Problem Solving II: Data and Uncertainty

For a brief introduction to uncertainty analysis, see this handout, or for a more introductory take, this handout and this comic. For some entertaining general discussion, see chapters I-5 and I-6 of the Feynman lectures. There is a total of **77** points.

1 Basic Probability

Idea 1

If a quantity X has the probability distribution $p(x)$, that means

$$\text{the probability that } a \leq X \leq b \text{ is } \int_a^b p(x)\, dx.$$

In particular, the total probability has to sum to one, so

$$\int_{-\infty}^{\infty} p(x)\, dx = 1.$$

Using the probability distribution, we can calculate expectation values, i.e. averages. For example, the expectation value of X, also called the mean, is

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\, dx$$

while the expectation value of an arbitrary function of X is

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) p(x)\, dx.$$

One especially important quantity is the variance of X, defined as

$$\operatorname{var} X = \langle X^2 \rangle - \langle X \rangle^2.$$

The standard deviation is defined by $\sigma_X = \sqrt{\operatorname{var} X}$. It describes how "spread out" the distribution of X is, and it will play an important role in uncertainty analysis.

[1] **Problem 1.** Suppose that x is a length. What are the dimensions of $p(x)$, $\langle X \rangle$, $\operatorname{var} X$, and σ?

Solution. Since $p(x)\, dx$ is dimensionless, we have

$$[p(x)] = L^{-1}$$

where L denotes length. Similarly,

$$[\langle X \rangle] = L, \quad [\operatorname{var} X] = L^2, \quad [\sigma_X] = L.$$

Example 1

Trains arrive at a train station every 10 minutes. If I arrive at a random time, and X is the number of minutes I have to wait, what is the standard deviation of X?

Solution

We see that X can be anywhere between 0 and 10, with all possibilities equally likely, so

$$p(x) = \begin{cases} 1/(10\,\text{min}) & 0 \leq x \leq 10, \\ 0 & \text{otherwise} \end{cases}$$

where the denominator guarantees the total probability is 1. For the rest of this example, we'll suppress the units. We have

$$\langle X \rangle = \int_{-\infty}^{\infty} x p(x)\, dx = \int_0^{10} \frac{x}{10}\, dx = 5$$

which makes sense, as I should have to wait half the maximum time on average, and

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \int_0^{10} \frac{x^2}{10}\, dx = \frac{100}{3}.$$

Then the standard deviation is

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{5}{\sqrt{3}}\,\text{min}.$$

[3] **Problem 2.** Consider an exponentially distributed quantity,

$$p(x) = \begin{cases} ae^{-ax} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Verify that the total probability is 1, and compute the mean and standard deviation. To perform the integrals, you will have to integrate by parts.

Solution. First, to check normalization,

$$\int_{-\infty}^{\infty} p(x) dx = \int_0^{\infty} ae^{-ax} dx = \int_0^{\infty} e^{-u} du = 1 - 0 = 1.$$

Now, the mean can be evaluated using integration by parts,

$$\langle x \rangle = \int_0^{\infty} xae^{-ax}\, dx = -xe^{-ax}\Big|_0^{\infty} + \int_0^{\infty} e^{-ax}\, dx = 0 - \frac{1}{a} e^{-ax}\Big|_0^{\infty} = \frac{1}{a}.$$

To calculate the standard deviation, we must evaluate

$$\langle x^2 \rangle = \int_0^{\infty} x^2 ae^{-ax} dx = 0 + \int_0^{\infty} (2x) e^{-ax} dx = \frac{2}{a} \langle x \rangle = \frac{2}{a^2}.$$

We thus conclude

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\frac{2}{a^2} - \frac{1}{a^2}} = \frac{1}{a}.$$

[2] **Problem 3.** The purpose of subtracting $\langle X\rangle^2$ in the variance is to make sure it doesn't change when a constant is added to x, since shifting something left or right on the number line shouldn't change its spread. Verify that for any constant c, $\operatorname{var} X = \operatorname{var}(X + c)$.

Solution. We have

$$\operatorname{var}(X + c) = \langle (X + c)^2\rangle - \langle X + c\rangle^2.$$

By the definition of the expectation value, we have

$$\langle A + B\rangle = \langle A\rangle + \langle B\rangle, \quad \langle cA\rangle = c\langle A\rangle$$

for any quantities A and B and any constant c. Thus,

$$\operatorname{var}(X + c) = \langle X^2\rangle + \langle 2Xc\rangle + \langle c^2\rangle - \langle X\rangle^2 - 2\langle X\rangle\langle c\rangle - \langle c\rangle^2 = \operatorname{var} X$$

as desired.

[3] **Problem 4.** We say X is normally distributed if

$$p(x) \propto e^{-a(x-b)^2}.$$

For simplicity, let's shift X so that it's centered about $x = 0$, so

$$p(x) \propto e^{-ax^2}.$$

You may use the result given in **P1**,

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Find the constant of proportionality in $p(x)$, the mean, and the standard deviation.

Solution. Let $p(x) = ke^{-ax^2}$. We fix the constant k by demanding normalization,

$$\int_{-\infty}^{\infty} ke^{-ax^2}\, dx = \int_{-\infty}^{\infty} \frac{k}{\sqrt{a}} e^{-u^2} du = 1.$$

Using the provided integral, we conclude

$$k = \sqrt{\frac{a}{\pi}}.$$

The mean is clearly zero, since the distribution is symmetric about that point. Thus, we have

$$\operatorname{var} X = \langle X^2\rangle = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du.$$

This remaining integral can be evaluated using integration by parts,

$$\int u^2 e^{-u^2} du = -\frac{1}{2} u e^{-u^2}\Big|_{-\infty}^{\infty} + \frac{1}{2} \int e^{-u^2} du = 0 + \frac{\sqrt{\pi}}{2}$$

from which we conclude

$$\operatorname{var} X = \frac{1}{a\sqrt{\pi}} \frac{\sqrt{\pi}}{2}, \quad \sigma = \frac{1}{\sqrt{2a}}.$$

[2] **Problem 5.** If two random variables X_1 and X_2 are independent, then

$$\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle.$$

Use this result to show that

$$\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2)$$

which implies that the standard deviation "adds in quadrature",

$$\sigma_{X_1+X_2} = \sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}.$$

This is an important result we'll use many times below.

Solution. By definition, we have

$$\text{var}(X_1 + X_2) = \langle (X_1 + X_2)^2 \rangle - \langle X_1 + X_2 \rangle^2$$

Using the properties listed in problem 3,

$$\begin{aligned}\text{var}(X_1 + X_2) &= \langle X_1^2 \rangle + 2\langle X_1 X_2 \rangle + \langle X_2 \rangle^2 - \langle X_1 \rangle^2 - 2\langle X_1 \rangle \langle X_2 \rangle - \langle X_2 \rangle^2 \\ &= \text{var}(X_1) + \text{var}(X_2) + 2(\langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle)\end{aligned}$$

When X_1 and X_2 are independent, the last term vanishes, giving

$$\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2).$$

2 Uncertainty Propagation

In this section, we'll establish the fundamental results needed to compute uncertainties.

