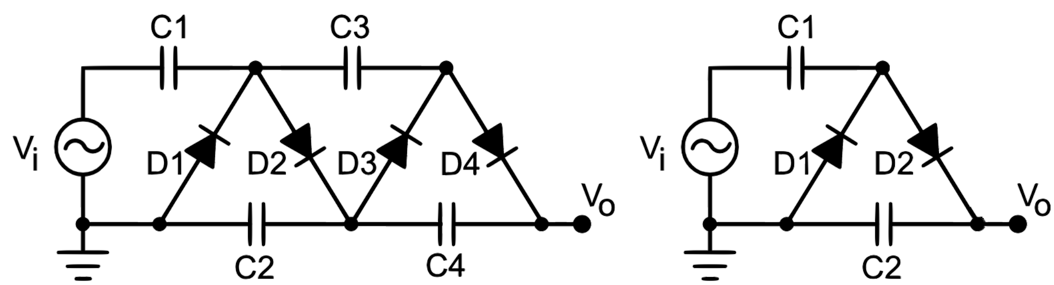
Electromagnetism VII: Electrodynamics

Chapter 9 of Purcell covers electromagnetic waves, and appendix H covers radiation by charges. For a pedagogical introduction with solved examples, see recitation 8 and recitation 9 of the MIT OCW 8.03 lectures. For more technical coverage, not necessarily relevant to the Olympiad, see chapters 7, 8, 10, 11 of Griffiths. For some lighter reading, see chapters I-28, I-32, II-18, II-20, II-21, and II-24 of the Feynman lectures. There is a total of **85** points.

1 More Nonlinear Circuit Elements

In this section we consider some more subtle applications of nonlinear circuit elements. First, we consider complex problems that use relatively familiar circuit elements.

[3] **Problem 1.** The setup below at left is called the Cockcroft–Walton voltage multiplier. It was used in 1932 to power the first particle accelerator.

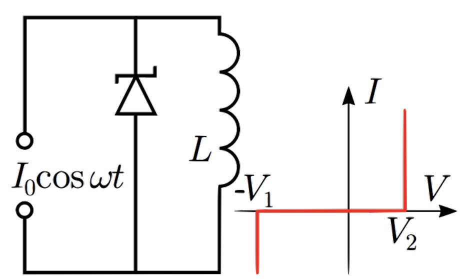


The four capacitors begin uncharged, and all have capacitance C. All four diodes are ideal. The voltage $V_i(t)$ alternates between V and $-V$. The output voltage is V_0.

(a) To warm up, consider the simpler setup shown at right above. Suppose the applied voltage V_i begins at $-V$. Describe how V_0 changes each time the applied voltage switches sign.

(b) Now consider the full setup, shown at left above. After a long time, what is V_0?

[3] **Problem 2** (NBPhO 2017)**.** A Zener diode is connected to a source of alternating current as shown.



The inductance L of the inductor is such that $L\omega I_0 \gg V_1, V_2$ where V_1 and V_2 are the breakdown voltages, and $V_1 > V_2$. The $I(V)$ characteristic of the Zener diode is shown above. Assume that a long time has passed since the current source was first turned on.

(a) Find the average current through the inductor.

(b) Find the peak-to-peak amplitude of the current changes ΔI in the inductor.

[4] **Problem 3.** NBPhO 2016, problem 4. A rather complicated problem involving several exotic circuit elements.

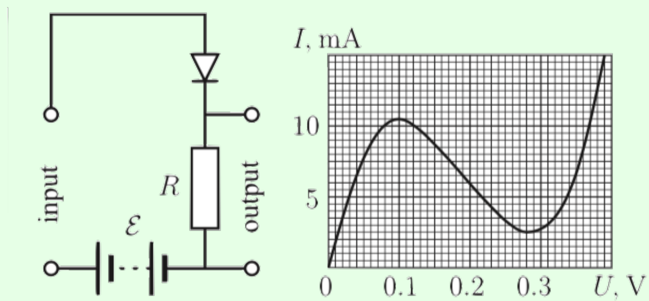
Next, we consider some qualitatively new behavior that can emerge from less familiar circuit elements, such as amplification, hysteresis, and instability.

Idea 1

Tunnel diodes are a variant of diodes, whose $I(V)$ rises, falls, and rises again. That is, they have a region with negative differential resistance, $dI/dV < 0$. This allows them to amplify signals, as we'll see below, and also can make them unstable.

Example 1: EFPhO 2003

The circuit below, containing a tunnel diode, acts as a simple amplifier.



