

# Electromagnetism VIII: Synthesis

Electromagnetism in matter is covered in chapters 10 and 11 of Purcell. For more on dielectrics, see chapters II-10 and II-11 of the Feynman lectures. Electromagnetism in matter is covered in greater detail in chapters 4, 6, and 9 of Griffiths, and chapters I-31 and II-32 through II-37 of the Feynman lectures. For an enlightening overview of the history of magnetism, see chapter 1 of *Magnetism and Magnetic Materials* by Coey. There is a total of **98** points.

## 1 Polarization

In **E2**, we introduced the basics of dielectrics. To review: when a dielectric is placed in an electric field, dipoles inside align with the field, reducing the field value. In very symmetrical situations, the field is simply reduced by a factor of the dielectric constant  $\kappa = \epsilon/\epsilon_0$ . The total energy density within a dielectric is  $\epsilon E^2/2$ . This section is about problems which require more than these few facts. To answer them, we need to think about the charge bound to the dielectric itself.

### Idea 1: Bound Charge

The polarization  $\mathbf{P}$  of a material is its electric dipole moment per unit volume. It corresponds to a “bound” charge density

$$\rho_b = -\nabla \cdot \mathbf{P}$$

within the dielectric, as well a bound surface charge density

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

on its surface.

### Example 1

Find the electric field of a sphere with uniform polarization  $\mathbf{P}$  and radius  $R$ .

### Solution

There is no bound charge density inside the sphere, but a bound surface charge density

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{r}} = P \cos \theta$$

on its surface. We could apply Coulomb’s law to this charge density, but an easier method is to recall that polarization just means an internal displacement of charge. This surface charge density precisely corresponds to having two uniformly charged balls of total charge  $\pm Q$  displaced by a tiny amount  $\mathbf{d}$  so that  $Q\mathbf{d} = (4\pi R^3/3)\mathbf{P}$ .

By the shell theorem, the resulting field inside is uniform, and points against  $\mathbf{P}$ ,

$$\mathbf{E} = -\frac{\mathbf{P}}{3\epsilon_0},$$

and the field outside is exactly a dipole field, with dipole moment  $\mathbf{p} = (4\pi R^3/3)\mathbf{P}$ .

[2] **Problem 1.** An infinite cylindrical rod of radius  $R$  has a uniform polarization  $\mathbf{P}$ .

- (a) If  $\mathbf{P}$  is perpendicular to the rod's axis, describe  $\mathbf{E}$  outside, and find the value of  $\mathbf{E}$  inside.
- (b) If  $\mathbf{P}$  is parallel to the rod's axis, find the electric field everywhere.

Now let's think about how polarization arises in the first place.

### Idea 2: Electric Susceptibility

A small number of insulators are “ferroelectrics”, whose crystal structure lets them maintain a preferred polarization  $\mathbf{P}$  in the absence of external electric fields. (You can suppose that the preceding example and problem were implicitly about ferroelectrics.) But the vast majority of insulators are dielectrics, whose polarization is related to the total electric field by

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0 \kappa$$

where  $\chi_e$  is the electric susceptibility and  $\kappa$  is the dielectric constant. The susceptibility is nonnegative, except in some very exotic materials. The key difficulty is that above,  $\mathbf{E}$  is the *total* electric field, including that due to the bound charge, which in turn depends on  $\mathbf{P}$ .

### Example 2

A point charge  $q$  is inside a dielectric sphere of radius  $R$  with dielectric constant  $\kappa$ . Find the electric field and charge density everywhere.

### Solution

This is one of the simple symmetric cases where the electric field in the dielectric is simply reduced by a factor of  $\kappa$ ,

$$\mathbf{E} = \frac{q \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \times \begin{cases} 1/\kappa & r < R \\ 1 & r > R \end{cases}.$$

Inside the sphere, this corresponds to an electric polarization

$$\mathbf{P} = \frac{q}{4\pi r^2} \frac{\chi_e}{\kappa} \hat{\mathbf{r}}.$$

To check that this solution is actually right, we need to ensure the original point charge  $q$ , plus the bound charge, indeed generates the claimed electric field.

The divergence of  $\mathbf{P}$  is zero everywhere besides the origin, where negative bound charge piles up to cancel some of the charge  $q$ . The charge at the origin is thus

$$q - q_b = q \left(1 - \frac{\chi_e}{\kappa}\right) = q \left(1 - \frac{\kappa - 1}{\kappa}\right) = \frac{q}{\kappa}$$

which is consistent with Gauss's law for  $\mathbf{E}$  there. At the surface of the sphere, there is a positive bound surface charge density

$$\sigma_b = \frac{q}{4\pi R^2} \frac{\chi_e}{\kappa}$$

which cancels the negative bound charge at the origin. Thus, by the shell theorem, the electric field outside the sphere is indeed that of the point charge  $q$  alone.

**Example 3**

A dielectric sphere of radius  $R$  and dielectric constant  $\kappa$  is placed in a uniform field  $\mathbf{E}_0$ , and as a result develops a uniform polarization  $\mathbf{P}$ . Find  $\mathbf{P}$  and the field inside the sphere.

