length is *larger* than  $L$ .

We can also derive this result using the relativistic Doppler shift. In the frame of the train, suppose light is emitted from the left end of the train with wavelength  $\lambda = L$ . Then when it passes by the right end, its phase has changed by  $2\pi$ . In the lab frame this must be remain true, so  $\lambda' = L_{\text{app}}$ . But the relativistic Doppler shift tells us that in the lab frame,  $\lambda' = L\sqrt{(1+v/c)/(1-v/c)}$ , giving the same result.

To check you understand these arguments, you can try them to the case where the person stands to the left of the train. In that case, we instead have  $L_{\text{app}} = L\sqrt{(1-v/c)/(1+v/c)}$ .

### Remark

In most of the problems below, we'll focus on how objects are measured in inertial reference frames, not on how they physically appear to an observer's eyes. This is a complicated but fascinating subject. For instance, it turns out that once one accounts for the light travel

time delay, moving objects [appear to be rotated](#). For an interactive simulation, check out the game [A Slower Speed of Light](#) (3D) and [Velocity Raptor](#) (2D only).

- [1] **Problem 6** (KK 12.6). A rod of proper length  $\ell_0$  oriented parallel to the  $x$  axis moves with velocity  $u\hat{x}$  in frame  $S$ . What is the length measured by an observer in frame  $S'$ , which, as usual, moves with velocity  $v\hat{x}$  with respect to  $S$ ?
- [4] **Problem 7**. An object at rest at the origin in frame  $S'$  emits a flash of light uniformly in all directions.

- (a) In frame  $S'$ , the expanding shell of radiation is a perfect sphere. Explain why it is also a perfect sphere, at any moment, in any other frame  $S$ .
- (b) Let frames  $S$  and  $S'$  be related as usual. Consider the light emitted at an angle  $\theta_0$  with respect to the  $x'$  axis in  $S'$ . Show that the angle  $\theta$  it makes with respect to the  $x$  axis in  $S$  obeys

$$\cos \theta = \frac{\cos \theta_0 + v/c}{1 + (v/c) \cos \theta_0}.$$

- (c) Therefore, if the object has an ultrarelativistic speed  $v \approx c$  in frame  $S$ , argue that in this frame, most of its radiation comes out in a narrow cone of opening angle  $1/\gamma$  along the direction of travel. This “relativistic beaming” effect is important in the Large Hadron Collider, where high-energy particles decay into lower-energy particles concentrated in narrow “jets”.

Now consider the case where the object is at rest, but the light is viewed by a very distant, slowly moving observer going in a circle, with momentarily comoving frame  $S$ . Because of your result in part (b), the observer will see the object perform an apparent circular motion. When the object is a star and the observer is a telescope on the Earth, this phenomenon is known as stellar aberration.

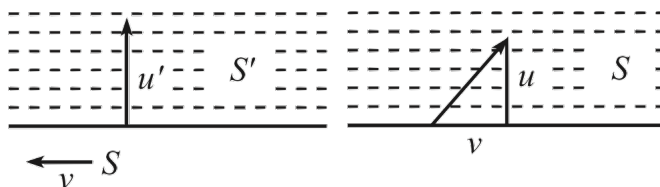
- (d) Suppose that the displacement from the sun to the distant star is perpendicular to the plane of orbit of the Earth. If the Earth performs a circular orbit with speed  $v \ll c$ , find the apparent angular radius  $\theta_A$  of the circle the star moves in.
- (e) There is another independent effect at play here, which is that the star will also seem to move in a circle due to parallax. Parallax exists even if the speed of light is taken to infinity; it is the result of the Earth moving in its orbit, and hence seeing the star from different angles. If the Earth orbits with radius  $r$ , and the star of part (d) is a distance  $d \gg r$  away, find the apparent angular radius  $\theta_P$  of the circle the star moves in.
- (f) For a typical star in the galaxy, which is larger,  $\theta_A$  or  $\theta_P$ ?

The fact that both aberration and parallax escaped detection over centuries of effort was a strong early piece of evidence against heliocentrism. Today we know that they are hard to observe because  $c$  and  $d$  are very large.

- [4] **Problem 8**. In his original 1905 paper on relativity, Einstein considered how light reflected from a moving mirror is Doppler shifted.
- (a) Find the frequency of light reflected directly back from a mirror which is approaching the observer with speed  $v$ , if the light originally had frequency  $f$ .

