Modern II: Atoms, Particles, and Nuclei

Chapters 48, 50, 51, and 52 of Halliday and Resnick are a useful introduction. For further reading, see chapters 12 through 14 of Krane for nuclear and particle physics, and section 5.2 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition) for atomic physics. If you'd like to learn a lot more about these subjects, see the MIT OCW 22.01 course on nuclear engineering, chapters 1 and 2 of Griffiths' *Introduction to Elementary Particles*, or David Tong's Lectures on Particle Physics. For some neat reading about symmetries in particle physics, see chapter I-52 of the Feynman lectures. For all problems, you can consult the periodic table. There is a total of **84** points.

1 Nuclear Decay

Idea 1

Atomic nuclei are written as ${}^{A}_{N}\mathrm{X}$ where X is the name of the element, A is the mass number (number of neutrons plus protons), and N is the atomic number (number of protons). Since N can be inferred from X, we often don't write it.

Idea 2

The most common nuclear decay channels are alpha decay,

$$ {}^{A}_{Z}\mathrm{X} \to {}^{A-4}_{Z-2}\mathrm{X}' + {}^{4}_{2}\mathrm{He} $$

and beta decay,

$$ {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z+1}\mathrm{X}' + e^- + \bar{\nu}_e. $$

Here, $\bar{\nu}_e$ is a light neutral particle called an anti-electron neutrino. A variant of beta decay, called β^+ decay or positron emission, is

$$ {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X}' + e^+ + \nu_e $$

where ν_e is called an electron neutrino, and e^+ is a positron. If electrons are present, the nuclei may also capture them, leading to the process

$$ {}^{A}_{Z}\mathrm{X} + e^- \to {}^{A}_{Z-1}\mathrm{X}' + \nu_e. $$

Finally, nuclei can decay from excited states by emitting photons, in gamma decay.

There are many more processes, such as inverse beta decay or double beta decay. However, the general principles underlying which decays are allowed are simple: baryon number, electric charge, and electron number are all conserved. In the restricted setting of nuclear processes,

$$ \begin{aligned} \text{baryon number} &= \text{number of protons and neutrons} \\ \text{electric charge} &= \text{number of proton and positrons} - \text{number of electrons} \\ \text{electron number} &= \text{number of electrons and electron neutrinos} \\ &\quad - \text{number of positrons and anti-electron neutrinos.} \end{aligned} $$

Idea 3

The amount of energy released in a nuclear decay can be inferred from the drop in mass energy, $\Delta E = (\Delta m)c^2$. A nuclear decay can only spontaneously occur if it lowers the energy of the *entire* nucleus. To emphasize this point, note that at the level of individual nucleons, $\beta^\pm$ decay involve the processes

$$n \to p + e^- + \overline{\nu}_e, \quad p \to n + e^+ + \nu$$

respectively. Either of these processes could be energetically favorable inside a nucleus, depending on its composition. But an isolated proton will never decay, because protons are heavier than neutrons.

[1] **Problem 1** (Krane 12.38)**.** Complete the following decays:

(a) $^{27}\mathrm{Si} \to {}^{27}\mathrm{Al} +$

(b) $^{74}\mathrm{As} \to {}^{74}\mathrm{Se} +$

(c) $^{228}\mathrm{U} \to \alpha +$

(d) $^{93}\mathrm{Mo} + e^- \to$

(e) $^{131}\mathrm{I} \to {}^{131}\mathrm{Xe} +$

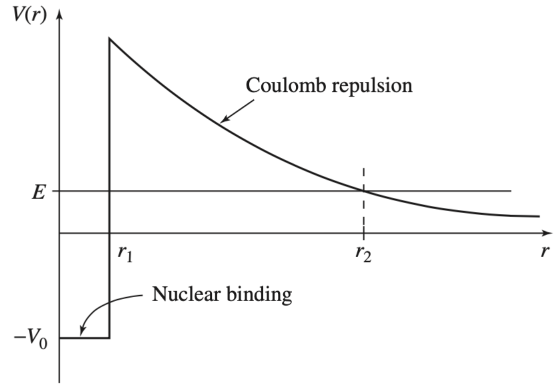
Example 1: PPP 189

^{7}Be is a radioactive element with a half-life of 53.37 days. When isotope 7 of beryllium is heated to a few thousand degrees, its half-life changes. This is strange, because nuclear processes typically involve much higher energy scales. What is the explanation for this?

