

Math 341: Midterm 2

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Spring 2020

§1

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (1)$$

- a. Suppose that $a \neq 0$, compute the solution of $\mathbf{Ax} = \mathbf{b}$ using row reduction and provide the conditions on a, b, c, d such that your computations are valid. Express the result as a simplified expression. (**Hint:** recall that you can not divide by zero)

Proof. We perform reduced row echelon form (rref) on the augmented matrix

$$\begin{aligned} (A|b) &= \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right] \\ R_2 &\leftarrow R_2 - \frac{c}{a}R_1 \left[\begin{array}{cc|c} a & b & e \\ 0 & d - \frac{cb}{a} & f - \frac{ce}{a} \end{array} \right] \\ &\left[\begin{array}{cc|c} a & b & e \\ 0 & \frac{ad-cb}{a} & \frac{af-ce}{a} \end{array} \right] \\ R_2 &\leftarrow \frac{a}{ad-cb}R_2 \quad \text{Assuming that } ad-cb \neq 0 \left[\begin{array}{cc|c} a & b & e \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array} \right] \\ R_1 &\leftarrow R_1 - bR_2 \left[\begin{array}{cc|c} a & 0 & e - b\frac{af-ce}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array} \right] \\ R_1 &\leftarrow \frac{R_1}{a} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}(e - b\frac{af-ce}{ad-cb}) \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array} \right] \\ &\left[\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array} \right] \\ x &= \begin{bmatrix} \frac{de-bf}{ad-cb} \\ \frac{af-ce}{ad-cb} \end{bmatrix} \quad \text{where } ad-cb \neq 0 \end{aligned}$$

□

- b. If $a = 0$, and $c \neq 0$, is your above computation still valid? How would you modify it? (explain briefly) (**Hint:** recall that you can swap the equations and the result is the same)

Proof. If $a = 0$, and $c \neq 0$, then the above computation will not be valid as we divided by a multiple times when we computed the rref. I would swap the first and second rows so that it would look like

$$\left[\begin{array}{cc|c} c & d & f \\ 0 & b & e \end{array} \right]$$

and compute the rref, assuming that $b \neq 0$. We obtain the rref,

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{bf-de}{bc} \\ 0 & 1 & \frac{e}{b} \end{array} \right]$$

□

- c. If $a = 0$, $c = 0$, but $b \neq 0$, $d \neq 0$, what are the conditions on e and f such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution? Is the solution unique? (**Hint:** recall that $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if \mathbf{b} can be written as a linear combination of the columns of \mathbf{A})

Proof. If $a = 0$, $c = 0$, $b \neq 0$, $d \neq 0$, we get the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & e \\ 0 & d & f \end{array} \right]$$

Performing row reduction,

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{e}{d} \\ 0 & 1 & \frac{f}{d} \end{array} \right]$$

So the condition of the solution is,

$$x_2 = \frac{e}{b} = \frac{f}{d}$$

Thus, there exists a infinite amount of solution.

□

- d. Solve the system

$$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}. \quad (2)$$

(**Hint:** You may want to use the formula you just deduced)

Proof.

$$\begin{aligned} x_1 &= \frac{de - bf}{ad - cb} \\ &= \frac{\sqrt{2}(5\sqrt{2}) - 3\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 30}{2 - 12} \\ &= \frac{-20}{-10} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\sqrt{2}(5\sqrt{2}) - 2\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 20}{-10} \\ &= 1 \end{aligned}$$

□

§2

Let

$$\mathbf{A} = \begin{bmatrix} 0 & -\alpha & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 3 \\ -2 & -2 & 4 & 2\alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 + \alpha \\ 2\beta + \alpha - 2 \end{bmatrix} \quad (3)$$

What are the conditions on α and β such that the system $\mathbf{Ax} = \mathbf{b}$:

a. Has no solution?

