Waves III: Specific Waves

Sound waves are covered in chapter 19 of Halliday and Resnick, while light waves are covered in chapters 39 and 44. For more about light waves, see chapter 9 of Purcell. For water waves, and many other neat wave phenomena, see chapter I-51 of the Feynman lectures and chapter 7 of Crawford's *Waves*. For more about polarization, see chapter I-33 of the Feynman lectures, or for more detail, chapter 8 of Hecht's *Optics*. Basic geometrical optics is covered in chapter 40 of Halliday and Resnick. There is a total of **79** points.

1 Sound and Longitudinal Waves

[4] **Problem 1.** In this problem, you'll work through Newton's slick derivation of the speed of sound. This derivation avoids thinking about how individual parcels of gas move, which can be confusing, by instead considering the motion of a piston at the end of the gas.

(a) Consider a cylinder of gas of length L and area A, closed on one end with a movable piston on the other end. Suppose the gas exerts a force F on the piston when in equilibrium. We may define an equivalent spring constant by $K = -dF/dL$. Show that for the gas,

$$K = -A^2 \frac{dp}{dV}.$$

(b) Argue that the speed v of longitudinal waves on a spring of mass M, spring constant K, and length L obeys

$$v^2 = \frac{KL^2}{M}.$$

(c) If we assume the sound waves are adiabatic, show that for the gas,

$$v^2 = \frac{\gamma p}{\rho}.$$

Check this answer is reasonable by evaluating the result for air. If each gas molecule has mass m, write the result in terms of γ, T, and m.

Next, we consider some limitations of this result.

(d) In an ideal gas, we assume the particles are noninteracting: they pass right through each other. But for sound waves to propagate, adjacent packets of ideal gas must exert pressure on each other. How is this possible? Use this observation to estimate the maximum possible frequency of sound in a gas in terms of the number density $n = N/V = P/k_BT$, the radius r of a gas molecule, and the speed of sound v.

(e) Our analysis also breaks down if the pressure variations are no longer adiabatic. The rate of heat conduction in a gas with thermal conductivity k_t across a surface of area A is

$$\frac{dQ}{dt} = -Ak_t \frac{dT}{dx}$$

Taking a sinusoidal temperature variation, show that the adiabatic approximation holds when $\omega \ll pk_B/mk_t$. Is this approximation good for audible sound in air, where $k_t \approx 25\,\mathrm{mW/m\,K}$?

(f) In a more traditional derivation of the speed of sound, such as the one in Halliday and Resnick, one would show that

$$v^2 = \frac{B}{\rho}$$

where B is the bulk modulus, the pressure per fractional change in volume,

$$B = -V \frac{dP}{dV}.$$

Check that this result is compatible to your result in (c) for adiabatic compression.

Liquids and solids typically have a much higher B and ρ than gases, and B is high enough so that the speed of sound in liquids and solids is typically greater as well.

Remark

Phase shifts upon reflection for sound waves can be a bit tricky. Recall from **W1** that a hard boundary for a transverse string wave $y(x, t)$ sets y to zero. As a result, upon reflection, y flips sign, but $v_y = \partial y/\partial t$ stays the same.

When a sound wave hits a hard wall, the wall sets the displacement $\xi(x, t)$ to zero. Then upon reflection, the displacement flips sign, while the pressure variation $\delta P(x, t) \propto \partial \xi/\partial x$ stays the same. In standing waves, a hard wall is thus a node for ξ and an antinode for δP. Similarly, when sound waves in a tube reflect off an open end, the end sets δP to zero, so it flips sign. An open end is thus a node for δP and an antinode for ξ.

The rule is always the same: whatever quantity gets fixed to zero by the boundary gets flipped in sign upon reflection, and for a standing wave, that quantity has a node at the boundary. But it's confusing enough that several common high school textbooks get it wrong. Some even state, in their confusion, that "hard boundaries flip transverse waves but not longitudinal ones", which is definitely not true in general.

[3] **Problem 2** (HRK)**.** Some conceptual questions about sound waves.

(a) What is larger for a sound wave, the relative density variations $\Delta\rho/\rho$ or the relative pressure variations $\Delta P/P$? Or does it depend on the situation?

