Electromagnetism I: Electrostatics

The material here is covered at the right level in chapters 1–3 of Purcell. For a separate introduction to vector calculus, see the resources mentioned in the syllabus, or chapter 1 of Griffiths. Electrostatics is covered in more mathematical detail in chapter 2 of Griffiths. For interesting general discussion, see chapters II-1 through II-5 of the Feynman lectures. There is a total of **79** points.

1 Coulomb's Law and Gauss's Law

We'll begin with some basic problems which can be solved with symmetry arguments.

Idea 1

Gauss's law is written in integral form as

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}.$$

In practice, you will only apply this form to situations with high symmetry, where

$$E = \begin{cases} Q/4\pi\epsilon_0 r^2 & \text{spherical symmetry,} \\ \lambda/2\pi\epsilon_0 r & \text{cylindrical symmetry,} \\ \sigma/2\epsilon_0 & \text{infinite plane.} \end{cases}$$

Example 1

Consider a spherical shell of uniform surface charge density σ. A small hole is cut out of the surface of the shell. What is the electric field at the center of this hole?

Solution

Consider a point P just outside the sphere, and another point P' nearby but just inside the sphere. If the shell didn't have a hole, then by Gauss's law and spherical symmetry, we have

$$\mathbf{E}_P = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{r}}, \qquad \mathbf{E}_{P'} = \mathbf{0}.$$

Next, suppose we *only* had the patch of the sphere near P and P'. When we zoom into P and P', this patch looks like an infinite plane dividing them, and from Gauss's law,

$$\mathbf{E}_P = \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{r}}, \qquad \mathbf{E}_{P'} = -\frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{r}}.$$

Now, a field of a spherical shell with a hole is just the difference of that of a full shell and that of a single patch, so subtracting these gives the answer,

$$\mathbf{E}_P = \mathbf{E}_{P'} = \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{r}}.$$

Note that $\mathbf{E}_P = \mathbf{E}_{P'}$ because there's no surface charge between P and P', so the value of the electric field can't jump discontinuously.

[2] **Problem 1** (Griffiths 2.18)**.** Some questions about uniformly charged spheres.

(a) Consider a sphere of radius R and uniform charge density ρ. Find the electric field everywhere.

(b) Now two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that they partially overlap. Call the vector from the positive center to the negative center $\mathbf{d}$. Find the electric field in the overlap region.

[2] **Problem 2.** Consider a cube with a corner at the origin, and sides parallel to the x, y, and z axes. If a charge q is placed at $(\epsilon, \epsilon, \epsilon)$ for some tiny ϵ, what's the flux through each face of the cube?

[2] **Problem 3** (BAUPC)**.** In both parts below, take the potential to be zero at infinity.

(a) Consider a solid sphere of uniform charge density. Find the ratio of the electrostatic potential at the surface to that at the center.

(b) Consider a solid cube of uniform charge density. Find the ratio of the electrostatic potential at a corner to that at the center. (Hint: use symmetry.)

Idea 2

If you follow an electric field line, the potential monotonically decreases along it.

[2] **Problem 4.** Two questions about electrostatic equilibrium.

(a) Prove that when a system of point charges is in equilibrium (i.e. the net force on *each* of the charges due to the others vanishes), the total potential energy of the system is zero.

(b) Show that for a positive point charge in the electric fields of fixed, positive point charges, there is a path along which the charge can be moved to infinity without ever needing positive external work, i.e. a path along which the potential only decreases.

Idea 3

Gauss's law is written in differential form as

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

The divergence of a vector field $\mathbf{F} = F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}} + F_z \hat{\mathbf{z}}$ is

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$

in Cartesian coordinates, where ∂_x stands for $\partial / \partial x$, and so on.

Example 2

Show that the two forms of Gauss's law are equivalent.

Solution

To do this, we need to establish the geometric meaning of the divergence. For simplicity we consider two dimensions; the proof for three dimensions is similar. Consider a small rectangle prism with one corner at the origin, with axes aligned with the Cartesian coordinate axes and side lengths Δx and Δy. To apply Gauss's law in integral form, we need to compute the flux through each side. The flux going out the top side is

$$\int_0^{\Delta x} E_y(x, \Delta y)\, dx$$

while the flux going out the bottom side is

$$-\int_0^{\Delta x} E_y(x, 0)\, dx.$$

The sum of these two terms is

$$\int_0^{\Delta x} (E_y(x, \Delta y) - E_y(x, 0))\, dx \approx \Delta y \int_0^{\Delta x} (\partial_y E_y)|_{(x,0)}\, dx$$

where we applied a tangent line approximation, and the subscript indicates where the function $\partial_y E_y$ is evaluated. Higher-order terms in the Taylor series would be proportional to higher powers of Δy, which is small, so we can ignore them.

