Special Topics Review

There is a total of **88** points.

1 Thermodynamics

[**3**] **Problem 1.** A photon of energy E bounces between two mirrors separated by a distance L.

(a) If the mirrors are slowly moved together to a distance $L/2$, find the final energy E by considering the blueshift the photon experiences at every collision.

(b) Check this result agrees with the adiabatic theorem.

(c) Using your result, infer the value of γ for a one-dimensional photon gas. Can you also infer the value of γ for a d-dimensional photon gas?

Solution.   (a) If the left mirror has speed v, then the Doppler shift factor on every reflection is $1 + 2v/c$, so $\Delta E = (2v/c)E$ because $E = \hbar\omega$. We also have $\Delta x = -(2x/c)v$, which implies

$$\frac{\Delta E}{\Delta x} = -\frac{E}{x}.$$

Treating this as a differential equation $dE/dx = -E/x$ shows that Ex is conserved. (Note that the adiabatic theorem works fine in this case, because the photon is always moving quickly relative to the walls, so every collision has a small effect.)

Another valid, equivalent approach would be to note that the energy gets an overall Doppler shift of $(1 + 2v/c)^N$ where N is the number of collisions with the moving wall, and compute N directly.

(b) The adiabatic invariant is

$$\oint p\,dx = \frac{E}{c}(2L) \propto Ex$$

giving the same result, using $E = pc$.

(c) The pressure exerted by each photon is

$$P = \frac{\Delta p}{\Delta t} = \frac{2E/c}{2L/c} = \frac{E}{x}.$$

Therefore, Px^2 is constant which gives $\gamma = 2$. This is exactly as expected, since $C_v = kT$.

In d dimensions, much of the same reasoning holds. The quantity Ex is still invariant, where x is the side length of, say, a cube. However, the pressure is now $P \propto E/V$. Then $PVx = PV^{1+1/d}$ is constant, so $\gamma = 1 + 1/d$. For example, when $d = 3$ we have $\gamma = 4/3$, as we found in **T1** and **T2**.

Note that the reasoning here works equally well if you don't know about photons, because electromagnetic waves redshift and blueshift in the same way. By using this reasoning, i.e. by thinking about how successive Doppler shifts change the frequency of light, Wien was able to derive the original form of Wien's displacement law.

[**3**] **Problem 2.** Let n be the local density of a gas of particles. If this density is nonuniform, it will tend to be smoothed out by diffusion, which produces a particle current

$$\mathbf{J} = -D\,\nabla n$$

with units of particles per second per unit area. This is a continuity equation, as explained in **T2**, and D is called the diffusion coefficient. In addition, we know from **M7** and **T1** that each particle in such a gas will experience a drag force from the others. As a result, when a constant force $\mathbf{F}$ is applied to one, its terminal velocity satisfies $\mathbf{F} = \mu\mathbf{v}$.

(a) Suppose the particles are placed inside a potential $V(\mathbf{r})$. Write down the contribution to the particle current due to the resulting force, neglecting diffusion.

(b) In thermal equilibrium, the particle current produced by the force, which pulls the particles to lower potential, balances the particle current produced by diffusion, which spreads them out. Assuming the temperature is T, find a relation between D and μ.

Solution.   (a) The force is $\mathbf{F} = -\nabla V$, which means $\mathbf{v} = -(\nabla V)/\mu$. The particle current is

$$\mathbf{J} = n\mathbf{v} = -\frac{n}{\mu}\,\nabla V.$$

(b) By setting the sum of the two currents to zero,

$$\frac{n}{\mu}\,\nabla V = -D\nabla n.$$

Rearranging gives

$$\frac{1}{\mu D}\,\nabla V = -\nabla(\log n).$$

On the other hand, we know that $n(\mathbf{r}) \propto e^{-V(\mathbf{r})/k_B T}$ because the steady state must obey the Boltzmann distribution. Then we have $\log n = -V/k_B T + \text{const}$, which means

$$k_B T = \mu D.$$

Incidentally, this same result is derived in IPhO 2023, problem 1, which you can see for comparison.