(b) What is larger, the velocity of a sound wave v or the amplitude of the velocity variations Δu of the underlying particles? Or does it depend on the situation?

(c) Bats and porpoises each emit sound waves of frequency about 100 kHz. However, bats can detect objects as small as insects but porpoises only small fish. Why the difference?

[2] **Problem 3.** A rubber rope with unstretched length L_0 is stretched to length $L \gg L_0$.

(a) Find the ratio of the speeds of transverse and longitudinal waves.

(b) Experimentally, it is found that the longitudinal waves are much more strongly damped. (You can check this at home, by making such a rope by tying together cut rubber bands.) Can you explain why, by considering the molecular structure of rubber?

Idea 1: Doppler Effect

Working in one dimension with speed of sound c, if a source of sound at frequency f_0 travels at velocity v_s while an observer to their right travels at velocity v_o, the observed frequency is

$$f = \frac{c - v_o}{c - v_s} f_0.$$

Example 1

A speaker is between two perfectly reflective walls and emits a sound of frequency f_0. If you carry the speaker and walk with small speed v towards one of the walls, what do you hear?

Solution

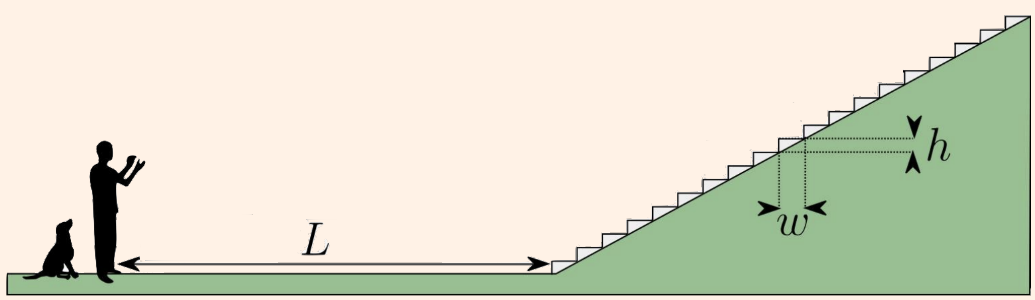
Let's work to lowest order in v/c everywhere. The wall you're walking toward experiences a sound of approximate frequency $f_0(1 + v/c)$ by the Doppler effect, and this is the frequency it reflects. Since you're walking towards the wall, a second Doppler effect occurs, causing you to hear frequency $f_0(1 + 2v/c)$. (We also saw this "double Doppler shift" back in **R1**.) By similar reasoning, you hear sound of frequency $f_0(1 - 2v/c)$ from the wall behind you. Since v/c is small, what this actually means is that you'll perceive a single tone of frequency f_0, but with "beats" of frequency $4v/c$.

Example 2

In my former college at Oxford, there is a long staircase that is said to "quack" when one claps at it. What is the explanation of this phenomenon?

Solution

A diagram of the staircase is given below, courtesy of Felix Flicker, fellow of New College.



The key is that each clap reflects off a stair individually. When the echoes arrive back at the listener, they arrive quickly enough to be heard as a pitch.

The width and height of the steps are $w = 30\,\text{cm}$ and $h = 16\,\text{cm}$. Suppose one claps at a distance $L \gg w, h$. The path length differences for reflections off the bottom few steps are approximately $2w$, giving the frequency

$$f = \frac{v}{2w} = 570\,\text{Hz}$$

where we used $v = 343\,\text{m/s}$. The quack then continues, due to reflections off higher and higher stairs. Once the stairs are much further away than L, path length differences for subsequent reflections are approximately $2\sqrt{w^2 + h^2}$, giving frequency

$$f = \frac{v}{2\sqrt{w^2 + h^2}} = 500\,\text{Hz}.$$

Hence the quack consists of a pitch that starts high and then falls slightly lower as it fades away. For further discussion, see the article *How the Mound got its Quack*.

[3] **Problem 4.** USAPhO 1998, problem B1. (The official solution has a qualitatively incorrect answer for the final part of the problem; see Stefan Ivanov's errata for the correct answer.)