The integrand is still a function of x, but we can Taylor expand it about the origin as

$$(\partial_y E_y)|_{(x,0,0)} = (\partial_y E_y)|_{(0,0,0)} + \Delta x(\ldots) + \ldots.$$

These extra terms are again higher-order in Δx and Δy, so we ignore them. The net flux through the top and bottom faces is hence, to lowest order,

$$\Delta y \int_0^{\Delta x} (\partial_y E_y)|_{(0,0,0)}\, dx = \Delta x \Delta y\, (\partial_y E_y)|_{(0,0,0)}.$$

By similar reasoning, pairing up the left and right faces gives

$$\text{flux} = \Delta x \Delta y\, (\partial_x E_x + \partial_y E_y)|_{(0,0,0)} = \Delta x \Delta y\, (\nabla \cdot \mathbf{E})|_{(0,0,0)}.$$

Thus the divergence is the outgoing flux per unit area, or volume in three dimensions.

This shows us why the two forms of Gauss's law area equivalent. For example, starting from the differential form, the left-hand side is the flux per volume, while the right-hand side is the charge per volume, divided by ϵ_0. Integrating both sides over some volume relates the total flux to the total charge divided by ϵ_0, which is Gauss's law in integral form.

If the above derivation was a bit abstract, we can also show the idea using specific examples.

Example 3

Suppose the region $0 < x < d$ has charge density $-\rho$, and the region $-d < x < 0$ has charge density ρ. Find the electric field everywhere.

Solution

By translational symmetry, the field always points along $\hat{\mathbf{x}}$ and only depends on x, $\mathbf{E}(\mathbf{r}) = E(x)\,\hat{\mathbf{x}}$. By applying the integral form of Gauss's law to a rectangular prism, with one side at x_l and another at x_r, we have

$$E(x_r) - E(x_l) = \frac{1}{\epsilon_0}\int_{x_l}^{x_r} \rho(x)\,dx, \quad E(x) = \frac{1}{\epsilon_0}\int_0^x \rho(x)\,dx + E_0.$$

Since the divergence of $\mathbf{E}(\mathbf{r})$ is just $\partial E(x)/\partial x$, this clearly satisfies the differential form of Gauss's law. To fix the undetermined constant E_0, we could demand the field be zero on both sides of the charge distribution, motivated by symmetry. Then we have

$$E(x) = \frac{\rho}{\epsilon_0} \times \begin{cases} d - x & 0 < x < d, \\ d + x & -d < x < 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Example 4

Find the electric field of a spherically symmetric charge density $\rho(r)$.

Solution

By spherical symmetry, the field always points radially and only depends on r, $\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}}$. By applying the integral form of Gauss's law to a sphere of radius r,

$$4\pi r^2 E(r) = \frac{1}{\epsilon_0}\int_0^r dr'\,4\pi r'^2 \rho(r'), \quad E(r) = \frac{1}{\epsilon_0}\frac{1}{r^2}\int_0^r dr'\,r'^2 \rho(r').$$

Let's check that this indeed satisfies the differential form of Gauss's law, using the divergence in spherical coordinates. For any vector field $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_\varphi\hat{\boldsymbol{\varphi}}$, the divergence is

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}.$$

This looks complicated, but things turn out simple because $\mathbf{E}$ only has a radial component, $E_r = E(r)$, which gives

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial(r^2 E(r))}{\partial r} = \frac{1}{r^2\epsilon_0}\frac{\partial}{\partial r}\int_0^r dr'\,r'^2\rho(r') = \frac{r^2\rho(r)}{r^2\epsilon_0} = \frac{\rho(r)}{\epsilon_0}$$

just as desired.

[3] **Problem 5.** Consider a vector field expressed in polar coordinates, $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$

are unit vectors in the radial and tangential directions. Gauss's law in differential form still works in these coordinates, but the form of the divergence is different.

By considering the flux per unit area out of a small region bounded by r and $r + dr$, and θ and $\theta + d\theta$, and applying Gauss's law in integral form, find what the divergence in polar coordinates must be for Gauss's law in differential form to hold. (Optional: try generalizing to spherical coordinates.)

[4] **Problem 6.** This subtle problem will expose a hidden assumption we've made in the previous two examples. Suppose that all of space is filled with uniform charge density ρ.

(a) Show that $\mathbf{E} = (\rho/\epsilon_0)x\hat{\mathbf{x}}$ obeys the differential form of Gauss's law.

(b) Show that $\mathbf{E} = (\rho/3\epsilon_0)r\hat{\mathbf{r}}$ also obeys Gauss's law.

(c) Argue that by symmetry, $\mathbf{E} = 0$. Show that this does not obey Gauss's law.

(d) $\star$ What's going on? Which, if any, is the actual field? If you think there's more than one possible field, how could that be consistent with Coulomb's law, which gives the answer explicitly? For that matter, what does Coulomb's law say about this setup, anyway?

Idea 4

A tricky, occasionally useful idea is to use Newton's third law: it may be easier to calculate the force of A on B than the force of B on A.

Example 5: Purcell 1.28

Consider a point charge q. Draw any imaginary sphere of radius R around the charge. Show that the average of the electric field over the surface of the sphere is zero.

Solution

Imagine placing a uniform surface charge σ on the sphere. Then the average of the point charge's electric field over the sphere times $4\pi R^2\sigma$ is the total force of the point charge on the charged sphere. But this is equal in magnitude to the force of the charged sphere on the point charge, which must be zero by the shell theorem. Thus the average field over the sphere has to vanish.

Example 6

Consider two spherical uniformly charged balls of charge q and radii a_i, with their centers separated by a distance $r > a_1 + a_2$. What is the net force of the first on the second?

