

# Relativity III: Fields

Relativity in electromagnetism is covered in chapter 5 of Purcell and then sprinkled in throughout the rest of the book, notably in sections 6.7 and 9.7, and appendix H. For a more advanced discussion, see chapter 3 of Schutz for tensors, and chapter 12 of Griffiths and chapters I-34, II-13, and II-25 through II-28 of the Feynman lectures for relativistic electromagnetism. For a brief taste of general relativity, see chapter 14 of Morin, and chapter II-42 of the Feynman lectures. For a great, accessible introduction to tests of general relativity, see *Was Einstein Right?* by Will. There is a total of **88** points.

## 1 Electromagnetic Field Transformations

### Idea 1: Field Transformations

If the electromagnetic field is  $(\mathbf{E}, \mathbf{B})$  in one reference frame, then in a reference frame moving with velocity  $\mathbf{v}$  with respect to this frame, the components of the field parallel to  $\mathbf{v}$  are

$$E'_{\parallel} = E_{\parallel}, \quad B'_{\parallel} = B_{\parallel}$$

while the components perpendicular are

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}), \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp}\right).$$

As alluded to in **R2**, this is the transformation rule for the components of a rank 2 antisymmetric tensor.

Under these transformations, Maxwell's equations remain true in all inertial frames, and the Lorentz force transforms properly as well. Furthermore, a Lorentz transformation does not change the total amount of charge in a system, where total charge is defined by Gauss's law via the electric flux through a surface containing the system.

### Remark

There are many ways of deriving the field transformations. The tensor method alluded to above is the mathematically cleanest, but the conceptually clearest is to think about how some simple setups must Lorentz transform, if Maxwell's equations are to remain true. For example, boosting a capacitor increases the charge density on the plates because of length contraction, which is why  $\mathbf{E}'_{\perp}$  contains  $\gamma\mathbf{E}_{\perp}$ . (Further examples are given in chapter 5 of Purcell, which is essential reading for this section.) Another method is to demand that the Lorentz force obeys the transformation of three-force derived in **R2**.

[4] **Problem 1.** Basic facts about the electric and magnetic fields of a moving charge.

(a) Show that the field of a point charge  $q$  at the origin moving with constant velocity  $v$  is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2}{(1 - v^2 \sin^2 \theta)^{3/2}} \hat{\mathbf{r}}$$

in units where  $c = 1$ , and  $\theta$  is the angle from  $\mathbf{v}$ . In particular, the field is still radial.

- (b) Verify that the charge of this moving charge is still  $q$ . It may be useful to consult the integral table in appendix K of Purcell.
- (c) Argue that the magnetic field of this point charge must be exactly

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

- (d) Verify that the previous result is correct in nonrelativistic electromagnetism (i.e. using Coulomb's law and the Biot–Savart law).

The result of part (a), first found by Heaviside in 1888, implies that the field lines of a moving charge “length contract” like they were rigid rods. In fact, since this result came first historically, it was one of the inspirations for length contraction in the first place! Since it's very hard to measure the Coulomb field of a relativistic electron, this prediction was first [directly verified](#) in 2022.

**Solution.** For concreteness, let the charge be moving along the  $z$  direction, and let the primed frame be the rest frame of the charge.

- (a) By rotational symmetry, it suffices to show this holds in the  $xz$  plane. Consider the point

$$(x, z) = (r \sin \theta, r \cos \theta)$$

at time  $t = 0$ . Lorentz transforming to the primed frame, this point corresponds to

$$(x', z') = (r \sin \theta, \gamma r \cos \theta).$$

By Coulomb's law, the electric field at that point is

$$E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{((x')^2 + (z')^2)^{3/2}}, \quad E'_z = \frac{q}{4\pi\epsilon_0} \frac{z'}{((x')^2 + (z')^2)^{3/2}}.$$

Applying the field transformations, the field in the original frame is

$$E_x = \gamma E'_{x'} = \frac{q}{4\pi\epsilon_0} \frac{\gamma r \sin \theta}{r^3 (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{3/2}}, \quad E_z = E'_z = \frac{q}{4\pi\epsilon_0} \frac{\gamma r \cos \theta}{r^3 (\sin^2 \theta + \gamma^2 \cos^2 \theta)^{3/2}}.$$

This is a radial field, as desired, and swapping the  $\gamma$ 's for  $v$ 's gives the desired result.

There's a simple explanation for why the field is radial. The fact that the field is radial in the charge's rest frame just means  $z/x = E_z/E_x$ . When we Lorentz transform,  $z$  is Lorentz contracted by a factor of  $\gamma$ , while  $E_x$  is enhanced by a factor of  $\gamma$ , so we still have the equality of ratios  $z'/x' = E'_z/E'_x$ . This should make it clear that the fact that the field stays radial here is a fortuitous coincidence, since  $x^\mu$  and  $(\mathbf{E}, \mathbf{B})$  transform totally differently in general.

- (b) The electric flux through a unit sphere surrounding the charge must still be  $q/\epsilon_0$ , so we need to show that

$$1 = \frac{1}{4\pi} \int \frac{1 - v^2}{(1 - v^2 \sin^2 \theta)^{3/2}} d\Omega.$$

Setting up spherical coordinates as usual, the  $d\phi$  integral gives  $2\pi$ , leaving

$$\frac{1 - v^2}{2} \int_0^\pi \frac{\sin \theta d\theta}{(1 - v^2 \sin^2 \theta)^{3/2}} = \frac{1 - v^2}{2} \left. \frac{-\cos \theta}{(1 - v^2) \sqrt{1 - v^2 \sin^2 \theta}} \right|_0^\pi$$

where we used equation (K.15) from Purcell. Plugging in the limits gives 1 as desired.

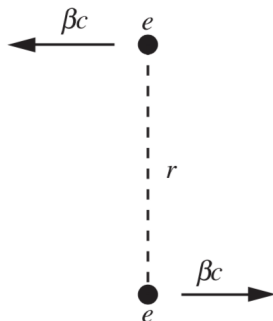
(c) We know  $\mathbf{B}' = 0$ , and using the field transformations immediately gives the result.

(d) Applying Coulomb's law, we have

$$\frac{1}{c^2} \mathbf{v} \times \mathbf{E} = (\mu_0 \epsilon_0) \frac{qv}{4\pi \epsilon_0 r^2} \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi} \frac{qv}{r^2} \hat{\boldsymbol{\theta}}$$

which is precisely what the Biot–Savart law predicts.

[3] **Problem 2** (Purcell 5.29). Two protons are moving antiparallel to each other, along lines separated by a distance  $r$ , with the same speed  $v$  in the lab frame, as shown.



Consider the moment the protons are a distance  $r$  apart.

(a) Show that the three-force experienced by each proton due to the electric field of the other is

$$F = \frac{\gamma e^2}{4\pi \epsilon_0 r^2}.$$

(b) Compute the three-force experienced by one of the protons by transforming to its rest frame, computing the force there, then transforming back to the lab frame. In particular, show that this is not equal to the result of part (a).

(c) Show that the discrepancy is resolved if the magnetic three-force is also included.

Recall from **R2** that the Lorentz three-force is  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ . You will also have to use the three-force transformation laws you derived there.

**Solution.** (a) Using part (a) of problem 1 gives

$$eE = \frac{e^2}{4\pi \epsilon_0 r^2} \frac{1 - v^2}{(1 - v^2)^{3/2}} = \frac{\gamma e^2}{4\pi \epsilon_0 r^2}$$

as desired.

(b) In the rest frame of one proton, we have

$$F' = eE' = \frac{e^2}{4\pi \epsilon_0 r^2} \frac{1 - v'^2}{(1 - v'^2)^{3/2}}, \quad v' = \frac{2v}{1 + v^2}.$$

Simplifying a bit yields

$$F' = \frac{e^2}{4\pi \epsilon_0 r^2} \frac{1 + v^2}{1 - v^2}.$$

Using our result from **R2** that transverse forces are redshifted,

$$F = \frac{F'}{\gamma} = \frac{\gamma e^2}{4\pi\epsilon_0 r^2} (1 + v^2).$$

This is more than the result we found in part (a).