[3] **Problem 5.** USAPhO 2016, problem A1.

[2] **Problem 6.** Some problems about sound waves in everyday life.

(a) Get a coffee cup with a handle and tap on the rim with a spoon. You will hear two distinct pitches, e.g. if you tap directly above the handle, or 45° away from this point. Investigate what happens for different angles. Can you explain why this happens?

(b) The octave key on an oboe forces the resonant mode from the fundamental to the first overtone, doubling the frequency of the note produced. It does this by opening a small hole on the back of the clarinet. Should this hole be placed at a pressure node or antinode for the fundamental, or somewhere else entirely?

(c) According to introductory textbooks, the fundamental mode for a pipe of length L and radius $r \ll L$, closed at one end and open at the other, has wavelength $4L$. In reality, it's a little bit different because the radius is nonzero. Is the wavelength actually higher or lower than $4L$?

(d) Find a way to produce beats in real life.

[3] **Problem 7.** BPhO 2008, problem 2.

2 Polarization

Now we'll introduce polarization for light waves, putting the results of **E7** to work.

Idea 2

The polarization of a light wave refers to the direction of its electric field; the light waves we saw in **E7** were linearly polarized. For example, a light wave traveling along $\hat{\mathbf{z}}$ with its polarization an angle θ from the x-axis has electric field

$$E_x(z,t) = (E_0 \cos\theta)\cos(kz - \omega t), \quad E_y(z,t) = (E_0 \sin\theta)\cos(kz - \omega t).$$

A polarizer lets only light of a certain linear polarization through; if light with a linear polarization an angle θ from this axis passes through it, then a fraction $\cos^2\theta$ of the energy is transmitted. Just as light can be incoherent, it can be unpolarized; unpolarized light hitting a polarizer loses half its energy.

[3] **Problem 8.** A simple polarizer contains many very thin, closely spaced wires. If the wires are vertical, they block vertical electric fields, allowing only horizontally polarized light to go through; this is a horizontal polarizer. One can similarly make diagonal and vertical polarizers.

(a) Suppose that perfectly monochromatic, but unpolarized light is incident on a double slit. (In this case, assume "unpolarized" means that at each instant in time, the polarization of the light passing through each slit is the same, but over longer timescales that polarization can vary.) What does the intensity pattern on the screen look like?

(b) Next, suppose a vertical polarizer is placed in front of one slit, and a horizontal polarizer is placed in front of the other slit. Now what does the intensity pattern look like?

(c) Finally, we further modify the setup of part (b) by placing many diagonal polarizers, at 45° to the vertical and horizontal, right in front of the *screen*. What does the intensity pattern on the screen look like now?

(d) On an unrelated note, suppose we wish to rotate the polarization of linearly polarized light by using $N \gg 1$ intermediate polarizers. What's the best way to do this, and what's the fraction of light that passes through the stack?

Idea 3

For a plane wave propagating along the z-axis with general polarization, it's useful to write

$$\mathbf{E}(z,t) = \mathrm{Re}\left(\mathbf{E}_0\, e^{i(kz-\omega t)}\right)$$

where $\mathbf{E}_0$ is a complex two-component vector, describing both its amplitude and polarization. For example, if $\mathbf{E}_0 = E_0\,\hat{\mathbf{x}}$, the light wave is horizontally polarized, if $\mathbf{E}_0 = iE_0\,\hat{\mathbf{x}}$, it's horizontally polarized with a phase shifted by $\pi/2$, if $\mathbf{E}_0 = E_0\,\hat{\mathbf{y}}$ it's vertically polarized, and if $\mathbf{E}_0 = E_0\,(\hat{\mathbf{x}}+\hat{\mathbf{y}})/2$ it's diagonally polarized.

When linear polarizations are combined with a relative phase, the result is circular (or more generally, elliptical) polarization. For example, when $\mathbf{E}_0 = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$, we have

$$E_x(z,t) = \frac{E_0}{\sqrt{2}}\cos(kz-\omega t), \quad E_y(z,t) = \frac{E_0}{\sqrt{2}}\sin(kz-\omega t)$$

which is a circularly polarized light wave; the electric field at a fixed point rotates in a circle over time, and if one draws the electric field vectors in a line along $\hat{\mathbf{k}}$, they trace out a spiral. Birefringent materials, which have different indices of refraction in different directions, cause such phase shifts, and thus can convert linear polarizations into other polarizations.