Solution

It might seem obvious that the answer is $q^2/4\pi\epsilon_0 r^2$, with no dependence on a_1 and a_2. In fact, if you've done any orbital mechanics, you've almost certainly assumed that the force between two spherical bodies (such as the Earth and Sun) is Gm_1m_2/r^2, which is equivalent.

This has a simple but slightly tricky proof. By the shell theorem, we can set $a_1 = 0$, replacing the first ball with a point charge, because this produces the same field at the second ball.

But the force on the second ball depends on the electric field at every point on it, which seems to require doing an integral. To avoid this, we use Newton's third law, which tells us it's equivalent to compute the force on the first ball. To compute *that*, we may set $a_2 = 0$ by the shell theorem again. This reduces us to the case of two point charges, giving the answer.

[3] **Problem 7** (Purcell 1.28)**.** Some extensions of the previous example.

(a) Show that if the charge q is instead outside the sphere, a distance $r > R$ from its center, the average electric field over the surface of the sphere is the same as the electric field at the center of the sphere.

(b) Show that for any overall neutral charge distribution contained within a sphere of radius R, the average electric field over the interior of the sphere is $-\mathbf{p}/4\pi\epsilon_0 R^3$ where $\mathbf{p}$ is the total dipole moment.

[3] **Problem 8.** There are two point charges, $q_1 > 0$ and $q_2 < 0$, in empty space. An electric field line leaves q_1 at an angle α from the line connecting the two charges. Determine whether this field line hits q_2, and if so, at what angle β from the line connecting the two charges. (Hint: this can be done without solving any differential equations.)

Idea 5

The integral $\int d\mathbf{S}$ over a surface with a fixed boundary is independent of the surface.

We proved this in the surface tension section of **M2** using a mechanical argument. If you want to see a proof using vector calculus, see problem 1.62 of Griffiths.

[3] **Problem 9.** A hemispherical shell of radius R has uniform charge density σ and is centered at the origin. Find the electric field at the origin. (Hint: combine the previous two ideas.)

[3] **Problem 10.** A point charge q is placed a distance $a/2$ above the center of a square of charge density σ and side length a. Find the force of the square on the point charge.

[4] **Problem 11** (Griffiths 2.47, PPP 113, MPPP 140)**.** Consider a uniformly charged spherical shell of radius R and total charge Q.

(a) Find the net electrostatic force that the southern hemisphere exerts on the northern hemisphere.

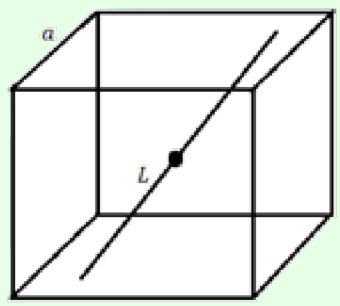
(b) Generalize part (a) to the case where the sphere is split into two parts by a plane whose minimum distance to the sphere's center is h.

(c) Generalize part (a) to the case where the two hemispherical shells have uniform charge density, opposite orientation, and the same center, but have different total charges q and Q, and different radii r and R, where $r < R$.

Hint: see example 10, and use superposition and symmetry when applicable.

Example 7: IdPhO 2020.1A

A point charge of mass m and charge $-q$ is placed at the center of a cube with side length a, whose volume has uniform charge density ρ. The point charge is allowed to slide along a straight line, which has an arbitrary orientation, so that the distance along the line from the center to one of the cube's faces is L.



Find the angular frequency of small oscillations.

Solution

The official solution goes as follows: consider displacing the point charge away from the origin by some small amount $\mathbf{r}$. The cube of charge can then be decomposed into (1) a slightly smaller cube of charge centered around the point charge's new position, and (2) three thin plates of charge on the faces opposite to the charge's motion. By symmetry, (1) contributes nothing, and we know what (2) contributes from the answer to problem 10. The result is a restoring force proportional to $-\mathbf{r}$, whose magnitude has no dependence on the orientation of $\mathbf{r}$, so the oscillation frequency doesn't depend on L. Once you know this, you can orient the line any way you want, so the problem is simple to finish.

Personally, I don't like this problem because the intended solution requires knowing the answer to problem 10, which itself is pretty tricky. That is, the difficulty of the problem depends mostly on whether you've seen that tough, but standard problem elsewhere. However, I'm including it as an example because there's another way to solve it, which is a bit more advanced, but quite illustrative.

Since this is a question about small oscillations, it suffices to expand the potential energy to second order about the center of the cube. The most general possible expression is

$$V(x,y,z) = a + b_1 x + b_2 y + b_3 z + c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 xy + c_5 yz + c_6 xz + O(r^3).$$

The constant a doesn't matter, so we can just ignore it. And since $\mathbf{E}$ vanishes at the center, the linear terms b_i are all zero as well. Because the x, y, and z axes are all equivalent by cubical symmetry (e.g. we can rotate them into each other, while keeping the cube the same),

$$c = c_1 = c_2 = c_3, \quad c' = c_4 = c_5 = c_6.$$

Thus, our complicated original expression reduces all the way down to

$$V(x,y,z) = c(x^2 + y^2 + z^2) + c'(xy + yz + xz) + O(r^3)$$

without even having to do any work! Finally, notice that the cube is symmetric under reflections $x \to -x$, $y \to -y$, or $z \to -z$. These reflections keep the c term the same, but flip the c' term. Therefore, we must have $c' = 0$, so

$$V(r) = cr^2 + O(r^3)$$

which is remarkably simple. The potential near the origin is spherically symmetric (to second order), even though the setup as a whole isn't! It's not automatic: it wouldn't be this simple if we had a slightly more complex shape. This "accidental" spherical symmetry is a consequence of the combination of cubical symmetry and the simplicity of Taylor series.

