

# Waves III: Specific Waves

Sound waves are covered in chapter 19 of Halliday and Resnick, while light waves are covered in chapters 39 and 44. For more about light waves, see chapter 9 of Purcell. For water waves, and many other neat wave phenomena, see chapter I-51 of the Feynman lectures and chapter 7 of Crawford's *Waves*. For more about polarization, see chapter I-33 of the Feynman lectures, or for more detail, chapter 8 of Hecht's *Optics*. Basic geometrical optics is covered in chapter 40 of Halliday and Resnick. There is a total of **87** points.

## 1 Sound and Longitudinal Waves

[4] **Problem 1.** In this problem, you'll work through Newton's slick derivation of the speed of sound. This derivation avoids thinking about how individual parcels of gas move, which can be confusing, by instead considering the motion of a piston at the end of the gas.

- (a) Consider a cylinder of gas of length  $L$  and area  $A$ , closed on one end with a movable piston on the other end. Suppose the gas exerts a force  $F$  on the piston when in equilibrium. We may define an equivalent spring constant by  $K = -dF/dL$ . Show that for the gas,

$$K = -A^2 \frac{dp}{dV}.$$

- (b) Argue that the speed  $v$  of longitudinal waves on a spring of mass  $M$ , spring constant  $K$ , and length  $L$  obeys

$$v^2 = \frac{KL^2}{M}.$$

- (c) If we assume the sound waves are adiabatic, show that for the gas,

$$v^2 = \frac{\gamma p}{\rho}.$$

Check this answer is reasonable by evaluating the result for air. If each gas molecule has mass  $m$ , write the result in terms of  $\gamma$ ,  $T$ , and  $m$ .

Next, we consider some limitations of this result.

- (d) In an ideal gas, we assume the particles are noninteracting: they pass right through each other. But for sound waves to propagate, adjacent packets of ideal gas must exert pressure on each other. How is this possible? Use this observation to estimate the maximum possible frequency of sound in a gas in terms of the number density  $n = N/V = P/k_B T$ , the radius  $r$  of a gas molecule, and the speed of sound  $v$ .
- (e) Our analysis also breaks down if the pressure variations are no longer adiabatic. The rate of heat conduction in a gas with thermal conductivity  $k_t$  across a surface of area  $A$  is

$$\frac{dQ}{dt} = -Ak_t \frac{dT}{dx}$$

Taking a sinusoidal temperature variation, show that the adiabatic approximation holds when  $\omega \ll pk_B/mk_t$ . Is this approximation good for audible sound in air, where  $k_t \approx 25 \text{ mW/m K}$ ?

- (f) In a more traditional derivation of the speed of sound, such as the one in Halliday and Resnick, one would show that

$$v^2 = \frac{B}{\rho}$$

where  $B$  is the bulk modulus, the pressure per fractional change in volume,

$$B = -V \frac{dP}{dV}.$$

Check that this result is compatible to your result in (c) for adiabatic compression.

Liquids and solids typically have a much higher  $B$  and  $\rho$  than gases, and  $B$  is high enough so that the speed of sound in liquids and solids is typically greater as well.

### Remark

Phase shifts upon reflection for sound waves can be a bit tricky. Recall from **W1** that a hard boundary for a transverse string wave  $y(x, t)$  sets  $y$  to zero. As a result, upon reflection,  $y$  flips sign, but  $v_y = \partial y / \partial t$  stays the same.

When a sound wave hits a hard wall, the wall sets the displacement  $\xi(x, t)$  to zero. Then upon reflection, the displacement flips sign, while the pressure variation  $\delta P(x, t) \propto \partial \xi / \partial x$  stays the same. In standing waves, a hard wall is thus a node for  $\xi$  and an antinode for  $\delta P$ . Similarly, when sound waves in a tube reflect off an open end, the end sets  $\delta P$  to zero, so it flips sign. An open end is thus a node for  $\delta P$  and an antinode for  $\xi$ .

The rule is always the same: whatever quantity gets fixed to zero by the boundary gets flipped in sign upon reflection, and for a standing wave, that quantity has a node at the boundary. But it's confusing enough that several common high school textbooks get it wrong. Some even state, in their confusion, that "hard boundaries flip transverse waves but not longitudinal ones", which is definitely not true in general.