**Solution**

In example 1, we found the electric field due to the polarized sphere itself, which we'll call  $\mathbf{E}_p$ . Here, we must remember that the polarization is produced in response to the *total* electric field inside the sphere,

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}, \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_p.$$

Using our previous result for  $\mathbf{E}_p$  and solving the system, we find

$$\mathbf{E} = \frac{3}{\kappa + 2} \mathbf{E}_0, \quad \mathbf{P} = 3 \frac{\kappa - 1}{\kappa + 2} \epsilon_0 \mathbf{E}_0.$$

The polarizability  $\alpha$  of each atom is defined as the dipole moment per applied field,

$$\mathbf{p} = \alpha \mathbf{E}_0$$

so we have shown above that

$$\alpha = \frac{3\epsilon_0}{n} \frac{\kappa - 1}{\kappa + 2}$$

where  $n$  is the number density of atoms. This is the Clausius–Mossotti formula; it relates the macroscopically measurable parameter  $\kappa$  to the microscopic parameter  $\alpha$ .

- [2] **Problem 2** (Purcell 10.10). Assume that the uniform field  $\mathbf{E}_0$  that causes the electric field in example 2 is produced by large capacitor plates very far away. The field lines tangent to the sphere hit each of the distant capacitor plates in a circle of radius  $r$ . Find  $r$  in terms of  $R$  and  $\kappa$ .

**Idea 3**

The “free” charge density  $\rho_f$  is the part of the charge density that isn't bound, so that

$$\rho = \rho_b + \rho_f.$$

If we take the divergence of  $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ , we get  $\rho_b = -\chi_e \rho$  inside a uniform dielectric, so

$$\rho = \rho_f / \kappa.$$

That is, a uniform dielectric “screens” charges embedded within it, reducing it by a factor of  $\kappa$ . That's exactly what we saw in example 2, and it also tells us that a conductor can be viewed as a dielectric with  $\kappa \rightarrow \infty$ , because conductors completely expel electric fields.

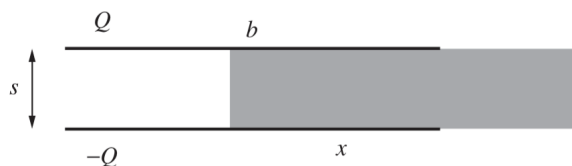
The difficulty in dealing with dielectrics is when  $\kappa$  changes in space, such as at the boundary of a dielectric, where a bound surface charge density  $\sigma_b$  can appear. If all the free charges in a problem are outside of uniform dielectrics, bound charges only appear on their surfaces.

- [3] **Problem 3.** A version of the method of images, introduced in **E2**, works for dielectrics. Let's suppose there is vacuum at  $z > 0$ , a dielectric  $\kappa$  at  $z < 0$ , and a point charge  $q$  a distance  $d$  above

the plane  $z = 0$ . We need to find the surface bound charge density  $\sigma_b$  that appears on the plane.

- Let  $E_0^z$  be the  $z$ -component of the electric field due to the point charge alone. At a given point just below the plane  $z = 0$ , find  $E^z$  in terms of  $E_0^z$  and  $\sigma_b$ .
- Use this result to solve for  $\sigma_b$  in terms of  $E_0^z$  and  $\kappa$ .
- Your answer will be exactly the same as what one gets for a conductor at  $z < 0$ , multiplied by a  $\kappa$ -dependent constant. Using this information, characterize the image charge and find the force on the real charge.

- [3] **Problem 4** (Purcell 10.2). A rectangular capacitor with side lengths  $a$  and  $b$  has separation  $s \ll a, b$ . It is partially filled with a dielectric with dielectric constant  $\kappa$ . The overlap distance is  $x$ .



The capacitor is isolated and has constant charge  $Q$ .

- What is the energy stored in the system?
  - Using the result of part (a), what is the force on the dielectric? Which direction does it point?
  - Is your answer to part (b) affected by the presence of fringe fields near the interface?
- [3] **Problem 5** (Griffiths 4.28). Two long coaxial cylindrical metal tubes of inner radius  $a$  and outer radius  $b$  stand vertically in a tank of dielectric oil, with susceptibility  $\chi_e$  and mass density  $\rho$ . The inner one is maintained at potential  $V$ , and the outer one is grounded. To what height  $h$  does the oil rise in the space between the tubes?

## 2 Magnetization

### Idea 4

As discussed in **E5**, materials contain two kinds of magnetic dipole moments: the “orbital” part, due to moving electrons, and the “spin” part, due to the electrons’ intrinsic magnetic moments. For most materials, in the absence of external magnetic fields, these dipole moments point in random directions, and thus sum to zero on average.

When such a material is placed in a magnetic field, two things happen at once:

- The spins partially align with the field, producing a net dipole moment along  $\mathbf{B}$ .
- The orbits are affected by the changing field in accordance with Lenz’s law, and thus produce a net dipole moment *against*  $\mathbf{B}$ .

These effects are often comparable in size. If the first is more important, the material is paramagnetic, and if the second is more important, it is diamagnetic.

This can be a bit tricky to remember, because it seems opposite to the definition of a dielectric, where the internal electric dipoles try to align *with* the external field. The reason it makes sense is that inside an electric dipole, the electric field points against the dipole moment, while inside a magnetic dipole, the magnetic field points with the dipole moment, as discussed in **E3**. So, both dielectrics and diamagnets try to reduce the applied field within them.

### Idea 5: Bound Currents

The magnetization  $\mathbf{M}$  of a material is its magnetic dipole moment per unit volume. It corresponds to a bound current density

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

as well as a surface bound current density

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

on its surface.

### Example 4

Find the magnetic field of a sphere with uniform magnetization  $\mathbf{M}$  and radius  $R$ .