Therefore, to finish the problem we only need to find the coefficient c. While there are simpler ways to do this, I'll do it in a way that introduces some useful facts. Combining the definition of V and Gauss's law, we have

$$\nabla \cdot (\nabla V) = -\nabla \cdot \mathbf{E} = -\frac{\rho}{\epsilon_0}.$$

This is a standard and fundamental result in electrostatics, called Poisson's equation, which we will see again later. The divergence of a gradient is also called a Laplacian, and written as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}.$$

Using this, we can easily compute the value of c, giving

$$V(\mathbf{r}) = -\frac{\rho r^2}{6\epsilon_0} + O(r^3).$$

Therefore, for a displacement $\mathbf{r}$ in *any* direction, the restoring force is $\rho q r/3\epsilon_0$ in the opposite direction, which means

$$\omega = \sqrt{\frac{\rho q}{3\epsilon_0 m}}$$

independent of the orientation of the line.

Remark

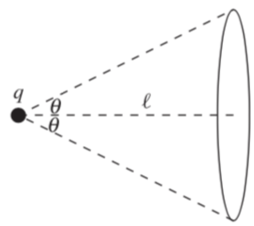
Accidental symmetry is important in modern physics. For example, protons are stable because of an accidental symmetry in the Standard Model, which ensures that baryon number is conserved. That explains why we often expect proton decay to occur in extensions of the Standard Model, such as grand unified theories, as explained in this nice article.

2 Continuous Charge Distributions

Idea 6

In almost all cases in Olympiad physics, there will be sufficient symmetry to reduce any multiple integral to a single integral. Remember that when using Gauss's law, the Gaussian surface may be freely deformed as long as it doesn't pass through any charges.

[2] **Problem 12** (Purcell 1.15). A point charge q is located at the origin. Compute the electric flux that passes through a circle a distance ℓ from q, subtending an angle 2θ as shown below.



[3] **Problem 13** (Purcell 1.8). A ring with radius R has uniform positive charge density λ. A particle with positive charge q and mass m is initially located in the center of the ring and given a tiny kick. If the particle is constrained to move in the plane of the ring, show that it exhibits simple harmonic motion and find the angular frequency.

[3] **Problem 14** (Purcell 1.12). Consider the setup of problem 9. If the hemisphere is centered at the origin and lies entirely above the xy plane, find the electric field at an arbitrary point on the z-axis. (This is a bit complicated, and is representative of the most difficult kinds of integrals you might have to set up in an Olympiad. For a useful table of integrals, see Appendix K of Purcell.)

[3] **Problem 15.** USAPhO 2018, problem B1.

Idea 7: Electric Dipoles

The dipole moment of two charges q and $-q$ separated by $\mathbf{d}$ is $\mathbf{p} = q\mathbf{d}$. More generally, the dipole moment of a charge configuration is defined as

$$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}\, d^3\mathbf{r}.$$

For an overall neutral charge configuration, the leading contribution to its electric potential far away is the dipole potential,

$$\phi(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where θ is the angle of $\mathbf{r}$ to $\mathbf{p}$.

Remark

Here's a trick to remember the dipole potential. Let $\phi_0(\mathbf{r}) = k/r$ be the potential for a unit charge at the origin. An ideal point dipole of dipole moment p consists of charges $\pm p/d$

separated by d, in the limit $d \to 0$. So the potential is

$$p \lim_{\mathbf{d}\to 0} \frac{\phi_0(\mathbf{r}) - \phi_0(\mathbf{r}+\mathbf{d})}{d}.$$

But this is precisely the (negative) derivative, so you can get the dipole potential by differentiating the ordinary potential! Indeed, for a dipole aligned along the $\hat{\mathbf{z}}$ axis,

$$-\frac{\partial}{\partial z}\frac{kp}{r} = \frac{kp}{r^2}\frac{\partial r}{\partial z} = \frac{kp}{r^2}\frac{z}{r} = \frac{kp\cos\theta}{r^2}$$

which matches the above result. You can use the same trick for quadrupoles and higher multipoles, which we'll see in **E8**.

[3] **Problem 16.** In this problem we'll derive essential results about dipoles, which will be used later.

(a) Using the binomial theorem, derive the dipole potential given above, for a dipole made of a pair of point charges $\pm q$ separated by distance d, oriented along the z-axis.

(b) Differentiate this result to find the dipole field,

$$\mathbf{E}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

where the expression above is in spherical coordinates. (Hint: feel free to use the expression for the gradient in spherical coordinates.)

(c) Show that this may also be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}).$$

You don't need to memorize these expressions, but it's useful to remember what a dipole field looks like, the fact that its magnitude is roughly $p/4\pi\epsilon_0 r^3$, and the fact that the numeric prefactor is 2 along the dipole's axis and 1 perpendicular to it.