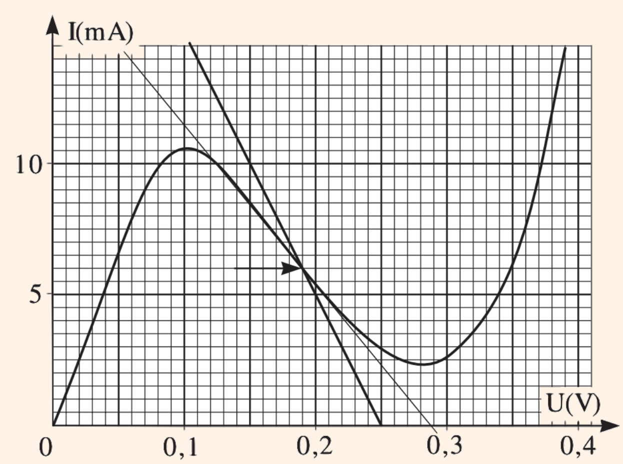
Here, $R = 10\,\Omega$ and $\mathcal{E} = 0.25\,\text{V}$. If a small signal voltage $V_{\text{in}}(t)$ is applied across the input, then an amplified and shifted version of the signal appears across the output. Find the amplification factor.

Solution

When a constant emf $\mathcal{E}$ is applied, Kirchoff's laws give

$$\mathcal{E} = I_0 R + V(I_0)$$

where $V(I)$ is the voltage characteristic of the diode.



By plotting $\mathcal{E} - IR$ on the graph above, we find I_0 at the intersection. (Notice the x-axis label: it is common to write a decimal point as a comma in Eastern Europe.) Now consider the effect of applying the signal voltage, which changes the current by ΔI,

$$\mathcal{E} + V_{\text{in}} = (I_0 + \Delta I)R + V(I_0 + \Delta I).$$

Since the signal voltage is small, we can Taylor expand the voltage characteristic, giving

$$V_{\text{in}} = \Delta I(R + V'(I_0)).$$

This in turn tells us that

$$V_{\text{out}} = (I_0 + \Delta I)R = V_{\text{out}}^0 + \frac{R}{R + V'(I_0)} V_{\text{in}}.$$

In other words, the change in V_{out} is just V_{in}, times the amplification factor

$$\frac{R}{R + V'(I_0)} = \frac{10}{10 - 16} = -\frac{5}{3}$$

where we read $V'(I_0)$ off the graph by drawing a tangent. The intuition here is that the circuit is like a voltage divider, but the tunnel diode acts like a negative resistance. If we had $V'(I_0)$ close to $-R$, for example, the amplification factor would have been huge. Since $V'(I_0)$ is more negative than $-R$, the signal ends up flipped.

[4] **Problem 4.** NBPhO 2020, problem 2. A comprehensive problem on the measurement and dynamics of tunnel diodes, which will give you a deeper understanding of negative resistance.

Idea 2

Op amps have four terminals, and output a voltage across the last two equal to the voltage across the first two, times a very large gain. Like tunnel diodes, op amps can be unstable: increasing the input increases the output, but this in turn could increase the input again. Thus, in practice, the output and input are always wired together in a way that produces negative feedback, with changes in the output acting to decrease the input. In this case, one can think of an op amp as a tool that tries to set the input voltages equal to each other.

The internals are somewhat complicated, consisting of a lot of resistors and transistors. In general, to analyze setups with multiple complex circuit elements like these, it's better to treat them as black boxes than to try to intuit what's going on at the level of individual subelements, or electric and magnetic fields. (Of course, engineers do need to understand circuit elements at these level to design them in the first place!)

[3] **Problem 5.** 🕓 USAPhO 2016, problem A2. This problem is a nice introduction to op amps.

Idea 3

In some nonlinear circuit elements, the function $I(V)$ is multivalued. This indicates hysteresis: given V, the actual value of I depends on the history of the system. The same goes for when

$V(I)$ is multivalued.

[5] **Problem 6.** IPhO 2016, problem 2. This problem illustrates the previous idea with a thyristor. Print out the official answer sheet and record your answers on it.

2 Displacement Current

Idea 4

In general, Ampere's law is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

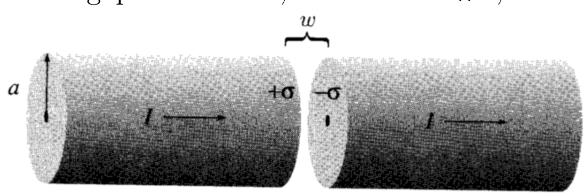
This is sometimes written in terms of a "displacement current" density $\mathbf{J}_d$, where

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_d), \quad \mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

In integral form, for a surface S bounded by a closed curve C,

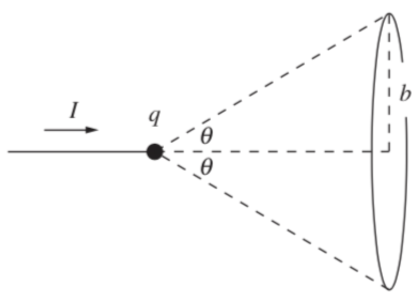
$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_S (\mathbf{J} + \mathbf{J}_d) \cdot d\mathbf{S} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

[2] **Problem 7** (Griffiths 7.34)**.** A fat wire of radius a carries a constant current I uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel plate capacitor.