### [3] Problem 2 (HRK). Some conceptual questions about sound waves.

- What is larger for a sound wave, the relative density variations  $\Delta\rho/\rho$  or the relative pressure variations  $\Delta P/P$ ? Or does it depend on the situation?
- What is larger, the velocity of a sound wave  $v$  or the amplitude of the velocity variations  $\Delta u$  of the underlying particles? Or does it depend on the situation?
- Bats and porpoises each emit sound waves of frequency about 100 kHz. However, bats can detect objects as small as insects but porpoises only small fish. Why the difference?

### [3] Problem 3. Consider a rubber rope with unstretched length $L_0$ , which is stretched to length $L$ .

- Find the ratio of the speeds of transverse and longitudinal waves.
- Experimentally, it is found that the longitudinal waves are much more strongly damped. (You can check this at home, by making such a rope by tying together cut rubber bands.) Can you explain why, by considering the molecular structure of rubber?

**Idea 1: Doppler Effect**

Working in one dimension with speed of sound  $c$ , if a source of sound at frequency  $f_0$  travels at velocity  $v_s$  while an observer to their right travels at velocity  $v_o$ , the observed frequency is

$$f = \frac{c - v_o}{c - v_s} f_0.$$

**Example 1**

A speaker is between two perfectly reflective walls and emits a sound of frequency  $f_0$ . If you carry the speaker and walk with small speed  $v$  towards one of the walls, what do you hear?

**Solution**

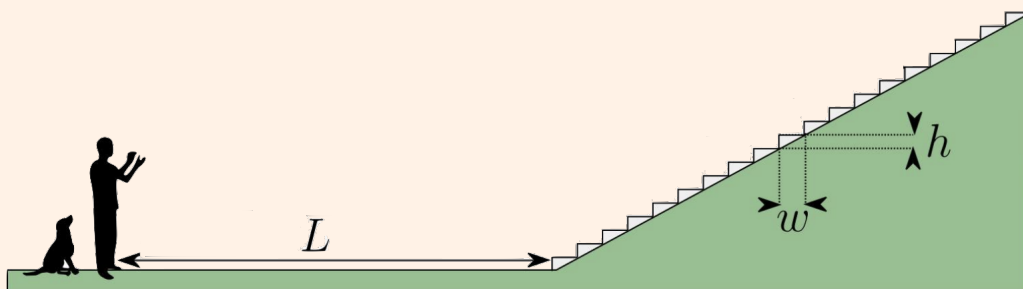
Let's work to lowest order in  $v/c$  everywhere. The wall you're walking toward experiences a sound of approximate frequency  $f_0(1 + v/c)$  by the Doppler effect, and this is the frequency it reflects. Since you're walking towards the wall, a second Doppler effect occurs, causing you to hear frequency  $f_0(1 + 2v/c)$ . (We also saw this “double Doppler shift” back in **R1**.) By similar reasoning, you hear sound of frequency  $f_0(1 - 2v/c)$  from the wall behind you. Since  $v/c$  is small, what this actually means is that you'll perceive a single tone of frequency  $f_0$ , but with “beats” of frequency  $4v/c$ .

**Example 2**

In my former college at Oxford, there is a long staircase that is said to “quack” when one claps at it. What is the explanation of this phenomenon?

**Solution**

A diagram of the staircase is given below, courtesy of Felix Flicker, fellow of New College.



The key is that each clap reflects off a stair individually. When the echoes arrive back at the listener, they arrive quickly enough to be heard as a pitch.

The width and height of the steps are  $w = 30$  cm and  $h = 16$  cm. Suppose one claps at a distance  $L \gg w, h$ . The path length differences for reflections off the bottom few steps are

approximately  $2w$ , giving the frequency

$$f = \frac{v}{2w} = 570 \text{ Hz}$$

where we used  $v = 343 \text{ m/s}$ . The quack then continues, due to reflections off higher and higher stairs. Once the stairs are much further away than  $L$ , path length differences for subsequent reflections are approximately  $2\sqrt{w^2 + h^2}$ , giving frequency

$$f = \frac{v}{2\sqrt{w^2 + h^2}} = 500 \text{ Hz}.$$

Hence the quack consists of a pitch that starts high and then falls slightly lower as it fades away. For further discussion, see the article [How the Mound got its Quack](#).