### Solution

In this case  $\mathbf{J}_b$  is zero in the sphere, while at the sphere's surface,

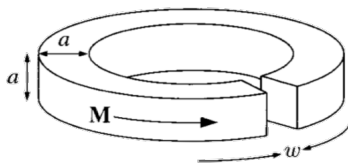
$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{r}} = M \sin \theta \hat{\boldsymbol{\phi}}$$

where we worked in spherical coordinates and aligned  $\mathbf{M}$  with the  $z$ -axis. However, this is precisely the current density of a rotating, uniformly charged sphere, as we discussed in **E3**. Scaling the constants appropriately, we find that inside,

$$\mathbf{B} = \frac{2}{3} \mu_0 \mathbf{M}$$

which should be compared with example 3. Outside, the field is exactly a magnetic dipole field, with  $\mathbf{m} = (4\pi R^3/3)\mathbf{M}$ .

- [2] **Problem 6.** An infinite cylindrical rod of radius  $R$  has a fixed, uniform magnetization  $\mathbf{M}$ .
- If  $\mathbf{M}$  is parallel to the rod's axis, find the magnetic field everywhere.
  - If  $\mathbf{M}$  is perpendicular to the rod's axis, describe  $\mathbf{B}$  outside, and find the value of  $\mathbf{B}$  inside.
- [2] **Problem 7** (Griffiths 6.10). A rod of length  $L$  and square cross section of side  $a$  is given a uniform longitudinal magnetization  $\mathbf{M}$  and then bent into a circle with a narrow gap of width  $w$ .



Find the magnetic field at the center of the gap, assuming  $w \ll a \ll L$ .

### Idea 6: Magnetic Susceptibility

Permanent magnets, such as the ones on your refrigerator, or the strong neodymium magnets popular in science toys, are made of “hard” ferromagnets. These are materials whose crystal structure lets them maintain a fixed magnetization  $\mathbf{M}$  in the absence of external magnetic fields. (The preceding example and problems were implicitly about hard ferromagnets.)

In most other materials, the magnetization is related to the magnetic field in the material by

$$\mathbf{M} = \frac{1}{\mu_0} \frac{\chi_m}{1 + \chi_m} \mathbf{B}, \quad \mu = \mu_0(1 + \chi_m) = \mu_0\mu_r$$

where  $\chi_m$  is the magnetic susceptibility,  $\mu$  is called the permeability, and  $\mu_r$  is called the relative permeability. We’ll explain later why  $\chi_m$  isn’t defined the same way as  $\chi_e$ .

- Diamagnets have  $-1 < \chi_m < 0$  and paramagnets have  $\chi_m > 0$ .
- Most materials are weakly diamagnetic ( $|\chi_m| \ll 1$ ), but some are weakly paramagnetic.
- As discussed in **E5**, a superconductor totally expels magnetic fields, and thus can be viewed as a “perfect diamagnet” with  $\chi_m = -1$  and hence  $\mu = 0$ .
- A “soft” ferromagnet (such as iron) is strongly paramagnetic, with  $\chi_m \gg 1$ .
- It is impossible to have  $\chi_m < -1$ , as then the energy density  $B^2/2\mu$  would be negative. The material would spontaneously develop arbitrarily large  $B$ , and blow itself up.

Because magnetization can arise from freely moving electrons, bound electrons orbiting, or the spin of electrons, these ideas can be applied to both conductors and insulators. As always, we must be careful to remember that  $\mathbf{B}$  is the total magnetic field, due to both whatever is outside the material, and the magnetization of the material itself.

### Remark: Estimating Susceptibility

Why is it that many common solids have  $|\chi_m| \ll 1$ , but  $\chi_e$  of order 1? Both the magnetic force on a moving electron, and the magnetic moment it produces are proportional to the electron’s speed  $v$ , so we would expect  $\chi_m \propto v^2$ . If one goes through the argument more carefully (which we won’t do here, as any proper derivation of  $\chi_m$  requires quantum mechanics), one finds that this is indeed the main factor at play. For a typical solid,

$$|\chi_m| \sim \left(\frac{v}{c}\right)^2 \sim \alpha^2 \sim 10^{-4}$$

where  $\alpha$  is the fine structure constant introduced in **P1**. You can check this is true from

standard tables. Of course, gases have even lower  $|\chi_m|$ , because they're mostly empty space.

### Example 5

An infinite solenoid with  $n$  turns per length and current  $I$  is filled with material with magnetic susceptibility  $\chi_m$ . Find the magnetic field inside.

### Solution

The magnetic field inside has contributions from the solenoid wire and the magnetization,

$$B = \mu_0(nI + M) = \mu_0 nI + \frac{\chi_m}{1 + \chi_m} B.$$

Solving for  $B$  yields

$$B = (1 + \chi_m)nI = \mu nI$$

which can be a significant enhancement if the material is a soft ferromagnet.

- [2] **Problem 8.** A sphere of magnetic susceptibility  $\chi_m$  is placed in a uniform field  $\mathbf{B}_0$ , and as a result develops a uniform magnetization  $\mathbf{M}$ . Find  $\mathbf{M}$  and the field  $\mathbf{B}$  inside the sphere. Then check the limiting cases of a superconductor and a soft ferromagnet. Do they make sense?