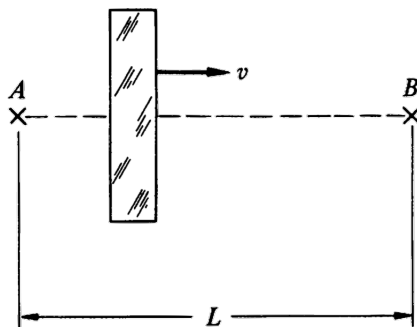
- (b) Show that this is the same as if the light were sourced with frequency  $f$  by an object moving at speed  $2v/(1 + v^2/c^2)$  towards the observer. Can you find an intuitive reason for this?
- (c) Confirm that the energy gained by the light is equal to the loss of the kinetic energy of the mirror. For simplicity, assume the mirror is very heavy. You'll have to use the fact that the energy carried by a pulse of light is related to its momentum by  $E = pc$ .
- (d) What happens if the mirror is moving along its own length? Concretely, suppose the mirror lies at  $x = 0$ , light hits it traveling in the  $\hat{x}$  direction, and the mirror has a velocity  $v\hat{y}$ . After reflection, which way does the light travel, and what is its new frequency?

[2] **Problem 9** (Morin 11.16). In frame  $S'$ , a particle moves with velocity  $(0, u')$  as shown at left.



Frame  $S$  moves to the left with speed  $v$ , so the situation in  $S$  is as shown at right, with the  $y$  speed now  $u$ . Consider a series of equally spaced dotted lines, as shown. By considering the rate at which the particle crosses the dotted lines in each frame, find  $u$  in terms of  $u'$  and  $v$ , and confirm the result agrees with the velocity addition formula.


[3] **Problem 10** (KK 12.9). A slab of glass moves to the right with speed  $v \ll c$ . A flash of light is emitted from  $A$  and passes through the glass to arrive at  $B$ , a distance  $L$  away.



In the rest frame of the glass, it has thickness  $D$  and the speed of light in the glass is  $c/n$ . Suppose  $n$  is a constant independent of light frequency.

- (a) If you were a 19<sup>th</sup> century physicist, who didn't know relativity but did know about the index of refraction and Galilean velocity addition, how long would you expect it to take the light to go from  $A$  to  $B$ ? Keep the lowest order term in  $v/c$ .
- (b) How long does it actually take the light to go from  $A$  to  $B$ , again to lowest order in  $v/c$ ?

This kind of setup could be part of an interference experiment, which would allow the tiny time difference to be effectively measured. Before the advent of special relativity, experiments like these which require relativistic velocity addition were very puzzling. They were interpreted by imagining that materials that slowed down light also partially “dragged” the ether along with it.

[3] **Problem 11.**  USAPhO 2021, problem A2. A simple, elegant problem with a useful punchline.

- [3] **Problem 12.** In relativity, objects that change their direction of motion also automatically *rotate*, even if they experience no torque in their own frames. Concretely, suppose an object is moving along the  $x$ -axis with speed  $v \ll c$ . In its own frame, it experiences an impulse along the  $y$ -axis, which doesn't rotate it, but does change its velocity in that direction by  $u \ll v$ . To keep things simple, you should set  $c = 1$ , and throw away terms smaller than either  $v^2$  or  $uv$ . Under this approximation, the final velocity of the object in the lab frame is just  $(u, v)$ .

- (a) Starting in the lab frame, with coordinates  $(t, x, y)$ , go into the object's frame by performing a Lorentz boost of  $v$  along  $\hat{\mathbf{x}}$ , and then of  $u$  along  $\hat{\mathbf{y}}$ . That is, express the object's coordinates  $(t_o, x_o, y_o)$  in terms of  $t$ ,  $x$ , and  $y$ .
- (b) To compare the orientation of this frame to that of the lab frame, start again in the lab frame and go into the object's frame using a *single* Lorentz boost of  $\mathbf{v} = (v, u)$ . You'll need the formula for a Lorentz transformation in an arbitrary direction, which is

$$t' = \gamma(t - \mathbf{v} \cdot \mathbf{r}), \quad \mathbf{r}' = \mathbf{r} - \gamma \mathbf{v} t + (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{r}) \hat{\mathbf{v}}.$$

- (c) Your two frames will differ in orientation by a small angle  $\Delta\theta$ . What is  $\Delta\theta$ ? More generally, if the object performs uniform circular motion with angular velocity  $\omega$  and speed  $v$  in the lab frame, what spin rotation rate  $\omega_s$  is induced by this effect?
- (d) Suppose the object accelerated by momentarily firing an array of rockets on its back. How would an observer in the lab frame explain why the object rotated?

