Math 341: Midterm 2

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§1

LATEXis cool! You can write things like:

> **Theorem** (Rank-nullity theorem)
> Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, and let $\mathbf{T} : \mathbf{V} \to \mathbf{T}$ be a linear

§2

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1}$$

a. Suppose that $a \neq 0$, compute the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ using row reduction and provide the conditions on a, b, c, d such that your computations are valid. Express the result as a simplified expression. (**Hint:** recall that you can not divide by zero)

Proof. We perform reduced row echelon form (rref) on the augmented matrix

$$
\begin{aligned}
(A|b) &= \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \\
R_2 \leftarrow R_2 - \frac{c}{a}R_1 &\left[\begin{array}{cc|c} a & b & e \\ 0 & d - \frac{cb}{a} & f - \frac{ce}{a} \end{array}\right] \\
&\left[\begin{array}{cc|c} a & b & e \\ 0 & \frac{ad-cb}{a} & \frac{af-ce}{a} \end{array}\right] \\
R_2 \leftarrow \frac{a}{ad-cb}R_2 \quad \text{Assuming that } ab - cd \neq 0 &\left[\begin{array}{cc|c} a & b & e \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow R_1 - bR_2 &\left[\begin{array}{cc|c} a & 0 & e - b\frac{af-ce}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow \frac{R_1}{a} &\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}(e - b\frac{af-ce}{ad-cb}) \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
&\left[\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]
\end{aligned}
$$

$$x = \begin{bmatrix} \frac{de-bf}{ad-cb} \\ \frac{af-ce}{ad-cb} \end{bmatrix} = \frac{1}{ad-cb}\begin{bmatrix} de - bf \\ af - ce \end{bmatrix} \quad \text{where } ad - cb \neq 0$$

□

b. If $a = 0$, and $c \neq 0$, is your above computation still valid? How would you modify it? (explain briefly) (**Hint:** recall that you can swap the equations and the result is the same)

Proof. If $a = 0$, and $c \neq 0$, then the above computation will not be valid as we divided by a multiple times when we computed the rref. I would swap the first and second rows so that it would look like

$$\left[\begin{array}{cc|c} c & d & f \\ 0 & b & e \end{array}\right]$$

and compute the rref, assuming that $b \neq 0$. We obtain the rref,

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{bf-de}{bc} \\ 0 & 1 & \frac{e}{b} \end{array}\right]$$

and the solution

$$x = \begin{bmatrix} \frac{bf-de}{bc} \\ \frac{e}{b} \end{bmatrix} \quad \text{where } b \neq 0$$

□

c. If $a = 0$, $c = 0$, but $b \neq 0$, $d \neq 0$, what are the conditions on e and f such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution? Is the solution unique? (**Hint:** recall that $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$)

Proof. If $a = 0$, $c = 0$, $b \neq 0$, $d \neq 0$, we get the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & e \\ 0 & d & f \end{array}\right]$$

Performing row reduction,

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{e}{b} \\ 0 & 1 & \frac{f}{d} \end{array}\right]$$

Having an infinite amount of solutions is by definition another way of saying that a system that is consistent and that the solutions are not unique.

It is clear that $\det(\mathbf{A}) = 0$ by multiplying the diagonal entries because it is an upper triangular matrix. This means that the solution, if it exists, is not unique by Theorem 3.10 and the corollary to Theorem 4.7. A system is consistent if and only if $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ by Theorem 3.11. For this condition to hold,

$$\mathbf{b} \in span(\mathbf{A}) \Leftrightarrow \begin{pmatrix} \frac{e}{b} \\ \frac{f}{d} \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ where } c_1, c_2 \in F$$

So the condition of the solution is,

$$c_2 = \frac{e}{b}$$
$$c_2 = \frac{f}{d}$$
$$\frac{e}{b} = \frac{f}{d}$$

Thus, there exists a infinite amount of solution. □

d. Solve the system

$$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}. \tag{2}$$

(**Hint:** You may want to use the formula you just deduced)

Proof.