Solution

Temperatures at this scale are not enough to affect nuclear physics, but are enough to affect atomic physics. The electrons gain enough energy to leave the atoms, then hit other nuclei, causing electron capture, changing the isotope of beryllium and hence its half-life.

[4] **Problem 2.** In Gamow's theory of alpha decay, alpha particles can escape from nuclei by quantum tunneling. The alpha particle is bound to the nucleus by a nuclear force, which we model as a finite square well, $V(r) = -V_0$ for $r < r_1$, and repelled by the Coulomb force, $V(r) = k(Ze)(2e)/r = \alpha/r$. The combination of the two creates a potential barrier the alpha particle must tunnel though. Let the alpha particle have mass m and energy E.



(a) Using classical mechanics, calculate the time between collisions with the wall. This is also correct in quantum mechanics; one can take the wavefunction to be a wavepacket, which really does collide with the walls with the same frequency.

(b) In quantum mechanics, each collision has an associated amplitude to escape by quantum tunneling. To compute this, recall from **X1** that the WKB approximation states that the wavefunction picks up a phase $e^{i\theta}$, where

$$\theta = \frac{1}{\hbar}\int p\,dx.$$

Calculate θ by integrating from r_1 to r_2, assuming that $r_1 \ll r_2$ for simplicity. You should find that θ is a complex number, indicating the wavefunction exponentially decays in the barrier. (Hint: you will find a tricky integral, for which you should use a trigonometric substitution.)

(c) The probability of escape scales as the amplitude squared. Write down an approximate expression for the timescale τ for decay to occur.

This model is very rough, so the numeric and slowly varying prefactors should not be expected to be accurate. But the exponential dependence of the timescale on the energy, which you should have found is due to the tunneling probability scaling as $e^{-\sqrt{E_g/E}}$ for some constant E_g, is by far the most important piece, and it fits experimental results.

(d) In nuclear fusion reactions in the Sun, the process above occurs in reverse: an incoming alpha particle (i.e. helium nucleus) needs to tunnel through the Coulomb barrier to fuse with another nucleus. The initial energy is Boltzmann distributed as e^{-E/k_BT}, so the fusion rate is

$$\Gamma \sim \int dE\, e^{-\sqrt{E_g/E}} e^{-E/k_BT}.$$

The integrand is the product of a rapidly rising exponential and a rapidly falling exponential. Estimate the exponential part of the dependence of Γ on T.

[3] **Problem 3.** Consider the process by which an electron absorbs a single photon, $e^- + \gamma \to e^-$.

(a) Show that this process is forbidden by energy-momentum conservation. By time reversal, emission of a single photon should be forbidden as well. This is quite puzzling, since we already know of many processes where something like absorption or emission seems to happen.

(b) Can an electron in an isolated atom absorb a single photon? If so, why doesn't the reasoning in part (a) work? If not, how can atoms absorb photons at all, as described in **X1**?

(c) Can isolated nuclei emit single photons? If so, why doesn't the reasoning in part (a) work? If not, how can gamma decay occur?

(d) Can isolated electrons absorb or emit *classical* electromagnetic radiation? If so, why doesn't the reasoning in part (a) work? If not, how can Thomson scattering (covered in **E7**) happen?

Idea 4

Radioactive decay is a memoryless process: in an infinitesimal time interval dt, any nucleus has a probability $\lambda\, dt$ of decaying, regardless of its previous history. As a result, the number of radioactive nuclei falls exponentially as

$$N(t) = N_0 e^{-\lambda t}.$$

The activity $A(t)$ is the rate of decay events, and also falls exponentially,

$$A(t) = A_0 e^{-\lambda t}.$$

The mean lifetime of the nuclei is $\tau = 1/\lambda$.

[3] **Problem 4.** This problem tests your understanding of memoryless processes. Below are several plausible ways to measure τ.

(a) We start a stopwatch at noon and stop it when the next decay happens, giving t_1.

