PHYS-330: QFT I — Solutions 4

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1. Using Wick's theorem. (7 points)

For both parts of this problem, you can use Wick's theorem for two real scalar fields.

a) Show that $T(\phi(x_1)\phi(x_2))$ and $:\phi(x_1)\phi(x_2):$ both remain the same when x_1 and x_2 are exchanged. Using these results, explain why $D_F(x_1 - x_2) = D_F(x_2 - x_1)$.

Solution: For the time ordered product this is true by definition; it puts the later operator on the left no matter what their original order was. As for the normal ordered product, we can split each field into parts with only creation and annihilation operators,

$$\phi(x_i) = \phi_i^+ + \phi_i^-, \qquad 0 = \phi_i^- |0\rangle = \langle 0|\phi_i^+. \tag{S1}$$

The normal ordered product is defined by

$$:\phi_1\phi_2: = \phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-. \tag{S2}$$

Since $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$, this expression is identical if we exchange x_1 and x_2. Finally, Wick's theorem tells us that the Feynman propagator is the difference of a time ordered product and a normal ordered product, so it is also symmetric in its arguments.

b) Prove Wick's theorem for a time ordered product of three scalar fields,

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = \,:\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)D_F(x_2 - x_3) + \phi(x_2)D_F(x_3 - x_1) + \phi(x_3)D_F(x_1 - x_2). \tag{1}$$

Solution: We simply work through the cases for the time ordering of x_1, x_2, and x_3. By definition,

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = \phi_1 T\{\phi_2\phi_3\}\Big|_{t_1>t_2,t_3} + \phi_2 T\{\phi_1\phi_3\}\Big|_{t_2>t_1,t_3} + \phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1}. \tag{S3}$$

Now let's apply Wick's theorem to the last term,

$$\phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1} = (\phi_3^+ + \phi_3^-)(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^- + D_F(x_2 - x_3))\Big|_{t_3>t_2,t_1}. \tag{S4}$$

A term is normal ordered if all the ϕ_i^+'s are to the left of all the ϕ_i^-'s, so

$$:\phi_1\phi_2\phi_3: = (\phi_3^+ + \phi_3^-)(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-)\Big|_{t_3>t_2,t_1} + [\phi_3^-, \phi_2^+]\phi_1 + [\phi_3^-, \phi_1^+]\phi_2\Big|_{t_3>t_2,t_1}. \tag{S5}$$

We have therefore shown that

$$\phi_3 T\{\phi_1\phi_2\}\Big|_{t_3>t_2,t_1} = \big(:\phi_1\phi_2\phi_3: + \phi_1[\phi_3^-, \phi_2^+] + \phi_2[\phi_3^-, \phi_1^+] + \phi_3 D_F(x_1 - x_2)\big) \times \theta(x_3^0 - x_2^0)\theta(x_3^0 - x_1^0). \tag{S6}$$

When we add the other two terms, the first terms of each one combine to give $:\phi_1\phi_2\phi_3:$ with no step functions. In addition, pairs of commutator terms combine into Feynman propagators by

$$D_F(x, y) = \theta(x^0 - y^0)[\phi^-(x), \phi^+(y)] + \theta(y^0 - x^0)[\phi^-(y), \phi^+(x)]. \tag{S7}$$

To see how this works, let's consider the terms that eventually combine to yield $\phi(x_3)D_F(x_1 - x_2)$,

$$\phi_3\Big(D_F(x_1 - x_2)\theta(t_3 - t_2)\theta(t_3 - t_1) + [\phi_1^-, \phi_2^+]\theta(t_1 - t_3)\theta(t_1 - t_2) + [\phi_2^-, \phi_1^+]\theta(t_2 - t_1)\theta(t_2 - t_3)\Big). \tag{S8}$$

The first term here is the desired term when t_3 is the latest time. The second and third terms combine to give the Feynman propagator for any ordering of t_1 and t_2, under the condition that t_3 is *not* the latest. So the sum is $\phi(x_3)D_F(x_1 - x_2)$, and the logic for the other terms is similar.