### Idea 7: The $\mathbf{H}$ Field

Historically, magnetism was formulated in terms of the field

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}.$$

Using this field gives a close analogy between electrostatics and magnetostatics. Note that

$$\nabla \times \mathbf{H} = \mathbf{J} - \mathbf{J}_b$$

so that  $\mathbf{H}$  has no curl if there are no currents around besides the bound current. (This extra current is sometimes called “free” current  $\mathbf{J}_f$ .) In addition,

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

which is analogous to how a polarization yields a charge density,  $\nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P}/\epsilon_0$ . Therefore, any magnetostatic problem without free current can be mapped to an electrostatic one via

$$\mathbf{E} \leftrightarrow \mathbf{H}, \quad \mathbf{P} \leftrightarrow \mu_0 \mathbf{M}, \quad \epsilon_0 \leftrightarrow \mu_0, \quad \rho_b \leftrightarrow \rho_m$$

where  $\rho_m = -\nabla \cdot \mathbf{M}$  is the “magnetic charge density”.

This is the mathematical formalization of the idea of Gilbert dipoles, introduced in **E3**, which replace a true magnetic dipole with a pair of fictitious magnetic charges. At the time, we remarked that this gives you the correct magnetic field outside of a magnet, but not inside. The underlying reason is this analogy is actually computing  $\mu_0 \mathbf{H}$ , not  $\mathbf{B}$ . To get the correct  $\mathbf{B}$  within a magnetized material, we have to compute  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ . Heuristically,

$\mu_0\mathbf{M}$  is the magnetic field due to dipole moments *right* at that location, while  $\mu_0\mathbf{H}$  is the contribution from all other currents and magnetic dipole moments.

Magnetic quantities were originally defined to be simple in terms of  $\mathbf{H}$ , with

$$\mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H}.$$

In fact, some even call  $\mathbf{H}$  “the” magnetic field, but we’ll just call it “ $\mathbf{H}$ ”. As always,  $\mathbf{H}$  here is the *entire*  $\mathbf{H}$  field, due to both the material itself and the field the material is in. Note that a soft ferromagnet can be thought of as a material within which  $\mathbf{H}$  is approximately zero.

### Remark: The History of $H$

If you learned high school physics in the United States, you might have found the preceding idea completely unfamiliar, but if you were educated elsewhere, such as in Eastern Europe or India, you might be surprised that this isn’t the *only* way to think about magnets.

The difference comes down to history. First, it’s worth noting that the choice between  $\mathbf{B}$  and  $\mathbf{H}$  isn’t obvious. Today we would say the  $\mathbf{B}$  field is more fundamental, because it is what determines the Lorentz force on a charge, and the force and torque on a dipole moment, and thereby the reading on a magnetometer. But in fact, for most of history, one could just as well argue that it is  $\mu_0\mathbf{H}$  that determines these things. After all, we can’t embed a measuring device within a magnet. Doing so would require hollowing out a hole, which would remove  $\mathbf{M}$  there, which would render  $\mathbf{B}$  just equal to  $\mu_0\mathbf{H}$ .

Therefore, choosing between  $\mathbf{B}$  and  $\mu_0\mathbf{H}$  requires measuring some interaction where particles pass *through* a magnet. In the 1930s, this became possible with neutron scattering, since neutrons are electrically neutral but carry a magnetic dipole moment  $\mathbf{m}$ . As discussed [here](#), the Nobel laureates Bloch and Schwinger disagreed on whether the interaction energy was  $-\mathbf{m} \cdot \mathbf{B}$  or  $-\mathbf{m} \cdot (\mu_0\mathbf{H})$ , which corresponds to treating  $\mathbf{m}$  as Amperian or Gilbertian, respectively. The controversy remained open for decades, but was eventually settled by data.

At the same time, there was a growing consensus in the United States that magnetic poles were a “[useless concept](#)” which should be banished from teaching entirely. So, when the modern American physics curriculum was formed in the 1960s, that’s exactly what happened. For example, Halliday, Resnick, and Krane spends only a few pages covering magnetization, and half of them are spent admonishing the reader that magnetic poles don’t exist. If you were educated in America, you probably only heard about them for a day in middle school.

Unfortunately, removing magnetic poles from the curriculum has some real costs. Experimentalists still think in terms of poles and  $\mathbf{H}$ , because it’s harder to visualize how currents source  $\mathbf{B}$ . Moreover, it’s harder to do anything with  $\mathbf{B}$  without vector calculus background. The Eastern European curriculum tends to value connection with experiment more, and defers calculus to later, and so it starts with magnetic poles. (Unfortunately, some teachers don’t clearly distinguish between  $\mathbf{B}$  and  $\mu_0\mathbf{H}$ , leading to confusion down the line.)



In this handout, I'll tell you only what you need to know about  $\mathbf{H}$  to solve theoretical problems. If you try to dive deeper into how experimentalists use it, you'll run into a lot more historical cruft. For instance, they tend to prefer the “Gaussian” system, where factors of  $4\pi$  are inserted into Maxwell's equations to make the Coulomb and Biot–Savart laws slightly simpler. Also, they use “cgs” (centimeter-gram-second) units, so all units need to be rescaled by some number of powers of 10. Worst of all, they have distinct units for  $\mathbf{B}$  (Gauss),  $\mathbf{H}$  (Oersted), and  $\mathbf{M}$  (emu/cm<sup>3</sup>), which have totally different conversions back to SI units. I wouldn't recommend learning any of this unless you have to for your job.