This subtle phenomenon goes by several names. When we think about it kinematically, as the result of composing Lorentz transformations, it's usually called Wigner rotation, while when we think about it dynamically, e.g. by tracking the orientation of an orbiting particle, it's usually called Thomas precession. In this problem, we considered the most concrete, straightforward derivation of this effect. For a beautifully geometric but more advanced derivation, see [this article](#). For a rather messy application of Wigner rotation, see [Physics Cup 2023, problem 4](#).

- [3] **Problem 13** (Morin 11.58). A person walks very slowly at speed  $u$  from the back of a train of proper length  $L$  to the front. The total time dilation effect in the train frame can be made arbitrarily small by picking  $u$  to be sufficiently small, so that if a person's watch agrees with a clock at the back of the train when he starts, then it also agrees with a clock at the front when he finishes, to arbitrary accuracy.

Now consider this setup in the ground frame, where the train moves at speed  $v$ . The rear clock reads  $Lv/c^2$  more than the front, so in view of the preceding paragraph, the time gained by the person's watch during the process must be  $Lv/c^2$  less than the time gained by the front clock. By working in the ground frame, explain why this is the case. Assume  $u \ll v$ .

### 3 Paradoxes

Now you're prepared to confront some classic relativistic paradoxes. They won't appear in competitions, but your understanding of relativity will be deeper if you grapple with them. (Also, now that we've got the basics out of the way, we'll start setting  $c = 1$  for most problems.)

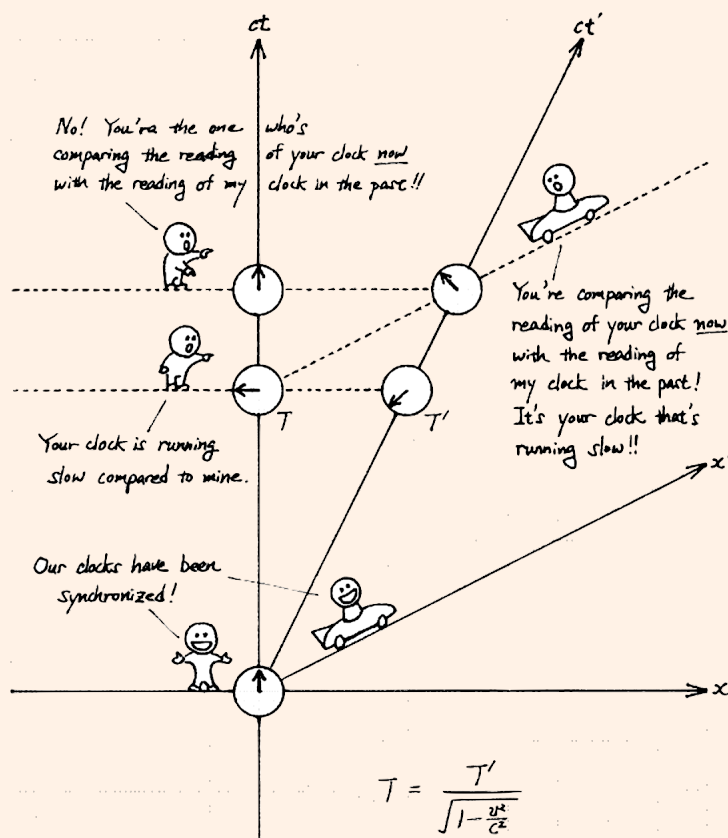
### Example 2

Bob moves away from Alice at constant speed. According to special relativity, each sees the other as aging slower. (This is true both in terms of their reference frames, and in terms of what they see with their eyes.) How can that possibly be self-consistent? Shouldn't time be running slower for one or the other?

### Solution

The first thing to point out about this paradox, and many other relativistic paradoxes, is that they rely on slipping in nonrelativistic assumptions using tricky wording. If you're fine with the idea of time being relative, there's nothing paradoxical about people disagreeing on whose clock runs slower. It's not really more confusing than the fact that when I walk away from you, I see you getting smaller, but you also see me getting smaller.

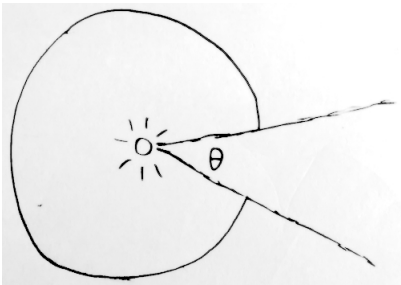
More seriously, though, the reason time dilation can be symmetric is the loss of simultaneity effect, as beautifully shown in Tatsu Takeuchi's Illustrated Guide to Relativity.



