PHYS-330: QFT I — Solutions 3

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1. Locality in relativistic quantum field theory. (20 points)

The vacuum two-point correlation function of a real scalar field is

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{1}$$

It only depends on the difference of positions $z = x - y$, by translational invariance, and it quantifies correlations between the field values at x and y in the vacuum state.

a) Show that

$$D(z) = \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\delta(p^2-m^2)\,\theta(p^0)\,e^{-ip\cdot z} \tag{2}$$

where the Heaviside step function θ is 1 if the argument is positive and 0 otherwise.

Solution: Using the same $đ$ and $\delta\!\!\!/$ notation as in the solutions to the second problem set,

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{đ\mathbf{p}\,đ\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\,\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\omega_p x^0}e^{-i\mathbf{q}\cdot\mathbf{y}}e^{i\omega_q y^0}\langle 0|a(\mathbf{q})a^\dagger(\mathbf{p})|0\rangle && \text{(S1)}\\
&= \int \frac{đ\mathbf{p}\,đ\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\,\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\omega_p x^0}e^{-i\mathbf{q}\cdot\mathbf{y}}e^{i\omega_q y^0}\delta\!\!\!/(\mathbf{p}-\mathbf{q}) && \text{(S2)}\\
&= \int \frac{đ\mathbf{p}}{2\omega_\mathbf{p}} e^{-i\omega_p z^0}e^{i\mathbf{p}\cdot\mathbf{z}}. && \text{(S3)}
\end{aligned}$$

On the other hand, starting from the desired expression, we have

$$\begin{aligned}
D(z) &= \int đ\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{z}}\int_0^\infty \frac{dp^0}{2\pi}\,(2\pi)\delta(p^2-m^2)e^{-ip^0z^0} && \text{(S4)}\\
&= \int \frac{đ\mathbf{p}}{2\omega_\mathbf{p}}\, e^{-i\omega_p z^0}e^{i\mathbf{p}\cdot\mathbf{z}} && \text{(S5)}
\end{aligned}$$

which matches.

b) Show that if $z'^\mu = \Lambda^\mu{}_\nu z^\nu$ for a proper orthochronous Lorentz transformation Λ, then $D(z') = D(z)$. (Hint: show that d^4p and $\delta(p^2-m^2)\,\theta(p^0)$ are each Lorentz invariant.)

Solution: For $D(z')$, let's change the integration variable from p to $p'^\mu = \Lambda^\mu{}_\nu p^\nu$. Then

$$\begin{aligned}
D(z') &= \int \frac{d^4p'}{(2\pi)^4}\,(2\pi)\delta(p'^2-m^2)\theta(p'^0)e^{-ip'\cdot z'} && \text{(S6)}\\
&= \int \frac{d^4p'}{(2\pi)^4}\,(2\pi)\delta(p'^2-m^2)\theta(p'^0)e^{-ip\cdot z} && \text{(S7)}
\end{aligned}$$

which is precisely the same thing as $D(z)$ provided that

$$d^4p'\,\delta(p'^2-m^2)\,\theta(p'^0) = d^4p\,\delta(p^2-m^2)\,\theta(p^0). \tag{S8}$$

Lorentz transformations always preserve inner products, $p^2 = p'^2$, so the two delta functions are the same. Furthermore, for timelike momenta (which are the only ones that contribute to the integral, because of the presence of the delta functions), orthochronous Lorentz transformations always preserve the sign of the time component, by definition. Physically this corresponds to the fact that you can't boost a particle from positive energy to negative energy, and you can check it by explicitly computing the effects of Lorentz boosts. (Of course, Lorentz boosts certainly can change the sign of the time component of *spacelike* vectors.)