- (c) The extra three-force due to the magnetic field adds to the electric force,

$$evB = ev^2 E = \frac{\gamma e^2}{4\pi\epsilon_0 r^2} v^2$$

where we used part (c) of problem 1. This is exactly the missing piece.

- [3] Problem 3** (Purcell 5.24). In the rest frame of a particle with charge  $q$ , another particle with charge  $q$  is approaching with relativistic velocity  $\mathbf{v}$ . Assume that both particles are extremely massive, and hence their velocities are nearly constant. The second particle passes a minimum distance  $b$  from the first.

- (a) Show that the impulse acquired by each particle is perpendicular to  $\mathbf{v}$  with magnitude  $q^2/2\pi\epsilon_0 vb$ . (Hint: you can avoid doing a nasty integral by using Gauss's law.)  
 (b) If the particles have mass  $m$ , roughly how large does  $m$  have to be for the above result to be a good approximation?

**Solution.** (a) Let the stationary particle be at the origin, and let the velocity of the moving particle be along the  $x$ -axis. Then the impulse experienced by the moving particle is

$$J = \int F dt = \frac{q}{v} \int_{-\infty}^{\infty} E_z(x, b) dx.$$

On the other hand, suppose we consider the electric flux through an infinite cylinder of radius  $b$ , oriented along the  $x$ -axis. Then

$$\Phi = \int \mathbf{E} \cdot d\mathbf{S} = 2\pi b \int_{-\infty}^{\infty} E_z(x, b) dx.$$

Therefore, we conclude

$$J = \frac{q}{v} \frac{\Phi}{2\pi b} = \frac{q^2}{2\pi\epsilon_0 vb}.$$

The great thing about this derivation is that exactly the same reasoning applies to the impulse experienced by the stationary charge. It can be written as a similar integral, except that  $E_z(x, b)$  is now the electric field of a moving charge. But the impulse is the exactly the same in magnitude because the Gauss's law argument still works. That's good to know, as it ensures momentum is conserved.

- (b) First off, we've ignored magnetic forces, even though the particles will pick up transverse velocity and hence begin to feel them. Since magnetic forces are small compared to electric forces when the (transverse) speeds of the charges are nonrelativistic, we should require

$$J \ll mc$$


to be safe. Now, given this assumption, we can focus on the electric forces. Here we have assumed the charges don't move transversely during the whole process. The final transverse velocity is of order  $J/m$ , and the total time the interaction takes is of order  $b/v$ , so we need

$$\frac{J}{m} \frac{b}{v} \ll b$$

which simplifies to

$$J \ll mv.$$

Since this is strictly stronger than the other condition, this is the only one we really need. In other words, this kind of calculation only works if the transverse speed  $J/m$  the charges pick up is small compared to the original speed, i.e. if the angular deflection is small.

- [3] **Problem 4.**  USAPhO 2014, problem B2. This isn't the clearest of problems, but it introduces and justifies the Galilean field transformations we first saw in **E4**.
- [3] **Problem 5** (Purcell 5.30). Consider an infinite wire oriented along  $\hat{\mathbf{x}}$  with linear charge density  $\lambda$  and current  $I$ . Show that under a Lorentz boost along  $\hat{\mathbf{x}}$ ,  $(\lambda, I)$  transforms like  $(ct, x)$ .

**Solution.** This can get kind of complicated if you think about a completely general set of charges in the wire. On the other hand, we already know that the way electromagnetic fields transform doesn't depend on what makes the fields, and we know that Maxwell's equations relate the fields to the charge density and current. Thus, the transformation of  $(\lambda, I)$  shouldn't depend on precisely what's responsible for the  $\lambda$  and  $I$ , so we can take a concrete choice that's easy to work with.

Specifically, let's suppose the wire is built entirely out of point charges of linear number density  $n$  and charge  $q$  moving with velocity  $u\hat{\mathbf{x}}$ . Then we have

$$\lambda = nq, \quad I = nqu.$$

Now we move to a frame moving to the right with speed  $v$ . In this frame, the charges are moving with speed

$$u' = \frac{u - v}{1 - uv}$$

and, by applying length contraction, the linear number density is

$$n' = \frac{\gamma_{u'}}{\gamma_u} n = \sqrt{\frac{1 - u^2}{1 - u'^2}} n = (1 - uv)\gamma_v n.$$

Putting things together, we have

$$\lambda' = n'q = (1 - uv)\gamma_v nq, \quad I' = n'qu' = (u - v)\gamma_v nq.$$

Writing this in terms of the unprimed quantities, we conclude

$$\lambda' = \gamma_v(\lambda - vI), \quad I' = \gamma_v(I - v\lambda)$$

which is precisely the desired result.

- [2] **Problem 6** (Purcell 6.22). A neutral wire carries current  $I$ . A stationary charge  $q$  is nearby; the Lorentz force on this charge is zero. Verify this remains true in a frame moving parallel to the wire with velocity  $\mathbf{v}$ , by using the Lorentz transformations of the fields.

**Solution.** Applying the transformations of problem 5, in that frame we have

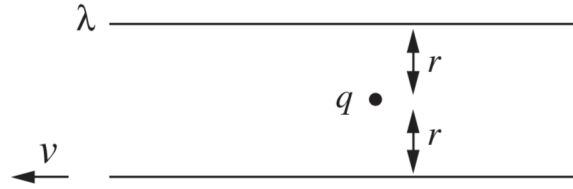
$$\lambda' = -\gamma_v v I, \quad I' = \gamma_v I.$$

In other words, the wire picks up a net charge. The electric and magnetic forces are

$$qE = \frac{\gamma_v v I q}{2\pi r \epsilon_0}, \quad qvB = \frac{\mu_0 \gamma_v v I q}{2\pi r}$$

which balance because  $\mu_0 \epsilon_0 = 1$ , in our units where  $c = 1$ .

- [3] **Problem 7** (Purcell 6.69). Two very long sticks each have uniform linear proper charge density  $\lambda$ . One stick is stationary in the lab frame, while the other moves to the left with speed  $v$ , as shown.



They are  $2r$  apart, and a stationary point charge  $q$  lies midway between them. Find the Lorentz three-force on the charge in the lab frame, and also in the frame of the bottom stick, and verify the forces relate properly.

**Solution.** Let upward-pointing forces be denoted with a positive sign. In the lab frame, we only have an electric force. The charge on the bottom stick is length contracted, so

$$F = q \left( \frac{\gamma \lambda}{2\pi \epsilon_0 r} - \frac{\lambda}{2\pi \epsilon_0 r} \right) = \frac{q \lambda}{2\pi \epsilon_0 r} (\gamma - 1).$$

In the frame of the bottom stick, the bottom stick has charge density  $\lambda$ , while the top stick has

$$\lambda' = \gamma \lambda, \quad I' = \gamma v \lambda$$

where  $I'$  is directed to the right. Now the charge experiences both an electric and a magnetic force. The electric force is

$$F'_E = -\frac{q \lambda}{2\pi \epsilon_0 r} (\gamma - 1)$$

by the same logic as in the lab frame. The magnetic force is

$$F'_B = qvB = \frac{\mu_0 \gamma v^2 q \lambda}{2\pi r} = \frac{q \lambda}{2\pi \epsilon_0 r} (\gamma v^2).$$

The sum of the two is

$$F' = \frac{q \lambda}{2\pi \epsilon_0 r} (\gamma v^2 - \gamma + 1) = \frac{q \lambda}{2\pi \epsilon_0 r} \left( 1 - \frac{1}{\gamma} \right) = \frac{F}{\gamma}$$

exactly as expected.

