Electromagnetism IV: Lorentz Force

The problems here mostly use material covered in previous problem sets, though chapter 5 of Purcell covers relativistic field transformations. For further interesting physical examples, see chapter II-29 of the Feynman lectures. There is a total of **84** points.

1 Electrostatic Forces

Idea 1: Lorentz Force

A charge q in an electromagnetic field experiences the force

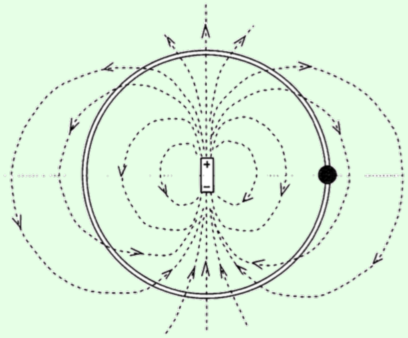
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

In particular, a stationary wire carrying current I in a magnetic field experiences the force

$$\mathbf{F} = I \int d\mathbf{s} \times \mathbf{B}.$$

Example 1: PPP 183

A small charged bead can slide on a circular, frictionless insulating ring. A point-like electric dipole is fixed at the center of the circle with the dipole's axis lying in the plane of the circle. Initially the bead is in the plane of symmetry of the dipole, as shown.



Ignoring gravity, how does the bead move after it is released? How would the bead move if the ring weren't there?

Solution

Set up spherical coordinates so that the dipole is in the $\hat{\mathbf{z}}$ direction. Then

$$V(r, \theta) = \frac{kp\cos\theta}{r^2}.$$

Since the ring fixes r, the potential on the ring is just proportional to $\cos\theta$, which is in turn proportional to z. But a potential linear in z is equivalent to a uniform downward field, so the bead oscillates like the mass of a pendulum, with amplitude $\pi/2$.

The answer remains the same when the ring is removed! Conservation of energy states that

$$\frac{kqp\cos\theta}{r^2} + \frac{1}{2}mv^2 = 0.$$

Let N be the normal force. Then accounting for radial forces gives

$$N + q\frac{\partial V}{\partial r} = \frac{mv^2}{r}.$$

However, plugging in our conservation of energy result for v^2 shows that $N = 0$, so the ring doesn't actually do anything, and it may be removed without effect.

Example 2

A parallel plate capacitor with separation d and area A is attached to a battery of voltage V. One plate moves towards the other with uniform speed v. Verify that energy is conserved.

Solution

The capacitance is $C = A\epsilon_0/d$. The power supplied by the battery is

$$P_{\text{batt}} = IV = V\,\frac{dQ}{dt} = V^2\frac{dC}{dt}.$$

On the other hand, the rate of change of the energy stored in the capacitor is

$$P_{\text{cap}} = \frac{d}{dt}\left(\frac{1}{2}CV^2\right) = \frac{1}{2}V^2\frac{dC}{dt}.$$

At first glance, there seems to be a problem. But then we remember that there is an attractive force between the plates, so the plates do work on whatever is moving them together,

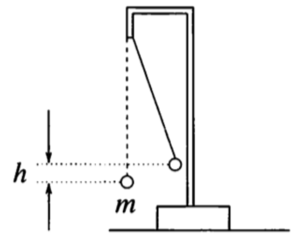
$$P_{\text{mech}} = Fv = \frac{QE}{2}\,v = \frac{QV}{2d}\,v = \frac{1}{2}CV^2\,\frac{v}{d} = \frac{1}{2}V^2\frac{dC}{dt}.$$

where E is the electric field inside the capacitor. Thus, $P_{\text{batt}} = P_{\text{cap}} + P_{\text{mech}}$ as required.

Technically there's energy in the magnetic field too, but it's smaller than the electric field energy by v^2/c^2, and thus negligible unless you're moving the plates so fast that relativity comes into play. Most problems in this problem set ignore such relativistic effects.

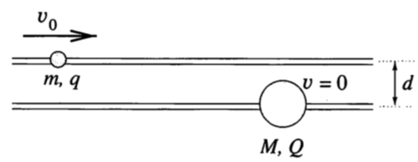
[2] **Problem 1** (PPP 193)**.** Two positrons are at opposite corners of a square of side a. The other two corners of the square are occupied by protons. All particles have charge q, and the proton mass M is much larger than the positron mass m. Find the approximate speeds of the particles much later.