- [2] **Problem 14.** The Lorentz transformations treat  $x$  and  $t$  completely symmetrically. So why is it that lengths *contract* while times *dilate*? Shouldn't both do the same thing?
- [3] **Problem 15.** A scientist is trying to drill through a piece of wood of thickness  $2L$ , but the longest drill bit they own has a length  $L$ . The scientist decides to move the wood relativistically fast, so that it length contracts to a thickness less than  $L$ . Then the drill can be held in the path of the

wood, and pulled out once it goes through. Can this really be done without harming the drill bit or ruining the wood? If not, what's wrong? If yes, then what does it look like in the rest frame of the wood?

- [3] **Problem 16.** A headlight is constructed by putting a light source inside a spherical cavity.



The opening of the cavity has angular width  $\theta$ , so a beam of light comes out with width  $\theta$ . The headlight is mounted on the front of a car, which then moves forward at a relativistic speed. The new width of the headlight's beam is  $\theta'$ , in the frame of the Earth. Consider the following two arguments.

The headlight length contracts, increasing the cavity opening angle. Therefore,  $\theta' > \theta$ .

By relativistic velocity addition, the light must have a greater forward velocity in the Earth's frame than in the car frame, because the car is moving forward. So the light must come out at a shallower angle in the Earth's frame. Therefore,  $\theta' < \theta$ .

Which argument is right?

- [3] **Problem 17.** In the lab frame, a horizontal stick of proper length  $L$  has horizontal speed  $v$ . There is a horizontal thin sheet which has a hole of length  $L$ . Since the stick's length is contracted to  $L/\gamma$ , it easily passes through the hole in the sheet, if the sheet is moved vertically. But in the frame of the stick, the sheet is moving horizontally, so the hole is length contracted instead. Qualitatively explain how the stick can still pass through the hole in this frame, in the following two cases:

- The sheet has a uniform vertical velocity in the lab frame.
- The sheet begins at rest at the lab frame, but is pushed upward a small amount when the stick passes over the hole, then ends at rest again.

- [4] **Problem 18.** Here is the statement of the traditional twin paradox.

Bob is an astronaut who leaves home on a rocket with speed  $v$ . Alice stays home. After time  $T$  in Alice's frame, Bob reverses direction and travels home with speed  $v$ . Who, if either, has aged more?

The obvious answer is that Alice has aged more by time dilation. The trouble is explaining why we can't just work in Bob's frame and conclude that Bob has aged more by time dilation.

- Draw a Minkowski diagram for Alice and Bob where Alice's worldline is  $x = 0$ .
- The reason that working in Bob's frame is subtle is that it is not a single inertial frame. Draw  $x'$  and  $t'$  axes for Bob at several points on Bob's worldline. Argue that when Bob turns around, thereby moving to a different inertial frame, Alice's age jumps upward. (Using the results of chapter 11 of Morin, you can even show that the amount of aging is exactly what is needed, using the Minkowski diagram alone.)



This illustrates why the situation is not symmetric between Alice and Bob. But this resolution of the twin paradox is a little unphysical. It does explain what goes wrong working in Bob's frame, but it's not related to what Bob actually physically *sees*, which is determined by when photons from Earth reach his eyes; nothing about that changes discontinuously when he turns around.

- (c) More physically, let us suppose that Bob continually emits radiation of frequency  $f$  (in his frame) towards Alice. Suppose that in Alice's frame, Bob travels with speed  $v$ , reaches a maximum distance  $L$  from Alice, and accelerates quickly to return with speed  $v$ . If Alice sees  $N_b$  wave crests in total during Bob's trip, then Bob has aged by  $N_b/f$ . Use the relativistic Doppler effect to compute  $N_b/f$ , working entirely from Alice's perspective.
- (d) Now suppose Alice continually emits radiation of frequency  $f$  (in her frame) towards Bob. If Bob sees  $N_a$  wave crests, use the relativistic Doppler effect to compute  $N_a/f$ , working entirely from Bob's perspective. If you're careful, this should differ from the answer to (c).
- (e) [A] Now consider a trickier example. Suppose Alice and Bob live on a torus, i.e. a spacetime where the point  $(x, y, z, t)$  is the same as the point  $(x + L, y, z, t)$ . Alice stays home, while Bob leaves on a rocket with velocity  $v\hat{x}$ . After a while, Bob returns home, without having done any acceleration along the way! It seems like the resolution above does not apply, so who, if either, has aged more? Can you explain the results from Bob's reference frame?