**Remark: Are Wires Neutral?**

The classic example in problem 6 starts by assuming the wire is neutral in the lab frame. But in problem 5, you showed that if a current-carrying wire is neutral in one reference frame, then it's *not* neutral in other reference frames. So how do we know which frame a *real* wire is neutral in? Is it the lab frame, the frame where the current vanishes, or something else?

It actually depends on the details, so for concreteness, let's consider two very long, parallel wires, connected at one end by a battery and at the other end by a resistor. Before current starts flowing, the whole system is neutral. So if the wires picked up a net charge density, the battery would have to have a large compensating charge, which would make it blow up. Or, to say it another way, if a net charge appears in the rest of the system, it pulls a compensating charge out of the battery, so the battery keeps the wires net neutral. You can then show that in a boosted frame, the wires stay net neutral, as you'd expected.

But this argument only shows that the wires have *opposite* charge densities  $\pm\lambda$  in the lab frame. Can we show that  $\lambda = 0$ ? Actually, we can't, because it's not true! As briefly discussed in a problem in **E2**, wires in circuits do carry charges in the lab frame, even if everything is ideal. One simple way to see this here is to note that the wires are at different electrical potentials. That's only possible if there's an electric field between them, which is created by the charge densities carried by the wires.

[3] **Problem 8.** The vectors  $\mathbf{E}$  and  $\mathbf{B}$  cannot go into four-vectors, as they transform among each other, but rather fit together into an antisymmetric rank two tensor. As a result, there is a different set of associated invariant quantities.

- Show that under the relativistic field transformations, the quantities  $\mathbf{E} \cdot \mathbf{B}$  and  $E^2 - B^2$  are both invariant. (Hint: this can be done using vector notation, using  $\mathbf{E}_\perp \cdot \mathbf{E}_\parallel = \mathbf{B}_\perp \cdot \mathbf{B}_\parallel = 0$ .) These are the two basic invariants, out of which all other invariants can be constructed.
- Suppose that in an inertial frame,  $\mathbf{E}$  is zero at a given point and  $\mathbf{B}$  is nonzero. Is it possible to find an inertial frame where  $\mathbf{B}$  is zero at that point?
- Recall from **E7** that, in units where  $\epsilon_0 = \mu_0 = 1$ , the energy density of the electromagnetic field is  $\mathcal{E} = E^2/2 + B^2/2$ , and the Poynting vector is  $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ . Show that  $\mathcal{E}^2 - |\mathbf{S}|^2$  is invariant. (Hint: don't use the field transformations for this part.)

**Solution.** (a) Setting  $c = 1$  for convenience as usual, we have

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E}'_\perp \cdot \mathbf{B}'_\perp + E'_\parallel B'_\parallel = \gamma^2(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp) \cdot (\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}_\perp) + E_\parallel B_\parallel.$$

The cross terms vanish (try an explicit example if you don't see why), which leaves

$$\mathbf{E}' \cdot \mathbf{B}' = \gamma^2(1 - v^2)\mathbf{E}_\perp \cdot \mathbf{E}_\perp + E_\parallel B_\parallel = \mathbf{E} \cdot \mathbf{B}$$

as desired. As for the other quantity, we have

$$\begin{aligned}
 E'^2 - B'^2 &= E_{\parallel}'^2 + \mathbf{E}_{\perp}' \cdot \mathbf{E}_{\perp}' - B_{\parallel}'^2 - \mathbf{B}_{\perp}' \cdot \mathbf{B}_{\perp}' \\
 &= E_{\parallel}^2 + \gamma^2(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp})^2 - B_{\parallel}^2 - \gamma^2(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}_{\perp})^2 \\
 &= E_{\parallel}^2 + \gamma^2(\mathbf{E}_{\perp} \cdot \mathbf{E}_{\perp} + v^2 \mathbf{B}_{\perp} \cdot \mathbf{B}_{\perp}) - B_{\parallel}^2 - \gamma^2(\mathbf{B}_{\perp} \cdot \mathbf{B}_{\perp} + v^2 \mathbf{E}_{\perp} \cdot \mathbf{E}_{\perp}) \\
 &= E_{\parallel}^2 + \mathbf{E}_{\perp} \cdot \mathbf{E}_{\perp} - B_{\parallel}^2 - \mathbf{B}_{\perp} \cdot \mathbf{B}_{\perp} \\
 &= E^2 - B^2
 \end{aligned}$$

as desired, where in the third equality the cross-terms canceled.

- (b) This is impossible, because then  $E^2 - B^2$  would have different signs in the two frames.
- (c) If we plugged in the field transformations, the algebra would get extremely messy. Instead, we use the hint that this invariant can be constructed out of the ones found in part (a), so

$$\begin{aligned}
 \mathcal{E}^2 - |\mathbf{S}|^2 &= \left( \frac{1}{2}E^2 + \frac{1}{2}B^2 \right)^2 - |\mathbf{E} \times \mathbf{B}|^2 \\
 &= \left( \frac{1}{2}E^2 + \frac{1}{2}B^2 \right)^2 + (\mathbf{E} \cdot \mathbf{B})^2 - E^2 B^2 \\
 &= \left( \frac{1}{2}E^2 - \frac{1}{2}B^2 \right)^2 + (\mathbf{E} \cdot \mathbf{B})^2.
 \end{aligned}$$

This is constructed out of the invariants in part (a), so it is invariant as well.

### Remark: Is Magnetism Real?

Purcell's electromagnetism textbook is exceptional because it shows that a force like magnetism must exist, if one believes Coulomb's law and relativity. The idea is simple. We know how forces transform between frames, and given some reasonable assumptions, can also deduce how electric fields transform between frames. If electric fields were all there were, then electric forces would have to transform just like three-forces, but they don't. So there must be some other force to make up the difference, and it turns out to be precisely the magnetic force. We saw an example of this in problem 2.

It is important not to misunderstand this beautiful idea. Many people, upon reading such arguments, believe that magnetism "doesn't exist" because it's "all just electric fields". Sometimes people even say that magnetic forces are a "mistake" caused by "forgetting about" relativistic corrections. This is all totally backwards. Sometimes time dilation in one frame can be explained in terms of length contraction in another, but that doesn't mean that length contraction doesn't exist, or is a mistake – it's perfectly real in that particular frame. (Furthermore, while you can always get rid of the magnetic *force* on one particle at one moment by going to that particle's rest frame, there are plenty of situations where you can't remove the magnetic field, as we saw in problem 8!)

The real lesson of relativity isn't that magnetic fields are a mistake, it's that electric and magnetic fields are as intertwined as space and time, as you can see from their transformation properties. Just as space and time combine into a four-vector, electric and magnetic fields



combine, in an equal footing, into the electromagnetic field tensor.

### Remark: Electromagnetism in Covariant Form

Problem 5 is a first step to showing that  $J^\mu = (\rho, \mathbf{J})$  is a four-vector, where  $\rho$  is the charge density and  $\mathbf{J}$  is the current density. Note that the continuity equation for charge, as mentioned in **T2**, can be simply written in four-vector notation as

$$\partial_\mu J^\mu = 0.$$

As another example, you can show that the four-current of a single charged particle  $q$  is  $J^\mu = qu^\mu$ . We can go even further and write the whole of electromagnetism in terms of four-vectors and tensors. Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

The invariant quantities found in problem 8 can be written in terms of the field strength tensor as  $F_{\mu\nu}F^{\mu\nu}$  and  $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$  where  $\epsilon_{\mu\nu\rho\sigma}$  is the Levi-Civita symbol. These are the only two ways to “contract all the indices” to get a scalar.