2. Diagrams in ϕ^3 theory. (18 points)

Consider a real scalar field ϕ of mass m with a cubic self-interaction, so that

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\phi^3. \tag{2}$$

In this problem, you will draw some diagrams in this interacting theory and compute some symmetry factors. As in lecture, $|0\rangle$ is the free vacuum, the interaction Hamiltonian density is $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, and H_I is the interaction Hamiltonian in interaction picture.

a) Using Wick's theorem, evaluate the vacuum correlation function

$$\langle 0| T \exp\left(-i \int dt\, H_I(t)\right) |0\rangle \tag{3}$$

up to and including terms of order λ^2. Your result should be an explicit expression in terms of integrals of the position-space Feynman propagator. (You will run into terms proportional to $D_F(0)$, which is infinite. Don't worry about this for now; we'll come back to this issue when we discuss renormalization.)

Solution: Let the desired correlation function be $c_0 + \lambda c_1 + \lambda^2 c_2 + \mathcal{O}(\lambda^3)$. Then

$$c_0 = \langle 0|0\rangle = 1 \tag{S9}$$

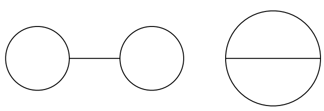
and

$$c_1 = \frac{i}{3!}\langle 0| T\left(\int d^4x\, \phi(x)\phi(x)\phi(x)\right)|0\rangle = 0 \tag{S10}$$

because we have an odd number of copies of ϕ. Finally,

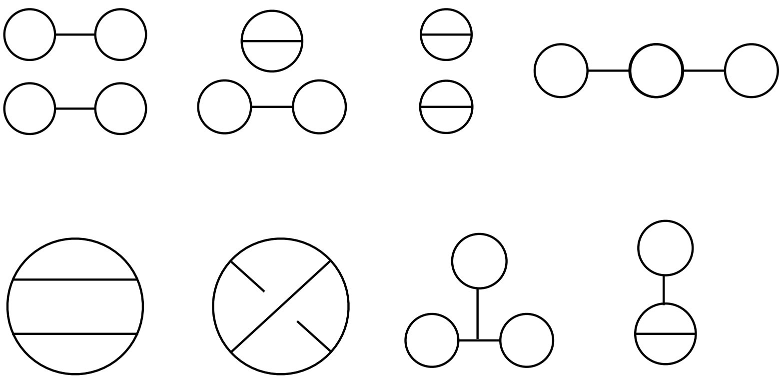
$$\begin{aligned} c_2 &= \frac{1}{2!}\left(\frac{i}{3!}\right)^2 \langle 0|T\left(\int d^4x\, d^4y\, \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\right)|0\rangle && \text{(S11)} \\ &= -\frac{1}{72}\int d^4x\, d^4y\, \left[3\cdot 3\cdot D_F(x-x)D_F(x-y)D_F(y-y) + 3!\cdot D_F(x-y)^3\right] && \text{(S12)} \\ &= -\int d^4x\, d^4y\, \left(\frac{1}{8}D_F(0)^2 D_F(x-y) + \frac{1}{12}D_F(x-y)^3\right). && \text{(S13)} \end{aligned}$$

The terms in c_2 correspond to the diagrams shown below.



b) Draw all distinct diagrams that contribute to the vacuum correlation function at order λ^3 and λ^4. For this part, you do not need to evaluate the diagrams or compute any symmetry factors, but you should neatly display all of your diagrams in one box.

Solution: There are no diagrams at any odd order. The 8 diagrams at order λ^4 are shown below.