(b) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay, giving t_2.

(c) We have an intern watch the sample continuously, then at noon, ask them how long it was since the last decay. We then set our stopwatch so that $t = 0$ when that decay happened, and stop the stopwatch when the next decay happens, giving t_3.

(d) We continuously watch the sample, start a stopwatch when the first decay happens, then stop it when the next decay happens, giving t_4.

We repeat procedure i many times, so the average of t_i is τ_i. Find the τ_i in terms of τ.

[2] **Problem 5** (Krane 12.37)**.** A radioactive sample contains N_0 atoms at time $t = 0$. It is observed that N_1 radioactive atoms remain at time t_1 and then decay by time t_2, N_2 remain at t_2 and then decay by time t_3, and so on. Show that if many observations are made, then τ can be measured as

$$\tau = \frac{1}{N_0} \sum_i N_i t_i.$$

Example 2

Radium can be found in trace quantities throughout the Earth, and has a half-life of 1620 years. Suppose that there is currently 1 kg of radium on the Earth. Then extrapolating backwards, there was $2^{4.5\times10^9/1620}$ kg of radius on the Earth when it was formed, which is greater than the mass of the observable universe! What's wrong with this calculation?

Solution

Nuclear decays don't happen in isolation; there are entire networks of nuclear decay chains. Radium decays quickly, but it is also constantly produced by the decay of other isotopes, which have much longer half-lives.

[3] **Problem 6.** USAPhO 2009, problem A2.

[3] **Problem 7.** IPhO 2000, problem 1c. Don't worry about the official answer sheet; treat this like a regular USAPhO problem.

[3] **Problem 8** (PPP 190)**.** Part of the series of isotopes produced by the decay of thorium-232, along with the corresponding half-lives, is given below:

$${}^{232}_{90}\mathrm{Th} \xrightarrow{1.4\times10^{10}\,\mathrm{y}} {}^{228}_{88}\mathrm{Ra} \xrightarrow{5.7\,\mathrm{y}} {}^{228}_{89}\mathrm{Ac} \xrightarrow{6.1\,\mathrm{h}} {}^{228}_{90}\mathrm{Th} \xrightarrow{1.9\,\mathrm{y}} {}^{224}_{88}\mathrm{Ra} \xrightarrow{3.6\,\mathrm{d}} {}^{220}_{86}\mathrm{Rn} \xrightarrow{56\,\mathrm{s}} \ldots.$$

Thorium-232 and thorium-228 in equilibrium are extracted from an ore and purified by a chemical process. Sketch the form of the variation in the number of atoms of radon-220 you would expect to be present in this material over a (logarithmic) range from 10^{-3} to 10^3 years.

2 Nuclear Processes

Example 3: PTD 45

Heavy nuclei can decay if struck by a neutron, releasing lighter nuclei and several more neutrons in the process. If each decay event causes, on average, more than one other decay event, then a runaway chain reaction occurs, causing a nuclear explosion. This happens in samples of mass greater than a given "critical mass". If the sample can be compressed, roughly how does the critical mass depend on density?

Solution

Let the sample have radius r, and let the cross-section of collision between neutrons and heavy nuclei be σ. Then for small r, the probability that a produced neutron will collide with another nucleus before exiting the sample is

$$p \sim n\sigma r$$

where n is the number density of nuclei. Critical mass is achieved when this reaches some fixed threshold value, which means $r_{\text{crit}} \propto 1/n \propto 1/\rho$. The critical mass is thus

$$m_{\text{crit}} \propto \rho r_{\text{crit}}^3 \propto 1/\rho^2.$$

Early nuclear weapons worked on the so-called implosion method, where a conventional explosive was used to compress a sphere of radioactive material.

[1] **Problem 9.** Nuclear reactions can occur when nuclei are collided. Find the missing particle in these reactions.

(a) ${}^4\text{He} + {}^{14}\text{N} \to {}^{17}\text{O} +$

(b) ${}^9\text{Be} + {}^4\text{He} \to {}^{12}\text{C} +$

(c) ${}^{27}\text{Al} + {}^4\text{He} \to n +$

(d) ${}^{12}\text{C} + \quad \to {}^{13}\text{N} + n$

In practice, many nuclear and particle physics problems boil down to "optimal collision" problems as you saw in **R2**, so we'll avoid repeating them.