Idea 2

When a physical quantity is measured in an experiment and reported as $x \pm \Delta x$, it is uncertain what the true value of the quantity is. If the quantity has a probability distribution $p(x)$, then the reported uncertainty Δx is essentially the standard deviation of $p(x)$.

Remark

In practice, you'll have to use intuition and experience to assign uncertainties for real measurements. For example, if you're using a clock that times only to the nearest second, you might take $\Delta t = 0.5\,\text{s}$. If you're using a good ruler, which has millimeter markings, you might take $\Delta x = 0.5\,\text{mm}$. Of course, the ultimate test is the results: if you assigned the uncertainties right, your final uncertainty should encompass the true result most (but not all) of the time.

[2] **Problem 6.** Suppose x has uncertainty Δx and y has uncertainty Δy, where x and y are independent. Explain why the uncertainty of $x + y$ is

$$\Delta(x + y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This is called "addition in quadrature". What is the uncertainty of $x - y$? How about $x + x$?

Solution. For independent variables, $\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2)$. Since our uncertainties represent the standard deviation, $\sigma_X = \sqrt{\text{var}(X)}$, we have

$$\Delta(x + y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

For $x - y = x + (-y)$, and $\Delta(-y) = \Delta y$, we get that $\Delta(x - y) = \Delta(x + y)$. Finally, by linearity we clearly have $\Delta(x + x) = 2\Delta x$. (The formula above doesn't apply, because x isn't independent of x.)

Remark

Note how this differs from "high school" uncertainty analysis. In school, you might be told to show uncertainty using significant figures, and when adding two things, to keep only the figures that are significant in both of them. That corresponds to

$$\Delta(x + y) = \max(\Delta x, \Delta y)$$

which is an underestimate. Or, you might be told that the uncertainty needs to encapsulate all the possible values, which implies that

$$\Delta(x + y) = \Delta x + \Delta y$$

which is an overestimate, since the uncertainties could cancel.

Example 2: $F = ma$ 2016 25

Three students make measurements of the length of a 1.50 m rod. Each reports an uncertainty estimate representing an independent random error applicable to the measurement.

- Alice performs a single measurement using a 2.0 m tape measure, to within 2 mm.
- Bob performs two measurements using a wooden meter stick, each to within 2 mm, which he adds together.
- Christina performs two measurements using a machinist's meter rule, each to within 1 mm, which she adds together.

Rank the measurements in order of their uncertainty.

Solution

The uncertainty in Alice's measurement is 2 mm. The uncertainty in Bob's is $2\sqrt{2}$ mm by quadrature, while the uncertainty in Christina's is $\sqrt{2}$ mm by quadrature. So the lowest uncertainty is Christina's, followed by Alice's, followed by Bob's.

[1] **Problem 7.** Given N independent measurements of the same quantity with the same uncertainty, $x_i \pm \Delta x$, find the uncertainty of their sum. Hence show the uncertainty of their average is $\Delta x/\sqrt{N}$.

This result is extremely important, since repeating trials is one of the main ways to reduce uncertainty. But it's important to remember that the results derived above hold only for independent measurements. For example, taking a single measurement, then averaging that single number with itself 100 times certainly wouldn't reduce the uncertainty at all!

Solution. The uncertainty of their sum ΔX can be found by adding in quadrature,

$$\Delta X = \sqrt{\sum_{i=1}^{N} (\Delta x_i)^2} = \sqrt{N} \Delta x.$$

Therefore, the uncertainty of the average is

$$\frac{\Delta X}{N} = \frac{\Delta x}{\sqrt{N}}.$$

Idea 3

If x has uncertainty Δx, and $f(x)$ can be approximated by its tangent line, $f(x') \approx f(x) + (x' - x) f'(x)$ within the region $x \pm \Delta x$, then $f(x)$ has approximate uncertainty $f'(x)\,\Delta x$.

[2] **Problem 8.** If x has uncertainty Δx, find the uncertainties of x^2, $\sqrt{x}$, $1/x$, $1/x^4$, $\log x$, and e^x.

Solution. Differentiate the functions and multiply by Δx to find the uncertainties. The sign isn't important, since uncertainties are always positive. The results are:

$$\Delta(x^2) = 2x\Delta x \qquad \Delta(\sqrt{x}) = \frac{\Delta x}{2\sqrt{x}} \qquad \Delta(1/x) = \frac{\Delta x}{x^2}$$

$$\Delta(1/x^4) = \frac{4\Delta x}{x^5} \qquad \Delta(\log(x)) = \frac{\Delta x}{x} \qquad \Delta(e^x) = e^x \Delta x$$

[2] **Problem 9.** The tangent line approximation doesn't always make sense. For example, suppose that x is measured to be zero, up to uncertainty Δx. Show that the above results for the uncertainties of x^2 and $\sqrt{x}$ give nonsensical results. What would be a more reasonable uncertainty to report?

Solution. The above uncertainties give 0, ∞ for the uncertainties of x^2 and $\sqrt{x}$ respectively. Since the uncertainties were found with $\Delta x \ll x$, now with $x \ll \Delta x$, we can get $(x + \Delta x)^2 - x^2 = 2x\Delta x + \Delta x^2 \approx \Delta x^2$ and $\sqrt{x + \Delta x} - \sqrt{x} \approx \sqrt{\Delta x}$. Thus, more reasonable uncertainties are $(\Delta x)^2$ and $\sqrt{\Delta x}$. There are numerical factors of order 1 because the shapes of the probability distributions will be distorted, but we won't worry about those, because we're just looking to get a reasonable result. (Of course, a professional would keep track of all these details.)

[2] **Problem 10.** Consider two quantities with independent uncertainties, $x \pm \Delta x$ and $y \pm \Delta y$.

(a) Show that the uncertainty of xy is

$$\Delta(xy) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

To do this, start by writing xy as $\exp(\log x + \log y)$.

(b) If we set $x = y$, then we find

$$\Delta(x^2) = x^2\sqrt{2\left(\frac{\Delta x}{x}\right)^2} = \sqrt{2}x\Delta x.$$

On the other hand, in a previous problem we found $\Delta(x^2) = 2x\Delta x$. Which result is correct?

(c) Find the uncertainty of x/y.

Solution. (a) We can write

$$xy = \exp(\log x + \log y)$$

which implies

$$\Delta(xy) = \exp(\log x + \log y)\Delta(\log x + \log y) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

(b) The result that $\Delta(x^2) = 2x\Delta x$ is correct, since the formula for $\Delta(xy)$ assumes x, y are independent, which fails when we set $y = x$.

(c) We have

$$\frac{x}{y} = \exp(\log x - \log y)$$

and by a very similar calculation to part (a), we conclude

$$\Delta(x/y) = \frac{x}{y}\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

[2] **Problem 11.** A student launches a projectile with speed $v = 5 \pm 0.1\,\text{m/s}$ in gravitational acceleration $g = 9.81 \pm 0.01\,\text{m/s}^2$. The resulting range is $d = 1.5 \pm 0.02\,\text{m}$. Given that the launch angle was less than $45°$, find the launch angle, with uncertainty, assuming all uncertainties are independent.

