Math 240: Homework 8

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§1 1

Prove that for all $k \in \mathbb{N}$, there is some graph $G = (V, E)$ such that $\sum_{v \in V} \deg(v) = 2k$

Proof. We proceed by induction on $k \in \mathbb{N}$. Let $P(k)$ be the predicate "there is some graph $G = (V, E)$ such that $\sum_{v \in V} \deg(v) = 2k$".
Base case: We prove that $P(0)$ holds. Let G_0 be an empty graph where $V = E = \varnothing$. It follows that

$$\sum_{v \in V} \deg(v) = 0 = 2k = 2(0)$$

as desired.
Inductive step: Suppose that $k \in \mathbb{N}$ such that $P(k)$ holds. We prove that $P(k+1)$ holds. Consider the graph, $G = (V, E)$ from $P(k)$. Let $V' = V \cup \{u\}$ and $E' = E \cup \{\{u, v_1\}, \{u, v_2\}\}$ where $u \notin V$, $v_1, v_2 \in V$, and $u \neq v_1 \neq v_2$. Let $G' = (V', E')$. It follows that

$$\begin{aligned}\sum_{v \in V'} \deg(v) &= \sum_{v \in V} \deg(v) + \deg(u) \\ &= 2k + 2 \\ &= 2(k+1)\end{aligned}$$

as desired. Thus there exists a graph, G', such that $P(k+1)$ holds. This completes the proof of the inductive step. We have proved by induction that for all $k \in \mathbb{N}$, there is some graph $G = (V, E)$ such that $\sum_{v \in V} \deg(v) = 2k$ □

§2 2

Let G be a connected graph. For all vertices u and v in G, we define the distance $d(u, v)$ to be the least number d such that there is a path between u and v of length d. Prove that for all vertices u, v and w

$$d(u, w) \leq d(u, v) + d(v, w)$$

Proof. For the sake of contradiction, assume that $d(u, w) > d(u, v) + d(v, w)$. This means that the shortest path $d(u, w)$ is greater than some path from u to v to w. This is a contradiction as $d(u, w)$ would be at most $d(u, v) + d(v, w)$. Therefore, $d(u, w) \leq d(u, v) + d(v, w)$ □

§3 3

Let G be a bipartite graph. Prove that if $v_1, \ldots, v_n$ is a cycle in G then n is odd (so the cycle has even length).

Proof. By definition, G can be partitioned into two disjoint sets V_1 and V_2. Let's assume that $v_1 \in V_1$. We know that $v_1 = v_n$ because the above sequence is a cycle. Since G is a bipartite graph, each following vertex in the sequence must in the opposite set. This means that $v_1, v_3, \cdots, v_n \in V_1$ and $v_2, v_4, \cdots, v_{n-1} \in V_2$. Every edge we take from V_1 leads us to V_2 because this graph is bipartite. Likewise, every edge we take from V_2 leads us to V_1. It follows that to go from a vertex in V_1 to a vertex in V_2 and back to a vertex in V_1 we must take an even number of steps. Thus any path from a vertex in V_1 back to itself must be an even length. It directly follows that n is odd because an even length path must be connected by an odd number of vertices. ☐

§4 4

a. Let T be a tree. Prove that if you remove a leaf l and the only edge incident with l from T, the resulting subgraph T' is still a tree.

   Proof. Since T is a tree, it is by definition acyclic and connected. We must show that T' holds the property of a tree.

   i. T' is connected

      From lecture, we know that a graph is connected if and only if there is a simple path between every two distinct vertices. If we removed a leaf and only its edge incident, we know that this cannot affect any simple path from $\{u, v\}$ in the tree. Thus, the resulting tree, T', is connected.

   ii. T' is acyclic

      If T is acylic, removing a vertex will not make it cyclic.

   Therfore, T' is a tree. ☐

b. Prove by induction on $n \geq 1$ that if a tree T has n vertices, then it has exactly $n-1$ edges.

   Proof. We proceed by induction on $n \geq 1$. Let $P(n)$ be the predicate "if a tree T has n vertices, then it has exactly $n-1$ edges".
   Base case: We prove that $P(1)$ holds. This means that T has 1 vertex. It trivially follows that T has 0, i.e. $n-1 = 1-1$ edges.
   Inductive step: Suppose that $n \geq 1$ such that $P(n)$ holds. We prove that $P(n+1)$ holds. Let $T = (V, E)$ be the tree with n vertices. We know that the tree with $n+1$ edges, T', will have a set of vertices $V' = V \cup \{u\}$, where u is a vertex. Since T' is a tree, u adds only one edge. If it is placed as a leaf, then u there is a new edge create from u to its parent. If it is placed in the middle of the tree, the existing edge between the adjacent vertex will be used as one of the edges connecting u and another edge must be added to finish connecting u to the tree. So in either case, adding a new vertex only adds one edge. It follows then by this fact and by our induction hypothesis that

$$\begin{aligned}\sum_{v \in V'} \deg(v) &= \sum_{v \in V} \deg(v) + \deg(u) \\ &= n - 1 + \deg(u) \\ &= n - 1 + 1 \\ &= n\end{aligned}$$

   Therefore, $P(n+1)$ holds and this concludes our inductive step. We have proved by induction that for all $n \geq 1$ that if a tree T has n vertices, then it has exactly $n-1$ edges. ☐

c. Prove that if G is a connected graph with n vertices, then it has at least $n-1$ edges. (Hint: Spanning tree.)

Proof. We know that a graph is connected if and only if it has a spanning tree (theorem given in the lecture). Thus, that G is a spanning tree. It directly follows from this fact and from part (b) that G has exactly $n-1$ edges. □

§5 5

Prove that in any group of n people ($n \geq 2$), there must be two people with the same number of friends within the group. Assume friendship is mutual, and nobody is their own friend.

Proof. Given that there is n people, the amount of friends each person will be in this set: $\{0, 1, \cdots, n-1\}$, excluding n because you cannot be friends with yourself. Consider the case when someone has $n-1$ friends. This means everyone in this group has at least one friend. Also consider the case when someone has 0 friends. This means that the most amount of friends one can have is $n-2$ because you cannot be friends with yourself and with the person with 0 friends. Thus the range of friends that a group has is actually $\{1, 2, \cdots, n-1\}$ or $\{0, 1, \cdots, n-2\}$. In both cases, the number possible combination of friends one can have is $n-1$. Using the pigeonhole principle, it directly follows that there must exist at least 2 people with the same number of friends in this group. □