Mechanics II: Statics

For review, read chapter 2 of Morin or chapter 2 of Kleppner and Kolenkow. Statics is covered in more detail in chapter 7 of Wang and Ricardo, volume 1. Surface tension is covered in detail in chapter 5 of *Physics of Continuous Matter* by Lautrup, which is an upper-division level introduction to fluids in general. There is a total of **88** points.

1 Balancing Forces

Idea 1

In principle, you can always solve every statics problem by balancing forces on every individual particle in the setup, but often you can save on effort by considering appropriate systems.

Idea 2

Any problem where everything has a uniform velocity is equivalent to a statics problem, by going to the reference frame moving with that velocity. Any problem where everything has a uniform acceleration $\mathbf{a}$ is also about statics, by going to the noninertial frame with acceleration $\mathbf{a}$, where there is an extra effective gravitational acceleration $-\mathbf{a}$. The same principle applies to uniform rotation, where a centrifugal force appears in the rotating frame, acting like an effective gravitational acceleration $\omega^2\mathbf{r}$.

Example 1

Six blocks are attached in a horizontal line with rigid rods, and placed on a table with coefficient of friction μ. The blocks have mass m and the leftmost block is pulled with a force F so the blocks slide to the left. Find the tension force in the rod in the middle.

Solution

There are six objects here and five rods, each with a different tension, so a direct analysis would involve solving a system of six equations. Instead, first consider the entire set of six blocks as one object; we can do this because the rigid rods force them to move as one. The total mass is $6m$, and applying Newton's second law gives

$$F - 6mg\mu = 6ma, \quad a = \frac{F}{6m} - \mu g.$$

Next, consider the rightmost three blocks as one object. Their total mass is $3m$, and their acceleration is the same acceleration a we computed above. This system experiences two horizontal force: tension and friction. Newton's second law gives

$$T - 3mg\mu = 3ma$$

and solving for T gives

$$T = \frac{F}{2}.$$

This is intuitive, because the differences of any two adjacent tension forces are the same; that's the amount of tension that needs to be spent to accelerate each block. So the middle rod, which has to accelerate only half the blocks, has half the tension.

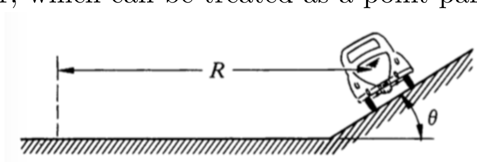
The reason we could ignore the tension forces in the other four rods is that the only thing they do is ensure the blocks move with the same acceleration. Once we assume this is the case, the specific values of the tensions don't matter; we can just zoom out and forget them. It's just like how *within* each block there is also an internal tension which keeps it together, but we rarely need to worry about its details.

Idea 3

To handle a problem where something is just about to slip on something else, set the frictional force to the maximal value μN and assume slipping is not yet occurring, so the two objects move as one. The same idea holds for problems which ask for the minimal force needed to make something move, or the minimal force needed to keep something from moving.

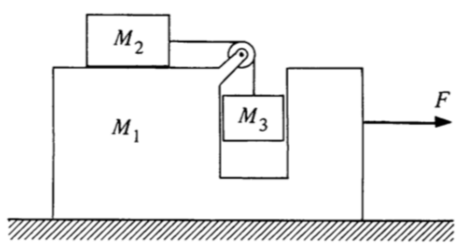
[1] **Problem 1** (KK 2.7)**.** A block of mass M_1 sits on a block of mass M_2 on a frictionless table. The coefficient of friction between the blocks is μ. Find the maximum horizontal force that can be applied to (a) block 1 or (b) block 2 so that the blocks will not slip on each other.

[2] **Problem 2** (KK 2.28)**.** A car, which can be treated as a point particle, enters a turn of radius R.



The road is banked at angle θ, and the coefficient of friction between the wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

[2] **Problem 3** (KK 2.19)**.** A "pedagogical machine" is illustrated in the sketch below.



All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

[2] **Problem 4.** Quarterfinal 2000, problem 2.

[3] **Problem 5.** USAPhO 2017, problem A1.

2 Balancing Torques

Idea 4

A static rigid body will remain static as long as the total force on it vanishes, and the total torque vanishes, where the torque about the origin is

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{r}_i$ is the point of application of force $\mathbf{F}_i$. If the total force vanishes, the total torque doesn't depend on where the origin is, because shifting the origin by $\mathbf{a}$ changes the torque by

$$\Delta\boldsymbol{\tau} = \sum_i \mathbf{a} \times \mathbf{F}_i = \mathbf{a} \times \left(\sum_i \mathbf{F}_i\right) = 0.$$

The origin should usually be chosen to set as many torques as possible to zero.

[1] **Problem 6.** The line of a force is defined to the line passing through its point of application parallel to its direction; then the torque of the force about any point on that line vanishes. Suppose a body is static and has three forces acting on it. Show that in two dimensions, the lines of these forces must either be parallel or concurrent. This will be useful for several problems later.

Idea 5

The center of mass $\mathbf{r}_{\text{CM}}$ of a set of masses m_i at locations $\mathbf{r}_i$ with total mass M satisfies

$$M\mathbf{r}_{\text{CM}} = \sum_i m_i \mathbf{r}_i.$$

If a system experiences no external forces, its center of mass moves at constant velocity.

Idea 6

A uniform gravitational field exerts no torque about the center of mass. Thus, for the purposes of applying torque balance on an *entire* object, the gravitational force $M\mathbf{g}$ can be taken to act entirely at its center of mass. (This is a formal substitution; of course, the actual gravitational force remains distributed throughout the object.)

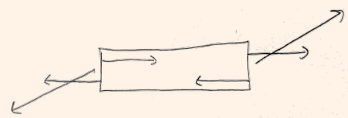
Torque balance works in noninertial frames, as long as one accounts for the torques due to fictitious forces. Thus, for an accelerating frame, the $-M\mathbf{a}$ fictitious force can be taken to act at the center of mass. In a uniformly rotating frame, the total centrifugal force is $M\omega^2\mathbf{r}_{\text{cm}}$, and for the purposes of balancing torques, can be taken to act entirely at the center of mass.

Example 2

Show that the tension in a completely flexible static rope, massive or massless, points along the rope everywhere in the rope.

Solution

Consider a tiny segment $d\ell$ of the rope. Since the rope is static, the tension forces on both ends balance, so they are opposite. Let them both be at an angle θ to the rope direction. Then the net torque on the segment is $(T d\ell) \sin\theta$. Since this must vanish for static equilibrium, we must have $\theta = 0$ and hence the tension is along the rope. In other words, flexible ropes can transmit force, but they can't transmit torque.