Solution. From the projectile range equation $d = v^2 \sin(2\theta)/g$, we get

$$\theta = \frac{1}{2}\arcsin\left(\frac{dg}{v^2}\right) = 18.03°.$$

To find the uncertainty, we write $\sin(2\theta) = gd/v^2$. The left-hand side is

$$2\cos(2\theta)\Delta\theta$$

by the tangent line approximation. By the result of problem 10, the right-hand side is

$$\frac{dg}{v^2}\sqrt{\left(\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta g}{g}\right)^2 + \left(\frac{2\Delta v}{v}\right)^2} = 0.0248$$

Combining the results, we have

$$\Delta\theta = 0.015 \text{ rad} = 0.9°$$

which means the final result should be written as

$$\theta = 18.0° \pm 0.9°$$

where we removed a superfluous significant figure.

[2] **Problem 12.** Two physical quantities are related by $y = xe^x$.

(a) If x is measured to be 1.0 ± 0.1, find the resulting value of y, with uncertainty.

(b) If y is measured to be 2.0 ± 0.1, find the resulting value of x, with uncertainty.

Solution. (a) To find the central value of y, we plug in to get $y = e = 2.7183$. To find the error, we use the tangent line approximation,

$$\frac{dy}{dx} = e^x(x+1)$$

which gives us

$$\Delta y \approx e^x(x+1)\Delta x = 0.54.$$

Thus, rounding to a reasonable number of significant figures, we have

$$y = 2.7 \pm 0.5.$$

Note that it would be incorrect to apply the "addition in quadrature" rule for products,

$$\Delta y = xe^x\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta(e^x)}{e^x}\right)^2}$$

because x and e^x aren't independent.

(b) To find the central value of x, we solve the equation $2 = xe^{-x}$ numerically. This can be done using the method of iteration introduced in **P1**. That is, we have $x = 2e^{-x}$, so by repeatedly plugging $2e^{-\text{Ans}}$ into the calculator, we get $x = 0.8526$.

Under the tangent line approximation,

$$\Delta x \approx \frac{\Delta y}{e^x(x+1)} = 0.023.$$

Rounding to a reasonable number of significant figures, we conclude

$$x = 0.85 \pm 0.02.$$

Idea 4

For practical computations, it is often useful to use relative uncertainties. The relative uncertainty of x is $\Delta x/x$, and can be expressed as a percentage.

[1] **Problem 13.** Some basic relative uncertainty results.

(a) Show that the relative uncertainty of the product or quotient of two quantities with independent uncertainties is the square root of the sum of the squares of their relative uncertainties.

(b) Show that averaging N independent trials as in problem 7 reduces the relative uncertainty by a factor of $\sqrt{N}$.

Solution. (a) Above we found that

$$\Delta(xy) = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

Dividing both sides by xy gives

$$\frac{\Delta(xy)}{xy} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

which is the desired result.

(b) We have $\Delta x_N = \Delta(\Sigma x)/N = \Delta x\sqrt{N}/N = \Delta x/\sqrt{N}$. Then

$$\frac{\Delta x_N}{x} = \frac{\Delta x}{x}\frac{1}{\sqrt{N}}$$

as expected.

Remark

There are many situations where the rules above can't be used. For example, consider the uncertainty of $x + y^2/x$, where x and y have independent uncertainties. You can calculate the uncertainty of either term with the standard rules, but you can't calculate the uncertainty of their sum, because the terms are not independent (both contain x).

In these cases, you can use the multivariable equivalent of the tangent line approximation,

$$f(x', y') \approx f(x, y) + (x' - x)\frac{\partial f}{\partial x} + (y' - y)\frac{\partial f}{\partial y}.$$

Adding the two contributions to the uncertainty in quadrature gives

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\,\Delta y\right)^2}.$$

This is the general rule that includes the rules you derived above as special cases. However, it shouldn't be necessary in Olympiad problems. If you run into such situations in an experiment, often one of the uncertainties is much smaller, and can be neglected entirely.

Remark

As you saw in problem 9, the tangent line approximation can sometimes fail. The proper way to handle situations like these would be to find the full probability distribution of the desired quantity, rather than just describing it crudely with its standard deviation. However, this can't be done analytically except in the simplest of cases. So when professional physicists run into situations like these, which are quite common, they often just numerically compute a few million or billion values, starting with randomly drawn inputs each time, and use that to infer the probability distribution. This technique is called Monte Carlo. It's very powerful, but certainly not needed for Olympiads! On Olympiads, you should just fall back to something reasonable, such as taking the minimum and maximum possible values.

3 Using Uncertainties

Example 3: $F = ma$ 2022 B21

Amora and Bronko are given a long, thin rectangle of sheet metal. (It has been machined very precisely, so they can assume it is perfectly rectangular.) Using calipers, Amora measures the width of the rectangle as 1 cm with 1% uncertainty. Using a tape measure, Bronko independently measures its length as 100 cm with 0.1% uncertainty. What are the relative uncertainties they should report for the area and the perimeter of the rectangle?

Solution

To compute the area, we multiply the two measurements, which means we add the relative uncertainties in quadrature,

$$\frac{\Delta A}{A} = \sqrt{(1\%)^2 + (0.1\%)^2} \approx 1\%.$$

Note that in this case, the relative uncertainty of Bronko's measurement is negligible; the relative uncertainty of the area is approximately the relative uncertainty of Alice's measurement.

Computing the perimeter involves adding the measurements, which means the absolute uncertainties are added in quadrature instead. These are 0.01 cm and 0.1 cm for Alice and Bronko's measurements, respectively, so the absolute uncertainty of Alice's measurement is negligible. Thus, the relative uncertainty of the perimeter is approximately the relative uncertainty of Bronko's measurement, 0.1%.

In simple Olympiad experiments, often only one uncertainty will really matter. This can dramatically simplify calculations, but it might take a little thought to tell which one.

[3] **Problem 14.** Solve $F = ma$ 2018 problems A12, A25, B19, and B25, and $F = ma$ 2019 problems A16, B18, and B25. Make sure to strictly adhere to the total time. Since these are $F = ma$ problems, you don't have to produce a writeup. If you find these questions difficult to finish in the allotted time, go back and review the earlier material!

[2] **Problem 15.** Suppose the goal of an experiment is to measure the ratio T_1/T_2 of the durations of two physical processes, where T_1 is about 15 seconds, and T_2 is about 3 seconds. Also suppose your stopwatch is only accurate to the nearest second. You have two minutes to perform measurements. Assume each measurement is independent.

(a) Using your instinct, figure out whether it's better to spend more total time measuring T_1, more total time measuring T_2, or an equal amount of time on both.

(b) To confirm this, qualitatively sketch the relative uncertainty of T_1/T_2 as a function of the fraction of time x spent measuring T_1, using explicit numeric examples if necessary.

Calculations of this sort are common when doing Olympiad experimental physics. You should be able to do them instinctively, getting the ballpark right answer without explicit calculation.