### Example 6

Using the  $\mathbf{H}$  field, recompute the magnetic fields inside a uniformly magnetized sphere, and a rod magnetized parallel to and perpendicular to its axis.

### Solution

In the first section, we found that inside these objects,

$$\mathbf{E} = \begin{cases} -\mathbf{P}/3\epsilon_0 & \text{sphere,} \\ -\mathbf{P}/2\epsilon_0 & \text{cylinder, perpendicular} \\ 0 & \text{cylinder, parallel} \end{cases}$$

Using the analogy above, we immediately conclude

$$\mathbf{H} = \begin{cases} -\mathbf{M}/3 & \text{sphere} \\ -\mathbf{M}/2 & \text{cylinder, perpendicular} \\ 0 & \text{cylinder, parallel} \end{cases}$$

The minus signs make sense, because the “magnetic charge” accumulates on the side that  $\mathbf{M}$  points to, and produces an  $\mathbf{H}$  field in the opposite direction. Finally, converting back to  $\mathbf{B}$  yields

$$\mathbf{B} = \begin{cases} 2\mu_0\mathbf{M}/3 & \text{sphere} \\ \mu_0\mathbf{M}/2 & \text{cylinder, perpendicular} \\ \mu_0\mathbf{M} & \text{cylinder, parallel} \end{cases}$$

which precisely matches what we found in example 4 and problem 6, with much less effort.

### Remark: Demagnetizing Fields

At a given point in an isolated magnet,  $\mu_0\mathbf{H}$  is the part of  $\mathbf{B}$  due to the rest of the magnet. However, the above example shows that  $\mathbf{H}$  always points *against* the direction of  $\mathbf{M}$ , so a permanent magnet is always trying to demagnetize itself! Similarly, electrically polarized materials carry an internal “depolarization” field. This is why, in the absence of external fields, the vast majority of materials have zero polarization and magnetization.

Before the advent of very effective “hard” ferromagnets, like neodymium magnets, magnets

had to be shaped to avoid this effect, e.g. by making them into long bars or horseshoes. Even so, the demagnetization effect would make the field produced by the magnet a little less than you would expect. For an average-shaped bar magnet made in the 1950s, the magnetic poles are effectively not at the ends, but rather 10% to 20% closer together.

This bit of historical trivia is irrelevant today, but strangely, it has stuck around in the Indian physics curriculum. Every Indian physics textbook demands its students memorize the ratio of the “magnetic length” and “geometric length” of a bar magnet, as if it were a fundamental constant of nature rather than an obsolete rule of thumb. But different Indian books don’t even agree on what the ratio is, with HC Verma giving 84%, various JEE prep sources stating 4/5, 5/6, or 7/8, and none whatsoever explaining where the number comes from.

### Idea 8: Magnetic Energy, Force, and Torque

The appropriate magnetic energy density depends on the material.

- If the magnetization is permanent, as in a hard ferromagnet, we should use the same potential energy introduced in **E4**. Specifically, the potential energy density is  $-\mathbf{M} \cdot \mathbf{B}$ .
- For all other materials, where the magnetization is induced by the presence of other fields, the total energy density is  $B^2/2\mu$ .

If you use one of these formulas where the other applies, you’ll typically be off by a factor of 2.

Regardless of how the magnetization arises, the resulting force and torque in an external field  $\mathbf{B}$  are given by the formulas introduced in **E4**. Specifically, the torque density is  $\mathbf{M} \times \mathbf{B}$ , and the force density is  $\nabla(\mathbf{M} \cdot \mathbf{B})$ , where the  $\nabla$  only acts on  $\mathbf{B}$ . Alternatively, if you’re using the magnetic pole trick, a magnetic charge  $q_m$  feels a force  $q_m \mathbf{B}$ .

[3] **Problem 9.** A version of the method of images works for magnetic materials. Let’s suppose there is vacuum at  $z > 0$ , and a material of relative permeability  $\mu_r$  at  $z < 0$ . When using the method of images, we only care about the field at  $z > 0$ , where  $\mathbf{B}$  and  $\mathbf{H}$  are proportional. So we can directly use the analogy between  $\mathbf{H}$  and  $\mathbf{E}$ .

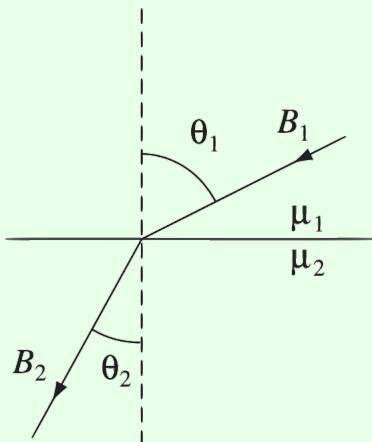
- Suppose a magnetic charge  $q_m$  is a distance  $d$  above the plane. By recycling your answer to problem 3, find the magnetic charge  $q'_m$  of the image. What does it become if the material is a superconductor, or a soft ferromagnet?
- Of course, magnetic charges don’t actually exist, so let’s instead suppose a permanent magnetic dipole moment  $\mathbf{m}$  was a distance  $d$  above the plane, with  $\mathbf{m}$  pointing towards the plane. Characterize the image dipole, and find the force on the real dipole.
- To be even more concrete, consider a very long permanent magnet of cross-sectional area  $A$  and uniform magnetization  $M$  along its length. When one end of the magnet is placed flat against an iron plate, what is the force between them?