Example 3

A plane wave with amplitude $\mathbf{E}_0 = E_0\,\hat{\mathbf{x}}$ enters a linear optical device, which does not absorb or reflect any energy. When the plane wave exits the device, it has circular polarization, $\mathbf{E}_0 = E_0\,(\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$. What does the device do to light with vertical polarization?

Solution

Vertical polarized light has to exit with circular polarization of the other handedness, i.e. with $\mathbf{E}_0 = E_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}})/\sqrt{2}$, because this is the only possibility consistent with energy conservation.

To see this, note that the energy of a light wave is proportional to the time-averaged value of $|\mathbf{E}|^2$, which is turn proportional to $|\mathbf{E}_0|^2$. Since horizontal and vertical polarizations are orthogonal, they don't interfere, so sending in both a horizontal and vertical light wave of amplitude E_0 at the same time just doubles the input energy. This must also double the output energy, and indeed, under the above ansatz we have

$$\hat{\mathbf{x}} + \hat{\mathbf{y}} \to \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}} = \sqrt{2}\,\hat{\mathbf{x}}$$

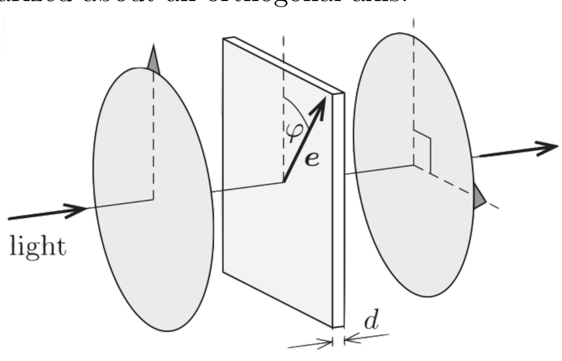
which indeed has double the energy of one wave by itself.

The more general principle here is that, since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ were orthogonal to each other, they must be mapped to two other unit vectors which are still orthogonal, as complex vectors. That is indeed true, because

$$(\hat{\mathbf{x}} + i\hat{\mathbf{y}})^\dagger(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) = \hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + i^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = 0.$$

By the way, now that we know what the device does to both horizontal and vertically polarized light, we also know what it does to any polarization of light, by superposition.

[3] **Problem 9** (MPPP 127)**.** A birefringent material is placed between two orthogonal polarizers. The material has thickness d, and has an index of refraction of n_1 for light linearly polarized along the axis $\mathbf{e}$, and n_2 for light polarized about an orthogonal axis.



If the system is illuminated with light of wavelength λ, give a value for d and orientation of $\mathbf{e}$ that maximizes the transmitted light.

[2] **Problem 10** (MPPP 128)**.** In the first 3D movies, spectators would wear glasses with one eye tinted blue and the other tinted red. This was quickly abandoned in favor of a system that used

the polarization of light.

(a) If you wear an old pair of 3D movie glasses, close one eye, and look in the mirror, then you can only see the open eye. Explain how these glasses employ light polarization. What disadvantages might this system have?

(b) If you wear a new pair of 3D movie glasses and do the same, then you can only see the closed eye. Explain why.

[2] **Problem 11** (HRK). A quarter-wave plate is a birefringent plate that causes a $\pi/2$ phase shift between light polarized along $\mathbf{e}$ and perpendicular to $\mathbf{e}$. Similarly, a half-wave plate causes a π phase shift. Suppose you are given an object, which may be a quarter-wave plate, a half-wave plate, a linear polarizer, or just a semi-opaque disk of glass. How can you identify the object? You can use an unpolarized light source, and any number of polarizers and quarter-wave and half-wave plates.

[2] **Problem 12** (HRK). A polarizer and a quarter-wave plate are glued together so that, if the combination is placed with face A against a shiny coin, the face of the coin can be seen when illuminated by light of appropriate wavelength. When the combination is placed with face A away from the coin, the coin cannot be seen. Which component is on face A and what is the relative orientation of the components?