It's important to note that the argument above doesn't work for a rigid rod, because the internal forces in a rigid object can look like the picture above. In other words, there can be extra shear forces from the adjacent pieces of the rod that provide the compensating torque. If one tried to set up forces like this in a rope, it would flex instead.

In general, the force distribution within a massless rigid rod can be quite complicated, but if we zoom out, we can replace it with a single tension which does not necessarily point along the rod. This transmits both a force and a torque through the rod, in the sense that a torque is eventually exerted by whatever holds the end of the rod in place. Note that if the rod's supports are free to rotate, then they can't absorb torque, so the rod acts just like a rope, with tension always along it.

Remark

Sometimes, problem writers will intentionally not introduce any variables that are irrelevant to the answer. This can occur in two ways. First, the variables might just cancel out, as one can often see by dimensional analysis. Second, the specific values of the variables might not matter in the limit when they are very large or small. For instance, if a problem simply states a mass is "very heavy" but doesn't give it a name like m, it is asking for the answer in the limit $m \to \infty$.

Idea 7

To handle problems where an object is just about to tip over, note that at this moment, the entire normal force will often be concentrated at a point. (For example, when you're about to fall forward, all your weight goes on your toes.) That often means it's a good idea to take torques about this point.

Example 3: Povey 5.6

In problem 2, we treated the car as a point particle, but in reality it can also tip over. Suppose that on level ground, a car has a distance d between its left and right tires, which are both thin, and its center of mass is a height h above the ground. Now suppose the car turns as in problem 2 on a vertical wall ($\theta = 90°$) with speed v. For what v is this possible?

Solution

Again working in the noninertial frame of the car, force balance gives

$$f_{\text{fric}} = mg, \quad N = \frac{mv^2}{R}$$

where f_{fric} and N are the total friction and normal forces on the four tires. Since $f_{\text{fric}}/N \leq \mu$,

$$v \geq \sqrt{gR/\mu}$$

which matches the general solution to problem 2. But in that problem, we only considered force balance. In this extreme situation, we also have to consider torque balance, i.e. the possibility that the car might topple over. When the car is about to topple over, all the normal and friction force is on the bottom tires. About this point, we have only torques from gravity and the centrifugal force, giving

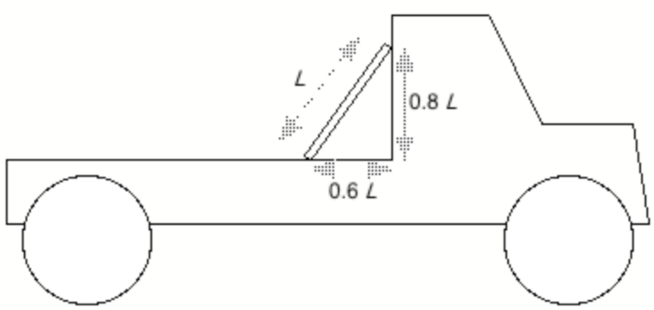
$$mgh = \frac{mv^2}{R}\frac{d}{2}$$

and solving for v gives $v = \sqrt{2gRh/d}$. Toppling is less likely the higher v is, so the answer is

$$v \geq \sqrt{gR}\,\max(1/\sqrt{\mu}, \sqrt{2h/d}).$$

Now here's a puzzle for you. A motorcycle only has one set of wheels, so it is effectively like a car with $d \to 0$. But motorcyclists can perform the motion described here, most famously in the Globe of Death, without toppling over. How is that possible?

[2] **Problem 7** (Quarterfinal 2004.3). A uniform board of length L is placed on the back of a truck.



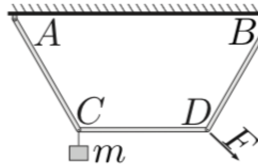
There is no friction between the top of the board and the vertical surface of the truck. The coefficient of static friction between the bottom of the board and the horizontal surface of the truck is $\mu_s = 0.5$. The truck always moves in the forward direction.

(a) What is the maximum starting acceleration the truck can have if the board is not to slip or fall over?

(b) What is the maximum stopping acceleration the truck can have if the board is not to slip or fall over?

(c) For what value of stopping acceleration is the static frictional force equal to zero?

[2] **Problem 8** (Kalda). Three identical massless rods are connected by freely rotating hinges.



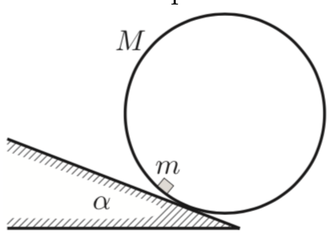
The rods are arranged so that CD is parallel to AB, and $\overline{AB} = 2\overline{CD}$. A mass m is hung on hinge C. What is the minimum force that must be exerted at hinge D to keep the system stationary?

Idea 8

An extended object supported at a point may be static if its center of mass lies directly above or below that point. More generally, if the object is supported at a set of points, it can be static if its center of mass lies above the convex hull of the points.

[2] **Problem 9.** N identical uniform bricks of length L are stacked, one above the other, near the edge of a table. What is the maximum possible length the top brick can protrude over the edge of the table? How does this limit grow as N goes to infinity?

[2] **Problem 10** (Kalda)**.** A cylinder with mass M is placed on an inclined slope with angle α so that its axis is horizontal. A small block of mass m is placed inside it.



The coefficient of friction between the block and cylinder is μ. Find the maximum α so that the cylinder can stay at rest, assuming that the coefficient of friction between the cylinder and slope is high enough to keep the cylinder from slipping.

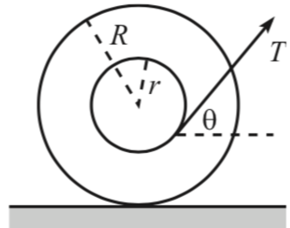
[2] **Problem 11** (PPP 11)**.** A sphere is made of two homogeneous hemispheres stuck together, with different densities. Is it possible to choose the densities so that the sphere can be placed on an inclined plane with incline 30° and remain in equilibrium? Assume the coefficient of friction is sufficiently high so that the sphere cannot slip.

[3] **Problem 12.** An object of mass m lies on a uniform floor, with coefficient of static friction μ.

(a) First, suppose the object is a point mass. What is the minimum force required to make the object start moving, if you can apply the force in any direction?

(b) Now suppose the object is a thin, uniform bar. What is the minimum force required to make the object start moving, if the force can only be applied horizontally? Assume the normal pressure on the floor remains uniform.