[3] **Problem 2** (PPP 114)**.** A small positively charged ball of mass m is suspended by an insulating thread of negligible mass. Another positively charged small ball is moved very slowly from a large distance until it is in the original position of the first ball. As a result, the first ball rises by h.



How much work has been done?

[3] **Problem 3** (PPP 71)**.** Two small beads slide without friction, one on each of two long horizontal parallel fixed rods a distance d apart.



The masses of the beads are m and M and they carry charges q and Q. Initially, the larger mass M is at rest and the other one is far away approaching it at a speed v_0. For what values of v_0 does the smaller bead ever get to the right of the larger bead?

[2] **Problem 4** (PPP 192)**.** Classically, a conductor is made of nuclei of positive charge fixed in place, and electrons that are free to move.

(a) Consider a solid conductor in a gravitational field $\mathbf{g}$. Argue that the electric field inside the conductor is *not* zero; find out what it is.

(b) Now suppose a positron is placed at the center of a hollow spherical conductor in a gravitational field $\mathbf{g}$. Find its initial acceleration.

[3] **Problem 5.** USAPhO 2008, problem B2. You may ignore part (c), which was removed in the final version of the exam, though you can also do it for extra practice.

[3] **Problem 6.** USAPhO 2019, problem B1.

[5] **Problem 7.** IPhO 2004, problem 1. A nice question on the dynamics of a multi-part system.

2 The Lorentz Force

Idea 2

Some questions below will involve special relativity. The Lorentz force law as written in idea 1 is still valid as long as $\mathbf{F}$ is interpreted as $d\mathbf{p}/dt$, where the relativistic momentum is

$$\mathbf{p} = \gamma m \mathbf{v}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The relativistic energy is also modified to

$$E = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \ldots.$$

We will return to this subject in more detail in **R2**, but for now this is all you need.

Example 3: Kalda 163

A beam of electrons, of mass m and charge q, is emitted with a speed v almost parallel to a uniform magnetic field $\mathbf{B}$. The initial velocities of the electrons have an angular spread of $\alpha \ll 1$, but after a distance L the electrons converge again. Neglecting the interaction between the electrons, what is L?

Solution

Consider an electron initially traveling at an angle α to the magnetic field. This electron has a speed $v_\parallel = v\cos\alpha \approx v$ parallel to the field, and a speed $v_\perp v \sin\alpha \approx v\alpha$ perpendicular to the field. The component $v_\parallel$ always stays the same, while $v_\perp$ rotates, so the electron spirals along the field lines.

The acceleration of the electron is

$$a_\perp = \frac{F}{m} = \frac{qv_\perp B}{m}.$$

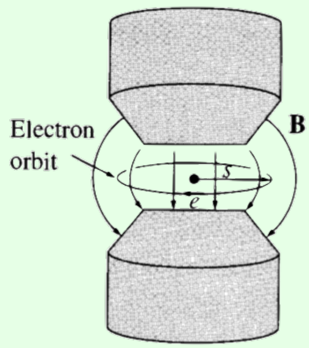
The perpendicular velocity component rotates through a circle in velocity space of circumference $2\pi v_\perp$. After one such circle, the total perpendicular displacement is zero, so the beam refocuses. Thus we have

$$L = \frac{2\pi v_\perp}{a} v_\parallel \approx \frac{2\pi m v}{qB}.$$

In other words, this setup acts like a magnetic "lens".

Example 4: Griffiths 7.50

In a "betatron", electrons move in circles in a magnetic field. When the magnetic field is slowly increased, the accompanying electric field will impart tangential acceleration.



Suppose the field always has the same spatial profile $B(r,t) = B_0(r)f(t)$. For what $B_0(r)$ is it possible for an electron to start at rest in zero magnetic field, and then move in a circle of constant radius as the field is increased?

