стр-1

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① 1) $a, x \in \mathbb{R}^n$. $a^T x$ — скалярная ф-я.

$$\frac{\partial}{\partial x}(a^T x) = \frac{\partial}{\partial x}(a_1 x_1 + \ldots + a_n x_n) = \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (a_1 \ldots a_n) = a^T \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

2) Применим 1) к каждой строке матр. A. $A = \begin{pmatrix} c_1^T \\ \vdots \\ c_m^T \end{pmatrix}$.

$$\frac{\partial (Ax)}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x}(c_1^T x) \\ \frac{\partial}{\partial x}(c_2^T x) \\ \vdots \\ \frac{\partial}{\partial x}(c_m^T x) \end{bmatrix} = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix} = A$$

3) Якобиан соотв-т линейному члену разложения в ряд Тейлора $\qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$g(x) = x^T A x$$

$$g(x + \tilde{x}) = (x^T + \tilde{x}^T) A (x + \tilde{x}) =$$

$$= x^T A x + \underbrace{\tilde{x}^T A x}_{= x^T A^T \tilde{x}} + x^T A \tilde{x} = x^T A x + \underbrace{x^T (A + A^T)}_{= \frac{\partial}{\partial x}(x^T A x)} \tilde{x},$$

4) $\frac{\partial}{\partial x}\|x\|^2 = \frac{\partial}{\partial x} x^T x =$ [применим 3) для $A = I$] $= 2x^T I = 2x^T$.

5) $x \in R^n$. $g(x) = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix}$

$$\frac{\partial g(x)}{\partial x} = \begin{bmatrix} \frac{\partial g(x_1)}{\partial x_1} & \cancel{\frac{\partial g(x_1)}{\partial x_2}} \to 0 & \dots & \cancel{\frac{\partial g(x_1)}{\partial x_n}} \to 0 \\ \vdots & & & \vdots \\ \cancel{\frac{\partial g(x_n)}{\partial x_1}} \to 0 & \dots & & \frac{\partial g(x_n)}{\partial x_n} \end{bmatrix} =$$

$= \operatorname{diag}(g'(x_1), \dots, g'(x_n)) = \operatorname{diag}(g'(x))$.

6) $h: R^n \to R^m$ $g: R^m \to R^p$

$x \in R^n$. Выразим лин. член разл-я в ряд Тейлора: $g(h(x+\tilde{x})) \simeq g(\underbrace{h(x)}_{y} + \underbrace{\frac{\partial h}{\partial x}\tilde{x}}_{\tilde{y}}) \simeq$

$\simeq g(y) + \frac{\partial g}{\partial y}\tilde{y} = g(h(x)) + \underbrace{\frac{\partial g}{\partial h}\frac{\partial h}{\partial x}}_{= \frac{\partial}{\partial x}(g(h(x)))}\tilde{x}$

② $g(\beta) = \|X\beta - y\|^2 = (X\beta - y)^T (X\beta - y)$

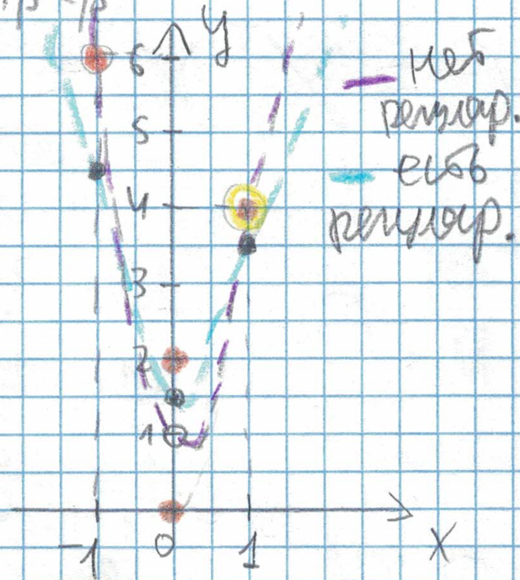
$$\frac{\partial g}{\partial \beta} = 2(X\beta - y)^T X = 2(\beta^T X^T X - y^T X)$$

$$\frac{\partial^2 g}{\partial \beta^T \partial \beta} = X^T X > 0 \text{ (положит. опр.)}$$