[3] **Problem 13** (Morin 2.17)**.** A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground.



A thread is wrapped around the axle and is pulled with tension T at an angle θ with the horizontal.

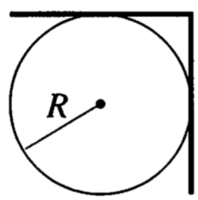
(a) Which way does the spool move if it is pulled with $\theta = 0$?

(b) Given R and r, what should θ be so that the spool doesn't move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(c) Given R, r, and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest?

(d) Given R and μ, what should r be so that you can make the spool slip from the static position with as small a T as possible? That is, what should r be so that the upper bound on T in part (c) is as small as possible? What is the resulting value of T?

[3] **Problem 14** (PPP 44)**.** A plate, bent at right angles along its center line, is placed on a horizontal fixed cylinder of radius R as shown.



How large does the coefficient of static friction between the cylinder and plate need to be if the plate is not to slip off the cylinder?

3 Trickier Torques

Idea 9

Sometimes, a clever use of torque balance can be used to remove any need to have explicit force equations at all. Rarely, the same situation can occur in reverse.

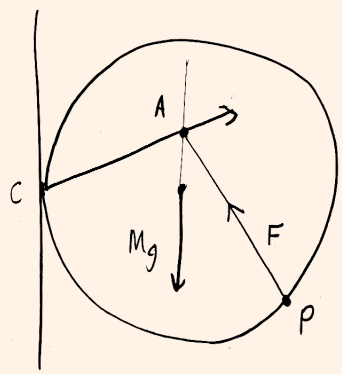
Example 4: EFPhO 2010.4

A spherical ball of mass M is rolled up along a vertical wall, by exerting a force F to some point P on the ball. The coefficient of friction is μ. What is the minimum possible force F, and in this case, where is the point P?

Solution

Following the logic of idea 3, when the minimum possible force is used, the frictional force with the wall must be maximal, $f = \mu N$, and directed upward. (If friction weren't pushing the ball up as hard as possible, we could get by using a smaller force F.) So even though we don't know the magnitude of the normal or the frictional force, we know the direction of the sum of these two forces, so we'll consider them as one combined force.

This reduces the number of independent forces in the problem to three: gravity (acting at the center of mass), the force F (acting at P), and the combined normal and friction forces (acting at the point of contact C with the wall). Therefore, by the result of problem 6, the lines of these forces must all intersect at some point A, as shown.



This ensures that the torques will balance, when taken about point A.

Next, we need to incorporate the information from force balance. Doing this directly will lead us to some nasty trigonometry, but there's a better way. There are in principle two force balance equations, for horizontal and vertical forces. However, one of these equations is just going to tell us the magnitude of the normal/frictional force, which we don't care about. So in reality, we just need one equation, which preferably doesn't involve that force.

The trick is to use torque balance *again*, about the point C, which says that the torques due to gravity and F must cancel. Now you might ask, didn't we already use torque balance? We did, but recall from idea 4 that taking the torque about a different point can give you a different equation if the forces don't balance. So by demanding the torque vanish about two different points, we actually are using force balance! (Specifically, we are using the linear combination of the horizontal and vertical force balance equations that *doesn't* involve the normal/friction force, which we don't need to find anyway.)

When taking the torque about C, we see that F is minimized if P is chosen to maximize the lever arm of the force. This occurs when $CA \perp PA$, in which case the lever arm is $R\sqrt{1+\mu^2}$, where R is the radius of the ball. So we have

$$MgR = FR\sqrt{1+\mu^2}, \quad F = \frac{Mg}{\sqrt{1+\mu^2}}$$

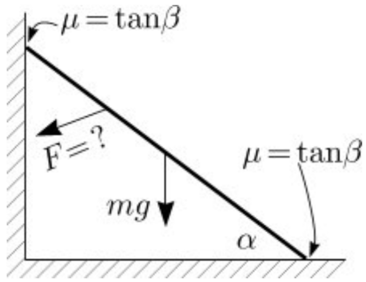
and P is determined as described above.

[2] **Problem 15.** NBPhO 2020, problem 4, parts (i) and (ii).

[3] **Problem 16.** EFPhO 2012, problem 3. The problem statement is missing some information: both the bars and rod have diameter d.

[3] **Problem 17.** EFPhO 2006, problem 6. You will need to print out the problem to make measurements on the provided figure.

[4] **Problem 18** (Physics Cup 2012)**.** A thin rod of mass m is placed in a corner so that the rod forms an angle α with the floor. The gravitational acceleration is g, and the coefficient of friction with the wall and floor is $\mu_s = \tan\beta$, which is not large enough to keep the rod from slipping.



What is the minimum additional force F needed to keep the rod static?

Next, we consider some questions that train three-dimensional thinking.

[2] **Problem 19** (PPP 10)**.** In Victor Hugo's novel *les Miserables*, the main character Jean Valjean, an escaped prisoner, was noted for his ability to climb up the corner formed by the intersection of two vertical perpendicular walls. Suppose for simplicity that Jean has no feet. Let μ be the coefficient of static friction between his hands and the walls. What is the minimum force that Jean had to exert on each hand to climb up the wall? Also, for what values of μ is this feat possible at all?

[3] **Problem 20** (PPP 69)**.** A homogeneous triangular plate has threads of length h_1, h_2, and h_3 fastened to its vertices. The other ends of the string are fastened to a common point on the ceiling. Show that the tension in each thread is proportional to its length. (Hint: with the origin at the point on the ceiling, let the vertices be at positions $\mathbf{r}_i$ and express everything in vector form.)

[4] **Problem 21** (KoMaL 2019, BAUPC 1998)**.** Two identical uniform solid cylinders are placed on a level tabletop next to each other, so that they are touching. A third identical cylinder is placed on top of the other two.

(a) Find the smallest possible values of the coefficients of static friction between the cylinders, and between a cylinder and the table, so that the arrangement can stay at rest.

(b) Repeat part (a) for spheres. That is, put three uniform solid spheres next to each other, with their centers forming an equilateral triangle, and put a fourth sphere on top.

(c) Now return to part (a), and suppose the setup is frictionless. A force is applied directly to the right on the leftmost cylinder, causing the entire setup to accelerate. Find the minimum and maximum accelerations so that all three cylinders remain in contact with each other.

Parts (a) and (b) demonstrate an interesting point: it is possible for a collection of objects to resist some force, even though a single one of those objects would begin moving even with an infinitesimal applied force! This is a simple example of how granular materials, like sand, can give rise to emergent phenomena that are hard to predict from analyzing individual grains alone. Understanding these materials is a whole field of applied research.