The tricky part is to show that proper orthochronous Lorentz transformations preserve spacetime volume, $d^4p' = d^4p$. There are a few different ways to show this. The most formal way is to just compute the Jacobian determinant. The Jacobian matrix for the transformations between the variables is just

$$\frac{\partial p'^{\mu}}{\partial p^{\nu}} = \Lambda^{\mu}{}_{\nu} \tag{S9}$$

and its determinant is

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}\epsilon^{\mu'\nu'\rho'\sigma'} \tag{S10}$$

as you can show by expanding out the definition of the determinant and using the antisymmetry of ϵ. On the other hand, we know the Levi–Civita symbol is an invariant tensor, so we just get

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 1. \tag{S11}$$

Another method is to note that in terms of matrix multiplications, $\eta = \Lambda^T\eta\Lambda$, which implies

$$\det\eta = (\det\Lambda^T)(\det\eta)(\det\Lambda) \tag{S12}$$

and therefore $J = \det\Lambda = \pm 1$. Since the proper orthochronous Lorentz transformations are connected to the identity, we must have $J = 1$. Finally, you could just manually show that $J = 1$ when $\Lambda^{\mu}{}_{\nu}$ is a rotation or a boost, i.e. by showing that the matrices

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\gamma & \sinh\gamma & & \\ \sinh\gamma & \cosh\gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix} \tag{S13}$$

have unit determinant. This is sufficient, since any proper orthochronous Lorentz transformation can be built out of rotations and boosts.

c) Show that for spacelike separation, $z^2 < 0$, the correlation function is exponentially decaying but nonzero. You can do this in two ways. First, you can show that

$$D(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\,K_1(\sqrt{-z^2}\,m) \tag{3}$$

where K_1 is a modified Bessel function of the second kind. Alternatively, you may numerically integrate $D(z)$ and graph the result. In both cases you may use any books or computer programs needed, such as Mathematica or Abramowitz and Stegun.

Solution: By our result from part (b), we can choose a frame where $z^{\mu} = (0, \mathbf{z})$, and set up spherical coordinates with the axis aligned with $\mathbf{z}$. Defining $r = \sqrt{-z^2} = |\mathbf{z}|$ and $p = |\mathbf{p}|$, we have

$$\begin{aligned} D(z) &= \int \frac{đ\mathbf{p}}{2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{z}} && \text{(S14)} \\ &= \frac{1}{(2\pi)^3}\int_0^{\infty} dp \int_{-1}^{1} d\,\cos\theta \int_0^{2\pi} d\phi\, \frac{p^2}{2\sqrt{p^2+m^2}} e^{ipr\cos\theta} && \text{(S15)} \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty} dp\, \frac{p}{\sqrt{p^2+m^2}}\frac{\sin(pr)}{r}. && \text{(S16)} \end{aligned}$$