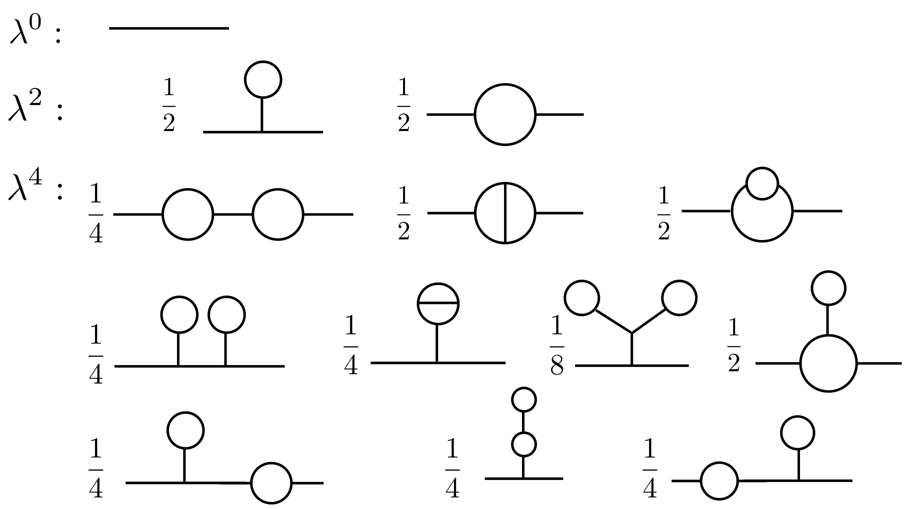


c) Now consider the two-point correlation function

$$\langle \Omega | T\left(\phi_H(x)\phi_H(y)\right)|\Omega\rangle = \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left(-i\int dt\, H_I\right)\right)|0\rangle}{\langle 0|T\exp\left(-i\int dt\, H_I\right)|0\rangle} \tag{4}$$

where ϕ_H and ϕ_I are Heisenberg and interaction picture fields. As discussed in class, only connected diagrams contribute to the left-hand side, i.e. diagrams where all fields are connected to at least one of the external points. Find all diagrams that contribute up to and including order λ^4, along with their symmetry factors, which you can compute with any method. Again, give your answer by drawing everything neatly inside one box. (Hint: there are 13 fully connected diagrams, i.e. diagrams where all fields are connected to *both* of the external points.)

Solution: Again, odd orders in λ always yield correlation functions on the right-hand side with an odd number of fields, which vanish since $\langle 0|\phi|0\rangle = 0$. The fully connected diagrams that contribute at orders λ^0, λ^2, and λ^4 are shown below, along with their symmetry factors.



These symmetry factors can all be found with the rules discussed in class:

- $1/2$ for every vertex connected to itself.
- $1/k!$ for k propagators connecting the same two vertices.

- 1/2 for two interchangeable vertices.

Of course, you can also compute them by hand if you want to make sure. There are also four non-fully connected diagrams, which are less interesting.

3. Recovering nonrelativistic quantum mechanics. (15 points)

A free complex scalar field Φ of mass m satisfies the Klein–Gordan equation, so its plane wave modes $e^{-ip\cdot x}$ satisfy $\omega^2 = |\mathbf{k}|^2 + m^2$. This implies we have plane waves with both positive and negative ω, which is universal in relativistic theories. Upon quantization, excitations of these modes correspond to matter and antimatter, with opposite charges.

The nonrelativistic limit is $|\mathbf{k}| \ll m$, but since antimatter is inherently a feature of relativistic theories, we should also throw away the negative ω modes. To do this, take

$$\Phi(x) = e^{-imt}\,\chi(x)/\sqrt{2m} \tag{5}$$

and assume $|\dot{\chi}| \ll m\chi$, so that only nonrelativistic positive ω modes are excited. The $\sqrt{2m}$ factor transfers us back to nonrelativistic normalization.

a) Simplify the complex scalar field action to

$$S = \int d^4x \left(i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi \right) \tag{6}$$

by dropping a term that is small for $|\dot{\chi}| \ll m\chi$.