Solution

The electrons experience a tangential force

$$\dot{p} = qE = q\frac{\dot{\Phi}_B}{2\pi r} = \frac{qr}{2}\dot{B}_{\text{av}}$$

where B_{av} is the average field over the orbit. Since the particles start from rest in zero field, we can integrate this to find

$$p = \frac{qr}{2}B_{\text{av}}.$$

On the other hand, the standard result for cyclotron motion is $p = qrB$, which means we must have $B = B_{\text{av}}/2$, i.e. the field at any radius is half the average magnetic field inside,

$$B(r) = \frac{1}{2}\frac{1}{\pi r^2}\int_0^r B(r')(2\pi r')\,dr'.$$

This rearranges slightly to give

$$r^2 B(r) = \int_0^r r' B(r')\,dr'.$$

Differentiating both sides with respect to r, we have

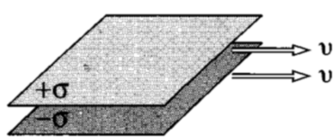
$$2rB(r) + r^2 B'(r) = rB(r)$$

which simplifies to

$$\frac{dB}{B} = -\frac{dr}{r}$$

which means the field profile should be $B_0(r) \propto 1/r$. (Of course, a real betatron might differ since it only needs to obey $B = B_{\text{av}}/2$ at the radii where electrons will be orbiting.)

[3] **Problem 8** (Griffiths 5.17). A large parallel plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v as shown.



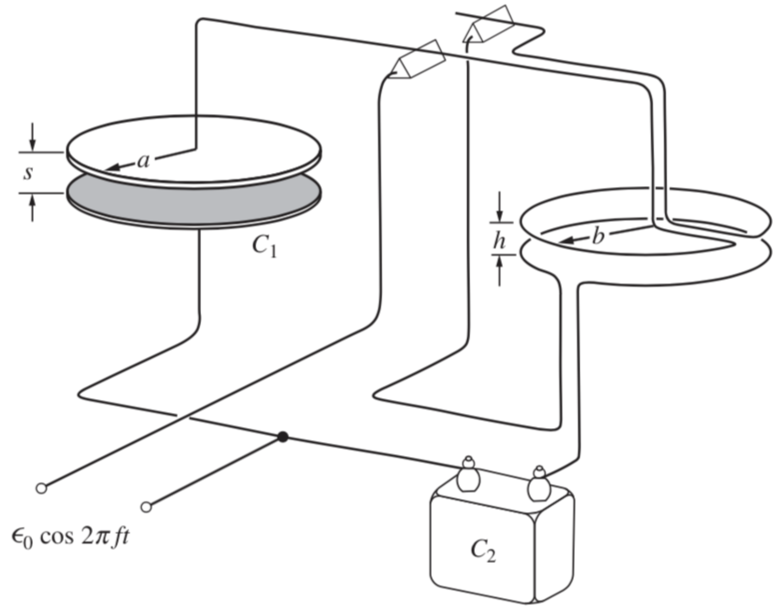
(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) What happens to the net force between the plates in the limit $v \to c$? Explain your result using some basic ideas from special relativity.

[3] **Problem 9.** EFPhO 2012, problem 7. An elegant Lorentz force problem with wires. (If you enjoy this problem, consider looking at IdPhO 2020, problem 1B, which has a similar setup but requires three-dimensional reasoning. The official solutions are here.)

[4] **Problem 10** (Purcell 6.35/INPhO 2008.6). Consider the arrangement shown below.



The force between capacitor plates is balanced against the force between parallel wires carrying current in the same direction. A voltage alternating sinusoidally with angular frequency ω is applied to the parallel-plate capacitor C_1 and also to the capacitor C_2, and the current is equal to the current through the rings. Assume that $s \ll a$ and $h \ll b$.

Suppose the weights of both sides are adjusted to balance without any applied voltage, and C_2 is adjusted so that the time-averaged downward forces on both sides are equal. Show that

$$\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{2\pi}\, a\omega \sqrt{\frac{b}{h}\frac{C_2}{C_1}}.$$

The left-hand side is equal to c, as we'll show in **E7**, so this setup measures the speed of light.

[3] **Problem 11.** An electron beam is accelerated from rest by applying an electric field E for a time t, and subsequently guided by magnetic fields. These magnetic fields are produced with a series of coils, which carry currents I_i.

Now suppose the apparatus is repurposed to shoot proton beams. Suppose a proton beam is accelerated from rest by applying an electric field E for a time t (in the opposite direction). Let an electron have mass m and a proton have mass M.

(a) Find the currents I_i needed so that the proton follows the same trajectory the electron did, assuming V is small enough that both the electron and proton are nonrelativistic.