$$
\begin{aligned}
x_1 &= \frac{de - bf}{ad - cb} \\
&= \frac{\sqrt{2}(5\sqrt{2}) - 3\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\
&= \frac{10 - 30}{2 - 12} \\
&= \frac{-20}{-10} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \frac{af - ce}{ad - cb} \\
&= \frac{\sqrt{2}(5\sqrt{2}) - 2\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\
&= \frac{10 - 20}{-10} \\
&= 1
\end{aligned}
$$

□

§3

Let

$$
\mathbf{A} = \begin{bmatrix} 0 & -\alpha & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 3 \\ -2 & -2 & 4 & 2\alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 \end{bmatrix}, \qquad \text{and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 + \alpha \\ 2\beta + \alpha - 2 \end{bmatrix} \tag{3}
$$

What are the conditions on α and β such that the system $\mathbf{Ax} = \mathbf{b}$:

a. Has no solution?

Proof. We begin by putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ in its reduced form.

$$(\mathbf{A}|\mathbf{b}) = \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & \alpha & -1 & 2\alpha+1/2 & 2\beta+\alpha-2 \end{array}\right]$$

$$R_5 \leftarrow R_5 + R_1 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_3 \leftarrow R_3 - 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 + 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2\alpha+2 & 4+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 - 2R_3, R_5 \leftarrow R_5 - \frac{1}{2}R_3 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 2\beta+\alpha+1 \end{array}\right]$$

$$R_5 \leftarrow R_5 - R_4 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta-1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad (\mathbf{A}'|\mathbf{b}') = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta-1 \end{array}\right]$$

By Theorem 3.11 and 3.13, a system is consistent if and only if $rank(\mathbf{A}') = rank(\mathbf{A}'|\mathbf{b}')$. Thus this system will have no solution if $2\beta - 1 \neq 0$ because then $\mathbf{b}' \notin span(\mathbf{A}')$.

This is when $\beta \neq \frac{1}{2}$. We observe that there will be no conditions on α. □

b. Has an unique solution? Find the solution. (**Hint:** you will need to row reduce the augmented system to echelon form, and then use the theorems seen in class to impose the conditions on α and β).

Proof. Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(\mathbf{A}) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &\neq 0 \\ -4\alpha^2 &\neq 0 \\ \alpha &\neq 0 \end{aligned}$$

and from (a), $\beta = \frac{1}{2}$. Combining these two conditions we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By performing back substitution we compute the unique solution

$$\begin{aligned} 2\alpha x_4 = \alpha &\Leftrightarrow x_4 = \frac{1}{2} \\ 2x_3 + x_4 = 2 &\Leftrightarrow x_3 = \frac{3}{4} \\ -\alpha x_2 + 2x_3 = 2 &\Leftrightarrow x_2 = -\frac{1}{2\alpha} \\ x_1 + x_2 + x_4 = 1 &\Leftrightarrow x_1 = \frac{1}{2} + \frac{1}{2\alpha} \end{aligned}$$

$$x = \begin{bmatrix} \frac{1}{2} + \frac{1}{2\alpha} \\ -\frac{1}{2\alpha} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

as desired. □

c. Has infinite amount of solutions? Find the solution set in parametric form. (**Hint:** You may have one equations for α and one for β that have to be satisfied simultaneously).

Proof. Having an infinite amount of solutions is by definition another way of saying that a system that is consistent and that the solutions are not unique. A system is consistent if and only if $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ by Theorem 3.11. If $\det(\mathbf{A}) = 0$, the solution, if it exists, is not unique by Theorem 3.10 and the corollary to Theorem 4.7.

Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &= 0 \\ -4\alpha^2 &= 0 \\ \alpha &= 0 \end{aligned}$$

Given that $\alpha = 0$, and that $\beta = \frac{1}{2}$ from part (a) we get the following system,

$$(\mathbf{A}'|\mathbf{b}') = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It is clear that $rank(\mathbf{A}') = rank(\mathbf{A}'|\mathbf{b}')$ because $\mathbf{b}'$ is a linear combination of the second and third column from $\mathbf{A}'$. By Theorem 3.13, we know that the solution set for $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ is equivalent to the solution set for $\mathbf{Ax} = \mathbf{b}$. Hence, both systems are consistent.