4 Paradoxical Reactions

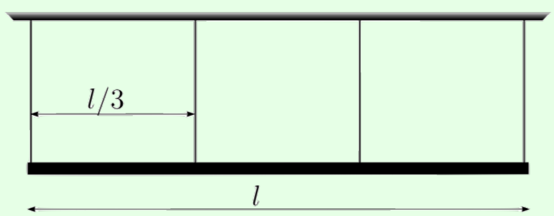
Idea 10

Physics is not fundamentally about solving tricky sets of idealized equations; that is just mathematics. Physics is also not fundamentally about describing common real-world situations as accurately as possible; that is just engineering. The heart of physics is to bridge the two effectively. A good physicist invents mathematical idealizations that decently describe as many things as possible. A great physicist figures out exactly when and why those idealizations break down, and how to replace them with better ones.

The point of this philosophical speech is that everything you've learned so far in this handout is an idealization. Real objects don't have single normal and friction forces applied at points. Instead, they are made of huge numbers of atoms connected by chemical bonds. Each atom applies forces to its neighbors, and each bond deforms in response to applied forces. Sometimes we can ignore these details, sometimes we can save our preferred idealizations with a clever adjustment, and sometimes the idealized picture breaks down completely. Each case is different, and requires thinking about the physics in play.

Example 5

A uniform bar with mass m and length ℓ hangs on four equally spaced identical light wires. Initially, all four wires have tension $mg/4$.



Find the tensions after the leftmost wire is cut.

Solution

This illustrates a common issue with setups involving rigid supports: there are often more normal forces than independent equations, so there is not a unique solution. In the real world, the result is determined by imperfect characteristics of the wires. A reasonable assumption here is that the wires are identical, very stiff springs. In equilibrium, the bar will tilt a tiny bit, so that the length of the middle wire will be the average of the lengths of the other two. By Hooke's law, the force in that wire will than be the average of the other two, so the tensions are $mg/3 - x$, $mg/3$, and $mg/3 + x$. Applying torque balance yields $7mg/12$, $mg/3$, and $mg/12$.

Of course, a real civil engineer designing a structure would use a sophisticated computer program which simulates all the complex internal forces, torques, and strains in play. To build some intuition for this, you could try building some structures yourself in Poly Bridge.

Example 6

In traditional rock climbing, it is often necessary to place tools in small cracks, which will catch the climber in the event of a fall. Suppose two parallel vertical walls are a distance L apart, and a rod of length L and mass m is placed horizontally between them. The coefficient of static friction between the rod and walls is μ. Does the rod stay static?

Solution

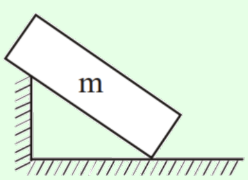
Clearly, there are solutions where the rod stays static. There can be an upward friction force $f = mg/2$ applied to the rod at each wall, and a normal force N at each wall of at least f/μ. But it would also be consistent with the laws of friction to have, for instance, $N = f = 0$, so that the rod falls down immediately.

In cases like this, the normal and friction forces depend on exactly how the rod was placed in contact with the walls. (In previous problems, you were able to resolve this ambiguity by considering the case where an object is about to slip, but here even the criterion for slipping is ambiguous.) For example, if you have to squeeze the rod very hard to fit it in, then it'll probably exert a comparable normal force once it's in. But exerting that much force would be very impractical, so rock climbers have an ingenious alternative, called a "cam". A cam contains parts that rotate, so that it grows wider when a rope pulls on it.

[2] **Problem 22.** AuPhO 2015, problem 12. An explanation of how a cam works. See the answer booklet for the diagrams for the final part.

Example 7

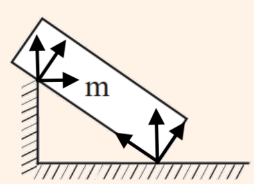
Here's an example which is taken from a real book.



The problem asks about the conditions for this perfectly rectangular block to stay static. Let's ask something even more basic. Which way do the normal forces on the block point?

Solution

If you think about it a bit, you'll see that the answer isn't well-defined.



At the bottom contact point, there are three different possible directions, depending on whether you take the normal to the floor, or either of the two sides of the block. The other contact point is even more ambiguous, because of the wall magically ending. Is the normal force perpendicular to the block, perpendicular to the vertical wall, or something else?

This is a case where the idealization of the normal force breaks down. What happens depends on the exact shape of the block and wall, and how deformable they are. For example, suppose the block was perfectly rigid, but had slightly rounded corners (not shown in the diagram). Then there's a definite normal direction at the bottom contact point, pointing up. Similarly, we could suppose that at the other contact point, the wall actually ends in a step with a rounded corner, in which case the normal direction points directly into the block.

Alternatively, suppose the block and step weren't rounded, but could deform. Then the answer depends on the relative hardness of the materials, and how they were placed in contact. For instance, if we suppose the block is much softer, then it could squash at the bottom contact point, again leading to a common upward normal direction. But then we would expect the step to dig into the block at the other contact point, which yields two separate normal forces at that point. Or perhaps the step is made of a softer material than the floor, so that it's the step rather than the block that deforms. Or maybe both deform!

To reiterate, the issue isn't that idealizations are unrealistic. Physics uses idealizations, like neglecting air resistance and friction, all the time, and they work in appropriate limits. The issue is that when you apply the idealizations implied by the diagram, the result is mathematically undefined – and you get completely different answers depending on how you adjust the idealization. That means the true answer depends crucially on the details.

Remark

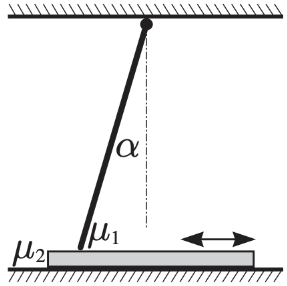
The above example illustrates why it's hard to write good physics questions if you don't know exactly what you're doing. The writers of thoroughly vetted competitions, like the IPhO, EuPhO, or NBPhO, or the national Olympiads of America or China, are perfectly aware of this issue and always make sure to avoid it. For example, you can see that in problems 18 and 23, and example 16, objects are clearly drawn with rounded corners.

But ill-defined problems are depressingly common in homework assignments and less carefully written exams, such as the JEE. If you personally encounter such a problem, your only option is to try to read the question writer's mind; that is, simply start guessing and go with whatever gives you tractable results. If you encounter this sort of thing often, in a book or competition, then it's not worth your time. We're in it to learn about nature, not to please examiners.