Solution: The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= (\partial_\mu\Phi)(\partial^\mu\Phi)^* - m^2\Phi^*\Phi & \text{(S14)}\\
&= \frac{1}{2m}(\partial_\mu(e^{-imt}\chi))(\partial^\mu(e^{imt}\chi^*)) - \frac{m}{2}\chi^*\chi & \text{(S15)}\\
&= -\frac{1}{2m}\nabla\chi\cdot\nabla\chi^* + \frac{1}{2m}(\partial_t(e^{-imt}\chi))(\partial_t(e^{imt}\chi^*)) - \frac{m}{2}\chi^*\chi & \text{(S16)}
\end{aligned}$$

where we split the summation over μ into a sum over $\mu = i$ and the $\mu = 0$ term. The middle term is

$$\frac{1}{2m}(\dot{\chi} - im\chi)(\dot{\chi}^* + im\chi^*) = \frac{\dot{\chi}^*\dot{\chi}}{2m} + \frac{i}{2}(\dot{\chi}\chi^* - \dot{\chi}^*\chi) + \frac{m}{2}\chi^*\chi. \tag{S17}$$

Of these terms, the first is negligible in the nonrelativistic limit, and the last term cancels with the last term of Eq. (S16). To get the second term in the desired form, we integrate the action by parts,

$$\frac{i}{2}\int d^4x\,(\dot{\chi}\chi^* - \dot{\chi}^*\chi) = i\int d^4x\,\chi^*\dot{\chi}. \tag{S18}$$

b) Find the Euler–Lagrange equations for χ and χ^*, and the conserved current $J^\mu = (\rho, \mathbf{J})$ corresponding to the symmetry $\chi \to e^{-i\alpha}\chi$ and $\chi^* \to e^{i\alpha}\chi^*$. (Be careful to account for minus signs from the relativistic metric.)

Solution: As usual, we treat χ and χ^* as independent variables. For both parts of this question, we need to compute the derivatives of $\mathcal{L}$ with respect to $\partial_\mu\chi$ and $\partial_\mu\chi^*$. These are

$$\frac{\partial\mathcal{L}}{\partial\dot{\chi}} = i\chi^*, \qquad \frac{\partial\mathcal{L}}{\partial(\nabla\chi)} = -\frac{1}{2m}\nabla\chi^* \tag{S19}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{\chi}^*} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\nabla\chi^*)} = -\frac{1}{2m}\nabla\chi. \tag{S20}$$

The Euler–Lagrange equation for χ is

$$\frac{\partial \mathcal{L}}{\partial \chi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\chi}} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \chi)}. \tag{S21}$$

The left-hand side is zero, and plugging in the results for the right-hand side gives

$$i\dot{\chi}^* = \frac{1}{2m} \nabla^2 \chi^*. \tag{S22}$$

Similarly, the Euler–Lagrange equation for χ^* is

$$\frac{\partial \mathcal{L}}{\partial \chi^*} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\chi}^*} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \chi^*)}. \tag{S23}$$

Now the left-hand side is nonzero, but the first term on the right-hand side is zero, giving

$$i\dot{\chi} = -\frac{1}{2m} \nabla^2 \chi. \tag{S24}$$

As expected, the two Euler–Lagrange equations are conjugates of each other, and the equation of motion for χ is just the Schrodinger equation.

For infinitesimal α, the change in the fields is $\delta\chi = -i\alpha\chi$ and $\delta\chi^* = i\alpha\chi^*$, so Noether's theorem gives

$$J^\mu = -i\chi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} + i\chi^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi^*)}. \tag{S25}$$

The density of the conserved charge is

$$\rho = -i\chi \frac{\partial \mathcal{L}}{\partial \dot{\chi}} + i\chi^* \frac{\partial \mathcal{L}}{\partial \dot{\chi}^*} = \chi^* \chi. \tag{S26}$$

The current density is

$$\mathbf{J} = -i\chi \frac{\partial \mathcal{L}}{\partial(\nabla \chi)} + i\chi^* \frac{\partial \mathcal{L}}{\partial(\nabla \chi^*)} = \frac{i}{2m}(\chi \nabla \chi^* - \chi^* \nabla \chi). \tag{S27}$$

These are the same form as the probability density and probability current in nonrelativistic quantum mechanics. (But here they don't mean the same thing: we're still talking about classical field theory, not quantum mechanics.)