We can compute a solution space to $\mathbf{Ax} = \mathbf{b}$ as outlined in Theorem 3.9. We start by first computing the solution set to $\mathbf{Ax} = 0$ denoted by K_H. It is clear that $rank(\mathbf{A}) = 3$ because the first two columns are the same and the rest of the columns are linearly independent from each other. By Theorem 3.8, $dim(K_H) = 4 - 3 = 1$. Thus any nonzero solution constitutes a basis for K. For example, since

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

is a solution to the $\mathbf{Ax} = 0$, it is a basis for K_H by Corollary 2 of Theorem 1.10. So a solution set to K_H would be

$$K_H = \left\{ t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

A solution to $\mathbf{Ax} = \mathbf{b}$ is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, by Theorem 3.9 we compute solution space when this system has infinite amount of solutions, which is when $\alpha = 0$ and $\beta = \frac{1}{2}$ as

$$K = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$\square$

§4

Let $A \in M_{n\times n}(F)$, for a field F. We want to prove that $rank(A^2) - rank(A^3) \leq rank(A) - rank(A^2)$. The solution to this exercise requires the notion of quotient spaces. Even though you should already be familiar with quotient spaces we will prove a few properties that will be useful.

Let W be a subspace of a vector space V over a field F. For any $v \in$ V the set $\{v\} + \mathrm{W} = \{v + w : w \in \mathrm{W}\}$ is called the coset of W containing v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$. Following this notation we write $\mathrm{V}/\mathrm{W} = \{v + \mathrm{W} : v \in \mathrm{V}\}$, which is usually called the quotient space V module W. Addition and scalar multiplication by scalars can be defined in the collection V/W as follows:

$$(v_1 + \mathrm{W}) + (v_2 + \mathrm{W}) = (v_1 + v_2) + \mathrm{W}$$

for all $v_1, v_2 \in$ V and

$$a(v + W) = av + W$$

for all $v \in$ V and $a \in F$

a. Prove that $v_1 + \text{W} = v_2 + \text{W}$ if and only if $v_1 - v_2 \in \text{W}$. (**Hint**: recall that $v_1 + \text{W}$ is a set, thus you need to prove equality between sets)

Proof.

i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$

If $v_1 + W = v_2 + W$, then $\exists w_1, w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$
$v_1 - v_2 = w_2 - w_1$
Since, $w_2 - w_1 \in W$ (clourse under addition)
Therefore, $v_1 - v_2 \in W$

ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$

This means $v_1 - v_2 = w$ where $w \in W$ $(*)$
Now let $x \in v_1 + W$
By definition, $\exists w_x \in W$ such that $x = v_1 + w_x$
By $(*)$ $v_1 = v_2 + w$
So, $x = v_2 + w + w_x$
Since, $w + w_x \in W$ (closure under addition)
We have $x \in v_2 + W$
So, $v_1 + W \subseteq v_2 + W$
Without loss of generality, we can show $v_2 + W \subseteq v_1 + W$
Therefore, $v_1 + W = v_2 + W$

Therefore, $v_1 + \text{W} = v_2 + \text{W}$ if and only if $v_1 - v_2 \in \text{W}$ □

b. Show that V/W with the operations defined above is a linear vector space.

VS 1: For all x, y in V/W, $x + y = y + x$ (commutativity of addition)

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} x + y &= (v_x + W) + (v_y + W) \\ &= (v_x + v_y) + W \\ &= (v_y + v_x) + W \\ &= (v_y + W) + (v_x + W) \\ &= y + x \end{aligned}$$

□

VS 2: For all x, y, z in V/W, $(x + y) + z = x + (y + z)$ (associativity of addition)

Proof. Let $x = v_x + W, y = v_y + W, z = v_z + W$, where $v_x, v_y, v_z \in V$.

$$\begin{aligned} (x + y) + z &= ((v_x + W) + (v_y + W)) + (v_z + W) \\ &= ((v_x + v_y) + W) + (v_z + W) \\ &= (v_y + v_x + v_z) + W \\ &= (v_x + W) + ((v_y + v_z) + W) \\ &= (v_x + W) + ((v_y + W) + (v_z + W)) \\ &= x + (y + z) \end{aligned}$$

□

VS 3: There exists an element in V/W denoted by $\mathbf{0}$ such that $x + \mathbf{0} = x$ for each x in V/W