Solution. (a) Since T_2 is smaller, a single measurement of T_2 has a much higher relative uncertainty. Furthermore, T_2 takes less time to measure. This means we definitely want more distinct measurements of T_2 than of T_1. As for how we split up the time, this is a bit harder to judge, but intuitively because uncertainty adds in quadrature, taking a single measurement of each makes T_2's uncertainty not 5 times as bad, but 25 times as bad. So T_2 really completely dominates the uncertainty here, and we should spent most of our time getting its uncertainty down.

(b) We have $\Delta T \approx 1$ s and $\Delta T_i = \Delta T/\sqrt{N_i}$, giving

$$\Delta(T_1/T_2) = \frac{T_1}{T_2}\sqrt{\left(\frac{\Delta T}{T_1\sqrt{N_1}}\right)^2 + \left(\frac{\Delta T}{T_2\sqrt{N_2}}\right)^2}$$

The total time T_t is constant, $N_1T_1 + N_2T_2 = T_t$ where $N_1T_1/T_t = x$. We want to minimize

$$f(x) = \frac{1}{T_1 x} + \frac{1}{(1-x)T_2}.$$

The derivative is

$$f'(x) = -\frac{1}{T_1 x^2} + \frac{1}{T_2(1-x)^2}$$

and setting this to zero gives

$$x^2(1 - T_1/T_2) - 2x + 1 = 0.$$

The smaller root is the desired one since $x < 1$, giving

$$x = \frac{1 - \sqrt{T_1/T_2}}{1 - T_1/T_2} = \frac{1}{1 + \sqrt{T_1/T_2}}$$

so we should spend 30% of our time measuring T_1. The graph of the uncertainty as a function of x is concave up, with vertical asymptotes at $x = 0$ and $x = 1$.

[3] **Problem 16.** In the preliminary problem set, you measured g using a pendulum. If you didn't do uncertainty analysis for it, as we covered above, then you should go back and estimate uncertainties more precisely. In this problem you'll do a different experiment: you will estimate g by finding the time needed for an object to roll down a ramp, with everything again made of household materials.

(a) Before starting, think about what the dominant sources of uncertainty will be, and how you can design the experiment to minimize them. In particular, do you think the result will be more or less precise than your pendulum experiment?

(b) Perform the experiment, taking at least ten independent measurements, and report the data and results with uncertainty.

Solution. Our formula for g is

$$g = \frac{2\ell(1+\beta)}{t^2 \sin(\theta)}$$

where $\beta = I/MR^2$ of the rolling object, and ℓ, t are the distance and time for the path. Let's assume you found a nice object, like a hollow can or a fully filled one, so that β is known relatively precisely. Then the uncertainty is

$$\Delta g = g\sqrt{\left(\frac{2\Delta t}{t}\right)^2 + \left(\frac{\Delta \ell}{\ell}\right)^2 + \left(\frac{\cos(\theta)\Delta\theta}{\sin(\theta)}\right)^2}.$$

Given the above, you definitely want a ramp as long as possible, and there's a tradeoff with the angle: if the angle is very large, t will be small so that the relative error on t will be large, while if the angle is very small, the relative error on θ will be large. So in practice you want to choose a moderately small, but not too small value of θ.

Some reasonable ballpark numbers are $\theta = (10 \pm 1)^\circ$, and $t = (3 \pm 0.3)\,\mathrm{s}$, so you probably can't easily get an uncertainty smaller than a few percent. The overall result will be less precise than the pendulum experiment, because for the pendulum there is no $\Delta\theta$ term, and you can measure $N \gg 1$ periods in a single trial so that the relative error on t falls as $1/N$. With the ramp, you can partially compensate by doing N separate trials, so that the relative error at best falls as $1/\sqrt{N}$, which isn't as good. It might not even be as good as $1/\sqrt{N}$, because your uncertainties may not be independent: you might systematically overestimate or underestimate the time or angle.

[3] **Problem 17.** [A] Consider N independent measurements of the same quantity, with results $x_i \pm \Delta x_i$. They can be combined into a single result by taking a weighted average. What is the optimal weighted average, which minimizes the uncertainty?

Solution. Let the weights be w_i, so we report the value

$$\overline{x} = \sum_i w_i x_i.$$

The uncertainty obeys

$$(\Delta \overline{x})^2 = \sum_i w_i^2 (\Delta x_i)^2.$$

A tempting but incorrect way to minimize this quantity to set the derivative with respect to w_i equal to zero. This doesn't work because the solution is just $w_1 = \ldots = w_N = 0$, which isn't a weighted average at all. To actually have a weighted average, we need the weights to sum to one,

$$\sum_i w_i = 1.$$

This is an optimization problem with a constraint, which can be solved with Lagrange multipliers.

However, for this particular problem, the constraint is simple enough to handle manually. Because of the constraint, if one increases some weight, then one must decrease others. At the minimum, the effect of increasing any weight infinitesimally and decreasing another the same amount must be zero, as if it weren't, we could just adjust those two weights to get a lower uncertainty. Setting the change in the uncertainty due to adjusting w_i and w_j in this way to zero gives

$$0 = d(w_i^2)(\Delta x_i)^2 + d(w_j^2)(\Delta x_j)^2 = (2\, dw)(-w_i(\Delta x_i)^2 + w_j(\Delta x_j)^2).$$

This tells us that $w_i \propto 1/\Delta x_i^2$, which means

$$w_i = \frac{1/(\Delta x_i)^2}{\sum_j 1/(\Delta x_j)^2}.$$

Note that all measurements are included in the optimal average, no matter how bad they may be.

All of the examples above involve combining continuous quantities, so we'll close this section with some applications to "counting" experiments, which work slightly differently.

Remark

In this problem set, we have given rules for calculating the mean and standard deviation of derived quantities. But in general, probability distributions can have all kinds of weird features, which aren't captured by those two numbers. The reason we focus on them anyway is because of the central limit theorem, which roughly states that if we have many independent random variables, the distribution of the sum will approach a normal distribution. As you saw in problem 4, normal distributions are characterized entirely by their mean and standard deviation, so we don't lose any information by reporting only those two quantities.

Example 4

A fair coin is tossed 1000 times, and the number of heads is counted. If this process is repeated many times, what is the standard deviation of the number of heads?

Solution

Consider one trial of 1000 tosses. The number of heads is $X = X_1 + X_2 + \ldots + X_{1000}$, where

$$X_i = \begin{cases} 1 & \text{heads on toss } i \\ 0 & \text{tails on toss } i \end{cases}.$$

Of course, the mean of each of these variables is $\langle X_i \rangle = 0.5$, so that the mean of X is 500. In addition, the X_i are independent of each other, so the variances add. The variance of each one of them is

$$\operatorname{var} X_i = \langle X_i^2 \rangle - \langle X_i \rangle^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus, the standard deviation of the number of heads is

$$\sqrt{\operatorname{var} X} = \sqrt{1000/4} \approx 16.$$

So getting 520 heads would not be surprising, but if you got 550, you might be justified in suspecting the coin isn't fair. (Also, the number of heads is very close to normally distributed, by the central limit theorem mentioned above.) To check whether you understand this, you can redo it with a general probability p of getting heads, where you should get $\sqrt{1000\, p(1-p)}$.