Proof. We begin by putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ in its reduced form.

$$\begin{aligned}
 (\mathbf{A}|\mathbf{b}) &= \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2 + \alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 & 2\beta + \alpha - 2 \end{array} \right] \\
 R_5 \leftarrow R_5 + R_1 &\left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2 + \alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array} \right] \\
 R_3 \leftarrow R_3 - 2R_2 &\left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ -2 & -2 & 4 & 2\alpha & 2 + \alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array} \right] \\
 R_4 \leftarrow R_4 + 2R_2 &\left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2\alpha + 2 & 4 + \alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array} \right] \\
 R_4 \leftarrow R_4 - 2R_3, R_5 \leftarrow R_5 - \frac{1}{2}R_3 &\left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 2\beta + \alpha + 1 \end{array} \right] \\
 R_5 \leftarrow R_5 - R_4 &\left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta - 1 \end{array} \right] \\
 R_1 \leftrightarrow R_2 &\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta - 1 \end{array} \right]
 \end{aligned}$$

Thus this system will have no solution if $2\beta - 1 \neq 0$, which is when $\beta \neq \frac{1}{2}$. explain more... \square

- b. Has an unique solution? Find the solution. (**Hint:** you will need to row reduce the augmented system to echelon form, and then use the theorems seen in class to impose the conditions on α and β).

Proof. Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(\mathbf{A}) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &\neq 0 \\ -4\alpha^2 &\neq 0 \\ \alpha &\neq 0 \end{aligned}$$

and from (a), $\beta = \frac{1}{2}$. Combining these two conditions we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

By performing back substitution we compute the generic solution as

$$x = \begin{bmatrix} \frac{1}{2} + \frac{1}{2\alpha} \\ -\frac{1}{2\alpha} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

□

- c. Has infinite amount of solutions? Find the solution set in parametric form. (**Hint:** You may have one equations for α and one for β that have to be satisfied simultaneously).

Proof. Having an infinite amount of solutions is by definition another way of saying that a system that is consistent and that the solutions are not unique. A system is consistent if and only if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$ by Theorem 3.11. If $\det(\mathbf{A}) = 0$, the solution, if it exists, is not unique by Theorem 3.10 and the corollary to Theorem 4.7.

Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &= 0 \\ -4\alpha^2 &= 0 \\ \alpha &= 0 \end{aligned}$$

Given that $\alpha = 0$, and that $\beta = \frac{1}{2}$ from part (a) we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

It is clear that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$ because \mathbf{b} is a linear combination of the second and third column from \mathbf{A} . Hence, this system is consistent.

By performing back substitution we compute the generic solution in parametric form as

$$x = \begin{bmatrix} \gamma \\ 1 - \gamma \\ 1 \\ 0 \end{bmatrix}$$

where $\gamma \in \mathbb{R}$. □

§3

Let $A \in M_{n \times n}(F)$, for a field F . We want to prove that $\text{rank}(A^2) - \text{rank}(A^3) \leq \text{rank}(A) - \text{rank}(A^2)$. The solution to this exercise requires the notion of quotient spaces. Even though you should already be familiar with quotient spaces we will prove a few properties that will be useful.

Let W be a subspace of a vector space V over a field F . For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is called the coset of W containing v . It is customary to denote this coset by $v + W$ rather than $\{v\} + W$. Following this notation we write $V/W = \{v + W : v \in V\}$, which is usually called the quotient space V module W . Addition and scalar multiplication by scalars can be defined in the collection V/W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$

- a. Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$. (**Hint:** recall that $v_1 + W$ is a set, thus you need to prove equality between sets)

Proof.

- i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$

If $v_1 + W = v_2 + W$, then $\exists w_1, w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$

$$v_1 - v_2 = w_2 - w_1$$

Since, $w_2 - w_1 \in W$ (closure under addition)

Therefore, $v_1 - v_2 \in W$

- ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$

This means $v_1 - v_2 = w$ where $w \in W$ (*)

Now let $x \in v_1 + W$

By definition, $\exists w_x \in W$ such that $x = v_1 + w_x$

By (*) $v_1 = v_2 + w$

So, $x = v_2 + w + w_x$

Since, $w + w_x \in W$ (closure under addition)

We have $x \in v_2 + W$

So, $v_1 + W \subseteq v_2 + W$

Without loss of generality, we can show $v_2 + W \subseteq v_1 + W$

Therefore, $v_1 + W = v_2 + W$

Therefore, $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$ □

- b. Show that V/W with the operations defined above is a linear vector space.