For the relativistic complex scalar field, we had to canonically quantize Φ, Π, Φ^*, and Π^* as two pairs of phase space variables. But for this nonrelativistic field, the canonical momentum for χ is just $i\chi^*$, so we only have one pair. This is because we threw away the negative ω solutions, and it implies the canonical commutation relations are

$$[\chi(\mathbf{x}), \chi^*(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{7}$$

with all other commutators vanishing. These can be satisfied with the mode expansion

$$\chi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \qquad \chi^*(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{8}$$

c) Show that $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})$. (Of course, the other commutators vanish.) Then show that the single-particle states $|\mathbf{x}\rangle = \chi^*(\mathbf{x})|0\rangle$ and $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ behave like nonrelativistic position and momentum states, obeying

$$\langle \mathbf{y}|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{y} - \mathbf{x}), \quad \langle \mathbf{q}|\mathbf{p}\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{p}), \quad \langle \mathbf{p}|\mathbf{x}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{9}$$

The factors of 2π are just due to our Fourier transform convention.

Solution: We use the same đ and $\not\delta$ notation as in the solutions to the second problem set. Inverting the Fourier transform gives

$$a(\mathbf{p}) = \int d\mathbf{x}\, \chi(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{S28}$$

which implies

$$\begin{aligned} [a(\mathbf{p}), a^\dagger(\mathbf{q})] &= \int d\mathbf{x}\, d\mathbf{y}\, [\chi(\mathbf{x}), \chi^*(\mathbf{y})]\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} && \text{(S29)} \\ &= \int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}} && \text{(S30)} \\ &= \not\delta(\mathbf{p}-\mathbf{q}). && \text{(S31)} \end{aligned}$$

Now, we have

$$\langle \mathbf{y}|\mathbf{x}\rangle = \langle 0|\chi^*(\mathbf{y})\chi(\mathbf{x})|0\rangle = \delta(\mathbf{y}-\mathbf{x}) \tag{S32}$$

where we applied the commutation relation and used $\chi^*(\mathbf{y})|0\rangle = 0$. Next,

$$\langle \mathbf{q}|\mathbf{p}\rangle = \langle 0|a(\mathbf{q})a^\dagger(\mathbf{p})|0\rangle = \not\delta(\mathbf{q}-\mathbf{p}) \tag{S33}$$

where we applied the commutation relation and used $a(\mathbf{q})|0\rangle = 0$. Finally,

$$\langle \mathbf{p}|\mathbf{x}\rangle = \langle 0|a(\mathbf{p}) \int \text{đ}\mathbf{q}\, a^\dagger(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}|0\rangle = e^{i\mathbf{p}\cdot\mathbf{x}} \tag{S34}$$

d) A general single-particle state takes the form

$$|\psi(t)\rangle = \int d^3\mathbf{x}\, \psi(\mathbf{x},t)|\mathbf{x}\rangle \tag{10}$$

where $\psi(\mathbf{x},t)$ is the particle's position-space wavefunction at time t. As always in quantum theories, the state evolves over time as $i\,\partial_t|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian. Write this equation as a partial differential equation involving $\psi(\mathbf{x},t)$.