Find the magnetic field in the gap, at a distance $s < a$ from the axis.

[3] **Problem 8** (Purcell 9.3)**.** A half-infinite wire carries constant current I from negative infinity to the origin, where it builds up at a point charge with increasing q. Consider the circle shown below.



Calculate the integral $\int \mathbf{B} \cdot d\mathbf{s}$ about this circle in three ways.

(a) Use the integrated form of Ampere's law, integrating over a surface which does not intersect the wire.

(b) Do the same, with a surface that does intersect the wire.

(c) Apply the Biot–Savart law to the current and displacement current.

In the previous problem, you should have found that the effect of the displacement current, in the Biot–Savart law, simply cancelled out everywhere. In fact, this cancellation is very general.

Idea 5

In any situation where $\mathbf{J}$ is constant, whether or not ρ is constant, Maxwell's equations are satisfied by applying Coulomb's law to ρ and the Biot–Savart law to $\mathbf{J}$. You can include displacement currents in the Biot–Savart integral too, but their contributions perfectly cancel.

To see why, note that

$$\nabla \times \mathbf{J}_d = \epsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

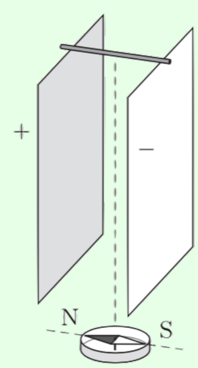
where we used Faraday's law. When the currents are constant, the magnetic fields are also constant, so the right-hand side vanishes. Then $\nabla \times \mathbf{J}_d = 0$. However, this means that $\mathbf{J}_d$ can always be written as a superposition of radial, spherically symmetric currents, and as we saw in the previous problem, such currents produce no magnetic fields.

This explains why we were able to get away with using Coulomb's law and the Biot–Savart law on previous problem sets, even in situations which were not exactly electrostatic or magnetostatic – all of these situations were "quasistatic". In general, the displacement current only matters in non-quasistatic situations involving rapid changes in $\mathbf{J}$, and hence rapid changes in $\mathbf{E}$ and $\mathbf{B}$. These are exactly the cases where significant electromagnetic radiation is produced, which is why radiation is covered in the last half of this problem set.

For more about this subtle point, see section 9.2 of Purcell, this paper and this paper.

Example 2: MPPP 190

A parallel plate capacitor is charged and positioned above a compass as shown.



The capacitor is discharged slowly when the tops of the plates are joined using a small conducting rod. Which way is the compass needle deflected during the discharge process?

Solution

Since the discharge is slow, the situation is quasistatic. (It would only be non-quasistatic in the case where the discharge time was comparable to the time it would take for light to cross the capacitor, a situation which is almost never achieved for RC circuits.) Then we know the magnetic field due to the displacement current cancels out everywhere, so only the current I in the rod matters. This current moves left to right, so by a straightforward application of the right-hand rule, we find that the compass is deflected east.

Things get more subtle if one insists on considering the displacement current anyway. A naive, incorrect argument would be to say that there is a total displacement current I going right to left inside the capacitor, and since this displacement current is closer than the current in the rod, it produces a larger magnetic field, deflecting the compass west. The problem with this reasoning is that it has ignored the displacement current due to the changes in the substantial fringe fields of the capacitor. When these are accounted for, the magnetic fields due to the displacement current cancel, as argued generally above.

[3] **Problem 9** (Griffiths 7.36). An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire along the $\hat{\mathbf{z}}$ axis and returns along a coaxial conducting tube of radius a.

(a) By neglecting displacement current, show that the electric field in the tube is

$$\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \log \frac{a}{r} \, \hat{\mathbf{z}}.$$

(b) Assuming this expression for the electric field, find the amplitude I_d of the total displacement current, and compute the ratio I_d/I_0.

(c) These results are approximately correct as long as $I_d/I_0 \ll 1$. Show that this corresponds to requiring that the speed-of-light travel time from the wire to the tube is much shorter than the current's oscillation period.

By contrast, when I_d/I_0 isn't small, this solution wouldn't be approximately correct. Instead, we would have to consider the magnetic fields induced by the changing electric fields associated with the displacement current, and then the displacement currents due to the changes in those magnetic fields, and so on. The full solution would contain a propagating electromagnetic wave.

[3] **Problem 10.** [A] Consider an infinite thin solenoid which initially carries no current, and a loop of wire around this solenoid of enormous radius, say one light year. At some moment, a current is suddenly made to flow through the solenoid. (This cannot be done by simply attaching a battery somewhere, because it will take a long time for the current to turn on throughout the solenoid. So instead, consider a situation where many batteries arranged around the solenoid are all attached in at once, which can be achieved by machines which have synchronized their clocks beforehand.)