(b) How does the answer change if relativistic corrections are accounted for?

[5] **Problem 12.** IPhO 2000, problem 2. A solid question on the Lorentz force with real-world relevance. Requires a little relativity, namely the expressions for relativistic momentum/energy.

[4] **Problem 13.** IPhO 1996, problem 2. An elegant problem on particles in a magnetic field. (There's a deeper principle behind the solution to this problem; see **R3** for more discussion.)

3 Magnetic Moments

[3] **Problem 14.** Consider a current loop I in the xy plane in a constant magnetic field $\mathbf{B}$.

(a) Show that the net force on the loop is zero.

(b) Show that the torque is
$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$$
where the magnetic moment is
$$\mathbf{m} = IA\hat{\mathbf{z}}$$
where A is the area of the loop. For simplicity, you can show this in the case where the current loop is a square of side length L, whose sides are aligned with the x and y axes. (The proof for a general loop shape requires some vector calculus, but you can attempt it for a challenge. You'll need the double cross product identity, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.)

Idea 3

The force on a small magnetic dipole $\mathbf{m}$ is
$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$
where the gradient acts only on the spatial dependence of $\mathbf{B}$. If there are no other currents at the dipole's location, so that $\nabla \times \mathbf{B} = 0$, this formula is equivalent to
$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B},$$
which is sometimes easier to evaluate.

As in problem 14, this can be shown relatively easily for a square loop, and requires some tricky vector calculus for a general current distribution. Both the force and torque on a magentic dipole can be found by differentiating the potential energy
$$U = -\mathbf{m} \cdot \mathbf{B}.$$
All of these results also hold for electric dipoles, if we replace $\mathbf{m}$ with $\mathbf{p}$ and $\mathbf{B}$ with $\mathbf{E}$.

Remark

The expression for the potential energy above is notoriously subtle. Here's the problem: we know the Lorentz force on a charge is $q\mathbf{v} \times \mathbf{B}$, which means magnetic fields never do work. So how can they be associated with a nonzero potential energy?

There are two levels of explanation. First, suppose the magnetic dipole is made of charges moving in a loop. When such a current loop is placed in a magnetic field, and moved or rotated, mechanical work can be done on the loop. But at the same time, there will be an induced emf in the loop, which speeds up or slows down the current. The work done by these two effects perfectly cancels, so that the energy of the loop stays constant. For this kind of dipole, the expression for U doesn't indicate the total energy, but only the "mechanical" potential energy, in the sense that differentiating it gives the right forces and torques. (Some further discussion of this point is in chapter II-15 of the Feynman lectures.)

On the other hand, the magnetic dipole moment of a common bar magnet doesn't come from charges moving in a loop! Instead, it comes from the intrinsic magnetic dipole moments of the unpaired electrons in the magnet. These kinds of dipole moments aren't composed of any moving subcomponents; they are an elementary and immutable property of the electron, like its mass or charge. In these cases, $U = -\mathbf{m} \cdot \mathbf{B}$ really is the total energy, and the magnetic field *can* do work. You won't hear much about these elementary dipole moments in introductory books, because they can only be properly understood by combining relativity and quantum mechanics, but they're responsible for most magnetic phenomena.

Example 5

If a magnet is held over a table, it can pick up a paper clip. If the paper clip is removed, it can pick up another paper clip just as well, and this process can seemingly continue forever without any effect on the magnet. Since the magnet does work on each paper clip, doesn't this mean a permanent magnet is an infinite energy source?

Solution

This is the kind of question that makes magnets feel so mysterious. They're basically the only everyday example of a long range force besides gravity (in fact, Kepler once thought the Sun acted on the planets like a giant magnet), and as such they've inspired countless attempts at perpetual motion machines. For centuries, many people have spent years of their lives trying to get elaborations of this example to work.

To see why this doesn't work for a bar magnet, just replace the word "magnet" with "charge". It's true that a positive charge can attract a negative charge to it. And if the negative charge is then removed, the positive charge can then attract another negative charge to it. But conservation of energy isn't violated, because the force from the positive charge is conservative: the work it does on the negative charge to draw it close is precisely the opposite of the work an external agent needs to do to pull it away. The force of a magnet on a paper clip is also conservative.