[3] **Problem 10.** EFPhO 2012, problem 6.

The following problems concern nuclear fusion processes in stars, an important topic.

[3] **Problem 11.** USAPhO 2010, problem A4. This covers the proton-proton chain in our Sun.

[2] **Problem 12.** In larger stars, energy is also produced by the CNO cycle. We start with a population of ${}^{12}\text{C}$, in an environment containing many protons. You are given that ${}^{13}\text{N}$ and ${}^{15}\text{O}$ quickly undergo β^+ decay, and that when ${}^{15}\text{N}$ is bombarded by a proton, the reaction

$${}^{15}\text{N} + {}^1\text{H} \to {}^{12}\text{C} + {}^4\text{He}$$

occurs. Write out the steps of the CNO cycle and find the net reaction.

Idea 5

A very basic model for the fission of large nuclei is the liquid drop model. We suppose the protons and neutrons are packed with uniform density; thus, the volume is proportional to A, the surface area to $A^{2/3}$, and the radius to $A^{1/3}$. The binding energy of the nucleus has several contributions:

- Each nucleon is bound to the others by the strong nuclear force. This force is short-ranged, so the binding energy for each nucleon is only due to its neighbors, not on how large the nucleus as a whole is, so it is proportional to A.
- There is a negative contribution scaling as $-A^{2/3}$ because nucleons at the surface don't have neighbors on one side.
- There is another negative contribution scaling as $-Z^2/A^{1/3}$ due to the Coulomb repulsion between protons. This scales quadratically with Z because the electromagnetic force is long-ranged, so every proton interacts with every other one.
- Depending on the sophistication of the model, there can be other terms added, whose origin can only be understood through quantum mechanics.

[3] **Problem 13.** INPhO 2014, problem 7. This is an instructive general application of the liquid drop model. The official solutions are already on the page, so you can check your work as you go.

[4] **Problem 14.** IPhO 1997, problem 2. This applies the liquid drop model to nuclear stability.

You can find a few more exercises on the liquid drop model in part A of IPhO 2023, problem 2, though I think the above problems are enough to get the general idea.

3 Basic Particle Physics

It's important to get a feeling for the basics of the Standard Model. To do this, read through chapter 14 of Krane or chapter 1 of Griffiths.

[3] **Problem 15.** After reading the chapter, do the following as well as you can without references. (You can peek if you need to, but try to do that as little as possible.)

(a) Write down the fundamental particles of the Standard Model, along with their electric charges.

(b) Order the particles from lightest to heaviest.

(c) Which particles make up most of what you see in the everyday world?

(d) Which particles participate in the strong interaction?

(e) Which particles participate in the weak interaction?

[3] **Problem 16** (Griffiths 1.19)**.** Your roommate is a chemistry major. She knows all about protons, neutrons, and electrons, and she sees them in action every day in the laboratory. But she is skeptical when you tell her about positrons, muons, neutrinos, pions, quarks, and intermediate vector bosons. Explain to her why each of these play no direct role in chemistry.

Olympiad questions about particle colliders boil down to questions from **E4**, **E7**, and **R2**, so they should be fairly straightforward if you know the principles.

[5] **Problem 17.** IPhO 2016, problem 3. This problem is about the physics of the LHC. Record your answers on the official answer sheet.

[5] **Problem 18.** IPhO 2018, problem 2. This problem covers LHC data analysis in more depth.

Remark

Now that you know the basics, can you tell the difference between the titles of real high energy physics papers, and randomly generated ones? Test your knowledge here!

4 Atomic Physics

There's not too much about atomic physics that can come up, because most quantitative results beyond the Bohr model need the full machinery of quantum mechanics. However, if you're given the atomic energy levels in advance, there's a bit of physics you can do with the resulting transitions.