[4] **Problem 13.** IZhO 2021, problem 3. A problem on the propagation of light through a waveguide, unifying material from **E7** and **W1**.

3 Water Waves

Water waves are the most familiar examples of waves in everyday life, but you won't find them mentioned often in introductory textbooks, because they're far more complicated than any other kind of wave we'll consider. In all the problems below, we will completely neglect viscosity, surface tension, and compressibility of the water. Despite this, our results will still only be approximate.

[4] **Problem 14.** In this problem we consider shallow water waves, the case where the water depth is much less than the wavelength. Let the water have density ρ and depth d. A wave travels along the x-direction with height $h(x,t)$ relative to the water level. For shallow waves, it turns out that the water has a horizontal velocity $v(x,t)$ which is independent of height. Assume that $h \ll d$, the vertical velocity of the water is negligible, and the hydrostatic pressure formula applies almost everywhere.

(a) Find a relation between the derivatives of $h(x,t)$ and $v(x,t)$ using conservation of mass. Using this result, show that the phase velocity v_w of the water waves obeys $v \ll v_w$.

(b) Find a relation between the derivatives of $h(x,t)$ and $v(x,t)$ using Newton's second law.

(c) Combining these results, find the phase velocity v_w of shallow water waves.

Now let's consider what happens when a shallow water wave created at sea approaches the shore, and the depth d slowly decreases.

(d) Explain why waves always arrive at the shore moving perpendicular to the shoreline.

(e) If the depth is halved, by what factor is the height of the wave multiplied? This phenomenon is known as shoaling.

The above problem considered the case $d \ll \lambda$. However, for $d \gtrsim \lambda$ the situation is much more complicated. In this case the motion of the water does depend on height, and it turns out that the individual wave molecules move in ellipses, which reduce to circles in the limit $d \gg \lambda$ of deep water waves. The next two problems consider deep water waves.

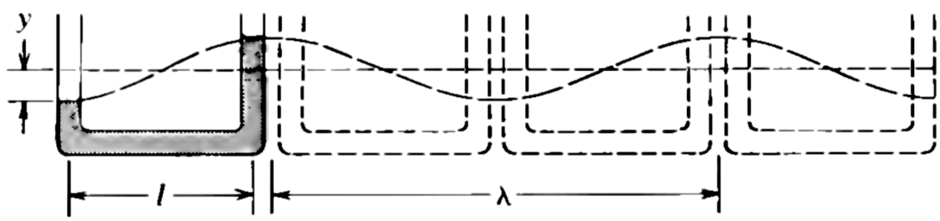
Remark

Textbooks commonly say that liquids can't support transverse waves, because they don't support shear stresses. But the waves considered in problem 14 are clearly transverse. This is possible because the textbook statement only applies to the internal forces of water *alone*. At the surface of the water, gravity provides the transverse restoring force; that's why these waves are also commonly called "gravity waves".

[2] **Problem 15** (French 7.20). In this problem we'll make a crude model for a deep water wave by using an artificial setup that forces the water to move in a simpler way.

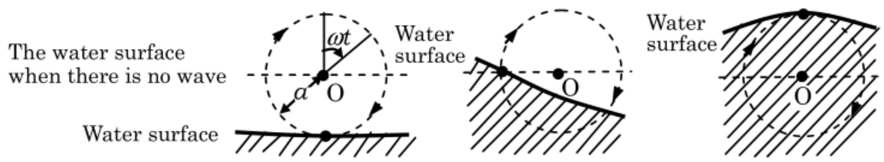
(a) Consider a U-tube of uniform cross section with two vertical arms, so that the horizontal section of the tube, with length l, is much longer than the water depth. Show that the period of oscillation is approximately $T = \pi\sqrt{2l/g}$.