Proof. Let $x = v_x + W$ where $v_x \in V$. Let $\mathbf{0}$ in V/W be defined as $0 + W$, where $0 \in V$.

$$\begin{aligned} x + \mathbf{0} &= x + (0 + W) \\ &= (v_x + W) + (0 + W) \\ &= (v_x + 0) + W \\ &= v_x + W \\ &= x \end{aligned}$$

□

VS 4: For each element x in V/W there exists an element y in V/W such that $x + y = \mathbf{0}$

Proof. Let $x = v_x + W$ where $v_x \in V$. Fix y such that $y = -v_x + W$.

$$\begin{aligned} x + y &= (v_x + W) + (-v_x + W) \\ &= (v_x - v_x) + W \\ &= 0 + W \\ &= \mathbf{0} \end{aligned}$$

□

VS 5: For each element x in V/W, $1x = x$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} 1x &= 1(v_x + W) \\ &= (1v_x + W) \\ &= (v_x + W) \\ &= x \end{aligned}$$

□

VS 6: For each pair of elements a, b in $\mathbb{F}$ and each element x in V/W, $(ab)x = a(bx)$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} (ab)x &= abv_x + W \\ &= a(bv_x + W) \\ &= a(bx) \end{aligned}$$

□

VS 7: For each element a in $\mathbb{F}$ and each pair of elements x, y in V/W, $a(x + y) = ax + ay$

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} a(x + y) &= a((v_x + v_y) + W) \\ &= ((av_x + av_y) + W) \\ &= (av_x + W) + (av_y + W) \\ &= a(v_x + W) + a(v_y + W) \\ &= ax + ay \end{aligned}$$

□

VS 8: For each pair of elements a, b in $\mathbb{F}$ and each element x in V/W, $(a+b)x = ax + bx$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned}(a+b)x &= (a+b)(v_x + W) \\ &= ((a+b)v_x) + W \\ &= ((av_x + bv_x) + W \\ &= (av_x + W) + (bv_x + W) \\ &= a(v_x + W) + b(v_x + W) \\ &= ax + bx\end{aligned}$$

□

Therefore, V/W is a vector space because it holds all the properties above.

c. Prove that if $dim(\mathrm{V}) < \infty$ then $dim(\mathrm{V/W}) = dim(\mathrm{V}) - dim(\mathrm{W})$. (Hint: Define a linear map $T : \mathrm{V} \to \mathrm{V/W}$ such that the range of T is V/W, and then use the rank-nullity theorem)

Proof. We define the linear map $T : \mathrm{V} \to \mathrm{V/W}$ by

$$T(v) = v + W$$

We first prove that T is in fact linear, where $v_1, v_2 \in V$ and $c \in F$.

$$\begin{aligned}T(cv_1 + v_2) &= (cv_1 + v_2) + W \\ &= (cv_1 + W) + ((v_2) + W) \\ &= c(v_1 + W) + ((v_2) + W) \\ &= cT(v_1) + T(v_2)\end{aligned}$$

We claim that $N(T) = W$ and $R(T) = \mathrm{V/W}$

1. $R(T) = \mathrm{V/W}$
   i. $R(T) \subseteq \mathrm{V/W}$
      By Theorem 2.1
   ii. $\mathrm{V/W} \subseteq R(T)$
      Let $y \in \mathrm{V/W}$ such that $y = v_y + W$ where $v_y \in V$.
      To prove that $y \in R(T)$ we need to show that $\exists x$ such that $T(x) = y$.
      Notice that we can fix $x = v_y$. Therfore, $y \in R(T)$ so $\mathrm{V/W} \subseteq R(T)$.
2. $N(T) = W$
   i. $N(T) \subseteq W$

Lemma 4.1

$w_1 + W = w_2 + W = W$ where $w_1, w_2 \in W$.

Proof.
Let $x_1 \in w_1 + W$ and $x_2 \in w_2 + W$
By definition, $w_1 + W = \{w_1 + w : w \in W\}$.
We know that $w_1 + w \in W$ because W is a subspace and has closure under addition. Hence, $x_1 \in W$ and similarly $x_2 \in W$. Therefore, $w_1 + W \subseteq W$ and $w_2 + W \subseteq W$.