Remark

The result of problem 2 is called the Einstein relation, and is one of the four major results Einstein derived in his "annus mirabilis" of 1905, the others being special relativity and $E = mc^2$, and explaining the photoelectric effect with photons. So why is this result so important? It's because at the time, it was not yet completely accepted that matter was made of atoms. As you saw in **T1**, the size of a single atom often drops out of kinetic theory calculations. At the time, many took this to mean that atoms were a fictitious calculational tool, like how one does integrals by summing over intervals of length Δx, then gets a result independent of this fiducial length by taking $\Delta x \to 0$.

But doesn't the number of atoms N appear all the time in basic thermodynamics, like the ideal gas law? Yes, but always in the combination $nR = Nk_B$. Thus, information about

the size of a single atom is equivalent to information about k_B. Einstein's relation is useful because it explicitly gives us k_B in terms of the separately directly measurable quantities D, μ, and T. It was one of the first unique, quantitative predictions of kinetic theory.

[**4**] **Problem 3** (Physics Cup 2018). Estimate the mean free path of a heavy black sphere of mass m and radius R in vacuum at temperature T. Here, we define the mean free path as the typical distance it takes for the velocity vector of the sphere to turn by an angle $\pi/2$. Assume that $k_B T R \gtrsim \hbar c$. (Hint: for a random walk where steps of size a are taken per time τ, the average overall displacement after time $t \gg \tau$ is approximately $a\sqrt{t/\tau}$.)

Solution. By the equipartition theorem, the typical speed v_0 of the sphere satisfies

$$mv_0^2 \sim k_B T.$$

The velocity vector is rotated by impulses from the photons, as they are absorbed or emitted by the sphere. Thus, we need to estimate the rate of collisions as well as the typical impulse of each one. The typical energy of a photon in blackbody radiation of temperature T is $E \sim k_B T$ by the equipartition theorem again, which means each one absorption of emission changes the velocity of the sphere by

$$\Delta v = \frac{\Delta p}{m} = \frac{E}{mc} \sim \frac{k_B T}{mc}.$$

On the other hand, we have

$$v_0 \sim \sqrt{\frac{k_B T}{m}}$$

which is much larger than v_0 since we are assuming the sphere is heavy. Using the provided hint, the number of photons required to change the velocity significantly is order

$$N \sim \left(\frac{v_0}{\Delta v}\right)^2 \sim \frac{mc^2}{k_B T}.$$

Now we need to find the rate at which photons are absorbed or emitted. In equilibrium, these two rates will be equal, so we can focus on emission, which is easier to handle. The power emitted from the sphere is

$$P = 4\pi R^2 \sigma T^4$$

and we already know the photons emitted have typical energy $k_B T$, so the time per photon is

$$\tau \sim \frac{k_B T}{P} \sim \frac{k_B}{\sigma R^2 T^3}.$$

Combining our results, the mean free path is

$$\lambda \sim v_0 N \tau \sim \frac{k_B^{1/2} m^{1/2} c^2}{\sigma R^2 T^{7/2}}.$$

Incidentally, you might wonder how we used the assumption $k_B T R \gtrsim \hbar c$. This assumption is equivalent to saying that the wavelength of a typical photon is smaller than the size of the sphere itself. We used this implicitly by thinking of the photons as little bullets hitting the sphere. If it weren't true, then we would have to account for their wave nature.

[3] **Problem 4.** Consider a layer of the atmosphere with density ρ, pressure P, adiabatic index γ, and density and pressure gradients $d\rho/dz$ and dP/dz. Suppose that a small parcel of air in this layer acquires a small upward velocity. Under certain conditions, the parcel of air will begin oscillating in height, performing simple harmonic motion. Neglecting drag and heat transfer between the parcel of air and its surroundings, find the angular frequency ω of this motion. This is called the Brunt–Vaisala or buoyancy frequency.