At this point we can straightforwardly evaluate the integral numerically, e.g. in Mathematica.

```
Dz[r_, m_] := NIntegrate[p Sin[p r] / (r Sqrt[p^2 + m^2]),
  {p, 0, Infinity}]

LogPlot[Dz[r, 1], {r, 0, 5}, AxesLabel → {"mr", "D(z)"}]
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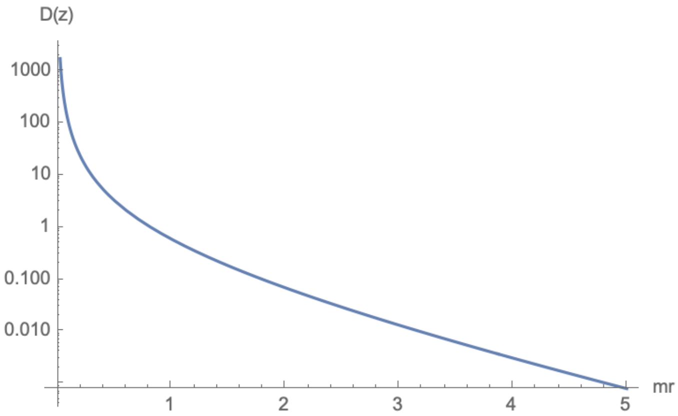

Alternatively, if we want to stay analytic to the end, a little more work is necessary. First, we replace the integral with an equivalent complex integral over the entire real axis,

$$D(z) = -\frac{i}{8\pi^2 r} \int_{-\infty}^{\infty} dp \, \frac{p \, e^{ipr}}{\sqrt{p^2 + m^2}}. \tag{S17}$$

To evaluate the integral, we deform the contour so that it wraps around the vertical branch cut that starts at $p = im$. Across this branch cut, the square root flips sign, so the contour integral going down the left end of the branch cut is equal to the contour integral going up the right end, giving

$$D(z) = \frac{1}{4\pi^2 r} \int_{m}^{\infty} dx \, \frac{x \, e^{-xr}}{\sqrt{x^2 - m^2}} \tag{S18}$$

where $p = ix$. Finally, looking up this integral in a book or Mathematica yields the final result,

$$D(z) = \frac{m}{4\pi^2 r} K_1(mr). \tag{S19}$$

A key requirement for a relativistic theory is that it is local, meaning that effects don't propagate faster than the speed of light. That means any change applied to the field at x should only affect observable results at y if x and y aren't spacelike separated, $(y-x)^2 \geq 0$.

You might thus be concerned about part (c), which implies a field in the vacuum "knows" about the values of the field at spacelike separation. But there's nothing wrong with this. For example, to define a reference frame in special relativity, one synchronizes the clocks by sending light pulses throughout all space. After synchronization, all the clocks "know" about the values on the other clocks, even at spacelike separation. But that doesn't mean that changes propagate faster than light; it's just a consequence of how we set up the system. Similarly, ensuring a field is in the vacuum state requires absorbing all particles throughout all space, and this process sets up correlations between field values.

To test if our theory is local, we must see whether *changes* in the state propagate faster than light; that is the subject of the rest of the question.

d) One way to interact with a quantum field is to measure its value at a point, a process which changes the state. From quantum mechanics, we know that measurements of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not affect each other if $[\mathcal{O}_1, \mathcal{O}_2] = 0$. Show that

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = 0 \tag{4}$$

for spacelike separation, $(x-y)^2 < 0$, as expected from locality.

Solution: We have

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = D(z) - D(-z). \tag{S20}$$

On the other hand, when z is spacelike, it can always be transformed into $-z$ by a Lorentz transformation. (A simple way to see this is to note that both z and $-z$ can be transformed into a standard four-vector like $(0, 0, 0, \sqrt{-z^2})$, by first boosting to get rid of the time component, and rotating to get the spatial component pointing the right way.) Therefore, $D(z) = D(-z)$, so the commutator vanishes.

Note that when z is timelike, you can't Lorentz transform it into $-z$, so the argument that $D(z) = D(-z)$ fails. That makes sense because the commutator *should* be nonzero for timelike separations; if it was zero for all separations, then the theory would be trivial, as no measurements would ever affect anything.

e) Another way to interact with a field is to couple it to a classical source $J(x)$, which for a real scalar field corresponds to taking the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \phi(x) J(x). \tag{5}$$