Now consider $x_1 \in W$ and $x_2 \in W$.
$x_1 \in w_1 + W$ because $\exists w$ s.t. $w = x_1 - w_1$. Similarly, $x_2 \in w_2 + W$.
Hence, $W \subseteq w_1 + W$ and $W \subseteq w_2 + W$.

Therefore, $w_1 + W = W$ and $W = w_2 + W$ so, $w_1 + W = w_2 + W = W$. □

Let $x \in N(T)$.
By definition of $N(T)$ and using the lemma above,

$$\begin{aligned} T(x) &= \mathbf{0} \\ &= 0 + W \\ &= w + W \end{aligned}$$

where $0, w \in W$ and $\mathbf{0} \in \mathrm{V/W}$.
This must mean that $x = w$ and since $w \in W$, $x \in W$. Therefore, $N(T) \subseteq W$

ii. $W \subseteq N(T)$
Let $w \in W$.

$$\begin{aligned} T(w) &= w + W \\ &= 0 + W \\ &= \mathbf{0} \end{aligned}$$

So $w \in N(T)$. Therefore, $W \subseteq N(T)$.

Notice that V/W $\subseteq$ V so V/W is finite dimensional. Since T is a linear map and that V and V/W are indeed finite dimensional vector spaces (by part b) we can use the rank-nullity theorem (Theorem 2.3).

$$\begin{aligned} dim(N(T)) + dim(R(T)) &= dim(V) \\ dim(W) + dim(\mathrm{V/W}) &= dim(V) \\ dim(\mathrm{V/W}) &= dim(V) - dim(W) \end{aligned}$$

as desired. □

d. Let $K = F^n$, define $AK = R(L_A)$, and $A^2K = R(L_{A^2})$. Show that AK/A^2K is a vector space of dimension $rank(A) - rank(A^2)$.

Proof. We begin by proving that AK/A^2K is a vector space. It is enough to show that A^2K is a subspace of AK to prove that AK/A^2K is a vector space, by part (b). It is clear that AK is a subspace because $AK = R(L_A)$ and Theorem 2.3.

We claim that $R(L_{A^2}) \subseteq R(L_A)$. Let $y \in R(L_{A^2})$. By definition, $\exists x \in F^n : A^2x = y$. We can rewrite $A^2x = y$ as $A(Ax) = y$ so it follows that $y \in R(L_A)$.

Thus, $A^2K \subseteq AK$ because $R(L_{A^2}) \subseteq R(L_A)$. Because $A^2K = R(L_{A^2})$, A^2K has the properties of a vector space because of Theorem 2.1. It follows from part (c) that

$$\begin{aligned} dim(AK/A^2K) &= dim(AK) - dim(A^2K)) \\ &= dim(R(L_A)) - dim(R(L_{A^2})) \\ &= rank(A) - rank(A^2) \end{aligned}$$

as desired. □

e. Show that A^2K/A^3K is a vector space of dimension $rank(A^2) - rank(A^3)$, where $A^3K = R(L_{A^3})$.

Proof. Similiar to what we did in part (d), we can prove that A^2K/A^3K is a vector space. Then it follows that,

$$\begin{aligned} dim(A^2K/A^3K) &= dim(A^2K) - dim(A^3K)) \\ &= dim(R(L_{A^2})) - dim(R(L_{A^3})) \\ &= rank(A^2) - rank(A^3) \end{aligned}$$

as desired. □

f. Define $T : AK/A^2K \to A^2K/A^3K$, by $T(v) = L_A(v)$, i.e, we left multiply each element of v by the matrix A. Show that $R(T) = A^2K/A^3K$.

Proof. To show that $R(T) = A^2K/A^3K$, is another way of saying show that T is onto. We must show that $\forall y \in A^2K/A^2K[\exists x \in AK/A^2K : T(x) = y]$. Let $y \in A^2K/A^3K$ such that $y = A^2k + A^3K$ where $k \in \mathbb{F}^n$. We claim that $T(x) = y$ where $x = Ak + A^2K$.

$$\begin{aligned} T(x) &= T(Ak + A^2K) \\ &= L_A(Ak + A^2K) \\ &= L_A(Ak) + L_A(A^2K) \\ &= A^2k + A^3K \\ &= y \end{aligned}$$

as desired. Therefore, T is onto i.e. $R(T) = A^2K/A^3K$. □

g. Use the rank-nullity theorem on T to conclude that $rank(A^2) - rank(A^3) \leq rank(A) - rank(A^2)$.