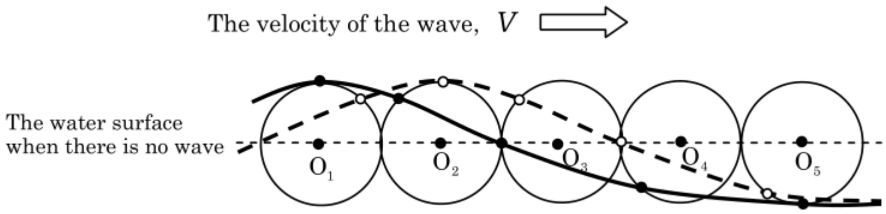
(b) Now suppose a succession of such tubes is placed next to each other and set oscillating to define a succession of crests and troughs, as shown.



Nothing is actually moving between the tubes. But if you squint, it kind of looks like there's a wave of wavelength $\lambda = 2l$. Using this picture, find the wave speed in terms of g and λ.

[2] **Problem 16** (Japan). In this problem, we'll give another heuristic treatment of deep water waves, accounting for the circular motion. For simplicity, we assume that the water molecules at the surface of the wave move in uniform circular motion with radius a and angular velocity ω, as shown.





(a) Consider the frame of reference moving to the right with velocity v. In this frame, the surface of the water is completely stationary, while molecules travel along the surface. Consider a small parcel of water which travels from a valley to a peak. By applying conservation of energy, derive a relationship between v, ω, and g.

(b) Find the phase and group velocity of the wave, in terms of g and the wavenumber k. In addition, find the condition on a and k for this derivation to make sense.

Showing that circular motion actually occurs takes more work, and involves solving partial differential equations; you can find a complete derivation here.

As you can see, water waves are quite complex. A diagram of the speeds of nine different limiting cases of water waves can be found in section 8.4 of The Art of Insight.

4 Reflection and Refraction

Now we'll introduce reflection and refraction with some real-world applications.

Idea 4

If a wave hits an interface, while traveling at an angle θ_1 to the normal to the interface, then it will generically both reflect and refract. The angle of the reflected ray is $\theta_2 = \theta_1$, and the angle of the refracted ray obeys $n_1 \sin\theta_1 = n_2 \sin\theta_2$. If there is no solution for θ_2 in the latter equation, then only reflection occurs.

These results follow directly from Huygens' principle, so they are very general, applying to light waves, sound waves, water waves, and so on, as long as the index of refraction n_i is always defined to be inversely proportional to the wave speed in each medium.

[2] **Problem 17.** Some conceptual questions about reflection and refraction.

(a) Does the index of refraction determine the phase velocity or the group velocity?

(b) Does a light beam of finite width get wider or narrower upon passing from air to water? Assume the light enters at an angle to the normal.

(c) Three mutually perpendicular mirrors intersect so as to form an internal right-angled corner. If a light ray strikes all three mirrors, show that it ends up traveling exactly opposite to its original direction. Can you think of a practical application of such a "corner reflector"?

Example 4

Let the index of refraction at height h above the Earth's surface be $n(h)$. In terms of $n(0)$ and the Earth's radius R, what should dn/dh be at the surface so that light rays orbit in circles around the Earth, with constant height?

Solution

First, let's ignore the curvature of the Earth. Consider a light ray moving slightly upward, at a small angle θ to the horizontal, experiencing index of refraction n. Over a horizontal distance L, it goes up by a height $L\theta$. At this point, it will have a different angle θ' to the horizontal, and experience index of refraction $n + L\theta\, dn/dh$. Snell's law says

$$n \cos\theta = \left(n + L\theta \frac{dn}{dh}\right) \cos\theta'$$

and expanding to lowest order in the small angles θ and θ' gives

$$\frac{n}{2}(\theta'^2 - \theta^2) = L\theta \frac{dn}{dh}.$$

Approximating again to lowest order gives

$$\theta - \theta' \approx -\frac{L}{n}\frac{dn}{dh}.$$

Thus, the light ray turns through an angle of $(1/n)\, dn/dh$ per unit horizontal distance. For the light ray to stay at a constant height over the curved Earth, this must equal $1/R$, giving

$$\frac{dn}{dh} = -\frac{n(0)}{R}.$$

More generally, this calculation shows that light bends towards the direction with higher n. In the case of air, where $n - 1 \ll 1$, we can rewrite this as

$$\frac{d(n-1)}{dh} \approx -\frac{1}{R}$$

which can plausibly occur on Earth, due to the nice coincidence that $n - 1$ and H/R (where H is the typical scale height of the atmosphere) are both of order 10^{-3}.