[3] **Problem 4.**  USAPhO 1998, problem B1.

[3] **Problem 5.**  USAPhO 2016, problem A1.

[2] **Problem 6.** Some problems about sound waves in everyday life.

- (a) Get a coffee cup with a handle and tap on the rim with a spoon. You will hear two distinct pitches, e.g. if you tap directly above the handle, or  $45^\circ$  away from this point. Investigate what happens for different angles. Can you explain why this happens?
- (b) The octave key on an oboe forces the resonance mode from the fundamental to the first overtone. It does this by opening a small hole on the back of the clarinet. Should this hole be placed at a pressure node or antinode for the fundamental, or somewhere else entirely?
- (c) According to introductory textbooks, the fundamental mode for a pipe of length  $L$  and radius  $r \ll L$ , closed at one end and open at the other, has wavelength  $4L$ . In reality, it's a little bit different because the radius is nonzero. Is the wavelength actually higher or lower than  $4L$ ?
- (d) Find a way to produce beats in real life.

[3] **Problem 7.**  BPhO 2008, problem 2.

## 2 Polarization

Now we'll introduce polarization for light waves, putting the results of **E7** to work.

### Idea 2

The polarization of a light wave refers to the direction of its electric field; the light waves we saw in **E7** were linearly polarized. For example, a light wave traveling along  $\hat{\mathbf{z}}$  with its polarization an angle  $\theta$  from the  $x$ -axis has electric field

$$E_x(z, t) = (E_0 \cos \theta) \cos(kz - \omega t), \quad E_y(z, t) = (E_0 \sin \theta) \cos(kz - \omega t).$$

A polarizing filter lets only light of a certain linear polarization through; if light with a linear polarization an angle  $\theta$  from this axis passes through it, then a fraction  $\cos^2 \theta$  of the energy is

transmitted. Just as light can be incoherent, it can be unpolarized; unpolarized light hitting a polarization filter loses half its energy.

- [3] **Problem 8.** A simple polarizing filter contains many very thin, closely spaced wires. If the wires are vertical, they block vertical electric fields, allowing only horizontally polarized light to go through; this is a horizontal polarizing filter. One can similarly make diagonal and vertical polarizing filters.
- Suppose that perfectly monochromatic, but unpolarized light is incident on a double slit. What does the intensity pattern on the screen look like?
  - Next, suppose a vertical polarizing filter is placed in front of one slit, and a horizontal polarizing filter is placed in front of the other slit. Now what does the intensity pattern look like?
  - Finally, we further modify the setup of part (b) by placing many diagonal polarizing filters, at  $45^\circ$  to the vertical and horizontal, in front of the *screen*. What does the intensity pattern on the screen look like now?
  - A stack of  $N$  polarizing sheets is arranged so that the angle between any two adjacent sheets is  $\theta/N$ . What is the fraction of light that passes through the stack in the limit  $N \rightarrow \infty$ ?

### Idea 3

It's useful to specify more complicated polarizations with the Jones unit vector,

$$\mathbf{e} = e_x \hat{\mathbf{x}} + e_y \hat{\mathbf{y}}$$

so that the electric field is

$$\mathbf{E}(z, t) = \text{Re}(\mathbf{e} e^{i(kz - \omega t)}).$$

For example, horizontal, vertical, and  $45^\circ$  diagonal polarizations are described by

$$\mathbf{e} = \hat{\mathbf{x}}, \quad \mathbf{e} = \hat{\mathbf{y}}, \quad \mathbf{e} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}.$$

We could also multiply any of these by a phase, which would just shift the phase of the wave.

When linear polarizations are combined with a relative phase, the result is circular (or more generally, elliptical) polarization. For example, when  $\mathbf{e} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$ , we have

$$E_x(z, t) = \frac{E_0}{\sqrt{2}} \cos(kz - \omega t), \quad E_y(z, t) = \frac{E_0}{\sqrt{2}} \sin(kz - \omega t)$$

which is a circularly polarized light wave; the electric field at a fixed point rotates in a circle over time, and if one draws the electric field vectors in a line along  $\hat{\mathbf{k}}$ , they trace out a spiral. Birefringent materials, which have different indices of refraction in different directions, cause such phase shifts, and thus can convert linear polarizations into other polarizations.