This is the scalar analogue of driving the electromagnetic field with a current $J^\mu(x)$. For simplicity, we'll treat $\phi(x)$ as a classical field for now, though similar conclusions will hold when it is a quantum field. Show that the classical equation of motion is

$$(\partial_\mu \partial^\mu + m^2)\, \phi(x) = J(x). \tag{6}$$

Solution: The Euler–Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{S21}$$

where we now have

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi + J \tag{S22}$$

which together give the desired result.

f) Show that the equation of motion is solved by

$$\phi(x) = i \int d^4 y\, G(x-y) J(y) \tag{7}$$

where G is a Green's function of the Klein–Gordan operator, which means

$$(\partial_\mu \partial^\mu + m^2)\, G(z) = -i\, \delta^{(4)}(z). \tag{8}$$

The factors of $-i$ here are purely conventional, and will simplify results later.

Solution: We have

$$(\partial^2 + m^2)\phi(x) = i(\partial^2 + m^2) \int d^4 y\, G(x-y) J(y) \tag{S23}$$

$$= i \int d^4 y\, (-i \delta^{(4)}(x-y)) J(y) \tag{S24}$$

$$= J(x) \tag{S25}$$

as desired. Note that there is no term from the derivatives acting on the current because the derivatives are with respect to x, while J is a function of y.

g) By taking Fourier transforms, we may heuristically write the Green's function as

$$G(z) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{p^2 - m^2} \tag{9}$$

which formally obeys Eq. (8). However, this expression is not mathematically well-defined because the integral blows up when $p^2 = m^2$. To get a definite result, we must add "$i\epsilon$" terms to the denominator to keep it from vanishing. There are multiple ways to do this, which physically corresponds to the fact that there are multiple possible Green's functions, depending on the field's boundary conditions. Three key examples are the retarded, advanced, and Feynman Green's functions,

$$G_R(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{(p^0 + i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{10}$$

$$G_A(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{(p^0 - i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{11}$$

$$G_F(z) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot z}}{p^2 - m^2 + i\epsilon} \tag{12}$$

where the limit notation means ϵ approaches zero from the positive end. By using the residue theorem to perform the integral over p^0, show that

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \tag{13}$$

$$G_A(z) = \theta(-z^0)(D(-z) - D(z)) \tag{14}$$

$$G_F(z) = \theta(z^0)D(z) + \theta(-z^0)D(-z) \tag{15}$$

Solution: Applying partial fractions, we have

$$G_R(z) = i \int \frac{d^3p\, dp^0}{(2\pi)^4} \frac{1}{2\omega_p} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)} - \frac{1}{(p^0 + \omega_p + i\epsilon)} \right] e^{-ip\cdot z} \tag{S26}$$

which implies both poles are shifted below the real p^0 axis. Now we perform the integral over p^0. When $z^0 > 0$, the integrand is exponentially damped below the real p^0 axis, so it makes no difference if we replace the contour over the real line with a closed contour which additionally includes a semicircle (of infinite radius) below the real axis, as shown.

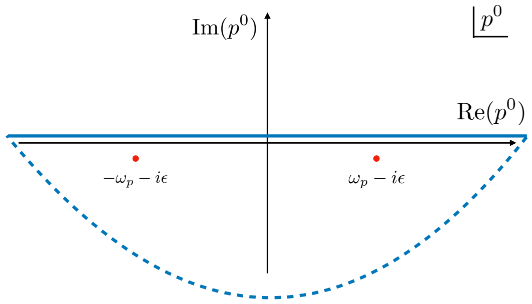

Figure 1: Contour for the retarded propagator for $z^0 > 0$.

This closed contour integral can be calculated using the residue theorem. Each pole contributes $-2\pi i$ times its residue (where the minus sign is because the contour goes around the poles clockwise), giving

$$G_R(z) = \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2\omega_p} e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - \frac{1}{2\omega_p} e^{i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} \right] \tag{S27}$$

Reindexing $\mathbf{p} \to -\mathbf{p}$ in the second term gives