Solution: First, we need to find the Hamiltonian. Applying the Legendre transform (or equivalently considering the element T^{00} of the stress-energy tensor),

$$\mathcal{H} = \dot\chi \frac{\partial \mathcal{L}}{\partial \dot\chi} - \mathcal{L} = \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi. \tag{S35}$$

Therefore, in terms of creation and annihilation operators, the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2m}\int d\mathbf{x}\, \nabla\chi^* \cdot \nabla\chi && \text{(S36)} \\ &= \frac{1}{2m}\int d\mathbf{x} \int \text{đ}\mathbf{p}\, \text{đ}\mathbf{q}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}\, (i\mathbf{p}\cdot(-i\mathbf{q}))\, a(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}} && \text{(S37)} \\ &= \frac{1}{2m}\int d\mathbf{x} \int \text{đ}\mathbf{p}\, \text{đ}\mathbf{q}\, (\mathbf{p}\cdot\mathbf{q})\, e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\, a^\dagger(\mathbf{p}) a(\mathbf{q}) && \text{(S38)} \\ &= \frac{1}{2m}\int \text{đ}\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) a(\mathbf{p}). && \text{(S39)} \end{aligned}$$

The Hamiltonian acts on position eigenstates as

$$\begin{aligned} H|\mathbf{x}\rangle &= \frac{1}{2m}\int \text{đ}\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) a(\mathbf{p}) \int \text{đ}\mathbf{q}\, a^\dagger(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}}|0\rangle && \text{(S40)} \\ &= \frac{1}{2m}\int \text{đ}\mathbf{p}\, |\mathbf{p}|^2\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle && \text{(S41)} \\ &= -\frac{\nabla^2}{2m}\int \text{đ}\mathbf{p}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle && \text{(S42)} \\ &= -\frac{\nabla^2}{2m}|\mathbf{x}\rangle. && \text{(S43)} \end{aligned}$$

Finally, the time evolution equation is

$$\int d\mathbf{x}\, i\dot{\psi}\, |\mathbf{x}\rangle = \int d\mathbf{x}\, \psi\, H|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\, \psi\, \nabla^2|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\, (\nabla^2\psi)|\mathbf{x}\rangle \tag{S44}$$

where we integrated by parts in the last step. Matching the coefficients of $|\mathbf{x}\rangle$ gives

$$i\dot{\psi} = -\frac{\nabla^2\psi}{2m} \tag{S45}$$

which is of course the nonrelativistic single-particle Schrodinger equation. Thus, our nonrelativistic quantum scalar field theory contains single-particle nonrelativistic quantum mechanics. In addition, it naturally contains states with arbitrarily many particles.

Notice that the single-particle quantum wavefunction ψ obeys the same equation of motion as the classical field χ, and both have a conserved current of the same form. This formal correspondence is a general structural feature of quantum field theory, and one of the original clues used to construct it. It is also the reason the pioneers of quantum field theory called it "second quantization," under the mistaken impression they were quantizing ψ a second time rather than χ for a first time.

e) Take the term in the action you dropped in part (a) and simplify it by applying the Euler–Lagrange equations you found in part (b). Find how the Hamiltonian in part (d) changes when this term is added. What is its physical interpretation?

Solution: We had dropped the term $\dot{\chi}^*\dot{\chi}/2m$, and plugging in the Euler–Lagrange equations gives

$$\mathcal{L} \supset \frac{1}{8m^3}\nabla^2\chi^*\, \nabla^2\chi. \tag{S46}$$

This contributes an additional term to the Hamiltonian,

$$H \supset -\frac{1}{8m^3}\int d\mathbf{x}\, \nabla^2\chi^*\, \nabla^2\chi \tag{S47}$$

and following the same steps as in part (d) yields

$$H \supset -\frac{1}{8m^3}\int đ\mathbf{p}\, |\mathbf{p}|^4\, a^\dagger(\mathbf{p})a(\mathbf{p}). \tag{S48}$$

This is simply the leading relativistic correction to the kinetic energy, which you often see, e.g. when computing the fine structure of the hydrogen atom. Applying a similar (but more technically complex) procedure to the Dirac equation, known as the Foldy—Wouthuysen transformation, one can derive the fine structure corrections for the hydrogen atom. You can continue turning the crank to get relativistic corrections at all orders.

