Modern III: Matter, Astro, and Cosmo

Chapters 35 and 36 of Blundell cover astrophysics, and chapter 15 of Krane covers cosmology. For solid state physics, see chapter 49 of Halliday and Resnick, chapter III-14 of the Feynman lectures, or section 5.3 of Griffiths' *Introduction to Quantum Mechanics* (3rd edition). For more on magnetism, see chapters II-34 through II-37 of the Feynman lectures. For a detailed introduction to the physics of stars and compact objects, see chapters 10 and 16 of *An Introduction to Modern Astrophysics* by Carroll and Ostlie. (Carroll and Ostlie is also a great introduction to astrophysics in general, accessible with just Olympiad physics knowledge.) There is a total of **82** points.

1 Condensed Matter

Condensed matter is an enormous field, touching everything from solid state physics to biophysics and atomic physics, and contains more than half of *all* physicists. However, you hear less about it in the news, and in Olympiad problems, because it requires a substantial amount of background to explain. The following problems cover some classic ideas in condensed matter, using a mix of modern physics and waves.

[5] **Problem 1.** APhO 2016, problem 3. A good question on the quantum mechanics of superconductivity.

[5] **Problem 2.** USAPhO+ 2021, problem 3. A nice question which derives the integer quantum Hall effect using classical electromagnetism and Bohr quantization.

[5] **Problem 3.** APhO 2015, problem 1. A question on the fractional quantum Hall effect, which is even subtler than the integer quantum Hall effect. This question is not very clearly written, and requires a good amount of educated guessing; however, I include it to give you practice with this type of Olympiad question, and some exposure to a very important topic of current research.

[5] **Problem 4.** IZhO 2021, problem 2. A problem on the thermodynamics of plasmas, reviewing some material in **T2**.

Solution. See the official solutions here.

[5] **Problem 5.** APhO 2021, problem 2. A very technical problem that fully explains how a famous test of quantum mechanics was conducted. Requires material from **E8**, **W1**, **W3**, and **X1**.

2 Stars

This section contains problems involving stars and star formation. You might think this is a lot about stars, but all of the problems below use a mix of mechanics, electromagnetism, thermodynamics, relativity, and modern physics; they are excellent review for the entire course. For a beautiful graphical overview of these objects, see this paper.

Example 1: PTD 44

The density of stars in the central region of the galaxy is about $n = 10^6\,\mathrm{pc}^{-3}$, and their speeds are about $v = 200\,\mathrm{kms}^{-1}$. Could an advanced civilization develop in this region?

Solution

Impacts between solar systems will occur frequently. For concreteness, suppose catastrophic effects will happen to an Earth-like planet if a different star passes within the equivalent of Jupiter's orbit, which has radius r. The typical time between such events is

$$t \sim \frac{1}{n(\pi r^2)v} \sim 2.4 \times 10^6 \text{ years.}$$

Some argue that this is too short a time for advanced civilization to develop, so the center of the galaxy is outside of the so-called galactic habitable zone.

Example 2: CPhO 2013.3

For stars not too much heavier than the Sun, the luminosity scale with mass as $L \propto M^{3.5}$. If all of these stars release the same fraction α of their rest mass energy by nuclear burning, then how does the lifetime of the star scale with M?

Solution

The amount of energy available is αM. The lifetime is thus

$$\tau = \frac{E}{L} \propto M^{-2.5}$$

so heavier stars live shorter lives. Incidentally, the luminosity scales as $L \propto R^2 T^4$ by the Stefan–Boltzmann law. By considering the details of the interior of the star, we can find how all of these quantities scale with mass, a principle known as stellar homology.

[3] **Problem 6.** USAPhO 2018, problem B3.

[5] **Problem 7.** IPhO 2012, problem 3. This elegant and tricky problem covers the early stages of star formation, and serves as a review of **T1**.

[5] **Problem 8.** GPhO 2017, problem 1. This problem covers the physics of fusion in main sequence stars, relying on **X1** and **X2**.

Solution. See the official solutions here.

[5] **Problem 9.** IPhO 2007, problem "pink". This problem covers binary stars, with a strong emphasis on data analysis methods, as covered in **P2**.

[5] **Problem 10.** APhO 2015, problem 2. A nice but somewhat hard to read problem on the aurora and the solar wind.

Solution. See the official solutions here. But also note that there are some minor typos in it, as pointed out here.

3 Compact Objects

Compact objects such as white dwarfs and neutron stars must be handled with quantum statistical mechanics, as introduced in **X1**.

[4] **Problem 11.** Do the following JPhO problem. This pedagogical problem reviews the physics of white dwarf stars, deriving the Chandrasekhar limit, using the techniques of **X1**.

Solution. See the official solutions here.

[3] **Problem 12.** USAPhO 2024, problem A2. Rough estimates of the dynamics of stars and white dwarfs.

Remark

The estimates performed in the previous problems are quite rough, basically treating the white dwarf as being homogeneous, with uniform density and pressure. In reality, we have $\nabla p = -\rho \mathbf{g}$ just like in any situation in hydrostatic equilibrium, where the degeneracy pressure p is determined by the local density n. It's like the gaseous atmospheres you dealt with in **T1**, but with a different equation of state.

If you additionally allow the white dwarf to have a net charge, then there is an additional contribution from the electrostatic force, and the resulting equations are called the Thomas–Fermi equations of structure. They can also be used to model many-electron atoms, when you can neglect the discreteness of the electrons.