[3] **Problem 18.** At any moment, a Geiger counter can click, indicating that it has detected a particle of radiation. Suppose that there is an independent probability $\alpha\, dt$ of clicking at each infinitesimal time interval dt. Let the number of clicks observed in a total time T be X.

(a) Find the expected value and standard deviation of X, and thereby compute its relative uncertainty. (Hint: split the total time into many tiny time intervals, and let X_i be the number of clicks in interval i, so $X = \sum_i X_i$.)

(b) Using a Geiger counter on a sample, you hear 197 clicks in 5 minutes of operation. Estimate the activity α of the sample (i.e. the expected clicks per second), with uncertainty. If you measure for longer, how does the uncertainty reduce over time?

(c) Now suppose that for a different sample, $N = 0$ after 5 minutes. Estimate the activity α of the sample (i.e. the expected clicks per second), with a reasonable uncertainty. If you measure for longer, and continue to hear no clicks, how does the uncertainty reduce over time?

Solution. (a) There are $N = T/dt$ time intervals. Using the hint and applying linearity of expectation,

$$\langle X \rangle = \sum_i \langle X_i \rangle = N(\alpha\, dt) = \alpha T.$$

Since the X_i are independent, their variances add. The variance of X_i is

$$\langle X_i^2 \rangle - \langle X_i \rangle^2 = \alpha\, dt - (\alpha\, dt)^2 = \alpha\, dt.$$

Thus, by adding the variances, we have

$$\operatorname{var} X = \alpha T$$

so the standard deviation is $\Delta X = \sqrt{\alpha T}$. The relative uncertainty is $\Delta X/\langle X \rangle = 1/\sqrt{\alpha T}$.

(b) Applying the formulas above, we estimate

$$\alpha = \frac{197}{T} = 0.66\,\mathrm{s}^{-1}$$

with an uncertainty of

$$\Delta\alpha = \frac{\alpha}{\sqrt{\alpha T}} = \sqrt{\frac{\alpha}{T}} = 0.05\,\mathrm{s}^{-1}.$$

The uncertainty falls as $1/\sqrt{T}$. Note that this is very similar to previous results we've found, where the uncertainty falls as $1/\sqrt{n}$ where n is the number of trials. In some sense, each instant of time we wait is another trial here.

(c) Of course, we estimate $\alpha = 0$, but then the formulas above imply $\Delta\alpha = 0$ and hence that we are absolutely certain $\alpha = 0$, which is absurd. (If you don't think that's absurd, note that the same result would have occurred if we had heard zero clicks in an *arbitrarily short* time interval, such as a nanosecond.)

This is a case where the basic rules of uncertainty propagation break down, and we need to think. The point of giving an uncertainty is to indicate the range of parameter values compatible with the data we observed. Now, the probability of having no clicks in time T is $e^{-\alpha T}$. If $\alpha T \gg 1$, then it would be very unlikely to have no clicks, so we can rule out $\alpha \gg 1/T$. But if $\alpha T \lesssim 1$, this isn't unlikely at all. Thus, your uncertainty window should be $\alpha \in [0, c/T]$ where c is an order-one number, whose value depends on the specific statistical procedure you use. (Note that the upper bound falls as $1/T$, not $1/\sqrt{T}$.)

[4] **Problem 19.** [A] This problem extends problem 18 to derive some canonical results.

(a) Let $\lambda = \alpha T$. Find the probability $p(X = k)$ of hearing exactly k clicks in terms of λ and k.

(b) To check your result, show that the sum of the $p(X = k)$ is equal to one.

(c) ⋆ In the limit $\lambda \gg 1$, show that the probabilities $p(X = k)$ approach that of a normal distribution with the mean and standard deviation calculated in problem 18, thereby providing an example of the central limit theorem at work. This is a rather involved calculation, which will use many of the techniques from **P1**. It will also require Stirling's approximation,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

for $n \gg 1$, which we mentioned in **P1**. (Hint: because the relative uncertainty falls as λ increases, start by writing $k = \lambda(1 + \delta)$ for $|\delta| \ll 1$, and expand in powers of δ. Be careful not to drop too many terms, as δ is small, but $\lambda\delta$ isn't.)

Solution. (a) Following the notation of problem 18, we have $X = \sum_i X_i$, and we get k clicks if precisely k of the X_i are equal to 1. Thus,

$$p(X = k) = \binom{N}{k}(\alpha\, dt)^k\,(1 - \alpha\, dt)^{N-k} \approx \frac{N^k}{k!}(\alpha\, dt)^k\,(1 - \alpha\, dt)^N = \frac{\lambda^k}{k!}e^{-\lambda}.$$

This is known as the Poisson distribution.

(b) This follows from the Taylor series of the exponential,

$$\sum_{k=0}^{\infty} p(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1.$$

(c) Using Stirling's approximation, we have

$$\begin{aligned} p(X=k) &= \frac{1}{\sqrt{2\pi k}} \left(\frac{\lambda e}{k}\right)^k e^{-\lambda} \\ &= \frac{1}{\sqrt{2\pi\lambda(1+\delta)}} \left(\frac{e}{1+\delta}\right)^{\lambda(1+\delta)} e^{-\lambda} \\ &\approx \frac{1}{\sqrt{2\pi\lambda}} e^{\delta\lambda}(1+\delta)^{-\lambda(1+\delta)} \end{aligned}$$

where we used the fact that $\delta \ll 1$.

Now we need to use a technique from **P1**. Letting the final term be equal to $1/y$, we have

$$\log y = \lambda(1+\delta)\log(1+\delta) = \lambda(1+\delta)\left(\delta - \frac{\delta^2}{2} + O(\delta^3)\right) = \delta\lambda + \frac{\delta^2\lambda}{2} + O(\delta^3\lambda).$$

In **P1**, we only expanded up to the first term, but here we need to keep the order δ^2 term. The reason is we want an approximation that works for the whole peak of the probability distribution, and we know it has relative uncertainty $1/\sqrt{\lambda}$, which means we need to take $\delta \sim 1/\sqrt{\lambda}$. That implies that $\delta^2\lambda$ is of order one and cannot be dropped, but $\delta^3\lambda$ is small and can be dropped. Anyway, plugging this in, we find

$$p(X=k) \approx \frac{1}{\sqrt{2\pi\lambda}} e^{-\delta^2\lambda/2} = \frac{1}{\sqrt{2\pi\lambda}} e^{-(k-\lambda)^2/2\lambda}$$

which is precisely a normal distribution with the appropriate mean and standard deviation.

[3] **Problem 20.** IPhO 2023, problem 1, parts A, B, and D.3. A short derivation of the key features of Brownian motion. It requires only the ideas of this problem set, and some basic mechanics.