Idea 11: The Painleve Paradox

Coulomb's laws for "dry" friction, $f \leq \mu_s N$ and $f = \mu_k N$, can lead to mathematical contradictions if the coefficients of friction are sufficiently high. For example, equations derived from these laws might have no solutions, or multiple solutions.

[2] **Problem 23** (Kalda). A rod is hinged to the ceiling, so that it makes an angle α with the vertical.



Underneath, a thin board is being dragged on the floor. The coefficient of (static and kinetic) friction is μ_1 between the board and rod, and μ_2 between the board and floor. The rod is meant to stop the board from being dragged to the right, no matter how hard or how quickly it is pulled. Is this possible? If so, what are the conditions on the parameters that allow this to occur?

Remark

In problem 23 you showed that for sufficiently strong friction, it is impossible for a static board to start moving to the right. But if we suppose the board was already moving to the right, then solving for the normal force will yield a mathematical contradiction. Specifically, the rightward friction force on the rod is so strong that it rotates the rod even harder into the board, requiring an even larger normal force to keep the rod from going through the board, which induces an even larger friction force, and so on. Technically, there is a solution for the normal force, but it's *negative*, which doesn't make any sense either.

Of course, you've probably seen what happens in real life. The board tends to move in fits and starts. The rod creaks and cracks, and might even visibly bounce up and down. But you can't understand this behavior through the idealized laws of friction. Instead, we need "contact mechanics", which studies how the rod and board dynamically deform when subject to stress. (In section 8, we'll consider some of the simplest ideas of contact mechanics.)

Of course, good Olympiad questions are carefully designed to avoid triggering Painleve paradoxes. For an excellent further discussion of these issues, with many examples, see this paper. More generally, real friction can be rather complicated even when the equations aren't paradoxical. For example, lubricated materials don't obey Coulomb's laws; instead the friction force has to be computed with fluid mechanics. The study of friction is called tribology. For an introduction, see *Tribology* by Hutchings and Shipway.

5 Extended Bodies

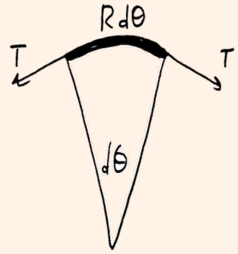
Next, we'll consider problems with continuous bodies, where one often needs to consider forces and torques acting on infinitesimal pieces.

Example 8

Find the tension in a circular rope of radius R spinning with angular velocity ω and mass per length λ.

Solution

Consider an infinitesimal segment of the rope, spanning an angle $d\theta$.



The mass of this segment is $dm = R\lambda\, d\theta$. The total force is downward, with magnitude

$$dF = 2T \sin \frac{d\theta}{2} \approx T\, d\theta$$

where we used the small angle approximation. This is the centripetal force, so

$$dF = (dm)\, \omega^2 R.$$

Combining these results yields $T = R^2\omega^2\lambda$.

Example 9

Find the distance d of the center of mass of a uniform semicircle of radius R to its center. (Note that a semicircle is half of a circle, not half of a disc.)

Solution

This can be done by taking the setup of the previous problem, and taking a subsystem comprising exactly half of the rope. In this case the net tension force is simply

$$F = 2T.$$

The total mass is $m = \pi R\lambda$, and the force must provide the centripetal force, so

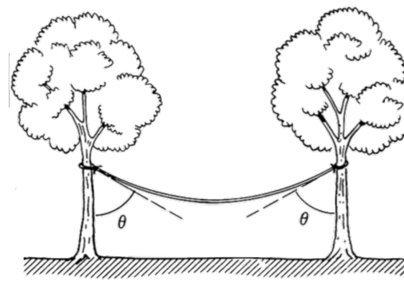
$$F = (\pi R\lambda)(\omega^2 d)$$

But we also know that $T = R^2\omega^2\lambda$ as before, so plugging this in gives

$$d = \frac{2}{\pi} R.$$

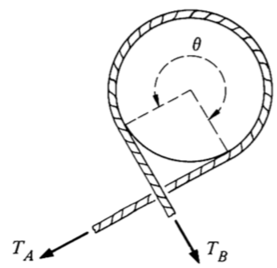
Alternatively, we could have worked in the frame rotating with the rope. The equations would be the same, but instead we would say the tension balances the centrifugal force.

[1] **Problem 24** (KK 2.22)**.** A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees.



Find the tension at either end of the rope, and the tension at the middle of the rope.

[3] **Problem 25** (KK 2.24)**.** A capstan is a device used aboard ships to control a rope which is under great tension.



The rope is wrapped around a fixed drum with coefficient of friction μ, usually for several turns. The load on the rope pulls it with a force T_A. Ignore gravity.

(a) Show that the minimum force T_B needed to hold the other end of the rope in place is $T_A e^{-\mu\theta}$, an exponential decrease.

(b) How does this result depend on the shape of the capstan, if we fix the angle θ between the initial and final tension forces? Would the answer be the same for an oval, or a square?

(c) If $\theta = \pi$, explain why the total normal and friction force of the rope on the drum is $T_A + T_B$.

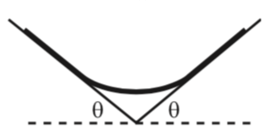
[2] **Problem 26** (***F*** $=$ ***ma*** 2018 B20)**.** A massive, uniform, flexible string of length L is placed on a horizontal table of length $L/3$ that has a coefficient of friction $\mu_s = 1/7$, so equal lengths $L/3$ of string hang freely from both sides of the table. The string passes over the edges of the table, which are smooth frictionless curves, of size much less than L. Now suppose that one of the hanging ends of the string is pulled a distance x downward, then released at rest. Neither end of the string touches the ground.

(a) Find the maximum value of x so that the string does not slip off of the table.

(b) For the case $x = 0$, draw a free body diagram for the string, indicating only the *external* forces on the entire string. Do the forces balance?

(c) Would the answer change significantly if the table's small edges had friction as well?

[3] **Problem 27** (Morin 2.25)**.** A rope rests on two platforms that are both inclined at an angle θ.



The rope has uniform mass density, and the coefficient of friction between it and the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum fraction?

Example 10

A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.

Solution

First, we note that the horizontal component of the tension T_x is constant throughout the chain; this just follows from balancing horizontal forces on any piece of it. Moreover, by similar triangles, we have $T_y = T_x y'$ everywhere.