### Remark

The above problem on the twin paradox is quite long. Every physics textbook that covers relativity mentions the twin paradox, and Morin even has a whole appendix with five different resolutions of it. But it's not *that* hard to resolve, so why spend so much energy on it?

The answer is that seemingly intelligent people really can get stuck on these things for years, or even decades. As an example, consider the honored English astronomer and philosopher Herbert Dingle. Dingle wrote several textbooks, served as president of the Royal Astronomical Society, and even wrote popular essays introducing relativity. But late in life, he started to believe that relativity could not explain the twin paradox.

[Dingle's arguments](#) contained thousands of words of vague, equation-free prose. Here's the simplest mathematical formalization of what he meant. In the Lorentz transformations

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx), \quad x = \gamma(x' + vt'), \quad t = \gamma(t' + vx'),$$

we notice that if we set  $x = 0$ , then  $t' = \gamma t$ , while if we set  $x' = 0$ , then  $t = \gamma t'$ . Therefore, aging in the twin paradox must be symmetric,  $\gamma = 1/\gamma$ . However, this implies  $\gamma = 1$ , so that time dilation doesn't happen at all, and relativity collapses.

Dingle continued pushing this for the rest of his life, writing endless letters and articles, and even publishing a rambling book, *Science at the Crossroads*, which warned of the grave dangers of trusting relativity. Today, it is a favorite of flat Earthers. And it's far from the only example. For instance, there was a book published in Nazi Germany called *A Hundred Authors Against Einstein*, where a vast array of philosophers argued that relativity had to be wrong, because it contradicted the metaphysical system of the native German, 18<sup>th</sup> century philosopher Immanuel Kant. Kant's ideas about space and time, they said, could be proven

true by verbal reasoning alone, so any theory or experiment saying otherwise had to be wrong.

If there's a lesson to be drawn from this bizarre history, it's that the ability to write or speak is not the same as the ability to think. People can churn out pages of flowing prose without ever having a single coherent thought. Physicists learn to think by solving well-defined problems mathematically. Many others never gain this skill, and spend their whole lives drunkenly stumbling from word to word. The real tragedy is that such people often grow to believe that no better method of reasoning can exist.

## 4 Four-Vectors

### Idea 4

A four-vector  $V^\mu$  is a set of four quantities  $(V^0, V^1, V^2, V^3)$  that transform in the same manner as  $(ct, x, y, z)$ . The inner product of two four-vectors is defined as

$$V \cdot W = V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3.$$

It is invariant under Lorentz transformations. By convention,  $V \cdot W$  is also written as  $V^\mu W_\mu$ .

- [2] **Problem 19.** Show explicitly that the norm of the displacement four-vector is invariant under Lorentz transformations, i.e. that

$$(\Delta s)^2 = \Delta s \cdot \Delta s = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is Lorentz invariant. Since all four-vectors transform the same way, this proves it for all of them.

### Example 3

Find a four-vector representing the velocity of a particle with position  $\mathbf{x}(t)$ .

### Solution

Just as multiplying an ordinary vector with a rotational invariant produces another vector, multiplying or dividing a four-vector with a Lorentz invariant gives another four-vector. In this case, the appropriate four-vector is found by dividing displacement by the proper time experienced by the particle,

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma, \gamma \mathbf{v})$$

where  $\mathbf{v} = d\mathbf{x}/dt$  is the spatial velocity and  $\gamma = 1/\sqrt{1-v^2}$  as usual. Since its spatial part reduces to the spatial velocity in the limit of low speeds, it is the relativistic generalization of the spatial velocity. Finally, we define the four-momentum as  $p^\mu = mu^\mu = (E, \mathbf{p})$ , where  $E = \gamma m$  and  $\mathbf{p} = \gamma m \mathbf{v}$  are the relativistic energy and momentum.

**Example 4**

Give a simple interpretation of the squared norm of a particle's four-velocity,  $u \cdot u$ , and its four-momentum,  $p \cdot p$ .

**Solution**

The answers have to be simple, because they must be invariants that only depend on the intrinsic properties of the particle, i.e. only on the invariant mass  $m$ . For the four-velocity,

$$u \cdot u = \gamma^2 - \gamma^2 v^2 = 1$$

which is clearly invariant. For the four-momentum we have  $p \cdot p = m^2$ .