4 Data Analysis

Idea 5

All data analysis for the USAPhO and IPhO can be done using extremely basic methods. Sometimes, it suffices to just calculate a value based on a single data point, or by cleverly using a pair of data points. When this isn't enough, you'll have to do graphical data analysis, which will usually correspond to drawing a line and measuring its slope and intercept. This is quite limited compared to modern statistical tools, but also can be surprisingly powerful.

Example 5

The activity of a radioactive substance obeys $A(t) = A_0 e^{-t/\tau}$. Using measurements of t and $A(t)$, plot a line to find A_0 and τ.

Solution

To handle exponential relationships, take the logarithm of both sides for

$$\log A(t) = \log A_0 - t/\tau.$$

Then a plot of $\log A(t)$ vs. t has slope $-1/\tau$ and y-intercept $\log A_0$.

[1] **Problem 21.** For a power law $y = \alpha x^n$ where y and x are measured, what line can be plotted to find α and n?

Solution. We have

$$\log(y) = \log(\alpha x^n) = \log(\alpha) + n \log(x).$$

Thus, if we plot $\log y$ against $\log x$, the slope will be n and the y-intercept will be $\log(\alpha)$.

[2] **Problem 22.** The rate R of electron emission from a solid in an electric field E is

$$R = \beta e^{-E/E_0}$$

for some constants β and E_0. The particular form is because the effect is due to quantum tunneling, and you will derive it in **X2**.

(a) If E and R are measured, what line can be plotted to find β and E_0?

(b) Your answer for part (a) should have formally incorrect dimensions, by the standards of **P1**. This often happens when one takes logarithms. What's going on? If the dimensions are wrong, how can the result be right?

(c) Suppose both β and E_0 have 1% uncertainty. For small E, which is more important for the uncertainty of R? What about for large E? Around where is the crossover point?

Solution. (a) Take the natural log of the equation to get

$$\log R = -\frac{E}{E_0} + \log \beta.$$

Plotting E on the x-axis and $\log R$ on the y-axis will give a line with slope $-\frac{1}{E_0}$ and y-intercept $\log \beta$.

(b) This gets into the details of what it even means to plot data. As a simpler case, consider the relationship $y = kx$ where y and x both have units of energy. We can plot y versus x to find the slope k, but in reality, you can't actually plot a dimensionful quantity: what would it even mean to move your pencil a distance of "3.7 J" on a page? Instead, we write y and x as dimensionless multiples of a standard unit of energy. That is, we are actually plotting

$$\frac{y}{E_0} = k \frac{x}{E_0}$$

where E_0 is some unit of energy, which is typically 1 J. But we don't bother to write E_0 explicitly because this step is kind of obvious.

Exactly the same thing is going on in this problem, but it looks strange because logarithms have the property $\log(xy) = \log x + \log y$. Both R and β have units of rate, so define a unit of rate R_0 and subtract $\log R_0$ from both sides to get an equation with correct dimensions,

$$\log \frac{R}{R_0} = -\frac{E}{E_0} + \log \frac{\beta}{R_0}.$$

This reflects what we actually do when constructing a log plot, though it is usually left implicit.

(c) The uncertainty in β alone always gives a 1% uncertainty in R. But the uncertainty in R due to the uncertainty in E_0 depends on the value of E. For $E \ll E_0$, we can expand the exponential as $(1 - E/E_0)$, and in this case the uncertainty in E_0 does almost nothing at all, so the uncertainty in β dominates. For $E \gg E_0$, the reverse is true. By dimensional analysis, the crossover must be around $E \sim E_0$.

Example 6

Suppose that y and x are related nonlinearly, as

$$y = bx + ax^2.$$

For example, this could model the force due to a non-Hookean spring. Using measurements of x and y, plot a line to find a and b.

Solution

If we divide by x, we find

$$\frac{y}{x} = ax + b.$$

Therefore, we can plot y/x versus x, which gives a line with slope a and intercept b. More generally, we can plot a line whenever we can rearrange a given relation into the form

$$(\text{known}) = (\text{unknown})(\text{known}) + (\text{unknown})$$

where all four terms can be arbitrarily complicated. In this way, it is possible to turn a lot of very nonlinear relations into lines.

[3] **Problem 23.** Some more examples of finding lines to plot.

(a) Suppose that you are given points (x, y) that lie on a circle centered at $(a, 0)$ with radius r. What line can be plotted to find a and r?

(b) Consider an Atwood's machine with masses m and $M > m$. The acceleration of the machine is measured as a function of M. However, since the pulley has mass, it slows the acceleration of the Atwood's machine, so that

$$a = \frac{M - m}{M + m + \delta m} g.$$

Find a line that can be plotted to find g and δm, assuming m, M, and a are known. This is an example of how plotting a line can separate out a systematic error, i.e. the value of δm, which would be impossible if only one value of M were used.

(c) Suppose an object is undergoing simple harmonic motion with amplitude A and angular frequency ω. Given measurements of the position x and velocity v, what line can be plotted to find A and ω?

Solution. (a) The equation of the circle is

$$(x-a)^2 + y^2 = r^2, \quad y^2 + x^2 = 2ax + r^2 - a^2$$

Plotting $y^2 + x^2$ vs. x will give a slope of $2a$ and a y-intercept of $r^2 - a^2$. Combining the two pieces of information yields a and r.

(b) The equation can be slightly rearranged to give

$$\frac{M-m}{a} = \frac{M+m}{g} + \frac{\delta m}{g}.$$

Therefore, a plot of $(M-m)/a$ vs. $M+m$ has slope $1/g$ and y-intercept $\delta m/g$.

(c) By conservation of energy, $A^2 = x^2 + v^2/\omega^2$, so

$$x^2 = A^2 - v^2/\omega^2.$$

Thus, a plot of x^2 vs. v^2 has y-intercept A^2 and slope $-1/\omega^2$.

[3] **Problem 24.** USAPhO 2012, problem A2. (This one requires basic thermodynamics.)

[3] **Problem 25.** USAPhO 2011, problem A2.

[3] **Problem 26.** INPhO 2018, problem 7. (This one requires basic fluid dynamics.)

Solution. See the official solutions here.

[3] **Problem 27** (USAPhO 2024)**.** An experimentalist drives a series RLC circuit with an sinusoidal voltage $V(t) = V_0 \cos\omega t$. In **E6**, you will learn how to show that the voltage across the capacitor, in the steady state, oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}$$

where ω_0 is the resonant angular frequency and Q is the circuit's quality factor. The experimentalist takes the following data near the resonance, for a fixed value of V_0:

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Find the values of ω_0 and Q as accurately as possible. Uncertainty analysis is not required. (Hint: this is the trickiest data analysis problem in the history of the USAPhO. It *can* be solved by drawing lines, but such a method is relatively inefficient. It is better to carefully approximate the given formula, and to consider just a few data points at a time.)

5 Estimation

Estimation is a useful skill for checking the answers to real-world problems.

Example 7

Estimate the circumference of the Earth.