$$G_R(z) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - e^{i\omega_p z^0 - i\mathbf{p}\cdot\mathbf{z}} \right] \tag{S28}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{-ip\cdot z} - e^{ip\cdot z} \right] = D(z) - D(-z) \tag{S29}$$

where here we are defining $p = (\omega_p, \mathbf{p})$. On the other hand, when $z^0 < 0$, the integral is exponentially damped above the real p^0 axis. We can thus close the contour above the real axis, and it encircles no poles, so the integral vanishes. Thus,

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \tag{S30}$$

as desired. The proof for $G_A(z)$ is extremely similar, but now with both poles above the real axis.

Finally, the Feynman propagator is

$$G_F(z) = i \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)(p^0 + \omega_p - i\epsilon)} \right] e^{-ip\cdot z} \tag{S31}$$

$$= i \int \frac{d^3p dp^0}{(2\pi)^4} \frac{1}{2\omega_p} \left[\frac{1}{(p^0 - \omega_p + i\epsilon)} - \frac{1}{(p^0 + \omega_p - i\epsilon)} \right] e^{-ip\cdot z} \tag{S32}$$

which has one pole above the real axis, and one pole below. In figure 2 we show the contours of integration

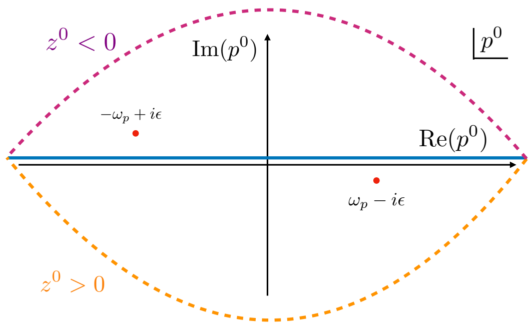

Figure 2: Contour for the Feynman propagator.

for the p^0 integral. If $z^0 > 0$ the dashed, orange line in fig. 2 represents a contour of a semi-circle with infinite radius in the lower half of the complex plane and does not contribute to the integral. The integral along the the real p^0 axis consequently yields minus one times the residue at $p^0 = \omega_p + i\epsilon$ and the other residue does not contribute. Conversely, if $z^0 < 0$, the purple dashed contour in the positive imaginary half-plane contributes zero to the integral and the integral along the real axis yields the residue at $p^0 = -\omega_p + i\epsilon$. We thus conclude

$$G_F(z) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \left[\theta(z^0) e^{-ipz} + \theta(-z^0) e^{ipz} \right] \tag{S33}$$

$$= \theta(z^0) D(z) + \theta(-z^0) D(-z). \tag{S34}$$

h) The retarded Green's function applies when the field is zero before the source acts. Show that $G_R(x - y)$ vanishes for spacelike separation, as expected from locality.

Solution: When z^0 is negative, $G_R(z)$ is just zero by the definition of the step function. When z^0 is positive, $G_R(z)$ is just the same thing as $\langle 0|[\phi(x), \phi(y)]|0\rangle$, which we showed was zero for spacelike separation in part (d).

i) The Feynman propagator will play a crucial role when we introduce Feynman diagrams because it is equal to $\langle 0|T\phi(x)\phi(y)|0\rangle$. Does it vanish for spacelike separation?

Solution: It doesn't vanish for spacelike separation. For example, when z^0 is positive, the Feynman propagator is just $D(z)$, and we already showed in part (c) that this can be nonzero for spacelike separations. For this reason, the Feynman propagator has little use in classical field theory (which is why it doesn't show up in electromagnetism classes, while the retarded and advanced propagators do). But it will be useful when we set up perturbation theory.

Some students tried to solve this problem by setting $z^0 = 0$, but this is a bit tricky because the step functions are discontinuous there. There's no need to do this, because we already showed that $D(z)$ can be nonzero for general spacelike separation, not just when $z^0 = 0$.

2. Recovering classical field theory. (10 points)

If an operator $\mathcal{O}$ is time-independent in Schrodinger picture, then in Heisenberg picture,

$$\frac{d\mathcal{O}(t)}{dt} = i\,[H(t), \mathcal{O}(t)]. \tag{16}$$

In quantum field theory, working in the Heisenberg picture allows fields to depend on spacetime, making results look more Lorentz invariant. For example, we saw that in Heisenberg picture, a real scalar field obeys the Klein–Gordan equation $(\partial^2 + m^2)\phi = 0$. Now consider the Lagrangian of Eq. (5), which additionally includes a source term $J(x)$. In this case, the Hamiltonian is explicitly time-dependent,