**Example 3**

A plane wave with horizontal polarization  $\hat{\mathbf{x}}$  enters an optical device, which does not absorb or reflect any energy. When the plane wave exits the device, it has circular polarization  $(\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$ . What does the device do to light with vertical polarization?

**Solution**

It seems at first that there isn't enough information to solve the problem. The key is to use energy conservation. If the device doesn't absorb or reflect any energy, all of it must come out. So if we put in a plane wave with horizontal polarization and energy  $E$ , a plane wave with circular polarization and energy  $E$  comes out. Similarly, if we put in a plane wave with vertical polarization and energy  $E'$ , a plane wave of unknown polarization and energy  $E'$  comes out.

Now consider superposing waves of energy  $E$  and  $E'$ . The total energy is

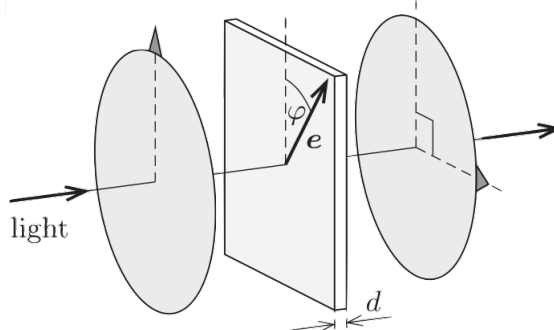
$$\sqrt{E^2 + E'^2 + 2EE' \cos^2 \theta}$$

where  $\theta$  is the angle between the Jones vectors. (This is just interference, as in **W2**.) Now consider passing a superposition of horizontal and vertical polarization through the device. These polarizations are orthogonal, so the energy put in is  $E + E'$ . And the energy that comes out must also be  $E + E'$ , so the outgoing polarizations must also be orthogonal. Thus, a vertical polarization must be converted into circular polarization of the opposite helicity,

$$\hat{\mathbf{x}} \rightarrow \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}, \quad \hat{\mathbf{y}} \rightarrow \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}}.$$

By superposition, this completely specifies what the device does.


- [3] **Problem 9** (MPPP 127). A birefringent material is placed between two orthogonal polarizing filters. The material has thickness  $d$ , and has an index of refraction of  $n_1$  for light linearly polarized along the axis  $\mathbf{e}$ , and  $n_2$  for light polarized about an orthogonal axis.



If the system is illuminated with light of wavelength  $\lambda$ , give a value for  $d$  and orientation of  $\mathbf{e}$  that maximizes the transmitted light.

- [2] **Problem 10** (HRK). A quarter-wave plate is a birefringent plate that causes a  $\pi/2$  phase shift between light polarized along  $\mathbf{e}$  and perpendicular to  $\mathbf{e}$ . Similarly, a half-wave plate causes a  $\pi$  phase shift. Suppose you are given a supply of quarter-wave plates, half-wave plates, and linear

polarizers. Given another object, which may be any one of these things, or an semiopaque disk of ordinary glass, describe how you could identify what the object is.

- [3] **Problem 11** (MPPP 128). In the first 3D movies, spectators would wear glasses with one eye tinted blue and the other tinted red. This was quickly abandoned in favor of a system that used the polarization of light.
- (a) If you wear an old pair of 3D movie glasses, close one eye, and look in the mirror, then you can only see the open eye. Explain how these glasses employ light polarization. What disadvantages might this system have?
  - (b) If you wear a new pair of 3D movie glasses and do the same, then you can only see the closed eye. Explain why.
- [2] **Problem 12** (HRK). A polarizing sheet and a quarter-wave plate are glued together so that, if the combination is placed with face A against a shiny coin, the face of the coin can be seen when illuminated by light of appropriate wavelength. When the combination is placed with face A away from the coin, the coin cannot be seen. Which component is on face A and what is the relative orientation of the components?
- [4] **Problem 13.**  IZhO 2021, problem 3. A problem on the propagation of light through a waveguide, unifying material from **E7** and **W1**.

### 3 Water Waves

Water waves are the most familiar examples of waves in everyday life, but you won't find them mentioned often in introductory textbooks, because they're far more complicated than any other kind of wave we'll consider. In all the problems below, we will completely neglect viscosity, surface tension, and compressibility of the water. Despite this, our results will still only be approximate.