Proof. We begin by proving that T is linear. We define $x_1 = Ak_1 + A^2K$ and $x_2 = Ak_2 + A^2K$ such that $x_1, x_2 \in AK/A^2K$ and $k_1, k_2 \in F^n$ and $c \in F$. We use properties of matrices as outlined in Theorem 2.12.

$$\begin{aligned} T(ck_1 + k_2) &= L_A(ck_1 + k_2) \\ &= A(ck_1 + k_2) \\ &= A(c(Ak_1 + A^2K) + (Ak_2 + A^2K)) \\ &= A(c(Ak_1 + A^2K)) + A(Ak_2 + A^2K) \\ &= c(A(Ak_1 + A^2K)) + A(Ak_2 + A^2K) \\ &= cL_A(k_1) + L_A(k_2) \\ &= cT(k_1) + T(k_2) \end{aligned}$$

Since T is linear, we can use rank nullity theorem. From part (d), (e), (f), and assuming that $nullity(T) \geq 0$ because the number of elements of a set cannot be negative

$$\begin{aligned} rank(T) + nullity(T) &= dim(AK/A^2K) \\ &= rank(A) - rank(A^2) \\ rank(T) &\leq rank(A) - rank(A^2) \\ rank(A^2) - rank(A^3) &\leq rank(A) - rank(A^2) \end{aligned}$$

as desired. □

§5

Let V be a finite-dimensional vector space. Let T and P be two linear transformations from V to itself, such that $T^2 = P^2 = 0$, and $T \circ P + P \circ T = I$, where I is the identity in V.

a. Denote $N_T = N(T)$ and $N_P = N(P)$, the null spaces of T and P, respectively. Show that $N_P = P(N_T)$, and $N_T = T(N_P)$, where $T(N_P) = \{T(v) : v \in N_P\}$ and $P(N_T) = \{P(v) : v \in N_T\}$.

Proof.

i. Show $N_P = P(N_T)$

1. $N_P \subseteq P(N_T)$

Let $x \in N_P$. By definition, $P(x) = 0$.

$$\begin{aligned} P \circ T(x) + T \circ P(x) &= x \\ P \circ T(x) + T(0) &= x \\ P \circ T(x) &= x \end{aligned}$$

$T(x) \in N_T$ because $T \circ T(x) = 0$. So $x \in P(N_T)$. Thus, $N_p \subseteq P(N_T)$.

2. $P(N_T) \subseteq N_P$

Let $P(x) \in P(N_T)$ where $x \in N_T$.
$P \circ P(x) = 0$ so $P(x) \in N_P$. Thus, $P(N_T) \subseteq N_P$.

Therfore, $N_P = P(N_T)$.

ii. Show $N_T = T(N_P)$

1. $N_T \subseteq T(N_P)$

Let $x \in N_T$. By definition, $T(x) = 0$.

$$\begin{aligned} P \circ T(x) + T \circ P(x) &= x \\ P(0) + T \circ P(x) &= x \\ T \circ P(x) &= x \end{aligned}$$

Notice that $P(x) \in N_P$, so $x \in T(N_P)$. Thus, $N_T \subseteq T(N_P)$.

2. $T(N_P) \subseteq N_T$

Let $T(x) \in T(N_P)$ where $x \in N_P$.
$T \circ T(x) = 0$ so $T(x) \in N_T$. Thus, $T(N_P) \subseteq N_T$.

Therfore, $N_P = P(N_T)$.

□

b. Show that $\mathrm{V} = N_T \oplus N_P$.

Proof. Need to prove the following two conditions.

i. $N_T \cap N_P = \{0\}$.

Let $x \in N_T$ and $x \in N_P$. This means that $T(x) = 0$ and $P(x) = 0$.

$$\begin{aligned} x &= P \circ T(x) + T \circ P(x) \\ &= P(0) + T(0) \\ &= 0 \end{aligned}$$

Thus, $N_T \cap N_P = \{0\}$.

ii. $N_T + N_P = \mathrm{V}$.