$$H(t) = H_0 - \int d^3\mathbf{x}\, \phi(\mathbf{x}, t) J(\mathbf{x}, t) \tag{17}$$

where H_0 is the free Hamiltonian.

a) Show that in Heisenberg picture, the field $\phi(x)$ obeys Eq. (6). (Thus, by the logic of problem 1, expectation values of a quantum field $\phi(x)$ respond locally to sources.)

Solution: Note that the new term in the Hamiltonian commutes with $\phi(\mathbf{y}, t)$ for any $\mathbf{y}$. Therefore, the Heisenberg equation of motion for ϕ is exactly the same as before,

$$\dot{\phi}(x) = \pi(x) \tag{S35}$$

As for the canonical momentum, we have

$$\dot{\pi}(x) = \nabla^2\phi(x) - m^2\phi(x) + i\left[-\int d^3\mathbf{y}\,\phi(\mathbf{y}, t) J(\mathbf{y}, t), \pi(\mathbf{x}, t)\right] \tag{S36}$$

$$= \nabla^2\phi(x) - m^2\phi(x) + (-i)(i)\int d^3\mathbf{y}\,\delta^{(3)}(\mathbf{y} - \mathbf{x}) J(\mathbf{y}, t) \tag{S37}$$

$$= \nabla^2\phi(x) - m^2\phi(x) + J(x). \tag{S38}$$

Combining these two equations gives the result.

With the source, quantum fields can evolve in time nontrivially. Suppose we start in the vacuum state $|0\rangle$, and at time $t = 0$ apply an impulse to the field via

$$J(x) = \delta(t)\, j(\mathbf{x}). \tag{18}$$

This is the scalar analogue of suddenly turning an electric current on and off in electromagnetism. It is also closely related to the last half of problem 2 of problem set 1.

b) The impulse causes operators to instantaneously shift in value at $t = 0$. Show that

$$\phi(\mathbf{x}, 0^+) = \phi(\mathbf{x}, 0^-), \qquad \pi(\mathbf{x}, 0^+) = \pi(\mathbf{x}, 0^-) + j(\mathbf{x}). \tag{19}$$

Here, 0^+ means a time right after $t = 0$, and 0^- means a time right before, i.e.

$$f(0^+) - f(0^-) = \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \frac{df(t)}{dt}\, dt \tag{20}$$

for any function of time.

Solution: Integrating the Heisenberg equation of motion for an infinitesimal time centered at $t = 0$,

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = -i\left[\int d^3\mathbf{y}\, \phi(\mathbf{y}, 0) j(\mathbf{y}), \phi(\mathbf{x}, 0)\right] = 0. \tag{S39}$$

On the other hand, we have

$$\pi(\mathbf{x}, 0^+) - \pi(\mathbf{x}, 0^-) = -i\left[\int d^3\mathbf{y}\, \phi(\mathbf{y}, 0) j(\mathbf{y}), \pi(\mathbf{x}, 0)\right] \tag{S40}$$

$$= -i \int d^3\mathbf{y}\, i\delta(\mathbf{y} - \mathbf{x}) j(\mathbf{y}) \tag{S41}$$

$$= j(\mathbf{x}). \tag{S42}$$

c) Show that this is equivalent to the annihilation operators shifting by

$$a(\mathbf{p}, 0^+) = a(\mathbf{p}, 0^-) + \frac{i}{\sqrt{2\omega_\mathbf{p}}}\, \tilde{j}(\mathbf{p}) \tag{21}$$

where $\tilde{j}$ is the Fourier transform of j.

Solution: First, note that

$$a^\dagger(\mathbf{p}, 0^+) = a^\dagger(\mathbf{p}, 0^-) - \frac{i}{\sqrt{2\omega_\mathbf{p}}}\, \tilde{j}^*(\mathbf{p}) \tag{S43}$$

Furthermore, because $j(\mathbf{x})$ is real, we have

$$\tilde{j}^*(\mathbf{p}) = \left(\int d\mathbf{x}\, j(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}}\right)^* = \int d\mathbf{x}\, j(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}} = \tilde{j}(-\mathbf{p}). \tag{S44}$$

The change in the field operator is

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = \int \frac{đ\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left(\frac{i\,\tilde{j}(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{i\,\tilde{j}^*(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{S45}$$

Substituting $\mathbf{k}' = -\mathbf{k}$ in the second term and renaming $\mathbf{k}'$ to $\mathbf{k}$ gives

$$\phi(\mathbf{x}, 0^+) - \phi(\mathbf{x}, 0^-) = i \int \frac{đ\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left(\frac{\tilde{j}(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{\tilde{j}^*(-\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}}\right) = 0. \tag{S46}$$