- [4] **Problem 14.** In this problem we consider shallow water waves, the case where the water depth is much less than the wavelength. Let the water has density  $\rho$  and depth  $d$ . Assume the hydrostatic pressure formula applies almost everywhere. Furthermore, assume that if the wave is traveling along the  $x$  direction, the velocity of the water molecules is solely directed along the  $x$  direction and only depends on  $x$ , almost everywhere. The height  $h(x, t)$  of the wave is a sinusoid, as is the horizontal velocity  $v(x, t)$  of the water.
- (a) Find a relation between  $h(x, t)$  and  $v(x, t)$  using Newton's second law.
  - (b) Find a relation between  $h(x, t)$  and  $v(x, t)$  using conservation of mass.
  - (c) Combining these two relations, find the speed of shallow water waves with wavenumber  $k$ .

Now let's consider what happens when a shallow water wave created at sea approaches the shore, and the depth  $d$  gradually decreases.

- (d) Explain why waves always arrive at the shore moving perpendicular to the shoreline.
- (e) If the depth is slowly halved, by what factor is the height of the wave multiplied? This phenomenon is known as shoaling.

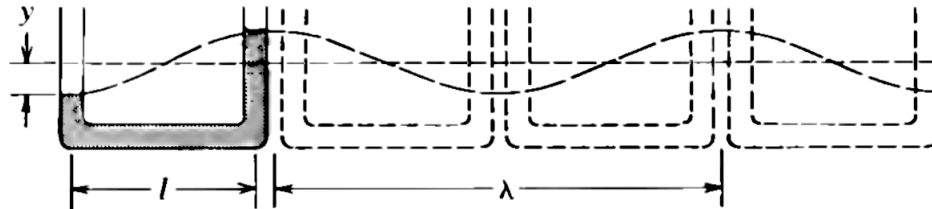
For general depths, the motion of the water molecules is much more complicated. The problem is that in general the waves are neither fully transverse nor fully longitudinal. We know there is a transverse component because the water surface moves up and down, while we can observe the longitudinal component by seeing how people in a wave pool move forward and backward as the wave passes by. In fact, in general the water molecules move in ellipses! In the limit of a deep water wave,  $d \gg \lambda$ , the ellipses reduce to circles.

### Remark

Textbooks commonly say that liquids can't support transverse waves, because they don't support shear stresses. But the waves considered in problem 14 are clearly transverse. This is possible because the textbook statement only applies to the internal forces of water *alone*. At the surface of the water, gravity provides the transverse restoring force; that's why these waves are also commonly called "gravity waves".

[3] **Problem 15** (French 7.20). In this problem we'll make a crude model for a deep water wave by using an artificial setup that forces the water to move purely vertically.

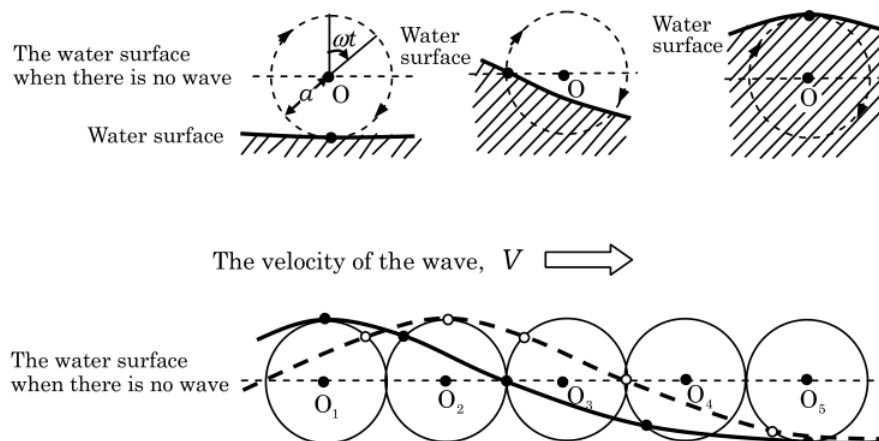
- Consider a U-tube of uniform cross section with two vertical arms, so that the horizontal section of the tube, with length  $l$ , is much longer than the water depth. Show that the period of oscillation is approximately  $\pi\sqrt{2l/g}$ .
- Now suppose a succession of such tubes is placed next to each other and set oscillating to define a succession of crests and troughs, as shown.