[3] **Problem 17.** USAPhO 2002, problem B2.

[3] **Problem 18.** USAPhO 2009, problem B2. This essential problem introduces useful facts about dipole-dipole interactions.

Idea 8

The potential energy of a set of point charges is

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i).$$

The sum over "$i < j$" indicates that we consider each pair of distinct point charges once. We don't include the energy of a single point charge due to its interaction with itself, which would be infinite. For a continuous charge distribution, the analogous equations are

$$U = \frac{1}{2}\int \rho(\mathbf{r})V(\mathbf{r})\,d^3\mathbf{r} = \frac{\epsilon_0}{2}\int |\mathbf{E}(\mathbf{r})|^2\,d^3\mathbf{r}.$$

Energy *doesn't* obey the superposition principle, because it's inherently quadratic, not linear.

[3] **Problem 19.** In this problem we'll apply the above results to balls of charge.

(a) Compute the potential energy of a uniformly charged ball of total charge Q and radius R.

(b) Show that the potential energy of two point charges of charge $Q/2$ separated by radius R is lower than the result of part (a).

(c) Hence it appears that it is energetically favorable to compress a ball of charge into two point charges. Is this correct?

[3] **Problem 20.** An insulating circular disc of radius R has uniform surface charge density σ.

(a) Find the electric potential on the rim of the disc.

(b) Find the total electric potential energy stored in the disc.

[3] **Problem 21.** Consider a uniformly charged ball of total charge Q and radius R. Decompose this ball into two parts, A and B, where B is a ball of radius $R/2$ whose center is a distance $R/2$ of the ball's center, and A is everything else. Find the potential energy due to the interaction of A and B, i.e. the work necessary to bring in B from infinity, against the field of A.

[2] **Problem 22** (PPP 149)**.** A distant planet is at a very high electric potential compared with Earth, say 10^6 V higher. A metal space ship is sent from Earth for the purpose of making a landing on the planet. Is the mission dangerous? What happens when the astronauts open the door on the space ship and step onto the surface of the planet?

Example 8

Since Newton's law of gravity is so similar to Coulomb's law, the results we've seen so far should have analogues in Newtonian gravity. What are they? For example, what's the gravitational Gauss's law?

Solution

The fundamental results to compare are

$$F = -\frac{Gm_1m_2}{r^2}, \quad F = \frac{q_1q_2}{4\pi\epsilon_0 r^2}$$

where the minus sign indicates that the gravitational force is attractive, while the electrostatic force between like charges is repulsive. Then we can transform a question involving (only positive) electric charges to one involving masses if we map

$$q \to m, \quad \frac{1}{4\pi\epsilon_0} \to -G, \quad \mathbf{E} \to \mathbf{g}.$$

Thus, while electrostatics is described by

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0},$$

the gravitational field is described by

$$\nabla \times \mathbf{g} = 0, \quad \nabla \cdot \mathbf{g} = -4\pi G \rho_m, \quad \oint \mathbf{g} \cdot d\mathbf{S} = -4\pi G M$$

where ρ_m is the mass density. Similarly, the potential energy can be written in two ways,

$$U = \frac{1}{2} \int \rho_m(\mathbf{r}) \phi(\mathbf{r}) \, d^3\mathbf{r} = -\frac{1}{8\pi G} \int |\mathbf{g}(\mathbf{r})|^2 \, d^3\mathbf{r}$$

where $\phi(\mathbf{x})$ is the gravitational potential. This result was first written down by Maxwell.

Remark

Here's a philosophical question: is potential energy "real"? You likely think the answer is obvious, but about half of your friends probably think the opposite answer is obviously correct! In fact, in the 1700s, there was a lively debate over whether the ideas of kinetic energy and momentum, which at the time were given various other names, were worthwhile. Which one of the two was the *true* measure of motion? In our modern language, proponents of energy pointed out that the momentum always vanished in the center of mass frame, which made it "trivial", while supporters of momentum replied that kinetic energy was clearly not conserved in even the simplest of cases, like inelastic collisions.

In the 1800s, thermodynamics was developed, allowing the energy seemingly lost in inelastic collisions to be accounted for as internal energy. But there still remained the problem that kinetic energy was lost in simple situations, such as when balls are thrown upward. By the mid-1800s, the modern language that "kinetic energy is converted to potential energy" was finally standardized, but it was still common to read in textbooks that potential energy was fake, a mathematical trick used to patch up energy conservation. After all, potential energy has some suspicious qualities. If a ball has lots of potential energy, you can't see or feel it, or even know it's there by considering the ball alone. It doesn't seem to be located anywhere in space, and its amount is arbitrary, as a constant can always be added.

In the late 1800s, a revolution on physics answered some of these questions. Maxwell and his successors recast electromagnetism as a theory of fields, and showed that the dynamics of charges and currents were best understood by allowing the fields themselves to carry energy and momentum. We'll cover this in detail in **E7**, but for now, it implies that electrostatic potential energy is fundamentally stored in the field, with a density of $\epsilon_0 E^2/2$. This implies that its location and total amount are directly measurable.