[2] **Problem 10.** [AuPhO 2019, problem 13](#). A neat explanation of how a fridge magnet works; for this problem it will be useful to consult the [answer sheet](#).

- [5] **Problem 11.** [Physics Cup 2024, problem 3.](#) This relatively straightforward problem reviews almost everything we’ve covered so far.

### Example 7: Griffiths 6.27

How does a magnetic field line bend when it passes from one medium to another?



### Solution

We say the field lines “bend” because of Gauss’s law for magnetism: they can’t start or end, so each one has to keep on going. Let’s suppose the figure above is drawn in the  $xz$  plane. Applying Gauss’s law for magnetism in a small pillbox spanning the interface gives

$$B_1^z = B_2^z.$$

On the other hand, since  $\nabla \times \mathbf{H}$  is zero (assuming no additional, “free” current is around), considering an Amperian loop spanning the interface gives

$$H_1^x = H_2^x.$$

Combining these results gives

$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{\mu_2}{\mu_1}.$$

In other words, when a field line enters a medium with higher  $\mu$ , it bends away from the normal, and when it enters a medium with lower  $\mu$ , it bends towards the normal.

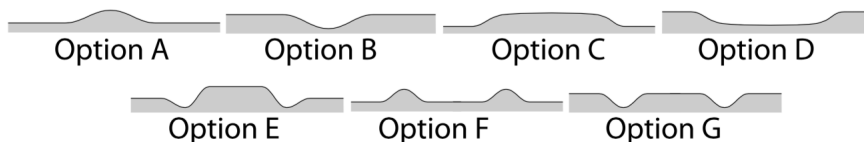
This statement has two limiting cases which will be important later.

- A magnetic field line can’t enter a superconductor ( $\mu_2 = 0$ ) at all, so field lines approaching a superconductor bend away, to become tangent to them ( $\theta_1 \rightarrow 90^\circ$ ).
- A magnetic field line entering a soft ferromagnet ( $\mu_2 \rightarrow \infty$ ) bends towards it to enter along the normal direction ( $\theta_1 \rightarrow 0^\circ$ ), similar to how electric field lines approach conductors. It’s also possible for  $\theta_1$  to be nonzero if  $\theta_2 \rightarrow 90^\circ$ , but we won’t see any examples of this.

You can see both of these behaviors in the limiting cases of problem 9. In general, we conclude that magnetic field lines are “attracted” to regions of higher  $\mu$ , which makes sense because it helps minimize the energy. Soft ferromagnets tend to keep magnetic field lines within themselves, which is why they’re used in transformers.

- [2] **Problem 12** (IPhO 2012 Experiment). Water is a diamagnetic substance. A powerful cylindrical magnet with field  $B$  is placed below the water surface.


(a) Which of the following shows the resulting shape of the water surface?



The magnet is roughly  $2/3$  as wide as each of these sketches.

- (b) Let  $\rho$  be the density of the water. If the maximum change in height of the water surface has magnitude  $h$ , find an approximate expression for the magnetic susceptibility  $\chi_m$  of water.

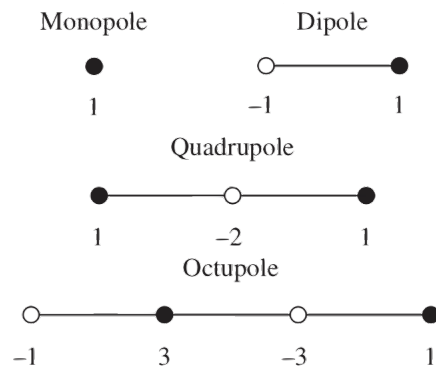
For a very closely related, but more extreme problem, see [EuPhO 2018, problem 2](#).

- [3] **Problem 13.** [EFPhO 2004, problem 6](#). A cute exercise with permanent magnets.
- [5] **Problem 14.**  IPhO 2022, problem 1. A series of exercises on spherical magnets, which uses almost everything covered in this section.
- [5] **Problem 15.** [Physics Cup 2012, problem 2](#). If you only know what’s taught in American introductory courses, this problem is basically impossible. If you only know what’s stated explicitly in Griffiths, it’s very hard. But if you’ve internalized the intuition of the above examples, and the relevant section of **E5** on superconductors, it should be relatively straightforward.
- [5] **Problem 16.** [Physics Cup 2018, problem 3](#). A substantially tougher problem which requires solving some differential equations. I would recommend starting from the [fifth hint](#).

### 3 Multipoles

In this section, we explore some of the physics of dipoles and higher multipoles.

- [3] **Problem 17** (Purcell 10.27). Two monopoles of opposite sign form a dipole, two dipoles of opposite sign for a quadrupole, and so on. Hence we can construct arbitrarily high multipoles using the rows of Pascal’s triangle.

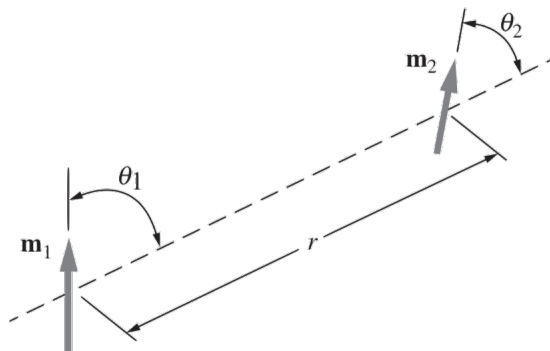


The field of a dipole falls as  $1/r^3$ , a quadrupole as  $1/r^4$ , and an octupole as  $1/r^5$ .