VS 1: For all x, y in V/W , $x + y = y + x$ (commutativity of addition)

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} x + y &= (v_x + W) + (v_y + W) \\ &= (v_x + v_y) + W \\ &= (v_y + v_x) + W \\ &= (v_y + W) + (v_x + W) \\ &= y + x \end{aligned}$$

□

VS 2: For all x, y, z in V/W , $(x + y) + z = x + (y + z)$ (associativity of addition)

Proof. Let $x = v_x + W, y = v_y + W, z = v_z + W$, where $v_x, v_y, v_z \in V$.

$$\begin{aligned} (x + y) + z &= ((v_x + W) + (v_y + W)) + (v_z + W) \\ &= ((v_x + v_y) + W) + (v_z + W) \\ &= (v_y + v_x + v_z) + W \\ &= (v_x + W) + ((v_y + v_z) + W) \\ &= (v_x + W) + ((v_y + W) + (v_z + W)) \\ &= x + (y + z) \end{aligned}$$

□

VS 3: There exists an element in V/W denoted by $\mathbf{0}$ such that $x + \mathbf{0} = x$ for each x in V/W

Proof. Let $x = v_x + W$ where $v_x \in V$. Let $\mathbf{0}$ in V/W be defined as $0 + W$, where $0 \in V$.

$$\begin{aligned} x + \mathbf{0} &= x + (0 + W) \\ &= (v_x + W) + (0 + W) \\ &= (v_x + 0) + W \\ &= v_x + W \\ &= x \end{aligned}$$

□

VS 4: For each element x in V/W there exists an element y in V/W such that $x + y = \mathbf{0}$

Proof. Let $x = v_x + W$ where $v_x \in V$. Fix y such that $y = -v_x + W$.

$$\begin{aligned} x + y &= (v_x + W) + (-v_x + W) \\ &= (v_x - v_x) + W \\ &= 0 + W \\ &= \mathbf{0} \end{aligned}$$

□

VS 5: For each element x in V/W , $1x = x$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} 1x &= 1(v_x + W) \\ &= (1v_x + W) \\ &= (v_x + W) \\ &= x \end{aligned}$$

□

VS 6: For each pair of elements a, b in \mathbb{F} and each element x in V/W , $(ab)x = a(bx)$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} (ab)x &= abv_x + W \\ &= a(bv_x + W) \\ &= a(bx) \end{aligned}$$

□

VS 7: For each element a in \mathbb{F} and each pair of elements x, y in V/W , $a(x + y) = ax + ay$

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} a(x + y) &= a((v_x + v_y) + W) \\ &= ((av_x + av_y) + W) \\ &= (av_x + W) + (av_y + W) \\ &= a(v_x + W) + a(v_y + W) \\ &= ax + ay \end{aligned}$$

□

VS 8: For each pair of elements a, b in \mathbb{F} and each element x in V/W , $(a + b)x = ax + bx$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} (a + b)x &= (a + b)(v_x + W) \\ &= ((a + b)v_x + W) \\ &= ((av_x + bv_x) + W) \\ &= (av_x + W) + (bv_x + W) \\ &= a(v_x + W) + b(v_x + W) \\ &= ax + bx \end{aligned}$$

□

Therefore, V/W is a vector space because it holds all the properties above.

c. Prove that if $\dim(V) < \infty$ then $\dim(V/W) = \dim(V) - \dim(W)$.

Proof. We will prove this fact by creating a basis for V/W and computing its dimension. Since V is a finite dimensional vector space, so W is also finite dimensional by Theorem 1.11. Let the dimension of V and W be denoted by k and j , respectively. We can construct a basis β_W for W , where $\beta_W = \{u_1, u_2, \dots, u_j\}$. We can extend β_W to be a basis for V by the corollary of Theorem 1.11. Let this basis be $\beta_V = \{u_1, u_2, \dots, u_j, u_{j+1}, u_{j+2}, \dots, u_k\}$. We claim that the basis for V/W is $\beta_{V/W} = \{u_{j+1} + W, u_{j+2} + W, \dots, u_k + W\}$.

i. $\beta_{V/W}$ is linearly independent

$$\begin{aligned}\alpha_{j+1}(u_{j+1} + W) + \alpha_{j+2}(u_{j+2} + W) + \cdots + \alpha_k(u_k + W) &= \mathbf{0} \\ (\alpha_{j+1}u_{j+1} + \alpha_{j+2}u_{j+2} + \cdots + \alpha_k u_k) + W &= 0 + W\end{aligned}$$

This means that $(\alpha_{j+1}u_{j+1} + \alpha_{j+2}u_{j+2} + \cdots + \alpha_k u_k) = 0 \in V$. We can rewrite this as $(0u_1 + 0u_2 + \cdots + 0u_j + \alpha_{j+1}u_{j+1} + \alpha_{j+2}u_{j+2} + \cdots + \alpha_k u_k) = 0$. Thus, $\alpha_{j+1} = \alpha_{j+2} = \cdots = \alpha_k = 0$ because it is written in terms of the basis for V . So, $\beta_{V/W}$ is linearly independent.