This may be interpreted as a water wave of wavelength  $\lambda = 2l$ . Conclude that  $v \approx \sqrt{2/\pi} \sqrt{g/k}$ . Is this a phase or group velocity?

[3] **Problem 16** (Japan). In this problem, we'll give another heuristic treatment of deep water waves, accounting for the circular motion. For simplicity, we assume that the water molecules at the surface of the wave move in uniform circular motion with radius  $a$  and angular velocity  $\omega$ , as shown.





- Express the velocity  $v$  of the wave in terms of the amplitude  $a$  and the angular frequency  $\omega$ .
- Now work in a frame of reference moving to the right with velocity  $v$ . In this frame, the surface of the water is completely stationary, while molecules travel along the surface. Consider a small parcel of water which travels from a valley to a peak. By applying conservation of energy, derive a relationship between  $v$ ,  $\omega$ , and  $g$ .
- Combine these results to show that  $v = \sqrt{g/k}$ , the *exact* result for deep water waves. Again, is this a phase or group velocity? Use this result to calculate the other velocity.

Of course, this derivation is quite incomplete since we haven't shown why circular motion occurs; you can find a complete derivation [here](#).

As you can see, water waves are quite complex. A diagram of the speeds of nine different limiting cases of water waves can be seen in section 8.4 of The Art of Insight.

## 4 Reflection and Refraction

Now we'll introduce reflection and refraction with some real-world applications.

### Idea 4

If a wave hits an interface, while traveling at an angle  $\theta_1$  to the normal to the interface, then it will generically both reflect and refract. The angle of the reflected ray is  $\theta_2 = \theta_1$ , and the angle of the refracted ray obeys  $n_1 \sin \theta_1 = n_2 \sin \theta_2$ . If there is no solution for  $\theta_2$  in the latter equation, then only reflection occurs.

These results follow directly from Huygens' principle, so they are very general, applying to light waves, sound waves, water waves, and so on, as long as the index of refraction  $n_i$  is always defined to be inversely proportional to the wave speed in each medium.

[2] **Problem 17.** Some conceptual questions about reflection and refraction.

- Does the index of refraction determine the phase velocity or the group velocity?

- (b) Does a light beam of finite width get wider or narrower upon passing from air to water? Assume the light enters at an angle to the normal.
- (c) Three mutually perpendicular mirrors intersect so as to form an internal right-angled corner. If a light ray strikes all three mirrors, show that it ends up traveling exactly opposite to its original direction. Can you think of a practical application of such a “corner reflector”?
- (d)

[3] **Problem 18** (Feynman). Practical measurements of the index of refraction.

- (a) How would you measure the index of refraction of a polished rectangular prism of glass?
- (b) How would you measure the index of refraction of a polished plate of black obsidian?

#### Example 4

Let the index of refraction at height  $h$  above the Earth’s surface be  $n(h)$ . In terms of  $n(0)$  and the Earth’s radius  $R$ , what should  $dn/dh$  be at the surface so that light rays orbit in circles around the Earth, with constant height?

#### Solution

First, let’s ignore the curvature of the Earth. Consider a light ray moving slightly upward, at a small angle  $\theta$  to the horizontal, experiencing index of refraction  $n$ . Over a horizontal distance  $L$ , it goes up by a height  $L\theta$ . At this point, it will have a different angle  $\theta'$  to the horizontal, and experience index of refraction  $n + L\theta \, dn/dh$ . Snell’s law says

$$n \cos \theta = \left( n + L\theta \frac{dn}{dh} \right) \cos \theta'$$

and expanding to lowest order in the small angles  $\theta$  and  $\theta'$  gives

$$\frac{n}{2}(\theta'^2 - \theta^2) = L\theta \frac{dn}{dh}.$$

Approximating again to lowest order gives

$$\theta - \theta' \approx -\frac{L}{n} \frac{dn}{dh}.$$

Thus, the light ray turns through an angle of  $(1/n) \, dn/dh$  per unit horizontal distance. For the light ray to stay at a constant height over the curved Earth, this must equal  $1/R$ , giving

$$\frac{dn}{dh} = -\frac{n(0)}{R}.$$

More generally, this calculation shows that light bends towards the direction with higher  $n$ . In the case of air, where  $n - 1 \ll 1$ , we can rewrite this as

$$\frac{d(n - 1)}{dh} \approx -\frac{1}{R}$$

which can plausibly occur on Earth, due to the nice coincidence that  $n - 1$  and  $H/R$  (where  $H$  is the typical scale height of the atmosphere) are both of order  $10^{-3}$ .