- (a) To warm up, verify explicitly that the quadrupole field along the axis of the quadrupole starts at  $1/r^4$ , i.e. that all lower terms cancel.
- (b) Prove that this cancellation occurs for general multipoles along their axis.
- (c) [A] The magnitude and orientation of a dipole is specified by a vector, with three components. How many numbers are necessary to specify the magnitude and orientation of a quadrupole? (The linear quadrupoles here are just a special case of a general quadrupole.) Try to generalize to arbitrary multipoles.

To learn how to decompose an arbitrary charge distribution into multipoles, see section 3.4 of Griffiths.

- [3] **Problem 18** (Purcell 11.23). Two magnetic dipoles are arranged as shown.



Show that the associated potential energy is

$$U = \frac{\mu_0 m_1 m_2}{4\pi r^3} (\sin \theta_1 \sin \theta_2 - 2 \cos \theta_1 \cos \theta_2).$$

For what orientations is this potential energy maximized or minimized?

- [2] **Problem 19** (Purcell 11.36). Three magnetic compasses are placed at the corners of a horizontal equilateral triangle. As in any ordinary compass, each compass needle is a magnetic dipole constrained to rotate in a horizontal plane. The Earth's magnetic field has been shielded. What orientation will the compass needles eventually assume? Does your result also hold for regular  $N$ -gons?
- [3] **Problem 20**. Some questions about forces between dipoles and other multipoles.

- (a) Above, you've shown that the force between permanent magnetic dipoles falls off as  $1/r^4$ . How about two permanent electric dipoles?
- (b) How about a permanent dipole and a permanent quadrupole?
- (c) How about two permanent quadrupoles?
- (d) Now consider an ion and a neutral atom. The electric field of the ion polarizes the atom; the field of that induced dipole then reacts on the ion. Show that the resulting force is attractive and falls as  $1/r^5$ .

## 4 Electromagnetic Waves in Matter

In this section, you will work out some of the theory of electromagnetic waves in matter.

### Idea 9

In the absence of any free charge or current, Maxwell's equations in matter are identical to Maxwell's equations in vacuum, except that  $\epsilon_0$  and  $\mu_0$  are related by  $\epsilon$  and  $\mu$ , so the waves propagate with speed  $1/\sqrt{\epsilon\mu} = c/n$ , with  $E = (c/n)B$ .

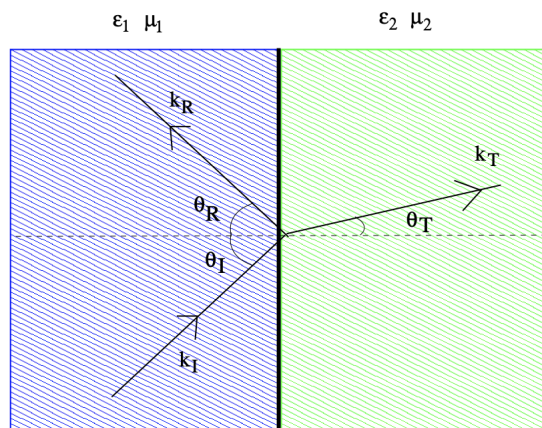
- [5] **Problem 21.** Suppose the regions  $x < 0$  and  $x > 0$  are filled with material with permittivities  $\epsilon_1$  and  $\epsilon_2$ , both with permeability  $\mu_0$ . (This is typical; if you don't count permanent magnets, most objects have permeability about  $\mu_0$ .) We send in an incident wave from the left with electric field

$$\mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)}.$$

The wave will be both transmitted and reflected at the interface, so the total electric field is

$$\mathbf{E} = \begin{cases} \mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)} + \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{x} - \omega_r t)} & x < 0, \\ \mathbf{E}_t e^{i(\mathbf{k}_t \cdot \mathbf{x} - \omega_t t)} & x > 0. \end{cases}$$

The angles with the normal are  $\theta_i$ ,  $\theta_r$ , and  $\theta_t$  as shown.



- (a) Argue that by continuity of the field at the boundary,

$$\omega_i = \omega_r = \omega_t.$$

- (b) Suppose the  $y$ -axis is oriented so that  $\mathbf{k}_i \cdot \hat{\mathbf{y}} = 0$ . Argue that

$$\mathbf{k}_r \cdot \hat{\mathbf{y}} = \mathbf{k}_t \cdot \hat{\mathbf{y}} = 0, \quad \mathbf{k}_i \cdot \hat{\mathbf{z}} = \mathbf{k}_r \cdot \hat{\mathbf{z}} = \mathbf{k}_t \cdot \hat{\mathbf{z}}.$$

From these conditions, derive the laws of reflection and refraction,

$$\theta_i = \theta_r, \quad n_1 \sin \theta_i = n_2 \sin \theta_t.$$

Note that neither this part nor the previous part require Maxwell's equations; they hold for *all* kinds of waves as long as we define  $n_i \propto 1/v_i$ .