Solution. Let the parcel have volume V and start at position z_0. After moving up by Δz, the weight of the parcel is $\rho_p(z_0 + \Delta z)gV$ and the buoyant force is $\rho(z_0 + \Delta z)gV$, so the net force is

$$F = (\rho(z_0 + \Delta z) - \rho_p(z_0 + \Delta z))gV$$

which means that for small Δz, the acceleration is

$$a = \frac{g}{\rho}\left(\frac{d\rho}{dz} - \frac{d\rho_p}{dz}\right)\Delta z.$$

To calculate $d\rho_p/dz$, note that since the parcel expands and contracts adiabatically, $P_p \propto \rho_p^\gamma$. Moreover, the pressure of the parcel is always equal to the surrounding atmosphere, so

$$P \propto \rho_p^\gamma$$

which for small displacements means

$$\frac{1}{P}\frac{dP}{dz} = \gamma\frac{1}{\rho}\frac{d\rho_p}{dz}.$$

Plugging this into the acceleration equation gives

$$a = g\left(\frac{1}{\rho}\frac{d\rho}{dz} - \frac{1}{\gamma P}\frac{dP}{dz}\right)\Delta z$$

which implies

$$\omega = \sqrt{g\left(\frac{1}{\gamma P}\frac{dP}{dz}\right) - \frac{1}{\rho}\frac{d\rho}{dz}} = \sqrt{g\frac{d\log(P^{1/\gamma}/\rho)}{dz}}.$$

When the argument of the square root becomes negative, there is no frequency of oscillation; instead, the parcel just keeps on going. That describes an atmosphere unstable against convection. As a check, marginal stability occurs for an adiabatic atmosphere, where $P \propto \rho^\gamma$. In realistic atmospheres, you wouldn't expect to see these kinds of oscillations because of damping effects, but they provide a characteristic timescale for dynamic processes.

[3] **Problem 5.** ◔ USAPhO 2021, problem B2. A conceptual problem on methods of heat transfer.

[3] **Problem 6.** ◔ USAPhO 2019, problem B2. A useful problem for getting comfortable with numbers and estimates in astronomy.

[3] **Problem 7.** ◔ USAPhO 2022, problem B1. A data analysis problem about a nonideal gas.

[3] **Problem 8.** ◔ INPhO 2019, problem 6. A thermodynamic cycle with a nonideal gas.

Solution. See the official solutions here.

[5] **Problem 9.** ◔ IPhO 2011, problem 2. A problem on an electrified soap bubble, which combines electrostatics, thermodynamics, and surface tension.

2   Relativity

[5] **Problem 10.** ⊘ IPhO 1998, problem 3. A great problem on a real controversy in physics, which also gives you practice working with real data.

[4] **Problem 11** (Physics Cup 2018). A spaceship travels with a constant proper acceleration g along a straight line. At a certain moment, it launches two missiles in the direction of its motion, with speeds v and $2v$. Find the proper time interval in the spaceship between catching up to the first and to the second missile.

Solution. See the official solutions here.

[2] **Problem 12.** In classical physics, light waves do not interact with each other, but they can due to quantum mechanical effects. Suppose two photons traveling in opposite directions scatter off each other. Initially the photons have wavelengths λ_1 and λ_2. One of the two outgoing photons exits at an angle θ to the first incoming photon. Find its wavelength λ in terms of λ_1, λ_2, and θ.

Solution. We aren't told anything about the other outgoing photon, so a direct solution would require introducing several variables to describe it, which is messy. We can avoid this using four-vectors. Let the incoming photons have four-momenta p_1^μ and p_2^μ, and let the outgoing photons have four-momenta p^μ and p'^μ. Then we can neatly eliminate p' using

$$p'^2 = 0 = (p_1 + p_2 - p)^2 = 2(p_1 \cdot p_2 - p_1 \cdot p - p_2 \cdot p)$$

since photons are massless. Using $E = h/\lambda$ and plugging in the four-momenta, we have

$$p_1 \cdot p_2 = \frac{2h^2}{\lambda_1 \lambda_2}, \quad p \cdot (p_1 + p_2) = \frac{h^2}{\lambda}\left(\frac{1 - \cos\theta}{\lambda_1} + \frac{1 + \cos\theta}{\lambda_2}\right)$$

and solving for λ yields

$$\lambda = \frac{\lambda_1(1 + \cos\theta) + \lambda_2(1 - \cos\theta)}{2}.$$

As a check, we get back λ_1 when $\theta = 0$, corresponding to no collision.