Now consider a small segment of chain with horizontal projection Δx. The length of the piece is $\Delta x \sqrt{1 + y'^2}$ which determines its weight, and this be balanced by the difference in vertical tensions. Thus

$$\Delta T_y = \lambda g \sqrt{1 + y'^2}\, \Delta x.$$

For infinitesimal Δx, we have $\Delta T_y = T_x d(y') = T_x y''\, dx$, so we get the differential equation

$$y'' = \frac{\lambda g}{T_x} \sqrt{1 + y'^2}.$$

Usually nonlinear differential equations with second derivatives are very hard to solve, but this one isn't because there is no direct dependence on y, just its derivatives. That means we can treat y' as the independent variable first, and the equation is effectively first order in y'.

Writing $y'' = d(y')/dx$ and separating, we have

$$\int \frac{dy'}{\sqrt{1 + y'^2}} = \int \frac{\lambda g}{T_x}\, dx.$$

Integrating both sides gives

$$\sinh^{-1}(y') = \frac{\lambda g x}{T_x} + C.$$

Choosing $x = 0$ to be the lowest point of the chain, the constant C is zero, and

$$y' = \sinh\left(\frac{\lambda g x}{T_x}\right).$$

Integrating both sides again gives the solution for y,

$$y = \frac{T_x}{\lambda g} \cosh\left(\frac{\lambda g x}{T_x}\right)$$

where we suppressed another constant of integration. This curve is called a catenary.

[1] **Problem 28.** To check that you understand the previous example, repeat it for a suspension bridge. In this case the cable is attached by vertical suspenders to a horizontal deck with mass λ per unit

length, and supports the weight of the deck. Assume the cable and suspenders have negligible mass.

Example 11

A uniform spring of spring constant k, mass m, and relaxed length L is hung from the ceiling. Find its length in equilibrium, as well as its center of mass.

Solution

Problems like this contain subtleties in notation. For example, if you talk about "the piece of the slinky at z", this could either mean the piece that's actually at this position in equilibrium, or the piece that was originally at this place in the absence of gravity. Talking about it the first way automatically tells you where the piece is now, but talking about it the second way makes it easier to keep track of, because then the z of a specific piece of the spring stays the same no matter where it goes.

In fluid dynamics, these are known as the Eulerian and Lagrangian approaches, respectively. If you don't use one consistently, you'll get nonsensical results, and it's easy to mix them up.

There are many ways to solve this problem, but I'll give one that reliably works for me. We're going to use the Lagrangian approach, and avoid confusion with the Eulerian approach by breaking the spring into discrete pieces. Let the spring consist of $N \gg 1$ pieces, of masses m/N, spring constants Nk, and relaxed lengths L/N.

The i^{th} spring from the bottom has tension $(i/N)mg$, and thus is stretched by

$$\Delta L_i = \frac{1}{kN}\,\frac{i}{N}mg = \frac{mg}{kN^2}\,i.$$

The total stretch is

$$\sum_{i=1}^{N} \Delta L_i = \frac{mg}{kN^2}\int_0^N i\,di = \frac{mg}{2k}.$$

This makes sense, since the average tension is $mg/2$. To find the center of mass, note that the j^{th} spring is displaced downward by a distance

$$\Delta y_j = \sum_{i=j}^{N} \Delta L_i = \frac{mg}{2k}\left(1 - \frac{j^2}{N^2}\right)$$

downward from its position in the absence of gravity. The center of mass displacement is

$$\Delta y_{\text{CM}} = \frac{1}{N}\sum_{j=1}^{N} \Delta y_j \propto \frac{1}{N}\sum_{j=1}^{N}\left(1 - \frac{j^2}{N^2}\right) = \frac{1}{N^3}\int_0^N N^2 - j^2\,dj = \frac{2}{3}$$

so restoring the proportionality constant gives

$$\Delta y_{\text{CM}} = \frac{mg}{3k}.$$

If you want to test your understanding of slinkies, you can also try doing this problem with the Eulerian approach. This would be best done without discretization. The first steps would be finding a relation between the density $\rho(z)$ and tension $T(z)$ from Hooke's law, and finding out how to write down local force balance as a differential equation.

[4] **Problem 29** (MPPP). A slinky is a uniform spring with negligible relaxed length, with mass m and spring constant k.

(a) Find the shape of a slinky hung from two points on the ceiling separated by distance d. (Hint: to begin, consider the mass and tension of a small piece of the spring with horizontal and vertical extent dx and dy. Don't forget that the slinky's density won't be uniform.)

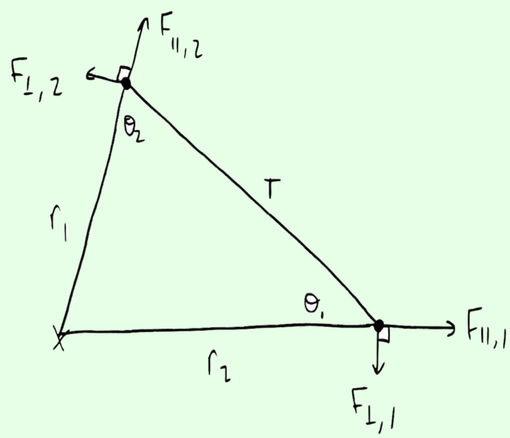
(b) Suppose a slinky's two ends are fixed, separated by distance d, and rotating uniformly with angular frequency ω like a jump rope in zero gravity. Find the values of ω for which this motion is possible, and the shape of the slinky in this case.

6 The Principle of Virtual Work

Let's motivate this section with a simple question: why use torque at all? In principle, everything in Newtonian mechanics can be derived by considering forces alone, so torques shouldn't even be necessary. This is illustrated with the following example.

Example 12

Consider the simplest possible nontrivial rigid body: a triangle with masses at the vertices, and sides made of very thin, very rigid, massless springs. The triangle is pivoted at one vertex, and experiences external forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at the other two vertices.



Find the criterion for this system to be in equilibrium, using force balance alone.

Solution

Consider force balance on the first marked vertex. The tension in the side of length r_2 takes whatever value is necessary to balance the horizontal force on the vertex, while the tension T in the other side has to balance the vertical force. Thus, $F_{\perp,1} = T \sin\theta_1$. Similarly, by considering the second marked vertex, we have $F_{\perp,2} = T\sin\theta_2$. Eliminating T and using the law of sines gives

$$r_2 F_{\perp,1} = r_1 F_{\perp,2}.$$

Of course, this is precisely the statement of torque balance about the pivot. And if you continue along this line of reasoning, letting the forces be arbitrary, you can also derive the rotational form of Newton's second law, $\tau = I\alpha$, for this system.