Solution

If you know that the United States is 3,000 miles wide, and there is a time zone difference of three hours between California and New York, then a reasonable estimate is 24,000 miles. Or, if you know the factoid that light can go about seven times around the Earth in a second, then a reasonable estimate is $(3/7) \times 10^8\,\text{m} \approx 4 \times 10^7\,\text{m}$.

Let's check these results are compatible. There are about 5 miles in 8 kilometers, a fact you can get by remembering how your car's speedometer looks, or by noting that 3 feet are about 1 meter. Then $4 \times 10^4\,\text{km} \approx (5/8) \times 4 \times 10^4\,\text{mi} = 2.5 \times 10^4\,\text{mi}$, so the two results are compatible. There are probably at least a hundred more ways to perform this estimation.

Example 8

Estimate the density of air, and compare this to the density of water.

Solution

We can directly use the ideal gas law, $PV = nRT$. The density is $\rho = \mu n/V$ where μ is the mass of one mole of air, so

$$\rho = \frac{\mu P}{RT}.$$

Atmospheric pressure is about 10^5 Pa, typical temperatures are about 300 K, and air is mostly N_2, which has a molar mass of $\mu = 28\,\text{g/mol}$, so

$$\rho = \frac{(0.028)(10^5)}{(8.3)(300)} \frac{\text{kg}}{\text{m}^3} \approx 1\,\frac{\text{kg}}{\text{m}^3}.$$

The density of water is, almost by definition,

$$\rho_w \approx 10^3\,\frac{\text{kg}}{\text{m}^3}.$$

Most liquids and solids have densities within an order of magnitude of this, since in all cases the atoms are packed close together. Evidently, air molecules are about a factor of $(10^3)^{1/3} = 10$ times further apart than typical water molecules.

Example 9

Estimate how much useful power you can produce in a short burst.

Solution

This is a bit tricky to test, because most exercises just burn energy against air resistance or friction, which is hard to estimate. However, a task that directly performs work is useful. I weigh about 75 kg and can run up a 3 m high staircase in around 3 s, so

$$P = mgv = (75)(10)(3/3)\,\mathrm{W} \approx 750\,\mathrm{W}.$$

This is a typical max power output, while typical steady state power outputs are several times smaller, and the corresponding numbers for elite athletes are several times larger.

For the below questions, feel free to look up specific numbers if you're stuck. In all cases, an answer to the nearest order of magnitude is good enough.

[3] **Problem 28.** Some questions about light energy.

(a) Estimate the number of photons emitted per second by a standard light bulb. (The energy of a photon is $E = hf$, and the frequency of a photon is related to the wavelength by $c = f\lambda$.)

(b) The Sun supplies power of intensity $1400\,\mathrm{W/m^2}$ to the Earth. The nearest star is about 4 light years away. Assuming this star is similar to the Sun, about how many of its photons enter your eye per second?

Solution. Before we continue, it's important to note that for estimation questions, one should only expect an answer to within an order of magnitude. Some teachers tweak their example calculations until they give almost exactly the right answer. This makes them look brilliant, but it's deceptive, because then when the student tries to do the same, their results will be much further off. So to combat this, in all solutions here, we've just presented our very first, simplest guesses. They can be up to an order of magnitude off from the real numbers, so if your numbers are within *two* orders of magnitude of ours, you're fine!

(a) We can estimate a standard light bulb to have around 50 W of power. The power $P = NE$ where N is the number of photons emitted per second, and the wavelength of visible light is from $400 - 700$ nm (we can use 500). Then

$$N = \frac{P\lambda}{hc} \approx 10^{20} \text{ photons/s}$$

(b) 1 AU is about 1.5×10^{11} m. (If you forget, you can use something like $GM_S/r^2 = (2\pi/T)^2 r$, where T is one year and $M_S \approx 2 \times 10^{30}$ kg). 1 light year is $c \times$ 1 year $\approx 9.5 \times 10^{15}$ m. Then the intensity from the star is reduced by a factor of $(1\text{ AU}/4\text{ ly})^2$ due to the inverse square law, so $I \approx 3.5 \times 10^{-7}\ \mathrm{W/m^2}$. The area of a human pupil depends on the light conditions, but is roughly $\pi r^2 = \pi(5\,\mathrm{mm})^2$. Then the number of photons that enter it per second is $P\lambda/hc$, which gives $N \approx 10^7$ photons/s. That's plenty, so it's very easy to see such a star at night, while it might be difficult during the day because of the background light from the Sun.

[2] **Problem 29.** Estimate the radius of the largest asteroid you could jump off of, and never return.

Solution. The escape velocity is $v = \sqrt{2GM/R}$, and we will assume a uniform spherical asteroid with density ρ. Rock is probably a few times denser than water, so $\rho \approx 3 \times 10^3\ \mathrm{kg/m^3}$ and $M \approx \frac{4}{3}\pi\rho R^3$. Humans can jump around half a meter, which determines $v = \sqrt{2gh}$. Thus

$$2gh = \frac{2G}{R}\frac{4}{3}\pi\rho R^3.$$

Since $g \approx \pi^2$ in SI units, this simplifies to

$$R \approx \sqrt{\frac{3\pi h}{4G\rho}} \approx 2 \text{ km}.$$

[4] **Problem 30.** Some questions about energy.

(a) Estimate the digestible energy content of a stick of butter. (A calorie is about 4000 J, and is also the energy needed to raise the temperature of a kilogram of water by 1 K.)

(b) Estimate the rate at which your body burns energy when at rest.

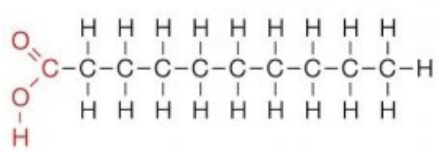
(c) Estimate the rate at which a human being radiates energy. (The Stefan–Boltzmann law states that the radiation power per unit area from a blackbody is σT^4, where $\sigma = 5.7 \times 10^{-8}\,\mathrm{W/m^2K^4}$.) Is radiation a significant source of energy loss for a human being, or is it negligible?

(d) A human being develops hypothermia, with their core body temperature dropping by 5 °F. Neglecting any heat transfer with the environment, estimate the number of calories required to raise their temperature back to normal.

Now let's verify the energy content of the butter microscopically. This will be a very rough estimate, so expect answers to be only within two orders of magnitude.

(e) A chemical bond typically involves two electrons, and a characteristic atomic separation distance of one angstrom, $r \sim 10^{-10}$ m. Estimate the binding energy of one chemical bond.

(f) The fats in butter are digested by inputting energy to break the bonds in the molecules, then harvesting energy by combining the atoms into CO_2 and H_2O, which have somewhat more stable bonds.



Estimate the energy content of a kilogram of butter. How close is this to the true result?

Solution. (a) Recall the usual "2000 calories per day diet" you see on the nutrition facts for food. Note that those calories are referring to kilocalories ($\sim$ 4000 J). Eating a few sticks of butter will probably make me feel quite full for a day (and disgusted), so a stick of butter probably has around 500-1000 kcal of digestive energy content (let's use 800, which is close to the actual value). Then $E = 800 \text{ kcal} \times 4000 \text{ J/kcal} \approx 3 \times 10^6$ J.