ii. $\beta_{V/W}$ generates V/W

Let $x = v_x + W$ where $x \in V/W$, $v_x \in V$.

$$\begin{aligned}x &= v_x + W \\ &= (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j + \alpha_{j+1} u_{j+1} + \alpha_{j+2} u_{j+2} + \cdots + \alpha_k u_k) + W \\ &= ((\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j) + (\alpha_{j+1} u_{j+1} + \alpha_{j+2} u_{j+2} + \cdots + \alpha_k u_k)) + W \\ &= (\alpha_{j+1} u_{j+1} + \alpha_{j+2} u_{j+2} + \cdots + \alpha_k u_k) + W \\ &= \alpha_{j+1}(u_{j+1} + W) + \alpha_{j+2}(u_{j+2} + W) + \cdots + \alpha_k(u_k + W)\end{aligned}$$

Since $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_j u_j \in W$, it can be removed because of W (the fourth line). So, $\beta_{V/W}$ generates V/W .

By definition, $\beta_{V/W}$ is a basis for V/W . Therefore,

$$\begin{aligned}\dim(V/W) &= k - j \\ &= \dim(V) - \dim(W)\end{aligned}$$

□

d. Let $K = F^n$, define $AK = R(L_A)$, and $A^2K = R(L_{A^2})$. Show that AK/A^2K is a vector space of dimension $\text{rank}(A) - \text{rank}(A^2)$.

Proof. We begin by proving that AK/A^2K is a vector space.

Not sure how to do this.

It follows from part (c) and Theorem 3.5 that

$$\begin{aligned}\dim(AK/A^2K) &= \dim(AK) - \dim(A^2K) \\ &= \dim(R(L_A)) - \dim(R(L_{A^2})) \\ &= \text{rank}(A) - \text{rank}(A^2)\end{aligned}$$

□

e. Show that A^2K/A^3K is a vector space of dimension $\text{rank}(A^2) - \text{rank}(A^3)$, where $A^3K = R(L_{A^3})$.

Proof. Using what we have proved in part (d)

$$\begin{aligned}\dim(A^2K/A^3K) &= \dim(A^2K) - \dim(A^3K) \\ &= \dim(R(L_{A^2})) - \dim(R(L_{A^3})) \\ &= \text{rank}(A^2) - \text{rank}(A^3)\end{aligned}$$

□

- f. Define $T : AK/A^2K \rightarrow A^2K/A^3K$, by $T(v) = L_A(v)$, i.e, we left multiply each element of v by the matrix A . Show that $R(T) = A^2K/A^3K$.

Proof. To show that $R(T) = A^2K/A^3K$, is another way of saying show that T is onto. We must show that $\forall x \in AK/A^2K [\exists y \in A^2K/A^3K : T(x) = y]$. Let $x = Ak + A^2K$ where $k \in \mathbb{F}^5$.

$$\begin{aligned} T(x) &= T(Ak + A^2K) \\ &= L_A(Ak + A^2K) \\ &= L_A(Ak) + L_A(A^2K) \\ &= A^2k + A^3K \in A^2K/A^3K \end{aligned}$$

Therefore, T is onto and $R(T) = A^2K/A^3K$. \square

- g. Use the rank-nullity theorem on T to conclude that $\text{rank}(A^2) - \text{rank}(A^3) \leq \text{rank}(A) - \text{rank}(A^2)$.

Proof. From part (d) and (e)

$$\begin{aligned} \text{rank}(T) + \text{nullity}(T) &= \dim(AK/A^2K) \\ &= \text{rank}(A) - \text{rank}(A^2) \\ \text{rank}(T) &\leq \text{rank}(A) - \text{rank}(A^2) \\ \text{rank}(A^2) - \text{rank}(A^3) &\leq \text{rank}(A) - \text{rank}(A^2) \end{aligned}$$

\square

§4

Let V be a finite-dimensional vector space. Let T and P be two linear transformations from V to itself, such that $T^2 = P^2 = 0$, and $T \circ P + P \circ T = I$, where I is the identity in V .

- a. Denote $N_T = N(T)$ and $N_P = N(P)$, the null spaces of T and P , respectively. Show that $N_P = P(N_T)$, and $N_T = T(N_P)$, where $T(N_P) = \{T(v) : v \in N_P\}$ and $P(N_T) = \{P(v) : v \in N_T\}$.