Remark

So why exactly are torques necessary? Torque isn't a necessary tool for single point particles or very simple rigid bodies. But in a general rigid body, the internal forces which maintain their rigidity are very complicated, and torques allow us to avoid having to think these forces.

For example, consider a rigid bar supported at its ends. The middle of the bar doesn't collapse, despite the force of gravity on it, because the bar contains internal, upward shear forces, which transmit the normal forces applied at its end throughout the rest of the bar. But to analyze such systems without using torque, one would have to account for all of these microscopic forces, acting on all of the rod's infinitely many pieces. With torque, we can compute useful information (such as the normal forces at each support) without much effort.

However, given how complicated internal forces can be, you might be wondering why torque balance even works in general. The simplest explanation is the principle of virtual work.

Idea 12: Principle of Virtual Work

To determine if a system is in static equilibrium, we consider each way the system could move. For each such way, we consider how much work would be done if the system moved a little bit in that way. (This motion is just in our heads, so we call it a "virtual displacement" which corresponds to a "virtual work".) The system is in static equilibrium if the virtual work vanishes for every possible virtual displacement.

If we apply the principle of virtual work to translational motion, then we get force balance, since $dW = F\,dx$. If we apply it to rotation about a pivot, then we get torque balance, since $dW = \tau\,d\theta$. However, as we'll see below and in **M4**, the principle of virtual work can also be applied to more exotic displacements. It is particularly useful when applied to systems with a lot of parts but also a lot of constraints, so that they can only move in a few ways. The converse of the principle of virtual work can also be useful: if you know a system is in static equilibrium, you can use it to deduce an unknown force.

Example 13: Roberval Balance

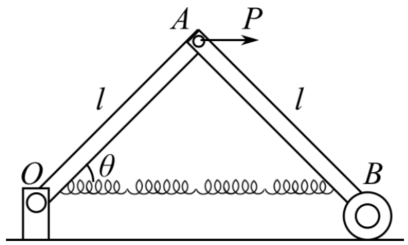
Consider the following scale made of rigid bars. The joints ensure that the quadrilateral in the middle always remains a parallelogram, with its left and right sides vertical.

If identical weights are placed on each horizontal arm as shown, can the system remain static?

Solution

There's only one way for the system to move: the rectangle can deform into a parallelogram so that the left horizontal arm moves up, and the right horizontal arm moves down by the same amount. Then the total virtual work done on the scale by the weights is zero, so the system can be in equilibrium no matter where on the arms the weights are placed.

[1] **Problem 30** (Wang)**.** Two massless rigid rods of length ℓ are connected by a joint A, which allows them to freely rotate with respect to each other. The left member is pinned to point O, while the right member is placed on a roller B which can roll frictionlessly on the ground.



A massless spring of zero relaxed length and spring constant k is stretched between O and B, and a rightward force P is exerted at A. Find the angle θ at equilibrium.

7 Pressure and Surface Tension

Example 14

A sphere of radius R contains a gas with a uniform pressure P. Find the total force exerted by the gas on one hemisphere.

Solution

The pressure provides a force per unit area orthogonal to the sphere's surface, so the straightforward way to do this is to integrate the vertical component of the pressure force over a hemisphere. However, there's a neat shortcut in this case.

Momentarily forget about the sphere and just imagine we have a sealed hemisphere of gas at pressure P. The net force of the gas on the hemisphere must be zero, or else it would just begin shooting off in some direction, violating conservation of momentum. So the force on the curved face must balance the force on the flat face, which is $\pi R^2 P$. The same logic must hold for the sphere, since the forces on the curved face are the same, so the answer is $\pi R^2 P$.

This trick will come in handy for several future problems; for example, it's the quick way to do $F = ma$ 2018 B24. It also generalizes to surfaces of arbitrary shape, as discussed in **E1**. Concretely, suppose a surface S has boundary C, and consider any other surface S' with the same boundary. Then by the same logic, the closed surface formed by S and S' together experiences no net pressure force, so the pressure forces on S and S' are equal in magnitude.

Idea 13

The surface of a fluid carries a surface tension γ. If one imagines dividing the surface into two halves, then γ is the tension force of one half on the other per length of the cut. Specifically, for a small segment $d\mathbf{s}$ along the cut, where the normal vector to the surface is $\hat{\mathbf{n}}$, the surface tension force is

$$d\mathbf{F} = \gamma \, d\mathbf{s} \times \hat{\mathbf{n}}$$

which means the force acts along the surface and perpendicular to the cut.

Example 15

A spherical soap bubble of radius R and surface tension γ is in air with pressure P, and contains air with pressure $P + \Delta P$. Compute ΔP.

Solution

We use the result of the previous problem to conclude that the force of one hemisphere on another is $\pi R^2 \Delta P$. This must be balanced by the surface tension force. By imagining cutting the surface of the bubble in half, the surface tension force is γL where L is the total length of the surface connecting the hemispheres.

At this point, we can write $L = 2\pi R$, giving

$$\Delta P = \frac{2\gamma}{R}.$$

This is called the Young–Laplace equation. However, in this particular case, this is not the right answer. The reason is that we should actually take $L = 4\pi R$ because the surface tension is exerted at both the inside and outside surfaces of the bubble wall, and thus the answer is

$$\Delta P = \frac{4\gamma}{R}.$$

The increased pressure inside balances the surface tension, which wants to collapse the bubble.

If you're confused about why $L = 4\pi R$, you can also think about it in terms of energy. Surface tension arises from the fact that it costs energy to take soapy water and stretch it out into a surface, because this breaks some of the attractive intermolecular bonds. The Young–Laplace equation would give the correct answer for a *ball* of soapy water. But for a *bubble* of soapy water, twice as much soapy water/air surface is created. So the energy cost is double, and the force is double.

[2] **Problem 31.** One can also derive the Young–Laplace equation using the principle of virtual work. Suppose the bubble radius changes by dr. The energy of the bubble changes for two reasons: first, there is net $\Delta P \, dV$ work from the two pressure forces, and there is the $\gamma \, dA$ surface tension energy cost. By setting the net virtual work to zero, find ΔP.

[2] **Problem 32** (Kalda)**.** Consider two soap bubbles which have stuck together. The part of the soap film that separates the interior of the first bubble from the outside air has radius of curvature R.

The part that separates the interior of the second bubble from the outside air has radius of curvature $2R$. What is the radius of curvature of the part which separates the bubbles from each other?