**Remark**

Pop-science books usually describe the result  $u \cdot u = 1$  by saying that “particles always move with the same speed through spacetime”, just like how a particle in uniform circular motion always has the same spatial speed. However, this is very misleading, because it makes people think that if  $dx/d\tau$  increases in magnitude, then  $dt/d\tau$  decreases. In fact it's the opposite: time dilation means more time passes for each tick of a moving clock, so  $dt/d\tau$  increases. The analogy can't work, because inner products of four-vectors have terms with minus signs, while ordinary inner products of three-vectors don't. The point, as always, is that there are a lot of simple things in physics which are almost impossible to explain properly with fuzzy math-free analogies. With math, relativity can make sense to high school students. Without math, it can't really make sense to anyone.

**Example 5**

Give a simple interpretation of the inner product of two momentum four-vectors,  $p_1 \cdot p_2$ .

**Solution**

By definition, this is equal to  $m_1 m_2 u_1 \cdot u_2$ , and since the inner product is invariant, we can evaluate  $u_1 \cdot u_2$  in any frame. Suppose we work in the frame of the first particle, where

$$u_1^\mu = (1, \mathbf{0}), \quad u_2^\mu = \left( \frac{1}{\sqrt{1-v^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2}} \right).$$

Carrying out the inner product, we have the relatively simple result

$$p_1 \cdot p_2 = \frac{m_1 m_2}{\sqrt{1-v^2}}$$

where  $v$  is the relative speed, meaning the speed of one particle in the frame of the other.

[2] **Problem 20.** In your inertial frame, there is a particle with four-momentum  $p^\mu$ , and an observer moving with four-velocity  $u^\mu$ . The observer measures the particle in *their* inertial frame.

(a) Show that the energy they measure is  $p \cdot u$ .

(b) Show that the momentum they measure has magnitude  $\sqrt{(p \cdot u)^2 - p \cdot p}$ .

(c) What is the speed that they measure?

Don't use Lorentz transformations here; everything can be done with four-vectors alone.

[3] **Problem 21.** In  $A$ 's frame,  $B$  has speed  $u$ , and  $C$  has speed  $v$ .

(a) Suppose  $B$  and  $C$  have velocities in opposite directions. Find the speed of  $B$  with respect to  $C$  using four-vectors, by computing the inner product  $v_B \cdot v_C$  in two different frames.

(b) The answer of part (a) should look familiar, but with four-vectors we can easily go further. Generalize part (a) to the case where  $B$  and  $C$  have velocities an angle  $\theta$  apart.

[3] **Problem 22.** Four-vectors provide a quick derivation of the relativistic Doppler effect. Given a plane wave, define  $k^\mu = (\omega, \mathbf{k})$ . Then the plane wave is proportional to  $e^{i\phi}$ , where the phase is

$$\phi = \omega t - \mathbf{k} \cdot \mathbf{x} = k \cdot x.$$

Since the phase  $\phi$  is Lorentz invariant, and we know  $x^\mu$  is a four-vector,  $k^\mu$  is a four-vector as well.

(a) Show that for light,  $k^\mu k_\mu = 0$ .

(b) Consider a light ray with angular frequency  $\omega$  traveling along the  $x$  axis, and an observer moving with speed  $v$  along the  $x$ -axis. Use an explicit Lorentz transformation to find the angular frequency  $\omega'$  the observer sees, thus rederiving the longitudinal Doppler shift for light.

(c) Now it's easy to go further. Repeat the previous part for a light ray traveling at an arbitrary angle  $\theta$  to the  $x$  axis. You can do this using either an explicit Lorentz transformation, or just properties of four-vectors.

(d) The angle  $\theta$  has different values in the source's frame and the observer's frame. In part (c), we defined it in the source's frame, but the most common form of the result defines  $\theta$  in the observer's frame. To get this formula, repeat part (c), but now suppose we're already in the observer's frame, where the source moves with velocity  $-v\hat{\mathbf{x}}$ , and the light ray is traveling at an angle  $\theta$  to the  $x$ -axis. Find the relationship between  $\omega'$  and  $\omega$ .

The answer to part (d) is also the final result of USAPhO 2021, problem A2. For more on the relativistic Doppler effect, see section 11.8.2 of Morin. (By the way, now that we have the four-vector formalism set up, it's not that much harder to compute the Doppler effect for waves that travel at general speeds. You probably won't need that result, but it's an example of something that's fairly annoying to derive without four-vectors.)