### Remark: Elegant Notation

Sometimes people dislike the index notation above because of all the little Greek letters floating around. If you *only* want to deal with vectors, vector notation is often better. It hides all the indices, at the cost of requiring you to introduce special symbols like  $\cdot$  and  $\times$  to specify the vector operations you want to do. The reason we don't use a vector-like notation for tensors is because there are too many operations you can do with them (e.g. “contract the 3rd index of a rank 4 tensor with the 1st index of a rank 2 tensor”) to define separate symbols for each one; indices are just more efficient. On the other hand, if you only work with totally antisymmetric tensors, then there are only a few possible operations, and one can use the elegant, index-free “differential form” notation. In this notation, Maxwell's equations are

$$d \star F = J$$

where  $d$  is called the exterior derivative,  $\star$  is the Hodge dual, and the fact that the electromagnetic fields are derivatives of potentials is expressed as

$$F = dA.$$

So is this the *best, most true* formulation of Maxwell's equations? Well, as Feynman once pointed out, you can easily do better. For example, you can define the “unworldliness”

$$U = |\mathbf{F} - m\mathbf{a}|^2 + (\nabla \cdot \mathbf{E} - \rho/\epsilon_0)^2 + \dots$$

Then *all* physical laws can be expressed in terms of the amazingly simple equation

$$\boxed{\boxed{U = 0.}}$$

But this doesn't actually help, because to use the equation for anything, you need to plug in the definition of  $U$ , and then you're back to where you were before. In general, more elegant notation is often more brittle: it only works well in a smaller set of situations. (For example, with differential form notation, you just can't write down the stress-energy tensor of the electromagnetic field, because that's symmetric rather than antisymmetric.) Index notation is great because it works as long as indices are contracted in pairs, which holds as long as you're dealing with laws that are independent of coordinate system. In general, there's no need to be ideological about notation; it's just a tool, and we should use the best tool for each job. If anyone tells you that their preferred notation for vectors or tensors will revolutionize physics, keep your hand on your wallet.

[4] **Problem 9.** Consider an electromagnetic wave of the form

$$\mathbf{E}(z, t) = E_0 \cos(kz - \omega t) \hat{\mathbf{x}}, \quad \mathbf{B}(z, t) = B_0 \cos(kz - \omega t) \hat{\mathbf{y}}.$$

As usual, you may work in units where  $c = 1$ .

- (a) What do Maxwell's equations imply about the relation between  $E_0$  and  $B_0$ , and  $k$  and  $\omega$ ?
- (b) Now consider a frame moving with velocity  $v$  along the  $\hat{\mathbf{z}}$  direction. Show that the electromagnetic wave continues to have the same basic functional form for  $\mathbf{E}'(z', t')$  and  $\mathbf{B}'(z', t')$ , but with new parameters  $E'_0$ ,  $B'_0$ ,  $k'$ , and  $\omega'$ . Using these results, show that the energy density of the wave is smaller by a factor of  $(1 - v)/(1 + v)$ .
- (c) The energy of a photon in an electromagnetic wave of frequency  $\omega$  is  $E = \hbar\omega$ . Show that for a finite-sized electromagnetic wave, the initial and boosted frames agree on the number of photons. This was one of the hints Einstein used to conclude light was made of photons.
- (d) Now consider another question Einstein pondered: what does the light wave look like if we try to "catch up" with it, taking  $v \rightarrow c$ ? Is this consistent with the invariants of problem 8?

**Solution.** (a) From **E7**, we know that  $E_0 = B_0$  and  $k = \omega$ .

- (b) The electromagnetic field only has perpendicular components. Using the field transformations,

$$\mathbf{E}'(z', t') = \gamma(E_0 \cos(kz - \omega t) - vB_0 \cos(kz - \omega t)) \hat{\mathbf{x}}.$$

In units where  $c = 1$ , we have  $E_0 = B_0$  for an electromagnetic wave, so this simplifies to

$$\mathbf{E}'(z', t') = \gamma(1 - v)E_0 \cos(kz - \omega t) \hat{\mathbf{x}}.$$

Repeating the reasoning for the magnetic field, we conclude

$$E'_0 = \gamma(1 - v)E_0, \quad B'_0 = \gamma(1 - v)B_0$$

which still obeys  $E'_0 = B'_0$  as expected. Thus, the energy density is reduced by a factor of

$$\gamma^2(1 - v)^2 = \frac{1 - v}{1 + v}$$

as stated. To find  $k'$  and  $\omega'$ , we can simply apply the Lorentz transformations,

$$kz - \omega t = k(\gamma(z' + vt')) - \omega(\gamma(t' + vz')) = \gamma(k - \omega v)z' - \gamma(\omega - kv)t'.$$

This indicates that  $\mathbf{E}'(z', t')$  is still a plane wave proportional to  $\cos(k'z' - \omega't')$ , where

$$k' = \gamma(k - \omega v), \quad \omega' = \gamma(\omega - kv)$$

which is of course just the statement that  $(\omega, \mathbf{k})$  is a four-vector, derived in **R1**. Using the fact that  $\omega = k$ , we conclude

$$k' = \omega' = \gamma(1 - v)\omega = \sqrt{\frac{1-v}{1+v}} \omega$$

which is of course just the usual Doppler shift.

- (c) The number of photons is the ratio of the total energy in the wave to the energy of each photon. Since the energy of each photon is reduced by a factor of  $\sqrt{(1-v)/(1+v)}$  in the boosted frame, we need to show that the total energy of the wave is reduced by the same factor. This results from the combination of two effects.

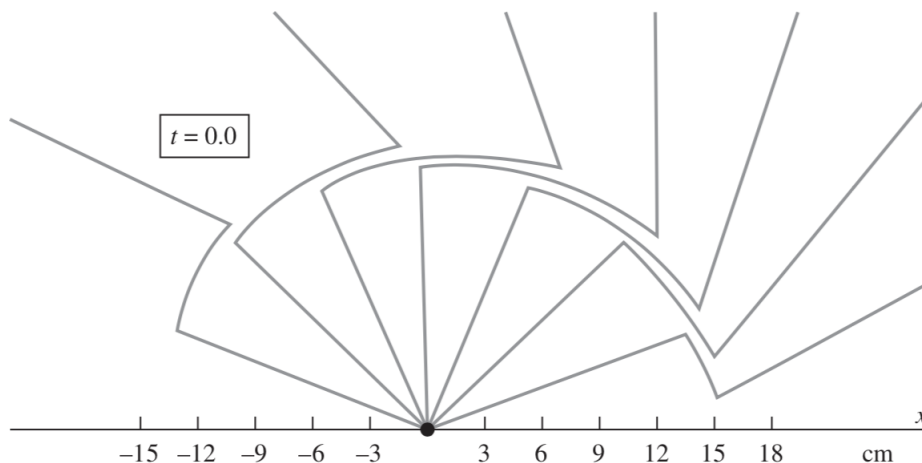
First, we know the energy density is reduced by the factor  $(1-v)/(1+v)$ . Second, the wavenumber is reduced by a factor of  $\sqrt{(1-v)/(1+v)}$ , which means the wavelength is increased by  $\sqrt{(1+v)/(1-v)}$ . Since the number of wavelengths contained in the wave is the same in every reference frame, this means the volume of the wave is increased by  $\sqrt{(1+v)/(1-v)}$ . Multiplying these factors gives the desired result.

- (d) In this case we have  $E'_0, B'_0, \omega', k' \rightarrow 0$ , so the light wave disappears! That is, you can never “catch up” to a light wave. This result is completely compatible with the invariants from part (a) of problem 8, which both vanish for a plane electromagnetic wave. The invariant in part (c) vanishes as well, since  $\mathcal{E} = |\mathbf{S}|$  for a plane wave.

### Idea 2

If a uniformly moving point charge suddenly stops moving, then the field outside a spherical shell, centered at the charge when it stopped moving, expanding at speed  $c$ , is precisely that calculated in problem 1. The same occurs if the point charge suddenly changes its velocity; information about the change only propagates at  $c$ .