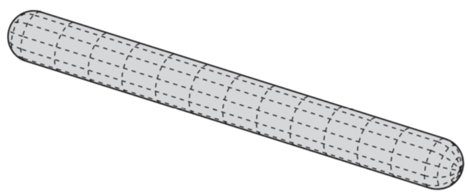
Remark

So far, we've only applied the Young–Laplace equation to spherical surfaces, which are characterized by a single radius of curvature. More generally, a surface has two principle radii of curvature R_1 and R_2 at each point. These are both equal to R for a sphere of radius R, while for a cylinder of radius R, one is equal to R and the other is infinity. For general surfaces, the Young–Laplace equation is

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where the R_i can each be positive or negative, depending on the direction of curvature.

[3] **Problem 33** (MPPP 67)**.** When a pipe bursts under pressure, it often splits "lengthwise" instead of "across". (One familiar example is the process of cooking a long, straight sausage.) The two modes of splitting are shown as dotted lines below.



Explain this observation, assuming the thickness of the sausage skin is uniform, and hence can support a constant surface tension before breaking. (Hint: model the sausage as a cylinder of length L capped by hemispheres of radius $R \ll L$, and consider the surface tension needed to prevent the two modes of splitting mentioned, once an excess pressure P builds up inside the sausage.)

[4] **Problem 34.** Two coaxial rings of radius R are placed a distance L apart from each other in vacuum. A soap film with surface tension γ connects the two rings.

(a) Derive a differential equation for the shape $r(z)$ of the film, and solve it.

(b) Show that for sufficiently large L, there are no solutions. If L is increased to this value, what happens to the film?

(c) Using a computer or calculator, find the largest possible value of L.

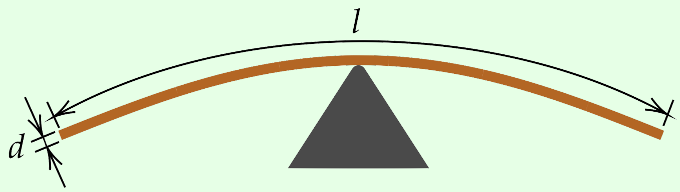
We'll consider surface tension in more detail in **T3**.

8 Deforming Solids

So far, the only continuous objects we've analyzed in detail have been ropes and bubbles. They are relatively simple because they can only support tension forces, and are one-dimensional and two-dimensional respectively. A three-dimensional solid is much more complex, as it can deform in many different ways, and can also support internal shear forces. A full treatment of this subject, which requires comfort with tensors, is given in chapters 6 through 11 of Lautrup, as well as chapters II-31, II-38, and II-39 of the Feynman lectures. In this problem set, we'll just give two simple examples.

Example 16: IPhO 2022 3A

A thin piece of spaghetti of diameter d is balanced horizontally from its middle.



It can have a length $\ell \gg d$ before it snaps under its own weight. How does ℓ scale with d?

Solution

Let the spaghetti rod have density ρ, and consider its right half. There must be a vertical normal force $F \sim \rho d^2 \ell$ to balance the weight. This vertical force is transmitted through the rod by a shear stress (i.e. an internal force per area, perpendicular to the rod) of order $\sigma_s \sim F/A \sim \rho\ell$. Each piece of the rod exerts such a shear stress on its neighbors, just like how pieces of a string exert tensions on their neighbors.

Now consider torques on the right half of the rod, about the pivot point. The torque $\tau \sim \rho d^2 \ell^2$ of the rod's weight has to be balanced by forces from the other half of the rod. Vertical forces don't work, since they don't provide any torque about the pivot. Instead, the torque is supplied by a horizontal compression force at the bottom, and a horizontal tension force at the top, which cancel out to maintain horizontal force balance. This combination of forces, which produces no net force but does produce a net torque, is a bending moment.

Let the associated normal stresses be of order $\pm\sigma_n$. Then the net compression and tension forces are of order $\pm d^2 \sigma_n$, and the lever arm is of order d, so balancing torques gives

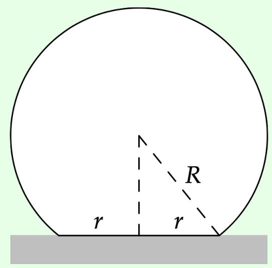
$$\rho d^2 \ell^2 \sim \sigma_n d^3$$

which implies $\sigma_n \sim \rho \ell^2/d$. This is much greater than σ_s, because of the miserably small lever arm, which is why thin rods usually break by snapping, not by shearing or pulling apart. Given a fixed maximum σ_n, we conclude the maximum length scales as $\ell \sim \sqrt{d}$.

[3] **Problem 35.** USAPhO 2022, problem A1. A practical bending moment problem.

Example 17

A solid ball of radius R, density ρ, and Young's modulus Y rests on a hard table. Because of its weight, it deforms slightly, so that the area in contact with the table is a circle of radius r.



Estimate r, assuming that it is much smaller than R.

Solution

Recall from **P1** that the Young's modulus is defined by

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}$$

and has dimensions of pressure. By dimensional analysis, you can show that

$$r = R\, f(\rho g R/Y)$$

but dimensional analysis alone can't tell us anything more about f. Moreover, an exact analysis using forces would be very difficult, because different parts of the ball are compressed in different amounts, and in different directions; there's little symmetry here.

Instead, we'll roughly estimate the stress and strain near the bottom of the ball. For the part directly in contact with the table, we have

$$\text{stress} \sim F/r^2 \sim \rho g R^3/r^2$$

because the normal pressure has to balance gravity. This is the pressure exactly at the bottom of the ball; at heights much greater than r, the pressure will be smaller because it can spread out over a wider horizontal surface area. Since stress is proportional to strain, that means the part of the ball that is significantly strained has typical height r. (This is an example of Saint–Venant's principle, which states that strain is generally confined near the location that external forces are applied.) So in that region, the strain must be

$$\text{strain} \sim \delta/r \sim r/R.$$

Using the definition of the Young's modulus, we conclude

$$r \propto R \left(\frac{\rho g R}{Y}\right)^{1/3}.$$

We can also phrase this result in terms of force and displacement. The ball's total vertical deformation is $d \sim r^2/R$ and the total force that pushes it into the table is $F \sim \rho g R^3$, so

$$F \propto Y R^{1/2} d^{3/2}.$$

The restoring force is not linear in d, so it doesn't obey Hooke's law.

As mentioned above, contact mechanics is the study of how normal and other forces behave for realistic, deformable solids. In this example, we considered "Hertzian contact". For much more, see *Contact Mechanics* by Johnson, and *Contact Mechanics and Friction* by Popov.

[4] **Problem 36.** EFPhO 2006, problem 5. A tough problem on a deforming object.