Math 341: Homework 7

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§1 A

For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution space.

a. $\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

It is clear that rank(A) = 1 because the two columns are a multiples of each other. If K is the solution set of this system, then dim(K) = 2 - 1 = 1. Thus any nonzero solution constitutes a basis for K. For example, since $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is a solution to the given system, $\left\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$ is a basis for K by Corollary 2 of Theorem 1.10. □

b. $\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

It is clear that rank(A) = 2 because there are two linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If K is the solution set of this system, then dim(K) = 3 - 2 = 1. Thus any nonzero solution constitutes a basis for K. For example, since $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is a solution to the given system, $\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right\}$ is a basis for K by Corollary 2 of Theorem 1.10. □

c. $\{x_1 + 2x_2 - 3x_3 + x_4 = 0$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}$$

It is clear that rank(A) = 1 because there are one linearly independent row (rank of a transposed matrix is the same as the rank of a matrix). If K is the solution set of this system, then dim(K) = 4 - 1 = 3. Note that, $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$ are linearly independent vectors in K. Thus they form a basis by Corollary 2 of Theorem 1.10. □

d. $\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ \qquad x_2 - x_3 + x_4 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

It is clear that rank(A) = 2 because there are 2 linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If K is the solution set of this system, then dim(K) = 4 - 2 = 2. Note that, $\begin{pmatrix}2\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-4\\2\\1\\-1\end{pmatrix}$ are linearly independent vectors in K. Thus they form a basis by Corollary 2 of Theorem 1.10. □

e. $\begin{cases} x_1 + x_2 - \ \ x_3 = 0 \\ 4x_1 + x_2 - 2x_3 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & -2 \end{bmatrix}$$

It is clear that rank(A) = 2 because there are 2 linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If K is the solution set of this system, then dim(K) = 3 - 2 = 1. Thus any nonzero solution constitutes a basis for K. For example, since $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is a solution for K, it forms a basis by Corollary 2 of Theorem 1.10. □

f. $\begin{cases} 2x_1 + \ \ x_2 - \ \ x_3 = 0 \\ \ \ x_1 - \ \ x_2 + \ \ x_3 = 0 \\ \ \ x_1 + 2x_2 - 2x_3 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

It is clear that rank(A) = 2 because there are 2 linearly independent columns (third column is second column multiplied by -1). If K is the solution set of this system, then dim(K) = 3 -

2 = 1. Thus any nonzero solution constitutes a basis for K. For example, since $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a solution for K, it forms a basis by Corollary 2 of Theorem 1.10. □

g. $\begin{cases} x_1 + 2x_2 = 0 \\ x_1 - \ \ x_2 = 0 \end{cases}$

Proof. Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

It is clear that rank(A) = 2 because there are 2 linearly independent columns. If K is the solution set of this system, then dim(K) = 2 - 2 = 0. This means the zero vector is the basis for K, i.e. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. □

§2 B

Using the results of Exercise 2, find all solutions to the following systems.

a. $\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 + 6x_2 = 10 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

b. $\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + \ \ x_2 + x_3 = 6 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

c. $\{x_1 + 2x_2 - 3x_3 + x_4 = 1$

Proof. A solution to the above system is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

□

d. $\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1 \\ \qquad\ \ x_2 - x_3 + x_4 = 1 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix} + t_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ -1 \end{pmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

□

e. $\begin{cases} \ \ x_1 + x_2 - \ \ x_3 = 1 \\ 4x_1 + x_2 - 2x_3 = 3 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

f. $\begin{cases} 2x_1 + \ \ x_2 - \ \ x_3 = 5 \\ \ \ x_1 - \ \ x_2 + \ \ x_3 = 1 \\ \ \ x_1 + 2x_2 - 2x_3 = 4 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

g. $\begin{cases} x_1 + 2x_2 = 5 \\ x_1 - \ \ x_2 = -1 \end{cases}$

Proof. A solution to the above system is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using Theorem 3.9 and following example 3 from the textbook, we know $K = \{s\} + K_H = \{s + k : k \in K_H\}$. So,

$$K = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

□

§3 C

Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in R(L_A)$.

Proof. Let A be an mxn matrix.

i. $Ax = b$ has a solution $\Rightarrow b \in R(L_A)$.

   Let s be a solution to $Ax = b$. This means $A_1 s_1 + \cdots + A_n s_n = b$. Notice that b is a linear combination of the columns of A, which is equivalent to $R(L_A)$ by the proof given in Theorem 3.5. Therefore, $b \in R(L_A)$.

ii. $b \in R(L_A) \Rightarrow Ax = b$ has a solution.

   $b \in R(L_A)$ means that b is a linear combination of the columns of A. So, $b = A_1 s_1 + \cdots + A_n s_n$. This means there exists an x, composed of the $s_1, \cdots, s_n$ as seen in the previous equation, such that $Ax = b$. Therfore, $Ax = b$ has a solution.

□

§4 D

Prove or give a counterexample to the following statement: If the comatrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

Proof. Let A be a mxn comatrix for the system $Ax = b$. By defintion, $L_A : \mathbb{F}^n \to \mathbb{F}^m$. Another way of saying that the system has a solution is to say that $b \in R(L_A)$, as we proved in problem C. Given that the rank$(A) = m$, it follows that that $\dim(R(L_A)) = m$ by Theorem 3.5. Since the range and the codomain of L_A are the same, it means that L_A is onto by definition. Thus, $\forall b \in R(L_A)[\exists s \in \mathbb{F}^n$ s.t. $As = b]$. Therefore, if the comatrix of a system of m linear equations in n unknowns has rank m, then system has a solution. □

§5 E

Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

a. Prove that rank$(A') \neq$ rank$(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

   Proof.

i. $\text{rank}(A') \neq \text{rank}(A'|b') \Rightarrow (A'|b')$ contains a row in which the only nonzero entry lies in the last column.