It's also interesting to consider a slightly different case. Unlike a bar magnet, an electromagnet (i.e. a magnet created by moving current in a loop) can be turned on and off with the flick of a switch. Therefore, we might suspect that the following is a perpetual motion machine:

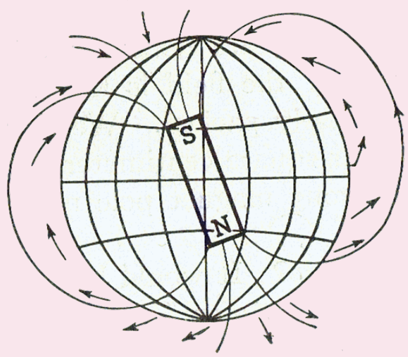
1. Turn on the electromagnet, which costs energy E_0.
2. Use it to lift a paper clip, increasing its potential energy by mgh.
3. Turn off the electromagnet, which costs energy E_0, while holding the paper clip.
4. Move the paper clip away; we've managed to raise it higher for free.

To see the problem, note that the attractive force between the magnet and paper clip arises because the magnet induces a magnetic dipole moment in the paper clip, leading to a $(\mathbf{m}\cdot\nabla)\mathbf{B}$

force. As the paper clip moves toward the magnet, its own dipole moment causes a changing magnetic flux through the electromagnet, and thus an emf against the current. Therefore, it costs extra energy to keep the current in the electromagnet steady. Since the $q\mathbf{v} \times \mathbf{B}$ Lorentz force doesn't do work, that energy must be precisely mgh, so nothing comes for free.

Remark

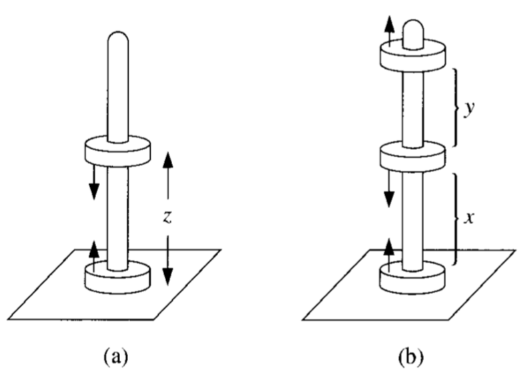
A compass needle is essentially a small magnetic dipole, whose dipole moment points towards the end painted red. We can also approximate the Earth's magnetic field as a dipole field.



Since the tangential component of this dipole field points north, the red end of the compass points towards the geographic north pole, which is the Earth's magnetic south pole.

By the way, a cheap compass calibrated to work in America or Europe won't work well in Australia. The reason is that the Earth's magnetic field also has a radial component, which acts to tip the compass needle up or down. The needle needs to be appropriately weighted to stay horizontal, so that it can freely rotate, but the side that needs to be weighted differs between the hemispheres.

[3] **Problem 15** (Griffiths 6.23). A familiar toy consists of donut-shaped permanent magnets which slide frictionlessly on a vertical rod.

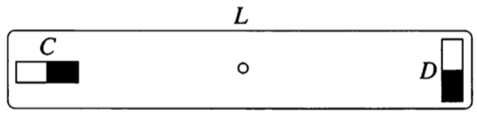


Treat the magnets as dipoles with mass m_d and dipole moment $\mathbf{m}$, with directions as shown above.

(a) If you put two back-to-back magnets on the rod, the upper one will "float". At what height z does it float?

(b) If you now add a third magnet parallel to the bottom one as shown, find the ratio x/y of the two heights, using only a scientific calculator. (Answer: 0.85.)

[3] **Problem 16** (PPP 89)**.** Two identical small bar magnets are placed on opposite ends of a rod of length L as shown.



(a) Show that the torques the magnets exert on each other are *not* equal and opposite.

(b) Suppose the rod is pivoted at its center, and the magnets are attached to the rod so that they can spin about their centers. If the magnets are released, the result of part (a) implies that they will begin spinning. Explain how this can be consistent with energy and angular momentum conservation, treating the latter quantitatively.

4 Point Charges

In this section we'll give a sampling of classic problems involving just point charges in fields; these will be a bit more mathematically advanced than the others in this problem set.

[3] **Problem 17.** A point charge q of mass m is released from rest a distance d from a grounded conducting plane. Find the time until the point charge hits the plane. (Hint: use Kepler's laws.)