**Remark: Mirages**

There are two classes of mirages.

- When  $dn/dh < 0$ , light rays bend down. If there is a distant object at the horizon, its image will appear *above* the horizon. This is called a superior mirage, or “fata morgana”.
- When  $dn/dh > 0$ , light rays bend up. Then a distant object at the horizon will appear *below* the horizon, forming an inferior mirage. This also applies to the sky near the horizon, producing the illusion of water on the ground sometimes seen in deserts.

In air, the refractive index is close to 1, and  $n - 1 \propto \rho \propto P/T$ , where  $\rho$  is the air density and the second step used the ideal gas law. Usually we have  $d\rho/dh < 0$ , since  $dP/dh < 0$  in hydrostatic equilibrium, but it depends on the value of  $dT/dh$ .

- In normal conditions, the Sun warms the ground and the hot air rises and adiabatically mixes the atmosphere (as discussed in **T1**), so that  $dT/dh < 0$ . This partially cancels the effect of the pressure variation, so that  $dn/dh$  is still negative but has small magnitude, so that mirage effects aren’t apparent.
- In rare “thermal inversion” conditions, we have  $dT/dh > 0$ , so that  $dn/dh$  is negative with large magnitude, leading to strong superior mirage effects. If  $dn/dh$  is negative enough, it can match the value computed in example 4, allowing an observer to see arbitrarily far along the horizon despite the curvature of the Earth. This was the reason the famous [Bedford Level experiment](#) concluded the Earth was flat.
- In hot deserts, the air near the ground is very hot, so that  $dT/dh < 0$  with a large magnitude. (A strongly negative  $dT/dh$  also occurs in cold days above water, since the water stays warmer than the air above it.) Here the temperature gradient overpowers the pressure gradient, so that  $dn/dh > 0$  and inferior mirages can occur.


Proponents of the flat Earth hypothesis claim that the Earth only seems curved due to atmospheric refraction. But they have it backwards: in almost all conditions  $dn/dh < 0$ , which makes the Earth look *less* curved than it actually is.

- [4] **Problem 19.** ⌚ IPhO 1995, problem 2. Refraction in the presence of a linearly varying wave speed. (This is a classic setup with a neat solution, also featured in IPhO 1974, problem 2.)
- [3] **Problem 20.** [INPhO 2019, problem 1](#). Another exercise on refraction, with an uglier solution.
- [3] **Problem 21.** ⌚ IPhO 2003, problem 3B. An exercise on refraction and radiation pressure.
- [4] **Problem 22.** ⌚ IPhO 1993, problem 2. Another exercise on the same theme.

## 5 Ray Tracing

### Idea 5

A pointlike object emits light rays in all directions. When those light rays subsequently converge at some other point, that point is the object's real image. If they don't actually converge, but all propagate outward with a common center, that point is the object's virtual image. In general, if we're given that an image exists, we can find its location by following the paths of selected rays from the object and looking for intersections.

- [2] **Problem 23.** [AuPhO 2020, section C](#). A cute series of real-world examples.
- [2] **Problem 24.** A pinhole camera is a simplified camera with no lens. It simply consists of a small hole with a screen behind it.
- Explain how the pinhole camera works by ray tracing.
  - What are the disadvantages of having an especially small hole, or an especially large hole?
  - Assuming the object being photographed is very bright, estimate the optimal aperture size for taking a clear picture with a pinhole camera.
- [2] **Problem 25.** [AuPhO 2013, problem 11](#). A simple problem which tests intuition for everyday optics. Write your answers on the official [answer booklet](#).
- [3] **Problem 26.** [AuPhO 2019, problem 12](#). Another nice question involving drawing good sketches. Write your answers on the official [answer booklet](#).
- [3] **Problem 27.**  [IZhO 2020, problem 1.3](#). A nice test of your intuition for 3D ray tracing.

### Idea 6

Conic sections have some simple properties under reflection.

- Light rays emitted from one focus of an ellipse will all be reflected to its other focus.
- Light rays emitted from one focus on a hyperbola will all be reflected so that the resulting rays all travel radially outward from the other focus.
- Parallel light rays entering a parabola along its symmetry axis (i.e. the axis perpendicular to the directrix) will all be reflected to its focus.