A magnetic field is hence quickly produced in the solenoid, so by Faraday's law,

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

there should quickly be an emf in the loop of wire. But this seems to violate locality, because the motion of charges in the solenoid is quickly affecting the motion of charges in the wire loop, which is very far away. What's going on? Could there be something wrong with Faraday's law?

[3] **Problem 11** (Griffiths 7.64). [A] Setting $\mu_0 = \epsilon_0 = 1$, Maxwell's equations read

$$\nabla \cdot \mathbf{E} = \rho_e, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mathbf{J}_e + \frac{\partial \mathbf{E}}{\partial t}$$

where ρ_e and $\mathbf{J}_e$ are the electric charge density and electric current density.

(a) Show that Maxwell's equations ensure the conservation of electric charge,

$$\dot{\rho}_e = -\nabla \cdot \mathbf{J}_e.$$

This is the continuity equation, and we saw versions of it for other conserved quantities in **T2**.

(b) Generalize Maxwell's equations to include a magnetic charge density ρ_m and a magnetic current density $\mathbf{J}_m$. Fix the signs by demanding that magnetic charge is conserved.

(c) Check that the resulting equations are invariant under the duality transformation

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \quad \begin{pmatrix} \rho'_e \\ \rho'_m \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}$$

which rotates electricity into magnetism with angle θ.

(d) Write down the Lorentz force law for a particle with electric and magnetic charge, using the fact that it should be invariant under the duality transformation above.

Remark

The peculiar name of $\mathbf{J}_d$ is because Maxwell thought of it as a literal displacement of a jelly-like ether. In that era, all electromagnetic quantities, such as fields, charges, currents, polarizations, and magnetizations, were thought to reflect properties of a mechanical ether, such as local strains, displacements, and rotations. However, making this picture precise was known to be difficult even before the advent of relativity, which rendered ether obsolete. The best way to understand why physicists abandoned ether models is to have a look at their daunting complexity. For a nice overview, with diagrams, see chapter 4 of *The Maxwellians.*

3 Field Energy and Momentum

Idea 6

The Poynting vector

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

gives the flux density of the energy of an electromagnetic field. That is, the flux of $\mathbf{S}$ into a closed surface is the rate of change of total field energy within that surface.

[3] **Problem 12.** USAPhO 2010, problem B2.

[3] **Problem 13.** USAPhO 2013, problem B2.

Remark

It's unlikely that you'll see any examples besides the ones in the above two problems, because in almost all other setups, the Poynting vector depends sensitively on the fringe fields, which are very hard to calculate. (For some work in this direction, see *Energy transfer in electrical circuits: A qualitative account*.) In any case, the examples above illustrate the important point that the energy of a circuit does not flow along the wire, carried by the charges; instead it flows into circuit elements from the sides. This was an important early clue of the importance of the electromagnetic field.

Remark

As proven in Poynting's theorem, the Poynting vector indeed tells us about the net flow of energy. However, this would remain true if we added a constant vector to it, or more generally any divergence-free vector field, since these wouldn't change the net flow. So which option is the "correct" one? According to everything we've learned so far, there's no absolute way to choose, and we just use the Poynting vector because it's the simplest option. However, in general relativity, the flow of energy directly influences the curvature of spacetime, so there is an unambiguous correct answer, which is indeed the Poynting vector.

Example 3

Consider two charges q, at positions $r\hat{\mathbf{x}}$ and $r\hat{\mathbf{y}}$ respectively, both moving with speed v towards the origin. Show that the magnetic forces between them are *not* equal and opposite. That is, electromagnetic forces do not obey Newton's third law.

Solution

In order to find the $\mathbf{B}$ field produced by each charge at the location of the other, we use the Biot–Savart law and the right-hand rule. Then we use the Lorentz force and the right-hand rule again to find the magnetic forces on each charge.

For example, the $\mathbf{B}$ field produced by the first charge at the location of the second is along $-\hat{\mathbf{z}}$. Then the magnetic force on the second charge is parallel to $\hat{\mathbf{x}}$. The magnetic force on the first charge is parallel to $\hat{\mathbf{y}}$. And the force are definitely nonzero, so they can't be equal and opposite.

To explain this, we recall that the point of Newton's third law is just momentum conservation. This still holds, as long as one remembers that the field carries momentum of its own. (If we want to save some version of Newton's third law, we could say that the real action-reaction pairs are the forces between the charges and the field, not the charges with each other. But the real lesson is that Newton's third law is not fundamental, momentum conservation is.)

Idea 7

The momentum density of the electromagnetic field is

$$\mathbf{p} = \frac{\mathbf{S}}{c^2}.$$

In other words, momentum density and energy flux density are just proportional. As you will see in **R2**, this is true in general in relativity. The angular momentum density is $\mathbf{r} \times \mathbf{p}$. For an explicit derivation that these definitions ensure the total momentum and angular momentum are conserved, see section 8.2 of Griffiths. (You might think the definitions come out of nowhere; the straightforward way to find them is to apply Noether's theorem, as you will learn in a more advanced class.)

