Math 341: Homework 8

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§1 A

Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$

Proof.

a. λ is an eigenvalue of $T \Rightarrow \lambda$ is an eigenvalue of $[T]_\beta$

   By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Using Theorem 2.14,

$$\begin{aligned} T(v) &= \lambda v \\ [T(v)]_\beta &= [\lambda v]_\beta \\ [T]_\beta [v]_\beta &= \lambda [v]_\beta \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of $[T]_\beta$.

b. λ is an eigenvalue of $[T]_\beta \Rightarrow \lambda$ is an eigenvalue of T

   By definition, there exists a eigenvector $v \in V$ such that $[T]_\beta [v]_\beta = \lambda [v]_\beta$. Using Theorem 2.14,

$$\begin{aligned} [T]_\beta [v]_\beta &= \lambda [v]_\beta \\ [T(v)]_\beta &= [\lambda v]_\beta \\ T(v) &= \lambda v \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. $\square$

§2 B

a. Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   Proof.

   i. Linear operator T on a finite-dimensional vector space is invertible $\Rightarrow$ zero is not an eigenvalue of T.

By the corollary of Theorem 4.7, $\det(T) \neq 0$. Assume, for the sake of contradiction, suppose zero is an eigenvalue of T. It follows from Theorem 5.2 that

$$\begin{aligned}\det(T - \lambda I) &= 0\\ \det(T - 0I) &= 0\\ \det(T) &= 0\end{aligned}$$

which is a contradiction. Thus, zero is not an eigenvalue of T.

ii. Zero is not an eigenvalue of $T \Rightarrow$ linear operator T on a finite-dimensional vector space is invertible.

By contrapositive, we will instead prove that if linear operator T on a finite-dimensional vector space is not invertible then zero is an eigenvalue of T. If T is not invertible then $\det(T) = 0$ by corollary of Theorem 4.7. It follows from Theorem 5.2 that

$$\det(T - \lambda I) = 0$$

It directly follows that zero is an eigenvalue of T.

Therefore, linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T. □

b. Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof.

i. A scalar λ is an eigenvalue of T $\Rightarrow$ λ^{-1} is an eigenvalue of T^{-1}.

By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Given that T is invertible and by definition eigenvalues are non zero,

$$\begin{aligned}T(v) &= \lambda v\\ T^{-1}(T(v)) &= T^{-1}(\lambda v)\\ v &= T^{-1}(\lambda v)\\ v &= \lambda T^{-1}(v)\\ \lambda^{-1} v &= T^{-1}(v)\end{aligned}$$

as desired. Thus, λ^{-1} is an eigenvalue of T^{-1}.

ii. λ^{-1} is an eigenvalue of T^{-1} $\Rightarrow$ a scalar λ is an eigenvalue of T.

By definition, there exists a eigenvector $v \in V$ such that $T^{-1}(v) = \lambda^{-1} v$. Given that T^{-1} is invertible linear operator,

$$\begin{aligned}T^{-1}(v) &= \lambda^{-1} v\\ T(T^{-1}(v)) &= T(\lambda^{-1} v)\\ v &= \lambda^{-1} T(v)\\ \lambda v &= T(v)\end{aligned}$$

as desired. Thus, λ is an eigenvalue of T.

Therefore, a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. □

c. State and prove results analogous to (a) and (b) for matrices.

(a) A matrix A is invertible if and only if zero is not an eigenvalue of A.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (a) that "matrix A is invertible if and only if zero is not an eigenvalue of A." is true. $\square$

(b) Let A be an invertible matrix. λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (b) that "λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}" is true. $\square$