Remark: Mirages

There are two classes of mirages.

- When $dn/dh < 0$, light rays bend down. If there is a distant object at the horizon, its image will appear *above* the horizon. This is called a superior mirage, or "fata morgana".

- When $dn/dh > 0$, light rays bend up. Then a distant object at the horizon will appear *below* the horizon, forming an inferior mirage. This also applies to the sky near the horizon, producing the illusion of water on the ground sometimes seen in deserts.

In air, the refractive index is close to 1, and $n - 1 \propto \rho \propto P/T$, where ρ is the air density and the second step used the ideal gas law. Usually we have $d\rho/dh < 0$, since $dP/dh < 0$ in hydrostatic equilibrium, but it depends on the value of dT/dh.

- In normal conditions, the Sun warms the ground and the hot air rises and adiabatically mixes the atmosphere (as discussed in **T1**), so that $dT/dh < 0$. This partially cancels the effect of the pressure variation, so that dn/dh is still negative but has small magnitude, so that mirage effects aren't apparent.
- In rare "thermal inversion" conditions, we have $dT/dh > 0$, so that dn/dh is negative with large magnitude, leading to strong superior mirage effects. If dn/dh is negative enough, it can match the value computed in example 4, allowing an observer to see arbitrarily far along the horizon despite the curvature of the Earth. This was the reason the famous Bedford Level experiment concluded the Earth was flat.
- In hot deserts, the air near the ground is very hot, so that $dT/dh < 0$ with a large magnitude. (A strongly negative dT/dh also occurs in cold days above water, since the water stays warmer than the air above it.) Here the temperature gradient overpowers the pressure gradient, so that $dn/dh > 0$ and inferior mirages can occur.

Proponents of the flat Earth hypothesis claim that the Earth only seems curved due to atmospheric refraction. But they have it backwards: in almost all conditions $dn/dh < 0$, which makes the Earth look *less* curved than it actually is.

[4] **Problem 18.** IPhO 1995, problem 2. Refraction in the presence of a linearly varying wave speed. (This is a classic setup with a neat solution, also featured in IPhO 1974, problem 2.)

[3] **Problem 19.** INPhO 2019, problem 1. Another exercise on refraction, with an uglier solution.

[3] **Problem 20.** IPhO 2003, problem 3B. An exercise on refraction and radiation pressure.

[4] **Problem 21.** IPhO 1993, problem 2. Another exercise on the same theme.

5 Ray Tracing

Idea 5

A pointlike object emits light rays in all directions. When those light rays subsequently converge at some other point, that point is the object's real image. If they don't actually converge, but all propagate outward with a common center, that point is the object's virtual image. In general, if we're given that an image exists, we can find its location by following the paths of selected rays from the object and looking for intersections.

[2] **Problem 22.** A pinhole camera is a simplified camera with no lens. It simply consists of a small hole with a screen behind it.

(a) Explain how the pinhole camera works by ray tracing.

(b) What are the disadvantages of having an especially small hole, or an especially large hole?

(c) Assuming the object being photographed is very bright, estimate the optimal aperture size for taking a clear picture with a pinhole camera.

The next three problems will exercise your intuition with real-world examples.

[2] **Problem 23.** AuPhO 2020, section C.

[2] **Problem 24.** AuPhO 2013, problem 11. Write your answers on the official answer booklet.

[3] **Problem 25.** AuPhO 2019, problem 12. Write your answers on the official answer booklet.

[3] **Problem 26.** IZhO 2020, problem 1.3. A test of your intuition for 3D ray tracing.

Idea 6

Conic sections have some simple properties under reflection.

- Light rays emitted from one focus of an ellipse will all be reflected to its other focus.
- Light rays emitted from one focus on a hyperbola will all be reflected so that the resulting rays all travel radially outward from the other focus.
- Parallel light rays entering a parabola along its symmetry axis (i.e. the axis perpendicular to the directrix) will all be reflected to its focus.