Maxwell believed that the dynamics of fields emerged from the microscopic motions and elastic deformations of an all-pervading ether, in the same way that, say, a fluid's velocity field emerges from the average motion of fluid molecules. This makes it manifestly positive, so he was disturbed to find that the energy density of a gravitational field is *negative*! He therefore concluded that gravity could *not* be described as a vector field.

A few decades later, the arrival of special relativity answered some questions and reopened others. On one hand, it demolished Maxwell's vision of the ether. On the other hand, it finally answered the question of whether all kinds of potential energy are "real", and it got rid of the freedom to add arbitrary constants. That's because in special relativity, the total energy of a system at rest is related to its mass by $E = mc^2$, and the mass is directly measurable. This finally puts thermal energy, elastic potential energy, and field energy on an equal footing.

Here's the most modern view of energy conservation. All particles and their interactions are fundamentally described by relativistic quantum fields. A famous result called Noether's theorem implies that whenever such a theory is time-translationally symmetric, there is a conserved quantity which we call the energy. (The distinction between kinetic and potential energy becomes irrelevant; it's all just energy.) The density of energy in space can be computed from the state of the fields, but it doesn't need to be explained, as Maxwell imagined, by the internal motion of whatever the fields are made of. The fields are fundamental: they aren't made of anything; instead, they make up everything!

What happens when we throw gravity into the mix? As we'll discuss further in **R3**, it turns out that at nonrelativistic velocities, the dynamics of gravitating particles can be described by "gravitoelectromagnetism", a theory closely analogous to electromagnetism, where moving masses also source "gravitomagnetic" fields $\mathbf{B}_g$, which result in $m\mathbf{v} \times \mathbf{B}_g$ forces. But the situation gets much more subtle when we upgrade to full general relativity. Here, the notion of a gravitational field disappears completely, and is replaced by the curvature of spacetime, making it hard to define an energy density for it at all. For an accessible overview of the debate, see this paper. Ultimately, though, it doesn't matter that much, since it doesn't impair our ability to use either Newtonian gravity or general relativity.

Example 9

For an infinite line of linear charge density λ, find the potential $V(r)$ by dimensional analysis.

Solution

This example illustrates a famous subtlety of dimensional analysis. The only quantities in the problem with dimensions are λ, ϵ_0, and r. To get the electrical units to balance, we have

$$V(r) = \frac{\lambda}{\epsilon_0} f(r)$$

where $f(r)$ is a dimensionless function. But there are *no* nontrivial dimensionless functions of a single dimensionful quantity r. The only possibilities are that $f(r)$ is a dimensionless constant, or that $f(r)$ is infinite. In the first case, the electric field would vanish, which can't be right. In the second case, it is unclear how to calculate the electric field at all.

In fact, the electric potential *is* infinite, if you insist on the usual convention of setting $V(\infty) = 0$. In that case, we have

$$V(r) = \int_r^\infty E(r)\, dr = \int_r^\infty \frac{\lambda}{2\pi\epsilon_0} \frac{dr}{r} = \infty$$

for any r. But this is useless; to get a finite result we can actually work with, we need to subtract off an infinite constant from the potential. Equivalently, we need to set the potential to be zero at some finite distance $r = r_0$. This process is known as renormalization, and it is extremely important in modern physics. After renormalization, we have

$$V(r) = \int_r^{r_0} E(r)\, dr = \frac{\lambda}{2\pi\epsilon_0} \log \frac{r_0}{r}$$

which is perfectly consistent with dimensional analysis.

Notice that in the process of renormalization, a new dimensionful quantity r_0 appeared out of nowhere. This phenomenon is known as dimensional transmutation. Nothing measurable depends on this new scale (e.g. the electric field is independent of r_0), but you can't write down quantities like the potential without it.

3 Conductors

Idea 9

In electrostatic conditions, $\mathbf{E} = 0$ inside a conductor, which implies the conductor has constant electric potential V. This further implies that $\mathbf{E}$ is always perpendicular to a conductor's surface. By Gauss's law, the conductor has $\rho = 0$ everywhere inside, so all charge resides on the surface.

Example 10

Consider a point on the surface of a conductor with surface charge density σ. Show that the outward pressure on the charges at this point is $\sigma^2/2\epsilon_0$.

Solution

Gauss's law tells us that the difference of the electric fields right inside and outside the conductor at this point is

$$E_{\text{out}} - E_{\text{in}} = \frac{\sigma}{\epsilon_0}$$

by drawing a pillbox-shaped Gaussian surface. But we also know that $E_{\text{in}} = 0$ since we're dealing with a conductor, so $E_{\text{out}} = \sigma/\epsilon_0$.

Let's think about how this electric field is made. If there were no charges around except for the ones at this surface, then the interior and exterior fields would have been $\pm\sigma/2\epsilon_0$. This means that all of the other charges, that lie elsewhere on the surface of the conductor, must provide a field $\sigma/2\epsilon_0$ here, so that E_{in} cancels out.

The pressure on the charges at this point on the surface is equal to the product of the surface charge density with the field due to the *rest* of the charges, since the charges at this point can't exert an overall force on themselves, so

$$P = \sigma \left(\frac{\sigma}{2\epsilon_0} \right) = \frac{\sigma^2}{2\epsilon_0}$$

as required. Equivalently, we can conclude that $P = \epsilon_0 E_{\text{out}}^2/2$.