[3] **Problem 13** (MPPP 195). The pion π^+ is a subatomic particle with mass m_π. In one of its possible decay modes, it decays into a positron e^+ of mass m_e and a massless electron-neutrino ν_e. What is the minimum speed of the pion if, following its decay, the positron and neutrino move at right angles to each other? Numerically evaluate this speed in the limit $m_e \ll m_\pi$.

Solution. Let the pion have energy E_π and momentum $\mathbf{p}_\pi$, with similar notation for the electron and neutrino. We know that

$$m_\pi^2 = E_\pi^2 - p_\pi^2.$$

On the other hand, by energy and momentum conservation,

$$E_\pi = E_e + E_\nu, \quad \mathbf{p}_\pi = \mathbf{p}_e + \mathbf{p}_\nu$$

and plugging this in above gives

$$m_\pi^2 = (E_e + E_\nu)^2 - (\mathbf{p}_e + \mathbf{p}_\nu) \cdot (\mathbf{p}_e + \mathbf{p}_\nu).$$

Since the positron and neutrino move at right angles to each other, this simplifies to

$$m_\pi^2 = E_e^2 + 2E_e E_\nu + E_\nu^2 - p_e^2 - p_\nu^2 = m_e^2 + m_\nu^2 + 2E_e E_\nu.$$

Since the neutrino is massless,

$$E_e E_\nu = \frac{m_\pi^2 - m_e^2}{2}.$$

Minimizing the speed of the pion is equivalent to minimizing its energy E_π. On the other hand, the equation above states that the geometric mean of E_e and E_ν is fixed. Thus, the minimum arithmetic mean occurs when they are equal, in which case

$$E_\pi = 2\sqrt{E_e E_\nu} = \sqrt{2(m_\pi^2 - m_e^2)}.$$

We also know that

$$E_\pi = \frac{m_\pi}{\sqrt{1 - v^2}}$$

so solving for v gives

$$v \geq \sqrt{\frac{m_\pi^2 - 2m_e^2}{2m_\pi^2 - 2m_e^2}}.$$

In the limit $m_e \ll m_\pi$, this reduces to $v \geq c/\sqrt{2}$.

[3] **Problem 14.** ⏸ USAPhO 2023, problem B3. A nice, qualitative problem on supernova neutrinos.

[3] **Problem 15** (Purcell 6.68). Consider two electrons moving side-by-side with parallel velocities, with speed v and separation r. We wish to compute the three-force between the electrons.

(a) Compute this force by working in the electrons' rest frame and Lorentz transforming back to the lab frame.

(b) Compute this force by using the electric and magnetic fields of a moving charge, and verify the answer agrees with that of part (a).

(c) What happens to the three-force as $v \to c$?

Solution. (a) The answer in the rest frame is ke^2/r, so transforming back gives $ke^2/\gamma r$. (For transformations of three-forces, it's helpful to remember that the force is always largest in the rest frame of the particle.)

(b) In this frame, the fields of one charge at the other are $E = \gamma ke^2/r$ and $B = vE/c^2$. Then

$$F = eE - evB = \frac{eE}{\gamma^2} = \frac{1}{\gamma}\frac{ke^2}{r},$$

which matches the result of part (a).

(c) In this limit, the force goes to zero. The electric and magnetic forces balance, as you saw in a question in **E4**.

3 Waves

[2] **Problem 16** (French 7.23). One end of a stretched string is moved transversely at constant velocity u for a time τ, and is moved back to its starting point with velocity $-u$ during the next interval τ. As a result, a triangular pulse is set up on the string and moves along it with speed v. Show that the total energy of the pulse is equal to the work done on the string, working to lowest order in u/v.

Solution. First let's compute the energy in the pulse. Recall from **W1** that the energy density is

$$u = \frac{dE}{dx} = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \frac{T}{2}\left(\frac{1}{v^2}\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2\right).$$

The triangular pulse has height $u\tau$, and the two halves of it have length $v\tau$. Thus, $|\partial y/\partial t| = u$ and $|\partial y/\partial x| = u/v$ across the pulse, so the total energy is

$$U = (2v\tau)u = \frac{2T\tau u^2}{v}.$$

When lifting the string to create the pulse, the string was at an angle of u/v to first order in u/v, so the transverse force that needed to be applied was Tu/v. The distance over which this force was applied was $u\tau$, for a total work of $T\tau u^2/v$. The same work was done when bringing it down with constant velocity, so the total work done was $2T\tau u^2/v$, as expected.