[3] **Problem 18.** A point charge of mass m and charge q is released from rest at the origin in the fields $\mathbf{E} = E\hat{\mathbf{x}}$, $\mathbf{B} = B\hat{\mathbf{y}}$. Find its position as a function of time by solving the differential equations given by Newton's second law, $\mathbf{F} = m\mathbf{a}$.

[3] **Problem 19** (Wang)**.** Two identical particles of mass m and charge q are placed in the xy plane with a uniform magnetic field $B\hat{\mathbf{z}}$. The particles have paths $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Neglect relativistic effects, but account for the interaction between the charges.

(a) Write down a differential equation describing the evolution of the separation $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

(b) Suppose that the initial conditions have been set up so that the particles orbit each other in a circle in the xy plane, with constant separation d. What is the smallest d for which this motion is possible?

[4] **Problem 20.** [A] Consider a point charge of mass m and charge q in the field of a magnetic monopole at the origin,

$$\mathbf{B} = \frac{g}{r^2}\hat{\mathbf{r}}.$$

In this problem we'll investigate the strange motion that results.

(a) Argue that the speed v is constant.

(b) Show that the angular momentum $\mathbf{L}$ of the charge is *not* conserved, but that

$$\mathbf{V} = \mathbf{L} - qg\hat{\mathbf{r}}$$

is. The second term is the angular momentum stored in the fields of the charge and monopole.

(c) Show that the charge moves on the surface of a cone. (Hint: in spherical coordinates where the z-axis is parallel to $\mathbf{V}$, consider $\mathbf{V} \cdot \hat{\boldsymbol{\phi}}$.) Sketch some typical trajectories.

One can do problem 18 slickly using field transformations, an advanced subject we will cover in **R3**.

Idea 4: Field Transformations

If the electromagnetic field is $(\mathbf{E}, \mathbf{B})$ in one reference frame, then in a reference frame moving with velocity $\mathbf{v}$ with respect to this frame, the components of the field parallel to $\mathbf{v}$ are

$$E'_{\parallel} = E_{\parallel}, \quad B'_{\parallel} = B_{\parallel}$$

while the components perpendicular are

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right).$$

Remark: Galilean Field Transformations

The nonrelativistic limit of the field transformation is useful, but one has to be careful in deriving it. You might think, what's the need for care? Can't we just send $c \to \infty$, Taylor expand the above expressions, and call it a day? The problem with this reasoning is that there's no such thing as setting $c \to \infty$. You can't change a fundamental constant, and moreover this statement isn't even dimensionally correct, as noted in **P1**. What we really mean by the nonrelativistic limit is restricting our attention to some subset of possible situations, within which relativistic effects don't matter.

For example, if we have a bunch of point charges with typical speed v, then the nonrelativistic limit is considering only situations where v/c is small. In other words, we are taking $v/c \to 0$, not $c \to \infty$. Since the magnetic field of a point charge is v/c^2 times the electric field, the magnetic field ends up small. Now if we also consider boosts with small speeds v, then expanding the field transformations to lowest order in v/c gives

$$\mathbf{E}' = \mathbf{E}, \quad \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

This is the nonrelativistic limit for situations where $E/B \gg c$, also called the electric limit.

However, there's another possibility. Suppose that we have a bunch of current carrying, approximately neutral wires. In this case, it's the electric fields that are small, $E/B \ll c$. Using this in the transformations above, we arrive at the distinct result

$$\mathbf{B}' = \mathbf{B}, \quad \mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

which apply for situations where $E/B \ll c$, also called the magnetic limit.

You might think we could improve the approximation by combining the two,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}$$

but this isn't self-consistent. For example, if you apply a Galilean boost with speed v, and then a boost with speed $-v$, you don't get back the same fields you started with! A sensible

Galilean limit is only possible if $E/B \gg c$ or $E/B \ll c$, which are called the electric and magnetic limits. It's only in relativity that E and B can be treated on an equal footing.

By the way, whenever relativity or similarly subtle physics is involved, internet sources will be generally poor. If you search for "Galilean electrodynamics", the first result will be an awful journal that only publishes rants from crackpots who don't understand relativity. If you actually want to learn more, ditch the search engines and just read this classic paper.

[3] **Problem 21.** Using the Galilean field transformations to solve problem 18.