Example 11

Is the charge density at the surface of a charged conductor usually greater at regions of higher or lower curvature?

Solution

We can't answer this question in general, because it is usually impossible to solve for the charge distribution of an irregularly shaped conductor. Charges at any point in the conductor will influence the charges everywhere else.

However, we can get some insight by considering the limiting case of a conductor made of two spheres of radii R_1 and R_2, connected by a very long rod. For the potential to be the same at both spheres, we must have $Q_1/R_1 = Q_2/R_2$, so the charge is proportional to the radius, and the charge density is inversely proportional to the radius. Thus, there's generally higher charge density at sharper points of the conductor, provided that those points are sharp enough, or far enough away from the rest of conductor for the charges on the rest of the conductor not to matter much. That's basically all we can say for sure.

[1] **Problem 23.** Show that any surface of charge density σ with electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ immediately on its two sides experiences a force $\sigma(\mathbf{E}_1 + \mathbf{E}_2)/2$ per unit area. (This is a generalization of the example above, where one side was inside a conductor.)

[2] **Problem 24.** Is it possible for a single solid, isolated conductor with a positive total charge to have a negative surface charge density at any point on it? If not, prove it. If so, sketch an example.

Idea 10: Existence and Uniqueness

Here's an important mathematical fact: suppose the function ϕ is defined on some volume V with boundary S, and obeys $\nabla^2 \phi = 0$ everywhere in V. Also suppose we specify either the value of ϕ or the value of $\nabla\phi \cdot \hat{\mathbf{n}}$ on every point of S, where $\hat{\mathbf{n}}$ is the normal vector to the surface; these are called Dirichlet and Neumann boundary conditions, repsectively. Then there always exists a solution for ϕ, and as long as at least one boundary condition is Dirichlet, the solution is *unique*. (If none are Dirichlet, we are free to add a constant to ϕ.) A proof of this statement is given in section 2.4.2 here.

To translate this mathematical statement to something useful for physics, suppose ϕ is the electric potential in an electrostatic problem. This determines the electric field by $\mathbf{E} = -\nabla\phi$. The condition $\nabla^2\phi = 0$ is just equivalent to $\nabla \cdot \mathbf{E} = 0$, which implies $\rho = 0$ by Gauss's law. In addition, $\nabla\phi \cdot \hat{\mathbf{n}}$ is equal to $-\mathbf{E} \cdot \hat{\mathbf{n}}$. Finally, adding a constant to ϕ doesn't affect $\mathbf{E}$.

Using these facts gives the following translation: suppose there is a volume V containing no charge density, and either the electric potential or the outward electric field $\mathbf{E} \cdot \hat{\mathbf{n}}$ is specified everywhere on its boundary S. Then there is a unique solution for $\mathbf{E}$ in V.

Example 12

Consider a conductor with nonzero net charge in an arbitrary environment, and suppose the conductor has an empty cavity inside. Show that the electric field is zero in the cavity.

Solution

If you don't know the previous idea, then it seems impossible to answer the question. Can't something arbitrarily complicated happen?

However, the problem is immediately solved by the uniqueness theorem. Let V be the cavity. Since every point on the conductor has the same electric potential, which we call ϕ_0, we know that $\phi = \phi_0$ on the boundary S. Therefore, the solution for ϕ inside the cavity is unique. We know the constant solution $\phi = \phi_0$ works, so it must be the only one, so the electric field vanishes inside the cavity, as desired.

Example 13

Consider an isolated spherical conducting shell of radius R with an arbitrary charge distribution inside, with total charge Q. Find the electric field outside the shell.

Solution

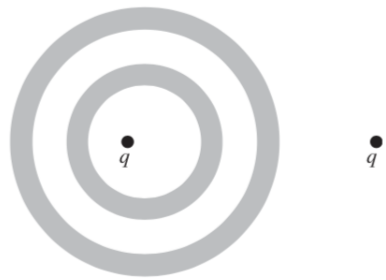
Let V be the volume outside the shell, and let $\phi = \phi_0$ on the shell's surface. For a given value of ϕ_0, one possible solution is that the potential is that of a point charge at the origin, $\phi(\mathbf{r}) = \phi_0 R/r$. So by the uniqueness theorem, this is the only solution. In this case, the value of ϕ_0 is related to Q by $\phi_0 = Q/(4\pi\epsilon_0 R)$, which implies $\mathbf{E}(\mathbf{r}) = Q\,\hat{\mathbf{r}}/(4\pi\epsilon_0 r^2)$.

The preceding two examples illustrate the general principle that a layer of conductor "shields" information from the other side. When you're inside, you can't know anything about what's going on outside, and when you're outside, you can only know the total charge inside.

Also, in this example, we weren't initially given Dirichlet *or* Neumann boundary conditions. We were told that the shell is a conductor, which tells us that $\phi = \phi_0$ without the specific value of ϕ_0, and the total charge inside, which tells us the surface integral of $\mathbf{E} \cdot \hat{\mathbf{n}}$ but not the value of $\mathbf{E} \cdot \hat{\mathbf{n}}$ at any particular point. However, this combination of information was still enough to fix a unique solution, and it turns out this is true in general: if all the surfaces in a problem are conductors, then specifying *either* the potential or total charge on each one fixes a unique solution. You'll give a heuristic proof of this in problem 28.