On the other hand, we have

$$\pi(\mathbf{x}, 0^+) - \pi(\mathbf{x}, 0^-) = -i \int đ\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{2}} \left(\frac{i\,\tilde{j}(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{-i\,\tilde{j}^*(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{S47}$$

$$= \frac{1}{2} \int đ\mathbf{k}\, \left(\tilde{j}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \tilde{j}^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{S48}$$

$$= \frac{1}{2} \int đ\mathbf{k}\, (\tilde{j}(\mathbf{k}) + \tilde{j}^*(-\mathbf{k})) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{S49}$$

$$= j(\mathbf{x}) \tag{S50}$$

as desired.

d) Show that the field is in a coherent state after the impulse, in the sense that the state is an eigenvector of $a(\mathbf{p}, 0^+)$. Then evaluate $\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle$, which gives the number of particles of momentum $\mathbf{p}$ produced, per unit volume of momentum space.

Solution: Because we are in the vacuum state before the impulse, $a(\mathbf{p}, 0^-)|0\rangle = 0$, we have

$$a(\mathbf{p}, 0^+)|0\rangle = \frac{i}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde{j}(\mathbf{p})|0\rangle \tag{S51}$$

which implies the field is in a coherent state after the impulse. Then we have

$$\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle = \langle 0|\,\frac{(-i)\,\tilde{j}^*(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}\,\frac{i\,\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}|0\rangle = \frac{|\tilde{j}(\mathbf{p})|^2}{2\omega_{\mathbf{p}}}. \tag{S52}$$

Incidentally, the fact that the field is in a coherent state implies that the number of particles in a given range of momenta is Poisson distributed.

As you saw in problem set 1, coherent states of the quantum harmonic oscillator have the same position and momentum uncertainties as the vacuum state. Similarly, in quantum field theory, coherent states of the field have the same ϕ and π uncertainties as the vacuum, while their expectation values behave classically. When the driving is strong, the uncertainties become negligible compared to the expectation values, and the number of particles becomes large so that we can no longer see their discreteness. We therefore recover a classical field.

3. Practice with time-ordered exponentials. (10 points)

Suppose the Hamiltonian is $H = H_0 + H_{\text{int}}$, where H_0 is a time-independent free Hamiltonian, and H_{int} is an interaction which could be time-dependent. In interaction picture, operators evolve under the free Hamiltonian H_0 alone, and the states are $|\psi(t)\rangle_I$. Let's first review some results derived in lecture.

a) Show that

$$i\,\partial_t|\psi(t)\rangle_I = H_{\text{int},I}(t)|\psi(t)\rangle_I. \tag{22}$$

where $H_{\text{int},I}(t)$ is the interaction Hamiltonian in the interaction picture.

Solution: By definition, matrix elements of operators are the same in all pictures, so for any states $|\psi\rangle$ and $|\psi'\rangle$, and any operator $\mathcal{O}$, we have

$${}_I\langle\psi'(t)|\mathcal{O}_I(t)|\psi(t)\rangle_I = {}_S\langle\psi'(t)|\mathcal{O}_S|\psi(t)\rangle_S. \tag{S53}$$

On the other hand, we know that

$$\mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t} \tag{S54}$$

so comparing the expressions gives

$$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S. \tag{S55}$$

Applying $i\,\partial_t$ to both sides gives

$$\begin{aligned}
i\,\partial_t|\psi(t)\rangle_I &= -H_0e^{iH_0t}|\psi(t)\rangle_S + ie^{iH_0t}\partial_t|\psi(t)\rangle_S && \text{(S56)}\\
&= -H_0|\psi(t)\rangle_I + e^{iH_0t}H_S|\psi(t)\rangle_S && \text{(S57)}\\
&= -e^{iH_0t}H_0e^{-iH_0t}|\psi(t)\rangle_I + e^{iH_0t}H_Se^{-iH_0t}|\psi(t)\rangle_I && \text{(S58)}\\
&= H_{\mathsf{int},I}(t)|\psi(t)\rangle_I && \text{(S59)}
\end{aligned}$$