(a) In the magnetic limit, show that the Lorentz force stays the same between frames, as it should. Then use the field transformations to find an appropriate reference frame where the problem becomes easy.

(b) In the electric limit, show that the Lorentz force stays the same up to terms that are order $(v/c)^2$ smaller, assuming $B/E \sim v/c^2$. (This is fine, since we're taking the limit $v/c \to 0$ anyway.) Then use the field transformations to find an appropriate reference frame where the problem becomes easy.

(c) You should have found two distinct behaviors in parts (a) and (b). One of them should look like what you found in problem 18, and the other should be very different. But the values of E and B in problem 18 were arbitrary, so why didn't you see the other type of behavior?

There are a number of other nice questions one can ask about the dynamics of point charges, which use more advanced concepts such as "hidden" momentum, canonical momentum, or adiabatic invariants. These ideas are collected in a section of **R3**.

5 Continuous Systems

Example 6: The Drude Model

Model a conductor as a set of electrons, of charge q, mass m, and number density n, which are completely free. Assume that in every small time interval dt, each electron has a probability dt/τ of hitting a lattice ion, which randomizes the direction of its velocity. Under these assumptions, compute the resistivity of the material.

Solution

First, suppose the electrons have some average momentum $\langle \mathbf{p} \rangle$ each. Because the collisions randomize the velocity, the average momentum falls exponentially with timescale τ,

$$\frac{d\langle \mathbf{p} \rangle}{dt} = -\frac{\langle \mathbf{p} \rangle}{\tau}.$$

On the other hand, if there is an applied field, a force term appears on the right,

$$\frac{d\langle \mathbf{p} \rangle}{dt} = -\frac{\langle \mathbf{p} \rangle}{\tau} + q\mathbf{E}$$

since $\mathbf{F} = d\mathbf{p}/dt$ for each individual electron. In the steady state,

$$\langle \mathbf{p} \rangle = q\mathbf{E}\tau.$$

The current density is

$$\mathbf{J} = nq\langle \mathbf{v} \rangle = \frac{nq\langle \mathbf{p} \rangle}{m} = \frac{nq^2\tau}{m}\mathbf{E}.$$

Thus, the resistivity in the Drude model is

$$\rho = \frac{m}{nq^2\tau}.$$

We can also compute the typical drift velocity,

$$v = \frac{qE\tau}{m}.$$

For values of m that give reasonable ρ, the value of v is a literal snail's pace, which is why people say that the electrons themselves move very slowly through a circuit. (Of course, a current can get started in a circuit much faster, because when a battery is attached, each moving electron pushes on the next one along the wire, and this wave of motion travels much faster than the electrons themselves.)

Remark: The Drude–Sommerfeld Model

Above we tacitly assumed there was a given probability of collision per unit time, but that's not right: when a particle flies through a medium, there is instead a given probability of collision per unit *length* it travels. These are equivalent for electrons moving at constant speed, but intuitively, we would expect electrons to have to accelerate starting from rest after each collision, in which case the two differ. To estimate this quickly, note that if the typical collision distance is ℓ, the kinetic energy picked up between collisions is $mv^2/2 \sim qE\ell$, giving typical speed $v \propto \sqrt{E}$. The analogue of Ohm's law would then be $I \propto \sqrt{V}$, completely contrary to observation!

The resolution is that electrons in solids really do effectively move with almost constant speed, even after collisions. This is a quantum mechanical effect, as explained in **X1**. The Pauli exclusion principle implies the electrons in the conductor have to occupy different quantum states, and the high density of electrons requires most of them to always have extremely high speeds, on the order of 1% of the speed of light! The drift velocity is merely the tiny amount by which their velocities are shifted on average.

[2] **Problem 22.** Consider Drude theory again, but now suppose there is also a fixed magnetic field $B\hat{\mathbf{z}}$. In this case, $\mathbf{J}$ is not necessarily parallel to $\mathbf{E}$, but the relation between the two can be described by the "tensor of resistivity". That is, the components are related by

$$E_i = \sum_{j \in \{x,y,z\}} \rho_{ij} J_j.$$

Calculate the coefficients ρ_{ij}. Express your answers in terms of the quantities

$$\rho_0 = \frac{m}{nq^2\tau}, \quad \omega_0 = \frac{qB}{m}$$

as well as the parameter τ.