### Example 6: Woodhouse 6.6

Four distant stars  $S_i$  are observed. Let  $\theta_{ij}$  denote the observed angle between the directions to  $S_i$  and  $S_j$ . Show that the ratio

$$\frac{(1 - \cos \theta_{12})(1 - \cos \theta_{34})}{(1 - \cos \theta_{13})(1 - \cos \theta_{24})}$$

is independent of the motion of the observer.

**Solution**

This Oxford undergraduate exam question is too technical to be relevant to Olympiads, but it shows how four-vectors can be essential. The  $\theta_{ij}$  depend on the motion of the observer because of the aberration effect in problem 7. That is, when you Lorentz transform to a moving observer's frame, it changes the direction of the incoming light. A direct attack on the question would thus require applying the full, four-dimensional Lorentz transformations to four vectors with arbitrary orientations, which would be a nightmare. Here's an alternative: let  $k_i^\mu$  be the wave vectors of an incoming photon from each star. Then

$$k_i \cdot k_j = \omega_i \omega_j - \mathbf{k}_i \cdot \mathbf{k}_j = \omega_i \omega_j (1 - \cos \theta_{ij})$$

where we used  $\omega_i = |\mathbf{k}_i|$ . Therefore, the ratio is

$$\frac{(k_1 \cdot k_2)(k_3 \cdot k_4)/\omega_1 \omega_2 \omega_3 \omega_4}{(k_1 \cdot k_3)(k_2 \cdot k_4)/\omega_1 \omega_2 \omega_3 \omega_4} = \frac{(k_1 \cdot k_2)(k_3 \cdot k_4)}{(k_1 \cdot k_3)(k_2 \cdot k_4)}$$

which is manifestly independent of frame.

- [4] **Problem 23.** In this problem we'll construct a four-vector  $a^\mu$  for the acceleration of a particle, and use it to derive the Lorentz transformation of the ordinary three-vector acceleration  $\mathbf{a} = d\mathbf{v}/dt$ .

- (a) Explain why  $a^\mu = du^\mu/d\tau$  is a four-vector, and why  $u \cdot a$  is always zero.  
 (b) Show that when  $\mathbf{v} = v\hat{\mathbf{x}}$ , the components of  $a^\mu$  are

$$a^\mu = (\gamma^4 v a_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z)$$

where  $\gamma = 1/\sqrt{1-v^2}$  as usual. As a check, what is the meaning of  $a \cdot a$ ?

- (c) Now let  $S'$  be the momentary rest frame of a particle, i.e. the inertial frame that, at a given moment, is moving with the same velocity as the particle. Let the particle have three-acceleration  $\mathbf{a}'$  in that frame. Show that in this frame,  $a^{\mu'} = (0, a'_x, a'_y, a'_z)$ .  
 (d) By Lorentz transforming to  $S$  and using part (b), show that the acceleration in frame  $S$  is

$$\mathbf{a} = (a'_x/\gamma^3, a'_y/\gamma^2, a'_z/\gamma^2).$$

As you can see, transformations of three-vector quantities can get quite nasty!

**Remark**

We can rewrite a lot of our results in terms of three-vectors. First, the Lorentz transformations for general  $\mathbf{v}$  are, using the same notation as in idea 1,

$$t' = \gamma(t - \mathbf{v} \cdot \mathbf{r}), \quad \mathbf{r}' = \mathbf{r} - \gamma \mathbf{v} t + (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{r})\hat{\mathbf{v}}.$$

The velocity addition formula for general  $\mathbf{v}$  and  $\mathbf{u}'$  is, using the same notation as in idea 2,

$$\mathbf{u} = \frac{1}{1 + \mathbf{v} \cdot \mathbf{u}'} \left( \mathbf{v} + \frac{\mathbf{u}'}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{u}') \right).$$

The first result of problem 23 is

$$a^\mu = (\gamma^4 \mathbf{a} \cdot \mathbf{u}, \gamma^4 (\mathbf{a} + \mathbf{u} \times (\mathbf{u} \times \mathbf{a})))$$

and the second result, for the transformation of acceleration, is

$$\mathbf{a} = \frac{\mathbf{a}'}{\gamma^2} - \frac{\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{a}')(\gamma - 1)}{\gamma^3}.$$

As you can see, these aren't very enlightening, and they don't tend to be useful in solving problems. The reason is that in relativity, there's nothing special about three-vectors. For concrete problems, you'll typically either want to do everything in terms of four-vectors, or descend all the way down to individual components – in which case you would align your axes so that  $\mathbf{v}$  points along one of them, rather than considering a completely general  $\mathbf{v}$ .