as desired.

b) Show that if the interaction picture time evolution operator is defined as

$$|\psi(t_f)\rangle_I = U(t_f, t_i)|\psi(t_i)\rangle_I \tag{23}$$

for $t_f > t_i$, then it is given by Dyson's formula,

$$U(t_f, t_i) = T \exp\left(-i \int_{t_i}^{t_f} dt\, H_{\text{int},I}(t)\right) \tag{24}$$

where T denotes time ordering.

Solution: Abbreviating $H_{\text{int},I}(t)$ as $H(t)$, note that the time-ordered exponential is

$$U(t_f, t_i) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t_f} dt_n \int_{t_i}^{t_f} dt_{n-1} \cdots \int_{t_i}^{t_f} dt_1\, T(H(t_n)H(t_{n-1}) \cdots H(t_1)). \tag{S60}$$

We can incorporate the effect of the time ordering by changing the integration bounds,

$$U(t_f, t_i) = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^{t_f} dt_n \int_{t_i}^{t_n} dt_{n-1} \cdots \int_{t_i}^{t_2} dt_1\, H(t_n)H(t_{n-1}) \cdots H(t_1) \tag{S61}$$

where the factor of $1/n!$ cancelled because the new integration region has $1/n!$ times the volume of the original one. Now, part (a) tells us that we want to prove

$$i\partial_{t_f} U(t_f, t_i) = H(t_f)U(t_f, t_i). \tag{S62}$$

When we differentiating Eq. (S61) with respect to t_f, we only pick up one term from the upper bound of the outer integral, giving

$$i\partial_{t_f} U(t_f, t_i) = H(t_f) \sum_{n=1}^{\infty} (-i)^{n-1} \int_{t_i}^{t_f} dt_{n-1} \cdots \int_{t_i}^{t_2} dt_1\, H(t_{n-1}) \cdots H(t_1). \tag{S63}$$

Shifting the sum in n downward by one recovers $U(t_f, t_i)$, as desired.

Now let's do some concrete calculations with the time evolution operator.

c) Suppose the interaction is only turned on for two moments, i.e. it has the form

$$H_{\text{int},I}(t) = g\,(h_1\, \delta(t - t_1) + h_2\, \delta(t - t_2)). \tag{25}$$

Write out $U(t_f, t_i)$ up to and including terms of order g^2, assuming $t_i < t_1 < t_2 < t_f$.

Solution: We have

$$\int_{t_i}^{t_f} dt\, H_{\text{int},I}(t) = g(h_1 + h_2) \tag{S64}$$

so the exponential without the time-ordering would be

$$1 - ig(h_1 + h_2) - \frac{g^2}{2}(h_1 + h_2)^2 + O(g^3). \tag{S65}$$

The time-ordering puts contributions that occur later on the left, which means all the h_2 terms are to the left of the h_1 terms, giving

$$1 - ig(h_1 + h_2) - \frac{g^2}{2}(h_1^2 + 2h_2h_1 + h_2^2) + O(g^3). \tag{S66}$$

d) Now consider a general $H_{\text{int},I}(t)$ which is proportional to a coupling g. It is generally true that $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. Explicitly show that this result is true, up to and including terms of order g^2, in the case $t_1 < t_2 < t_3$.

Solution: The desired right-hand side is

$$U(t_3,t_1) = 1 - ig\int_{t_1}^{t_3} dt\, H(t) - g^2\int_{t_1}^{t_3} dt\int_{t_1}^{t} dt'\, H(t)H(t') + O(g^3). \tag{S67}$$

On the left-hand side, expanding the product of the two terms gives

$$\begin{aligned} U(t_3,t_2)U(t_2,t_1) &= 1 - ig\int_{t_1}^{t_2} dt\, H(t) - ig\int_{t_2}^{t_3} dt\, H(t) \\ -g^2\int_{t_1}^{t_2} dt\int_{t_2}^{t_3} dt'\, H(t)H(t') &- g^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\, H(t)H(t') - g^2\int_{t_2}^{t_3} dt\int_{t_2}^{t} dt'\, H(t)H(t') + O(g^3). \end{aligned} \tag{S68}$$

Clearly, the constant and $O(g)$ terms agree, so the tricky part is showing the equality of the $O(g^2)$ terms,

$$\begin{aligned} \int_{t_1}^{t_3} dt\int_{t_1}^{t} dt'\, H(t)H(t') = &\int_{t_1}^{t_2} dt\int_{t_2}^{t_3} dt'\, H(t)H(t') \\ &+ \int_{t_1}^{t_2} dt\int_{t_1}^{t} dt'\, H(t)H(t') + \int_{t_2}^{t_3} dt\int_{t_2}^{t} dt'\, H(t)H(t') \end{aligned} \tag{S69}$$

The left-hand side integrates over all $t_1 < t' < t < t_3$, and t_2 is between t_1 and t_3. All terms on the right-hand side enforce $t' < t$. The second term includes the region where both t and t' are less than t_2, the third term includes the regions where both are greater than t_2, and the first term corresponds to when t_2 is between t and t'.