Remark

We have already seen an example of electromagnetic field momentum at work. Back in **E4**, you found that in the presence of a magnetic monopole, the mechanical angular momentum $\mathbf{L}$ of a point charge was not conserved, but $\mathbf{L} - qg\hat{\mathbf{r}}$ was. In fact, this second term turns out to be exactly the angular momentum of the field, so this conservation law is simply the conservation of total angular momentum. (If you'd like to verify this explicitly, it's easiest to use spherical coordinates with the monopole at the origin and the charge along the z-axis, but be warned, it's fairly messy.)

[3] **Problem 14** (Griffiths)**.** A long coaxial cable of length ℓ consists of an inner conductor of radius a and an outer conductor of radius b. The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current.

(a) Find the electromagnetic momentum stored in the fields.

(b) In part (a) you should have found that the fields contain a nonzero momentum directed along the cable. However, this is puzzling because it appears that no net mass is transported along the cable. How is this paradox resolved? (Hint: it doesn't make sense to consider the cable in isolation, as nothing would be keeping the current going. Consider attaching a battery across the left end and a resistor across the right end.)

[3] **Problem 15.** In the early 20$^{\text{th}}$ century, physicists sought to explain the $E = mc^2$ rest energy in terms of electromagnetic field energy. As a concrete example, model a charged particle as a uniform spherical shell of radius a and charge q.

(a) Find the radius a so that the total field energy equals the rest energy associated with the electron mass m. Up to an $O(1)$ factor, this quantity is called the classical electron radius.

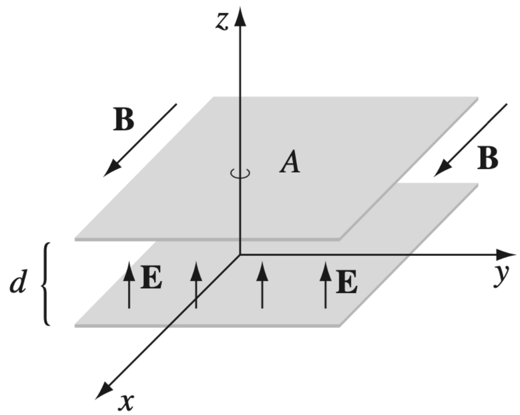
(b) If the shell moves with a small speed v, we expect to have $p = mv$, where p is the total field momentum. Show that instead, we have $p = (4/3)mv$. You may use the result

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}$$

which we will prove in **R3**. Many complicated ideas were put forth to explain this infamous "4/3 problem", as recounted in chapter II-28 of the Feynman lectures.

For more about the “radius” of an electron, see this blog post. For a modern discussion of the resolution of the 4/3 problem, see this paper.

[3] **Problem 16** (Griffiths 8.6)**.** A charged parallel plate capacitor is placed in a uniform magnetic field as shown.



(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z-axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; show the total impulse equals the stored momentum.

(c) Alternatively, suppose we slowly reduced the magnetic field. Show that the total impulse delivered to the plates equals the stored momentum.

This calculation is standard and given in many textbooks, but it is actually completely wrong: we have ignored the fringe field, and when it is included the total electromagnetic momentum is half of what was naively calculated in part (a). The answer in part (b) is correct, but the other half of the impulse corresponds to a change in non-electromagnetic “hidden momentum”. The most basic example of hidden momentum is covered in example 12.12 of Griffiths. For a detailed analysis of the hidden momentum in this setup, see this paper.

[3] **Problem 17.** USAPhO 2004, problem B2. (This is a classic setup which also appears on USAPhO 2020, problem A1, and INPhO 2020, problem 2. But note that the official solution to USAPhO 2020, problem A1 has typos.)

4 Electromagnetic Waves

Idea 8

Maxwell’s equations have propagating wave solutions of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where $\mathbf{E}$ and $\mathbf{B}$ are in phase, perpendicular in direction, and have magnitudes $E_0 = cB_0$. The propagation direction $\mathbf{k}$ is along $\mathbf{E} \times \mathbf{B}$, and the wave speed is

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Example 4

Verify explicitly that in the absence of charges and currents, the electromagnetic field above satisfies Maxwell's equations.