- [1] **Problem 10** (Purcell 5.18). In the figure below, you see an electron at time  $t = 0$  and the associated electric field at that instant.

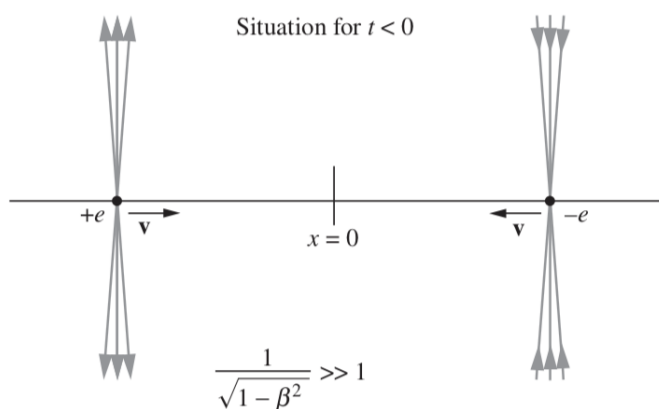


- (a) Describe what has been going on, as quantitatively as you can.
- (b) Where was the electron at the time  $t = -0.75$  ns?

**Solution.** (a) Clearly, the particle isn't moving at  $t = 0$ . Since there's a kink in the field lines at  $r = 15$  cm, it must have quickly stopped at  $t = -r/c = -0.5$  ns, since the speed of light is  $c = 30$  cm/ns. We also see that the field lines outside this shell are straight, and point towards the location  $x = 12$  cm. This implies that shortly before the charge stopped, it was moving with constant velocity  $v = |x/t| = 24$  cm/ns  $= 0.8c$ .

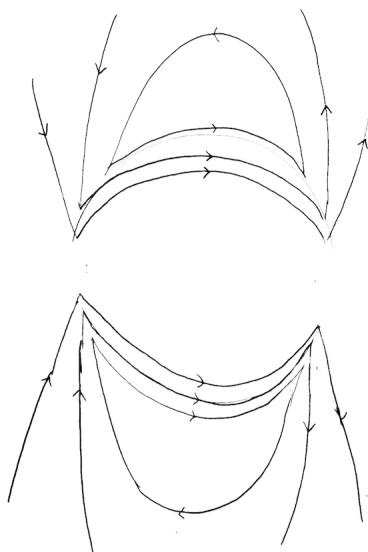
- (b) By combining the results from (a), it must have been at  $x = -(24 \text{ cm/ns})(0.25 \text{ ns}) = -6$  cm.

[2] **Problem 11** (Purcell 5.19). The figure below shows two highly relativistic particles with opposite charge approaching the origin.



They collide at the origin at time  $t = 0$  and remain there as a neutral entity. Sketch the field lines at some time  $t > 0$ .

**Solution.** Before the charges collide, we have a kind of distorted dipole field. After they collide, we still have a dipole field at  $r > ct$ , with the positive charge on the right and the negative charge on the left. For  $r < ct$  the field is zero, and at the shell  $r \approx ct$  there is a thin shell that connects up the field lines. This transverse pulse of radiation steadily moves outward over time.



Some crackpots claim that particle annihilation is impossible because that would imply the electric field has to “instantly vanish”, contradicting relativity. As you can see, that’s not the case. The resulting pulse of radiation travels outward at the speed of light. The electric field only vanishes inside the expanding shell.

- [3] **Problem 12.** Work through the derivation of the Larmor formula in Appendix H of Purcell.
- [3] **Problem 13** (Purcell H.4). The Larmor formula only applies to particles moving nonrelativistically. To get a result valid for faster particles, we can simply transform into an inertial frame  $F'$  where the particle is nonrelativistic, apply the Larmor formula, then transformed back to the lab frame.
- (a) Consider an relativistic electron moving perpendicularly to a magnetic field  $\mathbf{B}$ . Defining the radiation power as  $P_{\text{rad}} = dE/dt$ , find  $P'_{\text{rad}}$ , the power in a frame instantaneously comoving with the electron.
- (b) Argue that in this context,  $P_{\text{rad}} = P'_{\text{rad}}$ , and conclude that

$$P_{\text{rad}} = \frac{\gamma^2 v^2 e^4 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

Thus, the power increases rapidly as  $v \rightarrow c$ . Incidentally, a “relativistic” way to write the general result is

$$P_{\text{rad}} = \frac{q^2}{6\epsilon_0 c^3} \left( \frac{1}{m} \frac{dp^\mu}{d\tau} \right)^2$$

which clearly reduces to the Larmor formula in the nonrelativistic limit.

- (c) This radiation is also called synchrotron radiation. Qualitatively, how does its angular distribution differ from radiation from an accelerating nonrelativistic charge?

**Solution.** (a) In the frame comoving with the electron, it’s not relativistic, so we can just apply the Larmor formula,

$$P'_{\text{rad}} = \frac{e^2 a'^2}{6\pi\epsilon_0 c^3}.$$

In this frame, the only force is the electric force, so

$$a' = \frac{eE'}{m} = \frac{e\gamma v B}{m}.$$

Putting it together, we conclude

$$P'_{\text{rad}} = \frac{e^4 \gamma^2 v^2 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

- (b) We have  $P'_{\text{rad}} = dE'/dt'$ . Now, in the primed frame, the electron is just accelerating transversely, with no component along the unprimed frame’s  $\mathbf{v}$ . Thus, when we Lorentz transform back to the unprimed frame, we simply get  $dE = \gamma dE'$  and  $dt = \gamma dt'$ . The  $\gamma$  factors cancel out, giving the desired result.
- (c) In the primed frame, the radiation power comes out with a wide angular distribution, but none of it comes out along the direction of motion of the charge, and most of it comes out roughly transverse to the motion. But when we boost back to the original frame, where the charge is moving very quickly, the radiation’s direction gets a big component along the charge’s direction of motion. Thus, almost all the radiation is “beamed” in a narrow cone along the charge’s motion (as we saw in **R1**), though there still is zero radiation intensity *exactly* along the charge’s direction.

**Remark: Gravitoelectromagnetism**

As mentioned in **E1**, there's a close analogy between electrostatic fields, which are sourced by charge density  $\rho_e$ , and gravitational fields, which are sourced by energy density  $\rho$ . Therefore, if you apply the analogy and run the same arguments as in Purcell, you would expect there to be a “gravitomagnetic” field, which is sourced by momentum density  $\mathbf{J} = \rho\mathbf{v}$ . That's indeed correct! In the theory of gravitoelectromagnetism, the force on a point mass is

$$\mathbf{F} = m(\mathbf{E}_g + 4\mathbf{v} \times \mathbf{B}_g)$$

where the gravitoelectric and gravitomagnetic fields  $\mathbf{E}_g$  and  $\mathbf{B}_g$  satisfy

$$\nabla \cdot \mathbf{E}_g = 4\pi G\rho, \quad \nabla \cdot \mathbf{B}_g = 0, \quad \nabla \times \mathbf{E}_g = -\dot{\mathbf{B}}_g, \quad \nabla \times \mathbf{B}_g = 4\pi G\mathbf{J} + \dot{\mathbf{E}}_g.$$

From this you can draw some interesting conclusions. For example:

- Two masses moving parallel to each other will have an extra attraction due to the gravitomagnetic force.
- A rotating object will produce a gravitomagnetic field which can cause gyroscopes to precess; this is called the Lense–Thirring, or frame dragging effect, which has been measured by satellites such as [Gravity Probe B](#). (There is also a significantly larger “geodetic” effect caused by the curvature of spacetime around the Earth, but this isn't captured within gravitoelectromagnetism.)
- A mass at rest, inside a cylinder which suddenly starts to rotate, will pick up a small angular velocity in the same direction due to the induction of a gravitoelectric field.
- Gravitational waves are generated by accelerating masses and carry energy, just like electromagnetic radiation.