[3] **Problem 17.** EFPhO 2015, problem 2. A very nice combined interference and optics problem.

Solution. See the official solutions here.

[3] **Problem 18** (Crawford 4.15). Consider a jug with a large volume V, along with a thin neck of length ℓ and area A. The lowest standing wave frequency can be excited by blowing across the neck. Such a system is called a Helmholtz resonator. Naively, the corresponding wavelength would be four times the length of the jug, but it is observed to be much larger. (Try it at home!)

Since the jug has a neck, the standing wave profile looks very different from a standard profile. Most of the air motion is within the neck; the body of the jug serves as a large air reservoir that acts as a spring pushing back against this motion. The frequency can be quite low, because this reservoir is large.

(a) Show that if the air in the neck moves by a distance x, the restoring force is

$$F = -\frac{\gamma P_0 A^2 x}{V}$$

where P_0 is the original pressure in the jug, and γ is the adiabatic index.

(b) Show that this mode has a frequency of

$$\omega = \frac{v}{\ell}\sqrt{\frac{V_{\text{neck}}}{V}}, \quad V_{\text{neck}} = A\ell$$

where v is the speed of sound of air. Thus, the frequency is significantly lower than the naive answer $\omega \sim v/\ell$ one would get by considering standing waves in a cylinder shaped like the jug.

(c) See if this is roughly consistent with a real jug (e.g. a 1 liter soda bottle). Does the frequency vary as you'd expect as you add water to the jug? If you feel musically inclined, can you find how to excite higher frequencies? (I can't do it, but a wind instrument player probably can.)

Solution.   (a) Since PV^γ is constant, $\Delta P/P = -\gamma \Delta V/V$. Since $\Delta V = Ax$, we have

$$F = -A\Delta P = \frac{P_0 A^2 x}{V},$$

as desired.

(b) Consider the air originally in the neck. It oscillates back and forth, feeling an "effective spring constant" $k_{\text{eff}} = \gamma P_0 A^2/V$ by the result of part (a), and it has mass $m = \rho A \ell$. Thus,

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{\gamma P_0 A}{\rho V \ell}} = \frac{1}{\ell}\sqrt{\frac{\gamma P_0}{\rho}}\sqrt{\frac{A\ell}{V}}$$

which matches the desired result.

(c) In my personal experience, this does match pretty well with real jugs. Unfortunately, I can barely manage to excite the fundamental, but wind instrument players might have better luck.

> **Remark**
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> The results of problem 18 can also be used to describe window buffeting, the annoying "whuppa, whuppa, whuppa" sound you get when you slightly open one window of a rapidly moving car. Suppose the opening has area A and the car has volume V. Then the incoming air moves quickly up to a depth $\ell \sim \sqrt{A}$ in your car, and this region functions as the "neck". The resonant frequency is therefore $\omega \sim vA^{1/4}/V^{1/2}$. You can stop the buffeting by opening the window more, increasing A so the frequency is too high to be efficiently driven by the air.

> **Remark**
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> The Nigerian udu is a musical instrument which can be modeled as a Helmholtz resonator with two holes. The holes have different sizes, so you get a frequency f_1 if you cover one hole, and f_2 if you cover the other.
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> You can get a third frequency by hitting the instrument with both holes open. To find that third frequency, note that a Helmholtz resonator is like a spring-mass system, where the "spring" is the air in the instrument, and the "mass" m_i is the air near hole i, which moves back and forth. When both holes are open, we effectively have a spring with masses attached at both ends, which oscillates according to the reduced mass $1/\mu = 1/m_1 + 1/m_2$. Since $f_i \propto 1/\sqrt{m_i}$, the third frequency is $\sqrt{f_1^2 + f_2^2}$.

[**3**] **Problem 19.** 🕓 USAPhO 2008, problem A4. A neat Doppler shift problem that also tests your data analysis skills.