Solution

First let's consider Gauss's law, $\nabla \cdot \mathbf{E} = 0$. Splitting everything explicitly into components,

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= e^{-i\omega t}\left(\frac{\partial}{\partial x}(E_{0,x}e^{i\mathbf{k}\cdot\mathbf{r}}) + \frac{\partial}{\partial y}(E_{0,y}e^{i\mathbf{k}\cdot\mathbf{r}}) + \frac{\partial}{\partial z}(E_{0,z}e^{i\mathbf{k}\cdot\mathbf{r}})\right) \\
&= e^{-i\omega t}\left(E_{0,x}\frac{\partial}{\partial x}e^{i\mathbf{k}\cdot\mathbf{r}} + E_{0,y}\frac{\partial}{\partial y}e^{i\mathbf{k}\cdot\mathbf{r}} + E_{0,z}\frac{\partial}{\partial z}e^{i\mathbf{k}\cdot\mathbf{r}}\right) \\
&= e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left(iE_{0,x}k_x + iE_{0,y}k_y + iE_{0,z}k_z\right) \\
&= i\mathbf{k}\cdot\mathbf{E} = 0
\end{aligned}$$

since $\mathbf{k}$ is perpendicular to $\mathbf{E}_0$. This is another example of a lesson we saw in **M4**. Namely, when everything is a complex exponential, differentiation is very easy. For an complex exponential in time, $e^{i\omega t}$, differentiation with respect to time is just multiplication by $i\omega$. Similarly, for a field which is a complex exponential in space, $e^{i\mathbf{k}\cdot\mathbf{r}}$, the divergence $(\nabla\cdot)$ becomes $(i\mathbf{k}\cdot)$.

By similar reasoning, Gauss's law for magnetism is satisfied. Next, we check Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

By the same logic as above, the curl becomes $(i\mathbf{k}\times)$, while the time derivative becomes multiplication by $-i\omega$, giving

$$i\mathbf{k}\times\mathbf{B} = (-i\omega)\mu_0\epsilon_0\mathbf{E}.$$

Because $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{B}$ are all mutually perpendicular, the directions of both sides match. Then all that remains is to check the magnitudes,

$$kB_0 = \omega\mu_0\epsilon_0 E_0.$$

By plugging in results from above, this reduces to

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

which matches what we said above. (Or, if we didn't know what c case, this logic would have been a way to derive it, as Maxwell did.) The verification of Faraday's law is similar. Note that the displacement current term was essential; it wouldn't have been possible to get electromagnetic wave solutions without it.

[3] **Problem 18.** Consider the energy and momentum of the electromagnetic wave in idea 8.

(a) Show that the spatial average of the energy density is $\epsilon_0 E_0^2/2$. (Be careful with factors of 2.)

(b) Compute the spatial average of the momentum density $\langle \mathbf{p} \rangle$ using idea 7.

(c) Confirm that $E = pc$ for an electromagnetic wave.

[3] **Problem 19.** The intensity of sunlight at noon is approximately $1\,\mathrm{kW/m^2}$.

(a) Compute the rms magnetic field strength.

(b) Compute the radiation pressure acting on a mirror lying on the ground.

(c) In terms of the Lorentz force, how is this pressure exerted on the particles in the mirror?

[3] **Problem 20** (Purcell 9.7)**.** Consider the sum of two oppositely-traveling electromagnetic waves, with electric fields

$$\mathbf{E}_1 = E_0 \cos(kz - \omega t)\hat{\mathbf{x}}, \quad \mathbf{E}_2 = E_0 \cos(kz + \omega t)\hat{\mathbf{x}}.$$

(a) Write down the magnetic field.

(b) Draw plots of the energy density $U(z,t)$ for $\omega t \in \{0, \pi/4, \pi/2, 3\pi/4, \pi\}$.

(c) Plot the Poynting vector for the same values of ωt, and convince yourself that it describes how the energy sloshes back and forth.

Idea 9: Larmor Formula

An accelerating charge produces electromagnetic radiation, with power

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}.$$

We'll derive it properly in **R3**, but a lot of it can be motivated with the techniques of **P1**.

The power could only depend on q, ϵ_0, μ_0, and properties of the particle's motion. The only combinations of the first three parameters that get rid of the electromagnetic units are q^2/ϵ_0 and $1/\sqrt{\epsilon_0\mu_0} = c$. Since energy is proportional to the electric and magnetic fields squared, and these fields are proportional to q, the answer must be proportional to q^2/ϵ_0.

Radiation can't result from uniform velocity, by Lorentz invariance; another way to see this is that with only v and c, there is no way to write down an expression for power with the right units! The next simplest option is radiation from acceleration, from which the most general result is $P = (q^2 a^2/\epsilon_0 c^3) f(v/c)$. The fact that acceleration is squared is also natural, because acceleration is a vector, so this is the simplest way to get a rotationally invariant result. The proper derivation shows that $f(0) = 1/6\pi$. When v/c is substantial, there are relativistic corrections, which we will consider in **R3**.

[2] **Problem 21** (Purcell H.2)**.** A common classical model of an electron in an atom is to imagine it is a mass on a spring, where the spring force is due to the atomic nucleus. Suppose that such an electron, with charge e, is vibrating in simple harmonic motion with angular frequency ω and amplitude A.

(a) Find the average rate of energy loss by radiation.

(b) If no energy is supplied to make up the loss, how long will it take the oscillator's energy to fall to $1/e$ of its initial value?