Now you might be puzzled by two things: first, how does gravitoelectromagnetism relate to general relativity, and second, why is there an extra 4 in one of the equations above? Well, the truth is that Purcell's arguments don't really work for gravity. These arguments crucially depend on electric charge  $Q = \int \rho_e d\mathbf{x}$  being Lorentz invariant, which in our more sophisticated language was necessary to ensure  $j^\mu = (\rho_e, \rho_e\mathbf{v})$  is a four-vector. However, the total energy  $E = \int \rho d\mathbf{x}$  is not Lorentz invariant – instead it's itself a component of a four-vector. Thus,  $(\rho, \rho\mathbf{v})$  isn't a four-vector, so none of the arguments really work: the theory of gravitoelectromagnetism is just not Lorentz invariant at all.

Instead, gravitoelectromagnetism is properly derived as a limiting case of general relativity, valid when all the masses involved are moving slowly,  $v \ll c$ . The fact that general relativity is a theory of a rank 2 tensor field, the metric  $g_{\mu\nu}$ , is responsible for the extra factors of 2 above. Even though it's only approximately true, gravitoelectromagnetism is a very useful tool for analyzing precision tests of general relativity, since it's much easier to calculate with.

On the other hand, there's also a lot of nonsense written about it by people who don't understand it. For example, a lot of internet luminaries are certain that it can be used to replace dark matter, even though, using just the basic equations above, you can see that the

gravitomagnetic force is  $(v/c)^2$  times smaller than the usual gravitational force. That makes it about  $10^6$  times too small to explain the anomalous rotation of galaxies.

In fact, now is a good time to issue a warning. There's a concept called Lizardman's constant, which is the fact that in any survey, no matter how it's designed, about 3% of the answers will be complete nonsense. 3% of people will enthusiastically tell you that they were born on Mars, that the Moon landing was faked, or that the Earth is run by lizardmen. That's because there's an irreducible fraction of people that are mistaken, crazy, or just plain trolling.

The internet is a wonderful place to learn introductory physics, because it's relatively straightforward, so the sincere and competent outnumber the crazy. But as you go to more advanced topics, the fraction of people who know what's going on, and who have the time and energy to tell you, rapidly drops, while the 3% stays just as large. Now that you're at the end of this curriculum, you're also at the point where the *majority* of internet commentators on the topics you're learning are completely wrong. Fortunately, you're also learning what sources are good, and developing the knowledge needed to check things for yourself. As you continue learning tougher subjects, these skills will keep you on the right track.

## 2 Charges in Fields

- [5] **Problem 14.** ⌚ IPhO 1991, problem 2. A problem on the subtle relativistic “hidden momentum”.

### Idea 3: Scalar and Vector Potentials

In **E1**, we learned about the electric (or “scalar”) potential  $\phi(\mathbf{x})$ , which obeys  $\mathbf{E} = -\nabla\phi$ . More generally, the scalar potential can depend on both space and time, as can the vector potential  $\mathbf{A}(\mathbf{x}, t)$ , and these two quantities yield the electric and magnetic fields by

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Olympiad problems rarely require thinking about the vector potential, but it's essential to formulate the following idea.

### Idea 4: Canonical Momentum

Noether's theorem yields a conserved quantity for every symmetry. If a system is symmetric under time translations, then its energy is conserved. Spatial translational symmetry yields momentum conservation, and rotational symmetry yields angular momentum conservation.

We won't prove Noether's theorem, but we'll illustrate it for a nonrelativistic particle of mass  $m$  and charge  $q$ . First, if  $\phi$  and  $\mathbf{A}$  are both time-independent, then the conserved energy is

$$E = \frac{1}{2}mv^2 + q\phi.$$

This is quite familiar. Note that  $\mathbf{A}$  doesn't appear because in this case, the only role of  $\mathbf{A}$  is to determine the magnetic field, which does no work.

As for space-translational symmetry, if  $\phi$  and  $\mathbf{A}$  are both space-independent, then the conserved momentum, called the “canonical” momentum, is

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}.$$

This is less familiar, so let’s check it explicitly. Since we have assumed  $\phi$  and  $\mathbf{A}$  are both space-independent, we simply have  $\mathbf{B} = 0$  and  $\mathbf{E} = -\partial\mathbf{A}/\partial t$ , so

$$\frac{d\mathbf{p}}{dt} = m\mathbf{a} - q\mathbf{E} = 0$$

as desired. This tells us that  $q\mathbf{A}$  is like a “potential momentum”, similar to how  $q\phi$  is a potential energy. (Since the canonical momentum is such an important property, it is usually denoted by  $\mathbf{p}$  whenever it’s in play, while the Newtonian “mechanical”/“kinetic” momentum is demoted to  $\boldsymbol{\pi} = m\mathbf{v}$ .) However, in this case, the tool of canonical momentum doesn’t tell us much we didn’t already know.

Canonical momentum becomes useful in situations with only *partial* translational symmetry. For example, suppose that  $\phi$  and  $\mathbf{A}$  are both independent of  $x$ , but not  $y$  and  $z$ . Then the fields can be quite complicated, as can the particle’s motion, but  $p_x$  will still be conserved!

In addition, the canonical momentum is the building block used for more complex situations. For example, if  $\phi$  and  $\mathbf{A}$  are both invariant under rotations about the  $z$ -axis, then

$$J_z = (\mathbf{r} \times \mathbf{p}) \cdot \hat{\mathbf{z}}$$

is conserved. Moreover, the adiabatic invariant of **M4** must be written in terms of the canonical momentum. For example, for periodic motion along the  $x$ -axis, it is

$$I = \oint p_x dx$$

while for periodic circular motion in the  $xy$  plane, it is

$$I = \oint J_z d\theta.$$

Finally, though all the following problems will assume the particles are nonrelativistic, the results above go through unchanged in relativistic mechanics provided that  $\boldsymbol{\pi} = m\mathbf{v}$  is replaced with the relativistic momentum  $\gamma m\mathbf{v}$ .

- [4] **Problem 15.** Let’s check some of the statements made above in a simple case. Consider a situation with zero electric field and a constant uniform magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$ .

(a) Show that this situation can be described with

$$\phi = 0, \quad \mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) = \frac{Br}{2}\hat{\boldsymbol{\theta}}$$

which is symmetric under translations along the  $z$ -axis and rotations about the  $z$ -axis.



- (b) The symmetries of this problem tell us that  $p_z$  and  $J_z$  are conserved. Conservation of  $p_z = \pi_z$  is obvious, because there are no forces in the  $z$ -direction. As for  $J_z$ , find an explicit expression for it, in terms of the ordinary angular momentum  $L_z$ ,  $B$ , and  $r$ .

Back in **E4**, we encountered IPhO 1996, problem 2, which instructs the reader to solve a tricky problem by pulling out a magical conserved quantity; it's just  $J_z$ . Next, let's suppose the magnetic field's magnitude varies in time, corresponding to

$$\phi = 0, \quad \mathbf{A} = \frac{B(t)r}{2} \hat{\boldsymbol{\theta}}.$$

Of course, the changing magnetic field is accompanied by an induced electric field, as  $\mathbf{E} = -\partial\mathbf{A}/\partial t$ .

- (c) Suppose that  $B$  is initially equal to  $B_0$ , and the particle is orbiting in a circle of radius  $r_0$  about the  $z$ -axis. The field is *slowly* changed to  $B_1$ . What is the new radius  $r_1$  of the orbit?
- (d) What if instead the field is very quickly changed to  $B_1$ ?
- (e) Even more generally, let's suppose the field is  $\mathbf{B} = B(r, t) \hat{\mathbf{z}}$ , which can depend on both time and the distance  $r$  to the  $z$ -axis. Assuming the field changes slowly, find a compact expression for  $J_z$  in terms of  $r$ , the value of  $B$  at the particle's radius, and  $B_{\text{av}}$ , the average value of  $B$  within the circle formed by the particle. This will immediately yield the answer to the "betatron" example in **E4**.