[5] **Problem 13.** GPhO 2016, problem 3. This problem covers the physics of intense magnetic fields in neutron stars, using the results of **E8** and **X1**.

Solution. See the official solutions here.

[3] **Problem 14.** The Bekenstein–Hawking formula states that a black hole has an entropy of

$$S = \frac{A}{4}$$

where A is the area of its event horizon. The radius of an uncharged, nonrotating black hole is

$$R = 2M.$$

In these equations, $\hbar$, c, and G have all been set to one; you do not need to restore these factors.

(a) Compute the temperature and heat capacity of such a black hole.

(b) Two uncharged, nonrotating black holes begin very far apart from each other, then merge into a single black hole, emitting gravitational waves in the process; assume there is no initial angular momentum, so the final black hole is nonrotating as well. Find the maximum possible efficiency of this process, defined as the fraction of the initial energy that is converted into gravitational waves, for any set of initial black hole masses.

(c) The most interesting thing about the black hole entropy formula is that it scales with area, while most ordinary entropies scale with volume. Here's a *very* handwavy argument for this claim, due to Lenny Susskind. When a black hole absorbs a photon, its entropy increases because the photon could have been absorbed in different places on the event horizon. In the limiting case where the photon's wavelength matches the Schwarzschild radius, the absorption takes place everywhere, so the only entropy increase is by 1 "bit", namely whether absorption happened at all. Finish up this argument to conclude that $S \propto A$.

It may also be useful to review IPhO 2007, problem "blue", which we originally covered in **P1**.

Solution. (a) The temperature can be found with $T = dE/dS$, and the energy is simply $E = M$ since we're setting $c = 1$. We have $S = A/4 = \pi R^2 = 4\pi M^2$, giving $M = \sqrt{S/4\pi}$. Thus

$$T = \frac{1}{2\sqrt{4\pi S}} = \frac{1}{8\pi M}.$$

The heat capacity can be found with $C = dE/dT$. Using our answer for T, we have $E = 1/(8\pi T)$, giving a heat capacity of

$$C = -\frac{1}{8\pi T^2} = -8\pi M^2.$$

(b) By the second law of thermodynamics, the total change in entropy ΔS must be greater than or equal to zero. At the maximum efficiency, $\Delta S = 0$ so the total entropy of the two black holes must add up to the entropy of the final black hole:

$$S_1 + S_2 = S_f \implies M_1^2 + M_2^2 = M_f^2.$$

All the gravitational energy is included in the mass of the black holes, so the energy that went into the gravitational waves will be $E_w = M_1 + M_2 - M_f$ and $M_1 + M_2$ is the initial energy, which gives an efficiency of

$$\eta = 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2}.$$

This is maximized when $M_1 = M_2$, giving

$$\eta = 1 - \frac{\sqrt{2}}{2} = 0.29.$$

(c) If we have $\lambda \sim R$, then the energy of the photon must be $E \propto 1/R$. The change in radius of the black hole is $dR \propto dM = E$. So, to increase the entropy by one bit, we increment the radius by $dR \propto 1/R$, which is equivalent to incrementing the area $dA \propto R\,dR$ by a constant amount. We thus conclude that $S \propto A$.

[4] **Problem 15.** US TST 2022, problem 3. Rough estimates of gravitational wave emission.

Solution. See the official solutions here.

[3] **Problem 16.** NBPhO 2017, problem 4. Rough estimates of gravitational wave detection.

Solution. See the official solutions here.

Remark

The discovery of gravitational waves by LIGO has been one of the most important results this decade, so it's naturally a popular question topic. Once you finish the above problems, you can check out a few others with a different take on the same idea. GPhO 2016, problem 2 (solutions here) does a rougher treatment of gravitational wave emission, while IPhO 2018, problem 1 gives a more accurate treatment using more of the language of general relativity. For another way to estimate gravitational wave emission, see section 9.3 of The Art of Insight, and for some followup questions, see this paper.

Remark

In 1931, after building a sensitive short-wave radio receiver, Karl Jansky heard an unusual noise on his receiver from a direction that moved across the sky about once a day. He therefore initially thought it was from the Sun. However, over time he found that the direction of the noise moved across the sky only once every 23 hours and 56 minutes.

This was a huge difference! To understand why, note that the length of a day is the time it takes for the same side of the Earth to face the Sun again; it depends on both the Earth's spin and its orbital motion about the Sun. The period of the Earth's spin alone, the so-called sidereal period, is only 23 hours and 56 minutes. Thus, a signal with this period indicates an origin from outside the solar system. Jansky later found that the source was the center of the galaxy; today we know it is due to the supermassive black hole there, Sagittarius A*. The early days of radio astronomy were full of dramatic discoveries like this. To hear about the discovery of pulsars, see this talk.

4 Cosmology

Cosmology is a rather technical topic because a proper treatment requires general relativity, but one can derive special cases of some of the results using just Newtonian gravity.

[2] **Problem 17.** AuPhO 2014, problem 14. A quick problem on the basics of dark matter and galaxy measurements.

Solution. See the official solutions here.

[5] **Problem 18.** GPhO 2022, problem 1. More about dark matter.

Solution. See the official solutions here.

[5] **Problem 19.** APhO 2016, problem 2. This straightforward problem introduces the basic equations of cosmology, such as the Friedmann equation.