In the language of idea 5, if the foci of an ellipse/hyperbola are called F_1 and F_2, then an object at F_1 produces a real/virtual image at F_2. Note that a parabola is simply an ellipse in the limit where F_1 becomes very far away, so that rays coming in from F_1 become approximately parallel.

[2] **Problem 27** (Povey)**.** The mirascope is a toy consisting of two parabolic mirrors, pointing toward each other, so that the focus of each one is at the vertex of the other.

(a) When an object is placed at the bottom vertex, a real image appears at the top vertex. Why?

(b) How is the image oriented relative to the object?

The real image made by this setup is very convincing. There's a Michelin starred restaurant that uses it in a course: when you reach for what looks like the food, your hand just passes through air.

Idea 7: Paraxial Approximation

If a light ray hits a thin lens of focal length f at a shallow angle, and at a distance $y \ll f$ above the lens's center, then it will exit the lens bent vertically by an angle $\pm y/f$, where the sign depends on whether the lens is converging or diverging. (For example, any light ray going straight through the lens's center isn't bent at all.) This is the paraxial approximation, which only holds for light rays incident at shallow angles near the center of the lens.

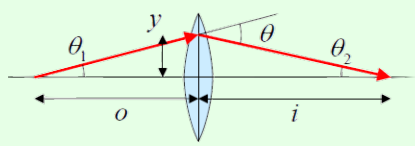
Conversely, if you don't know the focal length of a system, you can use this idea to find it. For example, the lensmaker's equation, giving the focal length of a lens of radii of curvature R_1 and R_2 and thickness d, is

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)\,d}{nR_1R_2}\right)$$

and can be derived by computing the bending of the light ray at each interface.

Example 5

An object is placed a distance o behind a thin converging lens with focal length f.



An image is formed a distance i in front of the lens. How are o, i, and f related?

Solution

The horizontal light ray goes straight through, so let's consider another light ray which emerges at a small angle θ_1 to the horizontal. Then we read off

$$\theta_1 \approx \frac{y}{o}, \quad \theta_2 \approx \frac{y}{i}$$

but their sum is the deflection y/f, from which we conclude

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}.$$

This is the familiar thin lens equation.

Example 6

A candle is placed behind a converging lens. An image is formed on a screen on the other side of the lens. Now suppose that the *top* half of the lens is covered with a black cloth. Describe how the image changes.

Solution

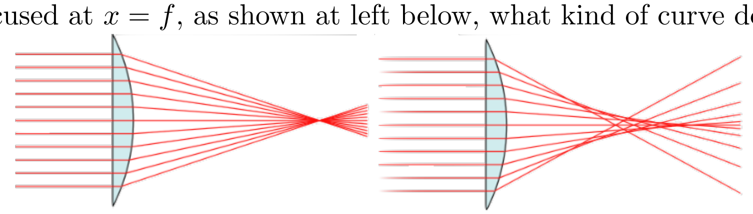
It is tempting to say that half of the candle's image disappears, but that's not right. Ray tracing shows that you can get a complete image of the candle, since there are always rays that pass through the bottom half of the lens. Instead, by blocking half the lens, the image gets half as bright.

[3] **Problem 28.** USAPhO 2024, problem A3. A series of optics exercises relevant for real cameras.

Idea 8: Fermat's Principle

For fixed starting and ending points, light always takes the path of least time. This implies that if light from point P is all focused at point P', then all the relevant paths from P to P' take the same time. This principle is completely equivalent to the laws of reflection and refraction above, but may be more useful in certain situations.

[2] **Problem 29.** Parallel light rays coming in along the $+\hat{\mathbf{x}}$ direction enter a lens of index of refraction n, whose left edge is at $x = 0$ and whose right edge is described by the function $x(y)$. If all the light beams are to be focused at $x = f$, as shown at left below, what kind of curve does $x(y)$ have to be?



You should find that $x(y)$ is not an arc of a circle, which implies that a spherical lens will fail to focus all incoming horizontal light to a point. Instead, we will get spherical aberration, as shown at right above. However, most lenses are spherical because it's easier to make them that way.