- (c) Argue that at the boundary,  $\mathbf{E}_{\parallel}$  and  $B_{\perp}$  must be continuous in general. In this case, because both sides have the same permittivity  $\mu_0$ , there is no bound current, so  $\mathbf{B}_{\parallel}$  is also continuous.
- (d) Now suppose the electric fields are polarized along the  $\mathbf{y}$  axis, so  $\mathbf{E}_i$ ,  $\mathbf{E}_r$ , and  $\mathbf{E}_t$  are all parallel to the  $y$ -axis. Then continuity of  $\mathbf{E}_{\parallel}$  gives

$$E_i + E_r = E_t.$$

Using continuity of  $\mathbf{B}_{\parallel}$ , show that

$$\frac{E_r}{E_i} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}, \quad \frac{E_t}{E_i} = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}.$$

These are the Fresnel equations for normal polarized light. (Hint: this is a bit messy, so you can warm up with the case  $\theta_i = 0$ .)

- (e) If  $n_1 > n_2$ , then total internal reflection occurs when

$$\sin \theta_i > \frac{n_2}{n_1}$$

and the wave is totally reflected. Nonetheless,  $E_t$  is nonzero in this regime. To make sense of this, show that  $\mathbf{k}_t \cdot \mathbf{x}$  is imaginary in this regime, indicating that the “transmitted” wave does not propagate in the region  $x > 0$ , but rather exponentially decays.

**[5] Problem 22.** In most common materials,  $\mu \approx \mu_0$  while  $\epsilon$  depends on frequency. We'll investigate the origin of this frequency dependence below.

- (a) Model an electron in an atom as a mass  $m$  with charge  $q$  attached to a spring, with natural angular frequency  $\omega_0$  and a damping force  $-m\gamma\mathbf{v}$ , in an electric field  $\mathbf{E}_0 e^{-i\omega t}$ . Write down the equation of motion for the electron.
- (b) The atomic polarizability is  $\mathbf{p} = \alpha\mathbf{E}$ . Show that

$$\alpha = \frac{q^2/m}{-\omega^2 + \omega_0^2 - i\gamma\omega}.$$

- (c) For a gas with small number density  $n$ , the Clausius–Mossotti formula reduces to

$$\epsilon = \epsilon_0 + n\alpha.$$

Therefore, the permittivity is generally a complex number. The wavevector and angular frequency are related by  $k^2 = \mu\epsilon\omega^2$ . Explain why the fact that  $\epsilon$  is complex indicates that waves can be absorbed.

- (d) What angular frequency maximizes the absorption rate of the electromagnetic waves? Roughly how many wavelengths does a wave propagate at this angular frequency before being absorbed?
- (e) What angular frequency maximizes the speed of the electromagnetic waves, and what is that speed?
- (f) Transparent objects such as glass can be modeled as having a very high resonant frequency, much higher than that of visible light. Does blue light or red light refract more when passing from air to glass?

The intuitive reason that these electrons can affect the propagation speed of light is because they emit secondary electromagnetic waves that are out of phase with the original wave; this “pushes” the phase of the composite wave forward or backward, affecting the phase velocity. A complete explanation can be found in chapter I.31 of the Feynman lectures.

- [5] **Problem 23.** ⌚ IPhO 2002, problem 1. A neat application of electromagnetic waves in matter.
- [5] **Problem 24.** ⌚ APhO 2007, problem 2. A problem on an exotic negative index of refraction.

### Remark

Above, we considered the response of a medium composed of atoms, obeying  $p = \alpha E$ . However, this relation is just an approximation, like Hooke’s law. For larger electric fields, higher order terms are necessary,

$$p = \alpha E + \alpha' E^2 + \dots$$

which lead to strange effects, studied in the field of nonlinear optics. For example, suppose we send in light of angular frequency  $\omega$ . Then

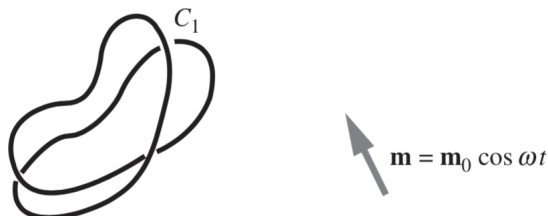
$$E^2 \propto \cos^2(\omega t) = \frac{1 + \cos(2\omega t)}{2}.$$

That means that a nonlinear medium can respond to light at angular frequency  $\omega$  by oscillating, and hence emitting light, at angular frequency  $2\omega$ . This phenomenon is called frequency doubling, or second-harmonic generation, and converts red light to blue. Similarly, for a cubic nonlinearity, you can use trigonometric identities to show that frequency tripling can occur.

## 5 Electromagnetic Systems


In this section we’ll consider problems that use everything we’ve covered, with a focus on technological applications and systems with multiple moving parts.

- [2] **Problem 25** (Purcell 11.19). A magnetic dipole  $\mathbf{m}$  oscillates so that  $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$ . Some of its flux links the nearby circuit  $C_1$ , inducing an electromotive force  $\mathcal{E}_1 \sin \omega t$ .





If a current  $I_1$  flowed in  $C_1$ , then the resulting field at the location of the dipole would be  $\mathbf{B}_1$ . Show that  $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$ . (Hint: recall the results involving mutual inductance in **E5**.)

- [3] **Problem 26.** [EFPhO 2007, problem 3](#). A problem on focusing particles with electric fields.
- [4] **Problem 27.**  IPhO 2004, problem 3. A practical problem which also reviews damped/driven oscillations.
- [4] **Problem 28.** [EFPhO 2014, problem 1](#). A challenging problem about a complex nonlinear circuit.
- [5] **Problem 29.** [Physics Cup 2020, problem 1](#).