   If $\text{rank}(A') \neq \text{rank}(A'|b')$, then we know that b' is a linearly independent vector to A'. This must mean that $(A'|b')$ contains a row in which the only nonzero entry lies in the last column, which is b'.

ii. $(A'|b')$ contains a row in which the only nonzero entry lies in the last column $\Rightarrow \text{rank}(A') \neq \text{rank}(A'|b')$

   $(A'|b')$ contains a row in which the only nonzero entry lies in the last column means that b' is a linearly independent vector to A'. Then this means that $\text{rank}(A') \neq \text{rank}(A'|b')$.

□

b. Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

   Proof.

   i. $Ax = b$ is consistent $\Rightarrow (A'|b')$ contains no row in which the only nonzero entry lies in the last column.
   In problem C, we proved that $Ax = b$ is consistent means that b is a linear combination of the columns in A, i.e. linearly dependent. This also must mean that b' is a linearly dependent of the columns in A' because row reduction is a rank and row space preserving operation. Thus, $(A'|b')$ contains no row in which the only nonzero entry lies in the last column because b' is a linearly dependent.

   ii. $(A'|b')$ contains no row in which the only nonzero entry lies in the last column $\Rightarrow Ax = b$ is consistent.
   From part (a), we know that the above premise is equivalent to $\text{rank}(A') \neq \text{rank}(A'|b')$. By Theorem 3.11, we know that $A'x = b'$ must be consistent. $Ax = b$ must also be consistent because row reduced echelon form preserves rank and row space.

□

§6 F

Let the reduced row echelon form of A be

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Determine A if the first, second, and fourth columns of A are

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Proof. Notice that $r_{j_3} = 2r_{j_1} - 5r_{j_2}$ and $r_{j_5} = -2r_{j_1} - 3r_{j_2} + 6r_{j_4}$ where r_{j_i} is the ith column of R. Using Theorem 3.16(c) and (d) we can compute A.

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{bmatrix}$$

□

§7 G

Let the reduced row echelon form of A be

$$R = \begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine A if the first, third, and sixth columns of A are

$$\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix}$$

Proof. Notice that $r_{j_2} = -3r_{j_1}$, $r_{j_4} = 4r_{j_1} + 3r_{j_3}$, and $r_{j_5} = 5r_{j_1} + 2r_{j_3} - r_{j_6}$ where r_{j_i} is the ith column of R. Using Theorem 3.16(c) and (d) we can compute A.

$$A = \begin{bmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{bmatrix}$$

□

§8 H

Let W be the subspace of $M_{2\times 2}(R)$ consisting of the symmetric 2×2 matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates W. Find a subset of S that is a basis for W.

Proof. We put S in an augmented matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix}$$

Then we compute reduced row echelon form of this matrix.

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the first, second, and fourth columns of R is linearly independent. Then, the cooresponding columns in A would be the linearly independent subset by Theorem 3.16.

$$S' = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}$$

forms a basis for W. □

§9 I

Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

Lemma 9.1

The pivot columns of matrix A's reduced row echelon form is unique

Proof. We proceed by induction on $n \in \mathbb{N}$. Let $P(n)$ be the following predicate "pivot columns of matrix A's reduced row echelon form is unique, where matrix A has n columns".

a. Base case:
   We prove $P(1)$. There is only one column so it must be unique.

b. Inductive step:
   Suppose $P(k)$ holds. We show that $\mathrm{P}(k+1)$ holds.
   Let $A = [A'|u_{k+1}]$ where A' is a matrix which as k columns and has unique pivot columns in its reduced row echelon form. Consider two cases: where u_{k+1} is linearly dependent to A' and where it is linearly independent to A'.

   Case 1: u_{k+1} is linearly dependent to A'
   If u_{k+1} is linearly dependent it means that $\operatorname{rank}(A) = \operatorname{rank}(A')$. So the reduced row echelon form of A is $[R'|b]$ where R' are the pivot columns from A'. That must mean that A has a unique amount of pivot columns because A' has a unique amount of pivot columns by our induction hypothesis.

   Case 2: u_{k+1} is linearly independent to A'
   If u_{k+1} is linearly independent it means that $\operatorname{rank}(A) = \operatorname{rank}(A') + 1 = r$, where r is just some fixed integer. This must mean that A has one more pivot column than A'. The reduced row echelon form of A is $[R'|b]$, and b must be e_r by Theorem 3.16(b) and the fact that it is the right most column. That must mean that A has a unique amount of pivot columns.

Therefore, by induction we have proved this lemma. □

Let's assume for the sake of contradiction that matrix A has two unique reduced row echelon forms, B and B'. Theorem 3.16(d) states that "For each $k = 1, 2, \cdots, n$, if column k of B is $d_1e_1 + d_2e_2 + \cdots + d_re_r$, then column k of A is $d_1a_{j_1} + d_2a_{j_2} + \cdots + d_ra_{j_r}$. Let's the kth column of B be $d_1e_1 + d_2e_2 + \cdots + d_re_r$ and the kth column of B' be $d_1'e_1 + d_2'e_2 + \cdots + d_r'e_r$ where kth column of B does not equal the kth column of B' because they are both unique. We know that the $e_1, e_2, \cdots, e_r$ are unique by our lemma. However, this would imply that the kth column of the original matrix A are different depending if you compute using B or B'. Thus, this is a contradiction and A must have a unique reduced row echelon form. □