Numerically, this is an extremely small time, so classical models of the atom are not realistic. We will see in **X1** that in quantum mechanics this problem is solved because in the ground state the electron does not move around the atom, but rather occupies a standing wave.

[3] **Problem 22** (Purcell H.3)**.** A plane electromagnetic wave with angular frequency ω and electric field amplitude E_0 is incident on an atom. As in problem 21, we model the electron as a simple harmonic oscillator, with mass m and natural angular frequency ω_0.

(a) First suppose that $\omega \gg \omega_0$. Argue that in this case, the "spring" force on the electron can be neglected. Find the average power radiated by the electron, and show that it is equal to the power incident on a disc of area

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2.$$

This is the Thomson scattering cross section. To an electromagnetic wave, each electron looks like it has this area.

(b) Now suppose $\omega \ll \omega_0$, yielding Rayleigh scattering, which describes the scattering of visible light by air. In this case, show that $\sigma \propto \omega^4$. This sharp frequency dependence explains why the sky is blue.

(c) Explain the meaning of the common phrase "red sky at night, sailor's delight; red sky in morning, sailor's warning". (Hint: in the cultures where this saying is used, weather patterns usually move from west to east.)

For some further discussion of Rayleigh scattering, see section 9.4 of The Art of Insight. For more about colors in the atmosphere, see this nice video.

[3] **Problem 23.** USAPhO 2016, problem B2.

Remark

We noted in **M7** that clouds are visible because the radiation scattered by a small droplet of n water molecules grows as n^2. To understand why, note that each of the molecules performs independent Rayleigh scattering, as computed above. For separated molecules, the energy scattered just adds. However, for nearby molecules the electromagnetic waves scattered interfere constructively, so the amplitude grows as n and hence the energy scattered as n^2.

This quadratic enhancement breaks down once the droplets exceed the wavelength λ of the light. This means the maximum possible enhancement is larger for larger wavelengths, acting against the ω^4 dependence of Rayleigh scattering. This is why clouds are white, not blue.

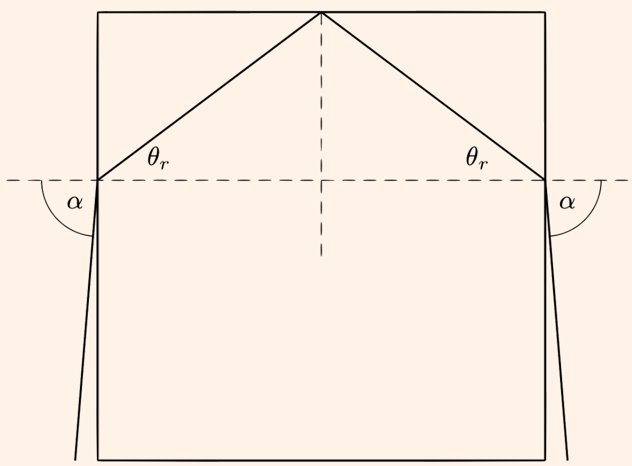
Radiation pressure can also have mechanical effects.

Example 5: NBPhO 2018.6

A laser pointer of power P is directed at a glass cube, with refractive index $n > \sqrt{2}$. The surface of the cube has an anti-reflective coating, so there is no partial reflection when light enters or exits it; the laser pointer only refracts. What is the maximum force the laser pointer can exert on the cube?

Solution

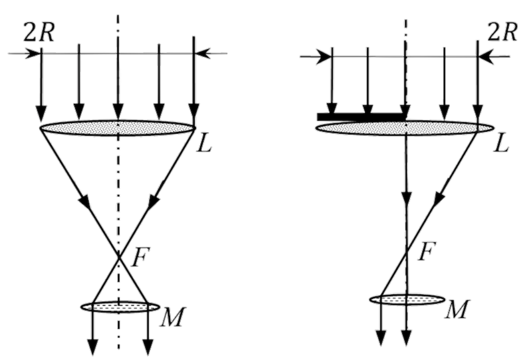
The force is due to a change in momentum of the light. The greatest possible force is attained if the direction of the light is reversed, which can occur as shown, in the limit $\alpha \to 90°$.



Assuming $n > \sqrt{2}$, we then have $\theta_r < 45°$, and then the laser internally reflects when it hits the top surface of the cube. It exits in the opposite direction it came in.

If the laser pointer has power P, then the momentum of the laser beam per time is P/c. The momentum is reversed, so the force is $2P/c$.

[3] **Problem 24** (IZhO 2022). In 2018, the Nobel Prize in physics was awarded to Arthur Ashkin for the creation of the "laser tweezer", a device that allows one to hold and move transparent microscopic objects with the help of light. In one such device, a parallel beam of light from a laser passes through a converging lens L and hits a microparticle M, which can also be considered a converging lens. Point F is the common focus of L and M.