[**4**] **Problem 20.** 😐 INPhO 2020, problem 4. A nice and tricky problem on the two-dimensional Doppler effect. (Incidentally, this problem was also on the 2013 Russian Physics Olympiad, but I'm linking to this one because it's in English.)

Solution. See the official solutions here.

[3] **Problem 21.** ⟲ USAPhO 2022, problem A3. A great problem on the rainbow caustic. For some brilliant rainbow graphics, see here and here.

4 Modern

[3] **Problem 22.** Consider a typical small LC circuit, with $L = 10^{-7}$ H and $C = 10^{-11}$ F.

(a) If the circuit is at zero temperature, estimate the voltage uncertainty across the capacitor.

(b) Do the same if the circuit is at room temperature.

Today, some of the leading quantum computing hardware is based on such circuits. Because thermal fluctuations would ruin the desired quantum mechanical effects, the circuit must be cooled so that $k_B T \ll \hbar\omega$, which corresponds to a small fraction of a degree. When you see pictures of quantum computers, most of what you're looking at is the fridge!

Solution. (a) In this problem, it's not clear how to use Heisenberg's uncertainty principle, because we don't know what the "position" and "momentum" for an LC circuit are. (It turns out the position variable is the flux through the inductor and the momentum variable is the charge on the capacitor, but this is a bit hard to guess.) But we do know that an LC circuit behaves just like a harmonic oscillator, so when it is quantized, it has energy $\hbar\omega/2$ in the ground state. This must manifest in terms of voltage and current uncertainties,

$$\hbar\omega \sim C(\Delta V)^2 + L(\Delta I)^2$$

where we drop all constants for brevity. We know from an example in **X1** that for the harmonic oscillator, these two contributions are equal. Therefore, by order of magnitude we have

$$\Delta V \sim \sqrt{\frac{\hbar\omega}{C}}, \quad \omega = \frac{1}{\sqrt{LC}}.$$

If you didn't manage to do the steps above, this final result can also be found by pure dimensional analysis. Plugging in the numbers gives $\Delta V \sim 10^{-7}$ V, which is surprisingly high!

(b) At zero temperature, the energy in the circuit is of order $\hbar\omega$, while at room temperature, it's $k_B T$ by the equipartition theorem. Plugging in the numbers, we have

$$\frac{k_B T}{\hbar\omega} = 4 \times 10^4.$$

Since energy is proportional to V^2, the voltage uncertainty is 200 times higher,

$$\Delta V \sim 2 \times 10^{-5} \text{ V}.$$

This is the typical noise you expect to see on a good oscilloscope. In electrical engineering, this kind of thermal noise is called "Johnson noise".

[2] **Problem 23.** INPhO 2013, problem 2. A short problem on X-ray diffraction.

Solution. See the official solutions here.

To finish up, here are three neat questions that each cover a broad range of topics.

[**5**] **Problem 24.** 🕑 IPhO 2009, problem 3. Estimating the size of a star from scratch.

[**5**] **Problem 25.** 🕑 IPhO 2021, problem 3. Molecular fluorescence and optical lattices.

[**5**] **Problem 26.** 🕑 USAPhO+ 2021, problem 2. A full analysis of a real dark matter detector.

> **Remark**
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> Problem 24 estimates the size of a star using a rough treatment of the nuclear physics we know applies at its core. When you think about it, it's amazing that this is possible at all. There is essentially no way to directly probe what's going on inside any star. We can basically only measure the size of the star, its rough age, and the temperature of its surface. But the models work! This was the result of generations of painstaking effort, which stimulated progress in both nuclear and particle physics. To learn the history of solar modeling, see this article for the early days and this article for the eventual triumph. For more estimates of the sizes of astronomical objects, see this article.
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> It is even more remarkable when you realize that this sort of story applies to essentially all of the dozens of known types of astronomical objects. For most of these, astronomers can directly measure only the electromagnetic spectrum and its variation over time, but that little is enough to develop and test detailed physical models. Most people think astronomy is very concrete, because they constantly see stars in the night sky, and flashy "artist's impression" graphics on the internet. But the field is actually extremely theoretical, with every insight won through extensive calculation.