$g(\beta)$ – дифф-ма $\Rightarrow \hat\beta = \operatorname{argmin} g(\beta)$ – стац. точка

$$\Rightarrow \left.\frac{\partial g}{\partial \beta}\right|_{\beta=\hat\beta} = 0 \Rightarrow \left.\frac{\partial g^T}{\partial \beta}\right|_{\beta=\hat\beta} = 0 \Rightarrow$$

$$X^T X \hat\beta = X^T y.$$



③

x	1	1	0	0	-1
y	4	4	0	2	6

1) ✓ 2)

$$X = \begin{array}{ccc} & & \\ \left[\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{matrix}\right] \\ \uparrow 1 \quad \uparrow x \quad \uparrow x^2 \end{array}$$

$$X^T X = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix} \qquad X^T y = \begin{bmatrix} 16 \\ 2 \\ 14 \end{bmatrix}$$

$$\begin{array}{ccc} \beta_0 & \beta_1 & \beta_2 \end{array}$$
$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & 16 \\ 1 & 3 & 1 & 2 \\ 3 & 1 & 3 & 14 \end{array}\right] \Rightarrow \beta_0 = 1 \quad \beta_1 = -1 \quad \beta_2 = 4$$

$$f(x) = 1 - x + 4x^2 = \left(2x - \frac{1}{4}\right)^2 + \frac{15}{16}$$

Регуляриз. сист.:

$$\begin{array}{ccc} \tilde\beta_0 & \tilde\beta_1 & \tilde\beta_2 \end{array}$$
$$\left[\begin{array}{ccc|c} 6 & 1 & 3 & 16 \\ 1 & 4 & 1 & 2 \\ 3 & 1 & 4 & 14 \end{array}\right] \quad \tilde\beta_0 = \frac{3}{2} \quad \tilde\beta_1 = -\frac{1}{2} \quad \tilde\beta_2 = \frac{5}{2} \quad f = \frac{3 - x + 5x^2}{2}$$

Стр. 4.

(17)

X_1	4	0	-1	3	4
X_2	2	-3	-2	1	2
X_3	3	2	2	1	-3

$$X = \begin{bmatrix} 4 & 2 & 3 \\ 0 & -3 & 2 \\ -1 & -2 & 2 \\ 3 & 1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$$

1. Центрируем данные:

$$X_c = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & -4 \end{bmatrix}$$

$\bar{x} = (2, 0, 1)$

2. Находим:

$$C = X_c^T X_c = \begin{bmatrix} 22 & 21 & -9 \\ 21 & 22 & -9 \\ -9 & -9 & 22 \end{bmatrix} \qquad N = 5$$

$$\frac{1}{N-1} C = \begin{bmatrix} \frac{11}{2} & \frac{21}{4} & -\frac{9}{4} \\ \frac{21}{4} & \frac{11}{2} & -\frac{9}{4} \\ -\frac{9}{4} & -\frac{9}{4} & \frac{11}{2} \end{bmatrix}$$ – выбор. матр. ковариации

3. Ищем главные напр-я и дисперсии по главным компонентам.

$$\det(C - \lambda I) = \begin{vmatrix} 22-\lambda & 21 & -9 \\ 21 & 22-\lambda & -9 \\ -9 & -9 & 22-\lambda \end{vmatrix} =$$

$$= -\lambda^3 + 66\lambda^2 - 849\lambda + 784 = -(\lambda-1)(\lambda-16)\cdot(\lambda-49)$$

$\lambda_1 = 1 \quad v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda_2 = 16 \quad v_2 = \frac{\sqrt{11}}{11}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

$\lambda_3 = 49 \quad v_3 = \frac{\sqrt{22}}{22}\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$

Стр. 5

Главные компоненты:

$$v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} \quad v_2 = \frac{\sqrt{11}}{11}\begin{pmatrix}1\\1\\3\end{pmatrix} \quad v_3 = \frac{\sqrt{22}}{22}\begin{pmatrix}3\\3\\-2\end{pmatrix}$$

Дисперсии:

$$\frac{1}{N-1}\lambda_1 = \frac{1}{4} = 0{,}25 \quad \frac{1}{N-1}\lambda_2 = \frac{16}{4} = 4 \quad \frac{1}{N-1}\lambda_3 = \frac{49}{4} = 12{,}25$$