**Solution.** (a) This follows immediately from evaluating the curl of  $\mathbf{A}$ , either in Cartesian or cylindrical coordinates.

- (b) The  $z$ -component of the canonical angular momentum is

$$J_z = (\mathbf{r} \times (m\mathbf{v} + q\mathbf{A})) \cdot \hat{\mathbf{z}} = L_z + \frac{qBr^2}{2}.$$

- (c) Because the field is slowly changed, the particle remains in a circular orbit centered on the  $z$ -axis, so we can simply use the conservation of  $J_z$  to find the answer. It is

$$J_z = -mvr + \frac{qBr^2}{2}$$

where the minus sign is because the two contributions to  $J_z$  have opposite signs. On the other hand, for circular motion we have  $mv^2/r = qvB$ , which tells us that the two terms are simply proportional to each other. So in this case, we just have

$$J_z = -\frac{qBr^2}{2}$$

and the answer to the question is

$$r_1 = r_0 \sqrt{B_0/B_1}.$$

We could also have gotten to this conclusion using the adiabatic theorem, but it's not any different, since here  $I = 2\pi J_z$ . And of course, you can also derive it directly using Newton's laws, in a manner similar to the betatron example in **E4**.

- (d) When the field is quickly changed, the particle simply receives a sharp impulse. The final orbit is still a circle, but it won't be centered on the  $z$ -axis. Conservation of  $J_z$  during the impulse gives

$$m \Delta v = \frac{qr_0 \Delta B}{2} = \frac{qr_0}{2}(B_1 - B_0)$$

which of course can also be deduced by Newton's laws. The final speed is

$$v' = \frac{qB_0 r_0}{m} + \Delta v = \frac{qr_0}{m} \frac{B_0 + B_1}{2}$$

which is related to the final radius by  $v' = qB_1 r_1/m$ . We thus conclude

$$r_1 = r_0 \frac{B_0 + B_1}{2B_1}.$$

For a more challenging problem which uses similar ideas, see [Physics Cup 2017, problem 3](#).

- (e) In this case, we have

$$J_z = -mvr + qrA_\theta$$

where  $A_\theta$  is independent of  $\theta$ . To evaluate this, we note that

$$A_\theta = \frac{1}{2\pi} \int_0^{2\pi} A_\theta d\theta = \frac{1}{2\pi r} \oint_C \mathbf{A} \cdot d\mathbf{r} = \frac{\Phi_B(r)}{2\pi r} = \frac{r}{2} B_{\text{av}}(r).$$

where  $C$  is the particle's momentary circular orbit, and we used Stokes' theorem and then the definition of  $B_{\text{av}}$ . We still have  $mvr = qB(r)r^2$ , so

$$J_z = qr^2 \left( -B + \frac{B_{\text{av}}}{2} \right).$$

If the quantity in parentheses doesn't vanish, then conservation of  $J_z$  implies that  $r$  has to change as the field is changed. As we found in **E4** using Newton's laws,  $r$  can stay the same if  $B = B_{\text{av}}/2$ , in which case  $J_z$  simply vanishes.

- [2] **Problem 16.** Let's consider one more simple application of canonical momentum. Suppose a point charge of mass  $m$  and charge  $q$  experiences the uniform constant fields  $\mathbf{E} = E\hat{\mathbf{x}}$  and  $\mathbf{B} = B\hat{\mathbf{y}}$ .

- (a) Write a corresponding  $\phi$  and  $\mathbf{A}$  which are independent of  $y$  and  $z$ .  
 (b) What are the associated conserved quantities?

**Solution.** (a) The simplest possible answer is  $\phi = -Ex$  and  $\mathbf{A} = -Bx\hat{\mathbf{z}}$ .

- (b) The conserved quantities are  $p_y = mv_y$  and  $p_z = mv_z - qBx$ . Conservation of  $p_y$  is trivial, as there are no forces in the  $y$ -direction. As for  $p_z$ , it's conserved because

$$\frac{dp_z}{dt} = ma_z - qBv_x$$

which is simply the  $z$ -component of the Lorentz force law, as we saw in the corresponding problem in **E4**. So in this case the canonical momentum doesn't tell you much new, though in a more subtle situation, such as when the particle is relativistic, it can be useful.

By now, we've covered most of the applications of canonical momentum for point charges. When it shows up on modern Olympiad problems, there's generally a twist. For instance, we can replace the point charge with an electric dipole, as illustrated in the following three tough questions.

[5] **Problem 17.** ⌚ APhO 2001, problem 2.

[5] **Problem 18.** ⌚ EuPhO 2022, problem 3.

**Solution.** See the official solutions [here](#).

[5] **Problem 19.** [Physics Cup 2021, problem 1](#). This one requires more electromagnetism background.

**Solution.** See the official solutions [here](#).

[5] **Problem 20.** ⌚ GPhO 2017, problem 3. A problem on a “shock wave” hitting an electron. Don't be intimidated by the language; you don't need to know anything about shock wave physics to do this question.

**Solution.** See the official solutions [here](#).

### 3 Gravitational Fields

#### Idea 5

In classical mechanics, you've seen that a uniform gravitational field behaves a lot like the fictitious force due to a uniform acceleration. The equivalence principle states that the two behave exactly identically, in all possible contexts; it was one of the key ideas that led to the development of general relativity.

[4] **Problem 21.** In this problem, we give one of the classic justifications for gravitational redshift, the fact that photons redshift when moving against a gravitational field. Suppose that point  $B$  is a height  $h$  above point  $A$ , in a gravitational field  $g$ . A set of electrons and positrons with total rest mass  $M$  are converted into photons of frequency  $f$  at point  $A$ . The photons fly upward to point  $B$ , where they are converted back into electrons and positrons. Assume throughout that  $g$  is small.

- Find the total mass  $M'$  at point  $B$ .
- Find the frequency  $f'$  of the photons measured at point  $B$ .
- Since the frequencies of photons can be used as a clock, the result of part (b) shows that gravitational fields cause time dilation, which applies to everything, not just photons. Show that your result in part (b) is equivalent to the statement that times are dilated by a factor of  $1 + \phi/c^2$ , where  $\phi$  is the gravitational potential and  $\phi/c^2 \ll 1$ .

We should also be able to understand part (b) using the equivalence principle. To confirm this, suppose that two observers  $C$  and  $D$  begin at rest, with  $D$  a distance  $h$  to the right of  $C$ . At a certain moment, both observers begin accelerating to the right with a small acceleration  $a$ .

- If  $C$  emits light of frequency  $f$  (in  $C$ 's rest frame), show that  $D$  observes light of frequency  $f'$ , where  $f'$  matches your answer to part (b).

- (e) The predicted frequency shift was observed in the 1959 Pound–Rebka experiment, where gamma rays were transmitted from the top to the bottom of a tower of height  $h = 22.5$  m. What is the fractional change in energy of the photons?
- (f) Gamma ray photons of energy 14 keV were used in the Pound–Rebka experiment. According to the energy-time uncertainty principle, what is the minimum time needed to detect the effect?

**Solution.** (a) By conservation of energy,

$$Mc^2 = M'c^2 + M'gh$$

from which we conclude, using the fact that  $g$  is small, that

$$M' = M \left( 1 - \frac{gh}{c^2} \right).$$

- (b) Each photon has its energy reduced by a factor of  $1 - gh/c^2$ , and since  $E = hf$ ,

$$f' = f \left( 1 - \frac{gh}{c^2} \right).$$

- (c) If we measure time through the frequency of light, then

$$\Delta t' = \Delta t \left( 1 + \frac{gh}{c^2} \right)$$

again to lowest order in  $g$ . This is the desired result, since  $\phi = gh$ . It implies that higher clocks tick faster.