Idea 6

Electrons in isolated atoms can spontaneously fall from energy level E_1 to E_0, releasing a photon of frequency $\omega = (E_1 - E_0)/\hbar$. Thus, since energy levels are discrete, light from such atoms will have a sharply peaked spectrum (i.e. frequency dependence). Since every atom has its own characteristic discrete energy levels, careful investigation of the spectrum can identify them.

Remark

If you like Olympiad number theory, you might want to chew on the following puzzle: in the hydrogen atom, it's possible that a transition from energy level $n \to m$ emits a photon of the same frequency as some other transition $n' \to m'$. How can you find all of the (n, m, n', m') for which this is true? The solution is given here.

[3] **Problem 19.** In this problem, we discuss how atomic physicists observe atomic energy levels.

(a) The discrete frequencies $(E_1 - E_0)/\hbar$ observed in the spectra are called "spectral lines". Why are they called lines? (Hint: if you were a 19^{th} century physicist, how would you sort light by frequency in the first place?)

Ideally, each spectral line has zero width. However, in practice, isolated atoms emit radiation in a range of frequencies centered about each spectral line. For concreteness, we'll consider the sodium doublet, a spectral line in sodium vapor which corresponds to yellow light with wavelength $\lambda = 589\,\text{nm}$. (Why specifically sodium vapor?)

(b) One contribution to spectral line width is the energy-time uncertainty principle: if an excited state survives for time Δt, then the resulting emitted energy must have a spread $\Delta E \Delta t \gtrsim \hbar$. In the case of the sodium doublet, the lifetime is 16 ns. Estimate the spread in frequencies $\Delta\lambda$ due to this "lifetime broadening".

(c) Another contribution to spectral line width is Doppler broadening: when a gas of atoms is at a nonzero temperature, the atomic motion causes the frequencies to be changed by the Doppler effect. Estimate the resulting spread in frequencies $\Delta\lambda$ at $T = 1000\,\text{K}$. (You can consult the tables in appendix D of Krane.)

(d) The spectrum of the Sun has a rather different form. Instead of having radiation at only a few frequencies, it has radiation at almost all frequencies, except for a few frequencies where the amount of radiation *decreases*. Why?

Idea 7

Conversely, when an atom is placed in an electromagnetic field of frequency ω, it may absorb a photon to go from energy level E_0 to E_1. The presence of such a field also increases the rate of decay from E_1 down to E_0 via stimulated emission, as we saw in **T1**.

Finally, an electron can be ejected from an atom entirely by absorbing a photon in the photoelectric effect; if the initial energy was $-E$, then the final kinetic energy of the electron is $\hbar\omega - E$.

[3] **Problem 20.** USAPhO 1997, problem A4.

[3] **Problem 21.** USAPhO 1998, problem A3.

[3] **Problem 22.** USAPhO 1998, problem B2.

[3] **Problem 23.** INPhO 2012, problem 5.

[5] **Problem 24.** IPhO 2009, problem 2. This relatively straightforward problem covers the neat application of Doppler laser cooling, a technique for creating ultracold gases that won the 1997 Nobel prize. (For a very similar problem, see APhO 2006, problem 1.)

[5] **Problem 25.** IdPhO 2020, problem 3. Another relatively straightforward problem focusing on chirped pulse amplification, which won the 2018 Nobel prize.

Remark

In a conventional refrigerator, cooling the inside requires the heating of a hot reservoir, which is usually a metal coil located at the back of the fridge. But in Doppler laser cooling, a sample of atoms is cooled without a hot reservoir heating up! This is actually allowed by the second law of thermodynamics because the entropy of the photons goes up. They begin by coming in by a definite direction (the laser beam) and come out in a random direction, so the entropy associated with their orientation increases.

To reach even lower temperatures, one uses the technique of evaporative cooling. The atoms are held in place by a trap, which you can think of as a static, attractive potential $U(r) \propto r^2$. If the trap has finite height, then only the most energetic atoms can escape. The remaining atoms have less energy on average, and hence are colder, just like how evaporating sweat cools people down. This doesn't violate the second law of thermodynamics because the atoms that escape the trap end up in some random place in the lab, so the entropy associated with their position increases.

[5] **Problem 26.** IZhO 2019, problem 3. A problem on the dynamics on a laser, which is arguably the most important invention for atomic physics in history.