On the other hand, you can get practice with three-vectors by staring at the above expressions until you see how they reduce to the component forms we had earlier. If you do this, you'll learn how to translate just about *any* component expression into three-vector notation.

## 5 Acceleration and Rapidity

### Idea 5

The geometry of special relativity is much like ordinary geometry, except that the dot product is replaced with an inner product, which has some minus signs. Lorentz transformations can be thought of as “generalized rotations” which mix up time and space, just as ordinary rotations mix up different spatial axes. The generalized angle is the rapidity  $\phi = \tanh^{-1} v$ .

[3] **Problem 24** (Morin 11.27). In this problem, we'll see the meaning of the rapidity more precisely.

(a) Show that a Lorentz transformation may be written as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}.$$

(b) Show that the composition of Lorentz transformations with rapidity  $\phi_1$  and  $\phi_2$  is a Lorentz transformation with rapidity  $\phi_1 + \phi_2$ . This makes rapidity extremely useful in kinematics problems with multiple boosts, such as problems involving acceleration.

(c) An ordinary rotation of spatial axes has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Show that a Lorentz transformation is essentially an ordinary rotation between space and time, if we treat time as like “imaginary space” and the rotation as by an imaginary angle. This was one of the ways the founders of relativity thought about it.

**Remark**

We know from **M8** that if we do multiple rotations in a row, the final result depends on the order the rotations are performed. The analogy between boosts and rotations gives intuition for the analogous result for boosts: the order matters. For example, the difference between boosting along the  $x$ -axis and then the  $y$ -axis, or vice versa, is a rotation in the  $xy$  plane. Therefore, if an object is boosted in a circle in the  $xy$  plane, it will have an extra rotation in that plane. This subtle phenomenon is called Thomas precession.

**Idea 6**

The next few questions will deal with accelerating objects. In Newtonian mechanics, a common strategy is to work in the accelerating frame of the object, but that's not a good idea at this stage of your education. (There's nothing wrong with doing so, but it brings in complications that one usually needs a course in general relativity to fully appreciate.)

Instead, we will describe accelerating objects using inertial frames. In principle we could do everything in the lab frame, but it is also often useful to work in a momentarily comoving frame, i.e. the inertial frame that, at some time  $t$ , moves with the same velocity as the object.

- [4] **Problem 25.** A rocket starts from rest in the lab frame at the origin, then accelerates in a straight line at constant rate  $a_0$  as measured by an accelerometer on the ship; that is, the proper acceleration is always  $a_0$ .
- Show that the acceleration measured in the lab frame is  $a_0/\gamma^3$ . (We already proved this more generally in problem 23, but try to do this more explicitly by working in the comoving frame, then going back to the lab frame.)
  - Find the speed of the rocket ship in the lab frame as a function of time  $t$  in the lab frame.
  - Find the speed of the rocket ship in the lab frame as a function of the proper time  $\tau$  elapsed on the rocket. Can you explain the simplicity of your result using rapidity?
  - To conclude, find expressions for  $t(\tau)$ ,  $x(t)$ , and  $x(\tau)$ , and comment on their limits.
- [3] **Problem 26.** ⌚ USAPhO 2020, problem A3. An unusual problem that tests your understanding of momentarily comoving frames, and higher-dimensional Lorentz transformations. As a warning, this question requires you to make an unstated assumption. The fact that uniformly moving clocks have their time dilated by a factor of  $\gamma$  follows directly from the postulates of special relativity. But here you'll have to assume this also holds for *accelerating* clocks, even though clocks can tell if they're accelerating, and may tick differently. This is called the clock hypothesis. For example, on a roller coaster, a pendulum clock doesn't obey the clock hypothesis, but a quartz watch does. Also, the solution is a bit misleading, so don't worry if you thought about the problem differently as long as you got the same final answers.
- [3] **Problem 27** (Morin 11.26). The following problem is called Bell's spaceship paradox. It caused a stir at CERN when many particle physicists could not agree on the answer.
- Two spaceships float in space and are at rest relative to each other. They are connected by a string. The string is strong, but it cannot withstand an arbitrary amount of stretching.



At a given instant, the spaceships simultaneously (with respect to their initial inertial frame) start accelerating in the same direction along the line between them, with the same constant proper acceleration. In other words, assume they bought identical engines from the same store, and they put them on the same setting. Will the string eventually break?

- [5] **Problem 28.** ⌚ APhO 2013, problem 2. This is a challenging question that ties together everything you've learned about kinematics.