4. ⋆ The force between sources. (5 points)

In this optional problem, we'll compute a directly measurable quantity with quantum field theory, for the first time in this course. So far we have only learned to compute correlation functions, which often do not have a direct physical interpretation. For example, the vacuum correlation function Eq. (3) can be used to find the energy of the interacting vacuum relative to the free vacuum, but this isn't directly measurable. (You can't turn off the interaction in real life.)

Instead, let's again consider a free real scalar field with source $J(x)$,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{11}$$

If we change the vacuum by including a static source $J(x)$, the resulting shift in vacuum energy can be directly interpreted as the energy of the source, which is measurable.

a) Using Wick's theorem, show that

$$\langle 0|\, T \exp\left(i \int d^4x\, \phi(x) J(x)\right) |0\rangle = \exp\left(-\frac{1}{2} \int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right). \tag{12}$$

Solution: Again, odd-order terms vanish, so the left-hand side is

$$\sum_n \frac{i^{2n}}{(2n)!} \int d^4x_1 \cdots d^4x_{2n}\, J(x_1) \cdots J(x_{2n}) \langle 0|\, T\phi(x_1) \cdots \phi(x_{2n}) |0\rangle. \tag{S49}$$

There are $(2n-1)(2n-3)\cdots(3)(1)$ possible Wick contractions, all of which yield the same thing, giving

$$\sum_n \frac{i^{2n}}{(2n)(2n-2)\cdots(2)} \left(\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)^n. \tag{S50}$$

Simplifying the numeric factors gives

$$\sum_n \frac{1}{n!} \left(-\frac{1}{2} \int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)^n \tag{S51}$$

which is indeed the Taylor series of the desired right-hand side.

b) Now consider the case where two point sources are turned on for a long time T,

$$J(x) = g\, f(t)\, (\delta^{(3)}(\mathbf{x}) + \delta^{(3)}(\mathbf{x} - \mathbf{R})), \qquad f(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

If you plug this into Eq. (12), you'll get infinite terms involving $D_F(0)$, reflecting the fact that point sources have infinite energy. Discarding these terms so we can focus on the sources' interaction energy, evaluate Eq. (12) assuming $T \gg R \gg 1/m$.

Solution: It's helpful to go to momentum space, where we get

$$\exp\left(-\frac{i}{2} \int d^4x\, d^4y\, J(x) J(y) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}\right). \tag{S52}$$

Taking only the two interaction terms and imposing the position-space delta functions gives

$$\exp\left(-ig^2 \int dx^0\, dy^0\, f(x^0) f(y^0) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip^0(x^0-y^0)} e^{-i\mathbf{p}\cdot\mathbf{R}}}{p^2 - m^2 + i\epsilon}\right). \tag{S53}$$

At this point, we can perform the temporal integrals; they are a representation of the delta function,

$$\int_{-T/2}^{T/2} dx^0 \int_{-T/2}^{T/2} dy^0\, e^{-ip^0(x^0-y^0)} = 2\pi T\, \delta(p^0) \tag{S54}$$

for large T, where the normalization can be found by integrating both sides against p^0. We thus get

$$\exp\left(ig^2 T \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{R}}}{|\mathbf{p}|^2 + m^2}\right) \tag{S55}$$

where we dropped the $i\epsilon$ since the denominator won't vanish. This is a standard integral you can look up or do with spherical coordinates and contour integration (we'll see it again later in the course), so

$$\exp\left(ig^2 T\, \frac{e^{-mr}}{4\pi r}\right). \tag{S56}$$

c) Relate your answer to the interaction energy $V(R)$ between the sources, and find it. Is the force between them attractive or repulsive?

Solution: If the interaction energy is V, then the correlation function should be e^{-iVT}, reflecting the change in energy of the vacuum. Comparing with above expression gives

$$V(r) = -\frac{g^2 e^{-mr}}{4\pi r}. \tag{S57}$$

The force falls off exponentially on the scale $1/m$, and it is attractive. (Incidentally, since we just have a free field theory and classical sources, we could have done this all much more easily in classical field theory. But the approach we've just taken is much more powerful and general.)