1. $N_T + N_P \subseteq \mathrm{V}$
   Let $x \in N_T + N_P$ such that $x = n_t + n_p$ where $n_t \in N_T \subseteq \mathrm{V}$ and $n_p \in N_P \subseteq \mathrm{V}$ by Theorem 2.11. Since V is a vector space $n_t + n_p \in V$ by closure under addition. Thus, $x \in \mathrm{V}$ and it follows that $N_T + N_P \subseteq \mathrm{V}$.
2. $\mathrm{V} \subseteq N_T + N_P$
   Let $x \in V$. We know that $x = P \circ T(x) + T \circ P(x)$.
   Notice that $P \circ T(x) \in N_P$ because $P \circ (P \circ T(x)) = 0$.
   Similarly, $T \circ P(x) \in N_T$ because $T \circ (T \circ P(x)) = 0$.
   Thus, $N_T + N_P \subseteq \mathrm{V}$.

Therefore, $N_T + N_P = \mathrm{V}$.

Therefore, by definition $\mathrm{V} = N_T \oplus N_P$. □

c. Prove that the dimension of V is even.

Proof.

Lemma 5.1

Let $W_1 + W_2 = V$. Then, V $= W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$

Proof. We begin by proving that $dim(W_1+W_2) = dim(W_1)+dim(W_2)-dim(W_1 \cap W_2)$
Let

$$B_{1\cap 2} = \{u_1, u_2, \cdots, u_k\}$$

be a basis for $W_1 \cap W_2$
By using the replacement theorem, we can extend $B_{1\cap 2}$ to be a basis for W_1.
So the basis for W_1 is B_1

$$B_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$$

Likewise, we can extend $B_{1\cap 2}$ to be a basis for W_2

$$B_2 = \{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_p\}$$

Basis for $W_1 + W_2$ will be $B_1 \cup B_2$, however they may contain the same vectors twice.
To prevent double counting, we must subtract $B_1 \cap B_2$ from $B_1 \cup B_2$
Thus the basis for $W_1 + W_2$ is

$$B_{1+2} = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$$

It follows that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Now we continue the proof for the lemma.

i. $V = W_1 \oplus W_2 \Rightarrow dim(V) = dim(W_1) + dim(W_2)$
From the definition of direct sum, $W_1 \cap W_2 = \{0\}$
This means $dim(W_1 \cap W_2) = 0$. Hence,

$$\begin{aligned} dim(V) &= dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) \\ &= dim(W_1) + dim(W_2) - 0 \\ &= dim(W_1) + dim(W_2) \end{aligned}$$

as desired.

ii. $dim(V) = dim(W_1) + dim(W_2) \Rightarrow V = W_1 \oplus W_2$
$V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$
$V = W_1 + W_2$ is a preconditon for this lemma.
We proved earlier that, $dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$
Our hypothesis is $dim(V) = dim(W_1) + dim(W_2)$
Setting the two equations equal to each other:

$$\begin{aligned} dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) &= dim(W_1) + dim(W_2) \\ dim(W_1 \cap W_2) &= 0 \end{aligned}$$

This means $W_1 \cap W_2 = \{0\}$. Thus, $dim(V) = dim(W_1) + dim(W_2)$. Therefore, by definition $V = W_1 \oplus W_2$.

Therefore, if $W_1 + W_2 = V$, then $V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$ □

We begin by claiming that $dim(T(N_P)) \leq dim(N_P)$. Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be a basis for N_P. We want to show that $\gamma = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ spans $T(N_P)$ so that the above statement holds. Let $w \in T(N_P)$. This means that $\exists x \in N_P : T(x) = w$. We can write x as a linear combination of β. $\exists c_1, \cdots, c_n \in F$ such that $x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

Applying T to both sides of this equation,

$$\begin{aligned} x &= c_1v_1 + c_2v_2 + \cdots + c_nv_n \\ T(x) &= T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) \\ &= T(c_1v_1) + T(c_2v_2) + \cdots + T(c_nv_n) \\ &= c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) \end{aligned}$$