Example 7: Griffiths 5.40

Since parallel currents attract, the currents within a single wire should contract. To estimate this, consider a long wire of radius r. Suppose the atomic nuclei are fixed and have uniform density, while the electrons move along the wire with speed v. Furthermore, assume that the electrons contract, filling a cylinder of radius $r' < r$ with uniform negative charge density, and that the wire is overall neutral. Find r'.

Solution

The contraction of the electrons produces an overall inward electric field, and hence an outward electric force on each electron, which balances the radially inward magnetic force. Specifically, equilibrium occurs when $E = vB$.

Let the charge densities of the nuclei and electrons be ρ_+ and ρ_-. The magnetic field at radius r is found by Ampere's law, which gives

$$(2\pi r)B = \mu_0(\rho_- v)(\pi r^2), \quad B = \frac{\mu_0 \rho_- v r}{2}.$$

The electric field at radius r is found by Gauss's law, which gives

$$(2\pi r)E = \frac{1}{\epsilon_0}(\rho_+ + \rho_-)\pi r^2, \quad E = \frac{1}{2\epsilon_0}(\rho_+ + \rho_-)r.$$

Note that both E and B are proportional to r. Then $E = vB$ can be satisfied at all r simultaneously, which confirms that our assumption that ρ_+ and ρ_- were uniform is self-consistent.

Plugging these results into $E = vB$ yields

$$\rho_+ + \rho_- = \rho_-(\epsilon_0 \mu_0 v^2) = \rho_- \frac{v^2}{c^2}.$$

This can be written in terms of the Lorentz factor of special relativity,

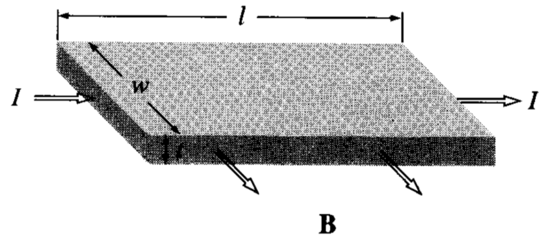
$$\rho_- = -\gamma^2 \rho_+, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Since the wire is overall neutral, $\rho_- r'^2 + \rho_+ r^2 = 0$, so

$$r' = \frac{r}{\gamma}.$$

For nonrelativistic motion, the contraction is extremely small. (However, in plasmas, where the positive charges are also free to move, this so-called pinch effect can be very significant.)

[2] **Problem 23** (Griffiths 5.41). A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\mathbf{B}$ pointing out of the page, as shown.



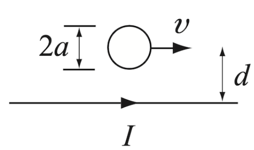
(a) If the moving charges are positive, in what direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the Hall effect.)

(b) Find the resulting potential difference, called the Hall voltage, between the top and bottom of the bar, in terms of B, the speed v of the charges, and the dimensions of the bar.

(c) How would the answer change if the moving charges were negative?

When measurements were performed in the early 20th century, some metals were found to have *positive* moving charges! This "anomalous Hall effect" was solved by the quantum theory of solids, as you can learn in any solid state physics textbook. (It is related to the strange behavior you will see in problem 27.) Today, extensions of the Hall effect, such as the integer and fractional quantum Hall effects, remain active areas of research, and could be used to build quantum computers. We'll return to these effects in **X3**.

[3] **Problem 24** (Zangwill 14.16). A conducting sphere of radius a is moving with speed v parallel to a straight wire which carries a current I. The distance between the wire and the center of the sphere is $d \gg a$.



Show that the force between the wire and the sphere scales as

$$F \sim \frac{v^2}{c^2} \frac{a^3}{d^3} \mu_0 I^2.$$

Is the force attractive or repulsive?

[3] **Problem 25.** USAPhO 1997, problem B1. A nice problem on the dynamics of a plasma. (Note that the assumption made in part (e) is somewhat arbitrary, without much physical meaning. It's just made to make part (f) a bit simpler.)

[3] **Problem 26.** USAPhO 2019, problem A3. This is a tough but useful problem. The first half derives the so-called Child–Langmuir law, covered in problem 2.53 of Griffiths.

[3] **Problem 27.** USAPhO 2022, problem B3. About the weird behavior of electrons in solids.