- (d) Since the acceleration is small, it takes about a time  $h/c$  for the light to arrive at  $D$ . By this time,  $D$  has picked up a velocity  $ah/c$ , so by the Doppler shift,

$$f' = f \sqrt{\frac{1 - ah/c^2}{1 + ah/c^2}} = f' \left( 1 - \frac{ah}{c^2} \right)$$

where we again work to lowest order in  $a$ .

- (e) Plugging in the numbers,  $gh/c^2 = 2.5 \times 10^{-15}$ .
- (f) The change in energy is  $\Delta E = h\Delta f$ , and the uncertainty principle says we need time

$$\Delta t \gtrsim \frac{h}{\Delta E} \sim \frac{h}{(2.5 \times 10^{-15})(14 \text{ keV})} = 10^{-4} \text{ s}.$$

In the real experiment, Pound and Rebka used two identical samples of iron as the emitted and receiver, and vibrated one of them vertically at a few tens of Hz. Whenever the relative velocity was just enough to cancel out the gravitational redshift effect, absorption occurred. Since the  $\Delta t$  required was substantially lower than the period of the vibration, the vibration didn't mess up the experiment.

**Remark**

You might be a little worried that the result of part (c) above does not seem to be invariant under a large, constant shift of  $\phi$ , even though in Newtonian mechanics we can always do this. In fact, in that case the same analysis is essentially valid, but the “extra” gravitational time dilation is canceled out by other effects, which unfortunately can’t be explained without full general relativity. In other words, the analysis above is only valid when  $\phi$  is small.

If you find this confusing, you’re not alone. In 2018, there was some excitement as researchers claimed to explain a long-standing anomaly in particle physics, making a mistake precisely along these lines. (A rebuttal is given [here](#).)

[3] **Problem 22.** In this problem we consider the effects of relativity on a clock on the surface of the Earth, which has mass  $M$  and radius  $R$ . It rotates about its axis in time  $T$ , as measured by an observer at infinity who is at rest relative to the center of the planet

- Consider a clock  $C$  that lies on the surface of the planet at a point on the equator. Compute the time measured by the clock  $C$  after a single rotation of the planet, incorporating both special relativity and gravitational time dilation. Which effect is bigger?
- Repeat part (a) for a clock  $C'$  on a satellite orbiting the planet, in a circular orbit a height  $h$  above the equator.
- Using the numbers  $M = 5.97 \times 10^{24}$  kg,  $R = 6.4 \times 10^6$  m, and  $h = 2 \times 10^7$  m, estimate the difference in time elapsed per day for the two clocks, counting only the effect of special relativity, or only the effect of gravitational time dilation.

[This paper](#) explains how the Global Positioning System accounts for both of these effects to work.

**Solution.** (a) The clock  $C$  always has speed  $v = 2\pi R/T$ , so if we only counted time dilation,

$$T_C = T \sqrt{1 - (2\pi R/cT)^2} \approx T \left( 1 - \frac{1}{2} \left( \frac{2\pi R}{cT} \right)^2 \right).$$

It is also at a lower gravitational potential than a clock at infinity, so counting only gravitational time dilation,

$$T_C = T (1 + \Delta\phi/c^2) = T (1 - GM/Rc^2).$$

Of course in reality both effects occur, and at leading order they just add, giving

$$T_C = T \sqrt{1 - (2\pi R/cT)^2} (1 + \Delta\phi/c^2) \approx T \left( 1 - \frac{1}{2} \left( \frac{2\pi R}{cT} \right)^2 - \frac{GM}{Rc^2} \right).$$

The two effects are equal when  $v^2 = 2GM/R$ , which describes escape velocity. Since the Earth is rotating a lot slower than that, the gravitational time dilation effect is much larger.

- We can just repeat the exercise, the only difference being that the clock  $C'$  has speed  $v = \sqrt{GM/(R+h)}$ , which implies

$$T_C = T \sqrt{1 - (2\pi R/cT)^2} \approx T \left( 1 - \frac{1}{2} \frac{GM}{(R+h)c^2} - \frac{GM}{(R+h)c^2} \right).$$

The gravitational time dilation effect is still larger, but only by a factor of 2.

(c) Plugging in the numbers, we have

$$\left(\frac{2\pi R}{cT}\right)^2 = 2.4 \times 10^{-12}, \quad \frac{GM}{Rc^2} = 7.0 \times 10^{-10}, \quad \frac{GM}{(R+h)c^2} = 1.7 \times 10^{-10}.$$




If we just consider the time dilation effect, the time difference per day is

$$\frac{T}{2} \left( \frac{GM}{(R+h)c^2} - \left(\frac{2\pi R}{cT}\right)^2 \right) = 7 \mu\text{s}.$$

If we consider just gravitational time dilation, the time difference per day is

$$T \left( \frac{GM}{(R+h)c^2} - \frac{GM}{Rc^2} \right) = -46 \mu\text{s}.$$

Not only is the gravitational effect important, it's more important than the time dilation effect!

- [5] **Problem 23.**  APhO 2014, problem 3. Gravitational fields bend light; this problem is about the geometry of gravitational lensing. Print out the official answer sheets and record your answers on them.
- [5] **Problem 24.**  IPhO 1995, problem 1. This problem is about the applications of gravitational redshift, and also serves as a nice review of **R2**.
- [3] **Problem 25.**  IPhO 2023, problem 2, parts C.1 through C.4. A neat problem on how the Shapiro delay, a classic test of general relativity, can be used to measure the masses of neutron stars.

### Remark: Visualizing Relativity

You've probably heard that in general relativity, gravity is explained by the curvature of spacetime. In other words, freely falling objects always move in straight lines through spacetime; they only look like they're accelerating downward because we are constantly being accelerated upward. This is nicely illustrated [here](#) and explained in greater detail in [this paper](#).

There is a common analogy for this involving picturing space as a distorted rubber sheet. It's a very bad analogy, because things will only accelerate towards the valleys in the sheets if you have gravity pointing down the sheet. In other words, the analogy tries to explain gravity by assuming you have spatial curvature *and* gravity. This misses the beautiful key point of relativity, which is that the gravity can be explained by *spacetime* curvature alone.

The fact that freely falling objects move in straight lines means that an object sitting on the surface of the Earth is actually being constantly accelerated. But this leads to a common followup question: in this picture, the surfaces of America and India are constantly accelerated in opposite directions, so why doesn't the Earth tear itself apart? Indeed, in special relativity this would make no sense. It's only possible because of spacetime curvature.

This can be explained with a spatial curvature analogy. Consider two people walking

east, side by side, with one just north of the equator and the other south. In order to stay a *constant* distance apart, the person walking on the north will constantly have to bear to the right, while the person walking on the south will have to bear to the left, because the Earth's surface is spatially curved. Similarly, in a situation with spacetime curvature, America and India need constant opposite accelerations to maintain the same distance.

There's a neat way to visualize this situation called the “[river model](#)”, which was rediscovered and animated [here](#). The basic idea is that we think of space as a river that is constantly flowing towards the center of the Earth. Observers in America and India constantly need to paddle in opposite directions against the river to stay in place. This is also a good way to think about the event horizon of a black hole, which is where the river starts to flow faster than light.

In this remark I've given three analogies about spacetime, so which of them is “correct”? None, really. The analogies don't tell us what spacetime is. They're just different ways of verbally describing what the equations of general relativity say. They each imperfectly describe some aspects of the equations, and fail to capture others. (Any simple analogy *must* fail to capture the content of a theory, because if it really were simpler and just as valid, then that analogy would be the theory instead!) There is no actual spacetime rubber or river; those are just stories we tell ourselves to make the mathematics more appealing to our animal-descended minds. Of course, philosophers debate over whether the attitude I've expressed in this paragraph is right. It's called “anti-realism”, and I wrote about it [here](#).