The light intensity in the beam is $I = 1.00\,\mu\text{W/cm}^2$, the beam radius is $R = 1.00\,\text{cm}$, and the focal length of the lens L is $F = 10.0\,\text{cm}$. Ignore the absorption and reflection of light.

(a) Calculate the force acting on the microparticle, in the setup shown at left above.

(b) Next, the left half of the lens L is covered by a diaphragm, as shown at right above. Calculate the force acting on the microparticle in the transverse direction of the beam.

[3] **Problem 25** (Feynman). In one proposed means of space propulsion, a spaceship of mass $10^3\,\text{kg}$ carries a thin sheet of area $100\,\text{m}^2$. The sheet is made of highly reflective plastic film, and can be used as a solar radiation pressure "sail". The spaceship travels in a circular orbit of radius r, which is initially equal to the Earth's orbit radius, where the intensity of sunlight is $1400\,\text{W/m}^2$. Assume the spaceship is moving nonrelativistically and the gravitational effect of the Earth is negligible.

(a) Find the angle at which the sail should be pointed to maximize dr/dt.

(b) Assuming the sail is pointed this way, find the numeric value of dr/dt.

(c) If this continues for a very long time, then r will grow as $r \propto t^n$. Find the value of n.

Finally, we'll consider electromagnetic wave propagation in transmission lines.

[4] **Problem 26** (Griffiths 7.62, Crawford 4.8). A certain transmission line is constructed from two thin metal ribbons, of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length $\mathcal{C}$, and the inductance per unit length $\mathcal{L}$.

(b) Argue that the speed of propagation of electromagnetic waves through this transmission line is of order $1/\sqrt{\mathcal{LC}}$, and evaluate this quantity.

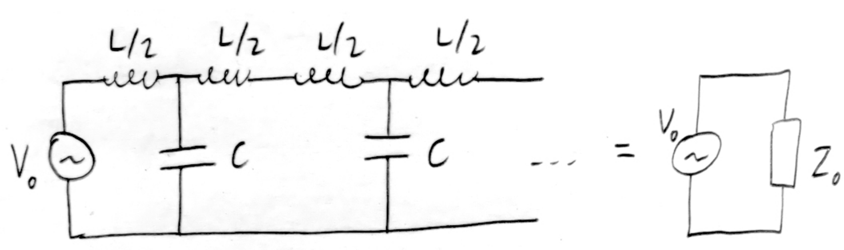
(c) Repeat the first two parts for a coaxial transmission line, consisting of two cylinders of radii $a < b$ with the same axis of symmetry.

(d) Repeat the first two parts for a parallel-wire transmission line, consisting of two wires of radius r whose axes are a distance $D \gg r$ apart.

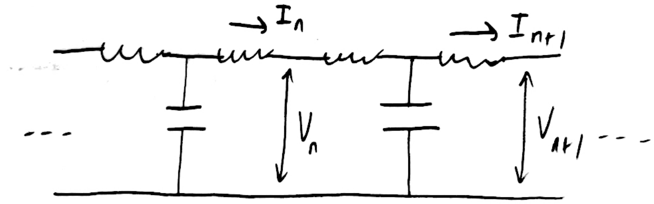
You should find that in all cases, $1/\sqrt{\mathcal{LC}}$ is the same, yielding the same speed for electromagnetic waves. This isn't a coincidence, and applies for transmission lines with conductors of any shape, though the general proof requires some elaborate vector calculus, as you can see here.

[4] **Problem 27.** In this problem, we treat electromagnetic wave propagation through a transmission line using a "lumped element" approach, where the line is replaced with discrete capacitors and inductors, as shown. (This is an example of a network synthesis, mentioned in **E6**.)

(a) Calculate the characteristic impedance $Z_0(\omega)$ of the entire network, as shown below.



(b) The diagram below shows two adjacent sections of the ladder.



Find the ratio of the complex voltage amplitudes V_{n+1}/V_n.

(c) The AC driving attempts to create electromagnetic waves which travel through the network, to the right. It turns out that above a certain critical angular frequency ω_c, waves will not travel through the ladder network. Find ω_c. (Hint: this can be done using either the result of part (a) or part (b).)

(d) For angular frequencies $\omega \ll \omega_c$, waves travel through the ladder with a constant speed. Find this speed, assuming each segment of the ladder has physical length ℓ. (Hint: the speed of a wave obeys $v = d\omega/dk$.)

(e) You should have found in one of the earlier parts that the impedance of this infinite network can be a real number, even though it's made of parts which all have imaginary impedance. That sounds strange, but what's even stranger is that we *should* be able to handle this infinite circuit by taking the limit of progressively larger finite circuits, just as we did for a similar network of resistors in **E2**. But for *any* finite LC network, the impedance will be imaginary, so the limit must be imaginary too! On one hand, we should trust the finite result because all real circuits are finite. On the other hand, the real impedance we get for the infinite result certainly can be measured in real life. So what's going on?