(b) Again, we will use what we see on the nutrition facts: 2000 cal/day$\approx$ 100 W. This energy is used to maintain homeostasis in your body, and it eventually gets exhausted as heat.

(c) First we approximate the surface area of a human, then assume a spherical human that's a perfect blackbody. Our height is about 1.7 m, and our width is around 0.25 m and negligible thickness. Then the surface area is around $2 * 1.7 * 0.25 \approx 1 \text{ m}^2$ (rounding up makes more sense for thickness and limbs). Then using the Stefan–Boltzmann law, $P = A\sigma T^4$. Humans skin is on the order of 300 K, so $P \approx 500$ W.

This is much too high, as it can't possible be higher than (b). The main difference is that the radiation output by the human body is almost completely cancelled by the radiation input by the environment, which is at almost the same temperature (in absolute terms). For example, in typical room-temperature conditions, the environment is at 70° F and human skin is at 90° F, for a difference of about 10 K. So the power is smaller by a factor of $1 - (290/300)^4 = 0.13$, giving a reasonable 65 W. It's still a significant contribution, but not unreasonably large. Of course, in colder environments one can reduce this contribution by, e.g. wearing clothes.

(d) 5°F is 10/9°C $\approx$ 1°C. Now we use $Q = mcT$, and since humans are mostly water, we'll approximate the specific heat to be the same as water. The mass of humans is usually around 60 kg. Since the "food calorie" is a kilocalorie (amount of energy needed to raise 1 kg by 1° C), we need 60 food calories to raise our temperature back to normal.

(e) A basic estimate for the binding energy is

$$E \sim \frac{e^2}{4\pi\epsilon_0 r} \sim 2 \times 10^{-18}\,\text{J}.$$

As a check, this is about 10 eV, and the binding energy of hydrogen is about 13.6 eV (one of those classic numbers you should remember), so this is in the right ballpark. Of course, the energy is actually *negative*, even though electrons repel, because it's due to how the electrons are attracted to the nuclei. We can, however, very roughly estimate this negative energy using the positive energy of repulsion $e^2/4\pi\epsilon_0 r$ because all energy scales in the problem should be roughly similar.

Actually, in reality the answer should be about an order of magnitude lower, for two reasons. The first is simply that atomic separations are a bit bigger, but this is cancelled by the fact that the nuclei have charge $Z_i > 1$. The main issue is that covalent bonds are a bit more subtle.

Naively, you could say that a covalent bond is attractive because the electrons in one atom are attracted to the nuclei of the other. But this is too naive, because at least parametrically, it's cancelled out by the repulsion of the nuclei with each other, and the repulsion of the electrons with each other, as all four of these terms are of order $\pm e^2/4\pi\epsilon_0 r$. Covalent bonds are stable because the electron orbitals can deform a bit, so that the negative contributions end up a bit bigger than the positive ones. So $e^2/4\pi\epsilon_0 r$ isn't really an estimate for the binding energy, but for the sizes of terms which *mostly* cancel out to give the binding energy, which is why the real answer is about 10 times smaller.

(f) Fats are mostly carbon. As a very rough estimate let's say that the carbon atoms end up in bonds that are twice as stable as before, so the energy released per carbon atom is on the order of magnitude of what we found in part (e). Then

$$\frac{\text{energy}}{\text{kilogram}} = \frac{\text{energy}}{\text{C atom}} \frac{\text{C atoms}}{\text{mole}} \left(\frac{\text{kilograms}}{\text{mole}}\right)^{-1} \sim (2 \times 10^{-18}\,\text{J})\, N_A \left(\frac{12\,\text{g}}{\text{mole}}\right)^{-1} = 10^8\,\text{J/kg}.$$

For comparison, the energy of one gram of fat is 9 calories, so the true answer is

$$(9)(4000)(1000)\frac{\text{J}}{\text{kg}} = 3.6 \times 10^7\,\text{J/kg}$$

which is not too far off!

[2] **Problem 31** (Povey). When human beings lose weight, most of it is by exhalation of carbon. About 20% of the air in the atmosphere is oxygen. When we breathe in and then out, about 25% of the oxygen is converted to carbon dioxide.

(a) Estimate the mass of air contained in a single breath.

(b) Estimate the amount of weight we lose every day by breathing alone.

Solution. (a) If I don't take a deep breath, I can barely blow up a crushed plastic water bottle (holds half a liter of volume), so I would estimate the volume in a single breath to be around 0.5 L. From chemistry class (or ideal gas law: $n = PV/RT$), we know that mole of gas takes up 22.4 liters of volume at STP (our body temperature, 310 K isn't that much more than 273 K but we can just use $22.4 \times 310/273 \approx 25$ L). Most of the air is nitrogen (N_2) with molecular mass 28 g/mol (oxygen, O_2, is 32 which is pretty close). Then one breath should have a mass of 0.5 L/25 L/mol $\times$ 28 g/mol ≈ 0.6 g.

(b) By counting, we can estimate humans to breathe around 10 to 15 times a minute, so let's use 12.5, giving around 20,000 breaths in a day. In each breath, 20% of the air is oxygen, and 25% of the oxygen is convert to carbon dioxide, for a net fraction of 5%. Carbon dioxide (CO_2) has a molecular mass of 48 g/mol, and oxygen is 32. Thus we lose a proportion of $(48/32 - 1) \times 0.05 = 0.5 = 0.025$ of the mass of the air we breathe in every day, which is about 0.3 kg. Most of the (non-water) mass of the food we eat leaves this way.

[2] **Problem 32** (Insight). How long a line can you write with a pencil?

Solution. Graphene, a layer of carbons arranged in a hexagonal way, famously can be made from using scotch tape to extract a few layers of graphite from pencil markings. It'll take plenty of tries to erase pencil from paper with tape probably (but progress is definitely noticeable), so we can estimate there to be around 100 layers of the hexagonal carbon from graphite.

We will assume that the line is drawn with the pencil perfectly vertical and the lead not sharpened. The diameter of the lead is around 2 mm, and the mass of a pencil should be around 2-10 grams, so the mass of the lead is on the order of 1 g. Assuming that the lead is almost all made out of carbon, we can estimate how many carbon atoms it has, and the surface density of carbon atoms.

The carbons are spread apart in a hexagonal fashion with a characteristic distance of $r \approx 10^{-10}$ m, and the centers of 3 adjacent hexagons will have a carbon atom at its center, giving a spacing of 1 carbon atom every r^2 square meters. There should be

$$6.022 \times 10^{23} \text{atoms/mole} \times 1 \text{ g} \times \frac{1}{12 \text{ g/mol}} \approx 5 \times 10^{22} \text{ atoms of C}$$

Thus that gives around 500 m^2 of a single layer of carbon, so around 5 m^2 of lead usage. The line will be approximately a rectangle with area $d\ell$, where d is the diameter of 2 mm.

Thus the pencil line should be around 2.5 km long. One can find other estimates of the order 50 km, i.e. a spread of an order of magnitude. The precise result within this order of magnitude of course depends on the details of the pencil.