Notice that $w = T(x)$. Since w was picked arbitrarily, γ spans $T(N_P)$. It directly follows that $dim(T(N_P)) \leq n$ because $dim(T(N_P))$ is only as large as its basis which is a minimal spanning set. Therefore, $dim(T(N_P)) \leq dim(N_P)$. Without loss of generality, we can show that $dim(P(N_T)) \leq dim(N_T)$

It follows from the claim above and from (a) that

$$\begin{aligned} dim(T(N_P)) &\leq dim(N_P) \\ dim(N_T) &\leq dim(N_P) \end{aligned}$$

$$\begin{aligned} dim(P(N_T)) &\leq dim(N_T) \\ dim(N_P) &\leq dim(N_T) \end{aligned}$$

This means that $dim(N_P) = dim(N_T)$

By using the lemma above,

$$\begin{aligned} dim(V) &= dim(N_T) + dim(N_P) \\ &= dim(N_T) + dim(N_T) = dim(N_P) + dim(N_P) \\ &= 2dim(N_T) = 2dim(N_P) \end{aligned}$$

as desired. Therefore, $dim(V)$ is even. □

d. Suppose that the dimension of V is two. Prove that V has a basis β, such that

$$[T]_\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad [P]_\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Proof. We claim that $T^2 = P^2 = 0$ implies that T and P are not invertible.

Assume for the sake of contradiction that T had an inverse U. Then

$$UT = I \Rightarrow UT(T) = IT \Rightarrow UT^2 = T \Rightarrow U0 = T \Rightarrow T = 0$$

which is a contradiction because the zero matrix is not left invertible. Therefore, T is not invertible and without loss of generality, P is not invertible.

This means that N_T, N_P are nontrivial null spaces. From (b), we know that $V = N_T \oplus N_P$. Asumming that $dim(V) = 2$, we get that $2 = dim(N_T) + dim(N_P)$ by our lemma from (c). Since N_T, N_P are nontrivial null spaces, this necessarily means that $dim(N_T) = dim(N_P) = 1$.

Since N_P is non trival, there exists a nonzero vector in N_P. Let $v \in N_P$ be nonzero. Since $dim(N_P) = 1$, any nonzero vector forms a basis for N_P So, $\{v\}$ is a basis for N_P.

Since $V = N_T \oplus N_P$, by definition $N_T \cap N_P = \{0\}$. So $v \notin N_T$. Thus, $T(v) \neq 0$. We know that $T(v) \in N_T$, is nonzero, and that $dim(N_T) = 1$, so $\{T(v)\}$ is a basis for N_T.

We claim that $\beta = \{T(v), v\}$ is a basis for V. Notice that $|\beta| = dim(V)$ and $T(v), v \in V$. So, it is enough to show that β is linearly independent for β to be a basis for V by Theorem 1.10 Corollary 2(b). Let $c_1, c_2 \in F$.

$$\begin{aligned}
c_1 T(v) + c_2 v &= 0 \\
c_1 T(v) &= -c_2 v \\
T(c_1 T(v)) &= T(-c_2 v) \\
c_1 T(T(v)) &= -c_2 T(v) \\
c_1 0 &= -c_2 T(v) \\
0 &= -c_2 T(v) \\
\frac{0}{T(v)} &= -c_2 \\
c_1 = c_2 &= 0
\end{aligned}$$

as desired. β is a basis for V.

Now we can compute $[T]_\beta$ and $[P]_\beta$.

$$\begin{aligned}
[T]_\beta &= \Big[[T(T(v))]_\beta \; [T(v)]_\beta \Big] \\
&= \Big[[0]_\beta \; [T(v)]_\beta \Big] \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

We claim that $P(T(v)) = v$. Given that $T \circ P + P \circ T = I$,

$$\begin{aligned}
P(T(v)) + T(P(v)) &= v \\
P(T(v)) &= v - T(P(v)) \\
&= v
\end{aligned}$$

because $v \in N_P$. We finish with computing $[P]_\beta$.

$$\begin{aligned}
[P]_\beta &= \Big[[P(T(v))]_\beta \; [P(v)]_\beta \Big] \\
&= \Big[[v]_\beta \; [0]_\beta \Big] \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

as desired. $\square$