In the language of idea 5, if the foci of an ellipse/hyperbola are called  $F_1$  and  $F_2$ , then an object at  $F_1$  produces a real/virtual image at  $F_2$ . Note that a parabola is simply an ellipse in the limit where  $F_1$  becomes very far away, so that the rays from the object which make it near  $F_2$  are approximately parallel.

- [2] **Problem 28** (Povey). The mirascope is a toy consisting of two parabolic mirrors, pointing toward each other, so that the focus of each one is at the vertex of the other.



- (a) When an object is placed at the bottom vertex, a real image appears at the top vertex. Why?
- (b) How is the image oriented relative to the object?

### Idea 7: Paraxial Approximation

If a light ray hits a thin lens of focal length  $f$  at a shallow angle, and at a distance  $y \ll f$  above the lens's center, then it will exit the lens bent vertically by an angle  $\pm y/f$ , where the sign depends on whether the lens is converging or diverging. (For example, any light ray going straight through the lens's center isn't bent at all.) This is the paraxial approximation, which only holds for light rays incident at shallow angles near the center of the lens.

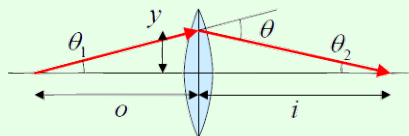
Conversely, if you don't know the focal length of a system, you can use this idea to find it. For example, the lensmaker's equation, giving the focal length of a lens of radii of curvature  $R_1$  and  $R_2$  and thickness  $d$ , is

$$\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} + \frac{(n - 1)d}{nR_1R_2} \right)$$

and can be seen by computing the bending of the light ray at each interface.

### Example 5

An object is placed a distance  $o$  behind a thin converging lens with focal length  $f$ .



An image is formed a distance  $i$  in front of the lens. How are  $o$ ,  $i$ , and  $f$  related?

### Solution

The horizontal light ray goes straight through, so let's consider another light ray which emerges at a small angle  $\theta_1$  to the horizontal. Then we read off

$$\theta_1 \approx \frac{y}{o}, \quad \theta_2 \approx \frac{y}{i}$$

but their sum is the deflection  $y/f$ , from which we conclude

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}.$$

This is the familiar thin lens equation.

### Example 6

A candle is placed behind a converging lens. An image is formed on a screen on the other side of the lens. Now suppose that the *top* half of the lens is covered with a black cloth. Describe how the image changes.

### Solution

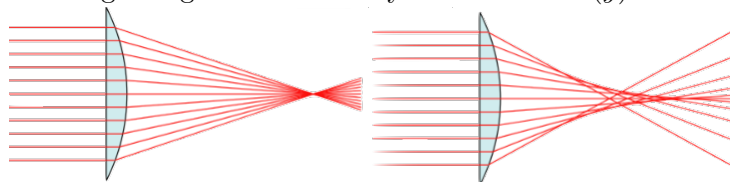
It is tempting to say that half of the candle's image disappears, but that's not right. Ray tracing shows that you can get a complete image of the candle, since there are always rays that pass through the bottom half of the lens. Instead, by blocking half the lens, the image gets half as bright.

- [3] **Problem 29.** ⌚ USAPhO 2024, problem A3. A series of optics exercises relevant for real cameras.

### Idea 8: Fermat's Principle

For fixed starting and ending points, light always takes the path of least time. This implies that if light from point  $P$  is all focused at point  $P'$ , then all the relevant paths from  $P$  to  $P'$  take the same time. This principle is completely equivalent to the laws of reflection and refraction above, but may be more useful in certain situations.

- [3] **Problem 30.** Find the shape of a lens that will focus parallel incoming rays to a point, as shown at left. Concretely, suppose that the rays coming in along the  $+\hat{x}$  direction, the left edge of the lens is the  $y$  axis, and the right edge is described by the function  $x(y)$ . What kind of curve is  $x(y)$ ?



The fact that  $x(y)$  isn't exactly a circle is responsible for spherical aberration. That is, a spherical lens will fail to focus all incoming horizontal light to a point, as shown at right. However, since spheres are much easier to deal with, most of the lenses we consider below will be spherical.