Finally, if you were reading very carefully, you might be wondering why we need to specify the shell is "isolated." The reason is that whenever the volume V is infinite, we also need to specify what is happening at the surface "at infinity." The default assumption is that the electric field and potential fall to zero at infinity. (As we've seen in problem 6 this becomes more subtle when the charge distribution itself is infinite.)

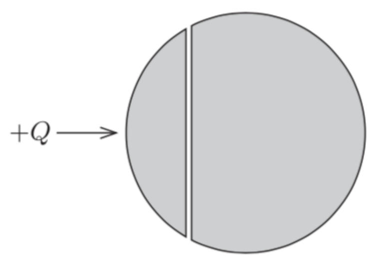
[2] **Problem 25** (Purcell 3.33)**.** The shaded regions represent two neutral conducting spherical shells.



Carefully sketch the electric field. What changes if the two shells are connected by a wire?

[3] **Problem 26.** USAPhO 2014, problem A4.

[4] **Problem 27** (MPPP 150)**.** A solid metal sphere of radius R is divided into two parts by a planar cut, so that the surface area of the curved part of the smaller piece is πR^2. The cut surfaces are coated with a negligibly thin insulating layer, and the two parts are put together again, so that the original shape of the sphere is restored. Initially the sphere is electrically neutral.



The smaller part of the sphere is now given a small positive electric charge Q, while the larger part of the sphere remains neutral. Find the charge distribution throughout the sphere, and the electrostatic interaction force between the two pieces of the sphere.

[3] **Problem 28.** In this problem we'll work through a heuristic proof of a version of the uniqueness theorem. In particular, we will show that for a system of conductors in empty space, specifying the total charge on each conductor alone specifies the entire surface charge distribution.

(a) Suppose for the sake of contradiction that two different charge distributions can exist, and consider their difference, which has zero total charge on each conductor. Argue that at least one conductor must have electric field lines both originating from and terminating on it.

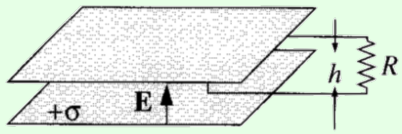
(b) Show that at least one of these field lines must originate from or terminate on another one of the conductors.

(c) By generalizing this reasoning, prove the desired result. (Hint: consider the conductors with nonzero surface charges that have the highest and lowest potentials.)

For a rigorous proof of this theorem using vector calculus, see section 3.1 of Griffiths.

Example 14: Griffiths 7.6

A wire loop of height h and resistance R has one end placed inside a parallel plate capacitor with electric field $\mathbf{E}$, as shown.

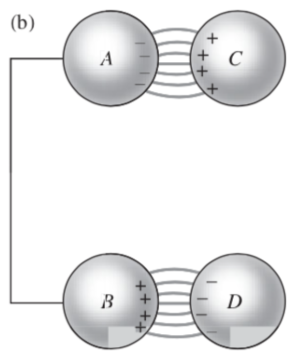


The other end of the loop is far away, where the field is negligible. Find the emf in the loop.

Solution

This is a trick question: if the answer were nonzero, the current would run forever, yielding a perpetual motion machine. Electrostatic fields always produce zero total emf along any loop. The $\sigma h/\epsilon_0$ voltage drop inside the capacitor is canceled out by the voltage drop due to the fringe fields, which are small, but accumulate over a long distance. The point of this example is that, while we can ignore fringe fields for some calculations, they are often essential to get a consistent overall picture. We'll revisit the subtleties of fringe fields in **E2**.

[2] **Problem 29** (Purcell 3.2)**.** Spheres A and B are connected by a wire; the total charge is zero. Two oppositely charged spheres C and D are brought nearby, as shown.



The spheres C and D induce charges of opposite sign on A and B. Now suppose C and D are connected by a wire. Then the charge distribution should not change, because the charges on C and D are being held in place by the attraction of the opposite charge density. Is this correct?

Remark

Here's a seemingly simple question. So far, we've considered a lot of problems involving uniformly charged spheres, cylinders, and planes. In these cases, all the electric field lines outside the charged region are straight. More generally, is it possible to have a situation where all the electric field lines in some charge-free region are straight, but the field lines *don't* have spherical, cylindrical, or planar symmetry?

The answer isn't obvious. For example, you might imagine we could simply deform a set of spherically symmetric field lines so that, e.g. the equipotential surfaces look like ellipses.

On the other hand, it's hard to write down an example that works, so you might think one of the three symmetries above is necessary. But how could you prove that mathematically?

It turns out that the field lines indeed must have spherical, cylindrical, or planar symmetry, but the simplest proof I know of requires a bit of differential geometry. We need to consider the principal curvatures k_1 and k_2 of adjacent equipotential surfaces. After some analysis, it turns out that the field lines can be straight only if

$$k_1 k_2 (k_1 - k_2) = 0$$

which precisely corresponds to allowing spherical ($k_1 = k_2$), cylindrical ($k_1 = 0$), or planar ($k_1 = k_2 = 0$) symmetry. For a full derivation, see this paper.