Math 341: Final

Daniel Ko

Spring 2020

§1

For $c \in \mathbb{R}$, we define the matrix $\mathbf{A}_c \in \mathbb{R}^{3\times 3}$ by

$$\mathbf{A}_c = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & c & 2 \end{bmatrix}$$

a. Compute $\det(\mathbf{A}_c)$. Does it depend of c?

Proof.

$$\begin{aligned} \det(\mathbf{A}_c) &= 1 \cdot \det \begin{bmatrix} 2 & 0 \\ c & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & 2 \\ 3 & c \end{bmatrix} \\ &= (4) + (4) + (2c - 6) \\ &= 2c + 2 \end{aligned}$$

Yes, it depends on c. □

b. For which c is the matrix $\mathbf{A}_c$ invertible?

Proof. Corollary to Theorem 4.7 states that a matrix is invertible if and only its determinant does not equal 0. From (a),

$$\begin{aligned} \det(\mathbf{A}_c) = 2c + 2 &= 0 \\ c &= -1 \end{aligned}$$

$\det(\mathbf{A}_c) = 0$ when $c = -1$. Hence, $\mathbf{A}_c$ is invertible for all c except $c = -1$. □

c. Compute $\mathbf{A}_0^{-1}$ (i.e. when $c = 0$).

Proof. Using the proof to Theorem 3.2, we compute the inverse by constructing an agumented matrix $(\mathbf{A}_0|I_3)$ and applying elementary row operations to transform it into the form of

$(I_3|\mathbf{A}_0^{-1})$.

$$(\mathbf{A}_0|I_3) = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \leftarrow R_2 - 2R_1 \quad R_3 \leftarrow R_3 - 3R_1 = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 3 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 + \frac{1}{4}R_2 \quad R_3 \leftarrow R_3 - \frac{3}{4}R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_3 \quad R_2 \leftarrow R_2 + 4R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 4 & 0 & -8 & -2 & 4 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right]$$

$$R_2 \leftarrow \frac{1}{4}R_2 \quad R_3 \leftarrow 2R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -3 & -\frac{3}{2} & 2 \end{array}\right]$$

Therefore,

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -\frac{1}{2} & 1 \\ -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

□

d. Let $b = (1, -4, 2)^t$, find the solution of $\mathbf{A}_0 x = b$

Proof. Let $x = (x_1, x_2, x_3)^t$.

$$\mathbf{A}_0 x = b$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + 2x_2 = -4 \\ 3x_1 + 2x_3 = 2 \end{cases}$$

$$x_1 = -4$$
$$x_2 = 2$$
$$x_3 = 7$$

□

e. Compute $\det(\mathbf{A}_c^2)$

Proof. From Theorem 4.7, we know that

$$\begin{aligned}\det(\mathbf{A}_c^2) &= \det(\mathbf{A}_c\mathbf{A}_c)\\ &= \det(\mathbf{A}_c)\cdot\det(\mathbf{A}_c)\\ &= (2c+2)(2c+2)\\ &= 4c^2+8c+4\end{aligned}$$

□

f. Compute $\det(5\mathbf{A}_c)$

Proof. Using the second "Properties of the Determinant" on pg 234,

$$\begin{aligned}\det(5\mathbf{A}_c) &= 5^3\det(\mathbf{A}_c)\\ &= 5^3(2c+2)\\ &= 250c+250\end{aligned}$$

□

g. Compute $\det(\mathbf{E}_k\mathbf{A}_c)$ where,

$$\mathbf{E}_k = \begin{bmatrix}1 & 0 & 0\\ k & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Proof.

$$\begin{aligned}\det(\mathbf{E}_k) &= 1\cdot\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 0\cdot\det\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix} + 0\cdot\det\begin{bmatrix}k & 1\\ 0 & 0\end{bmatrix}\\ &= 1\end{aligned}$$

From Theorem 4.7, we know that,

$$\begin{aligned}\det(\mathbf{E}_k\mathbf{A}_c) &= \det(\mathbf{E}_k)\det(\mathbf{A}_c)\\ &= (1)(2c+2)\\ &= 2c+2\end{aligned}$$

□

h. Compute $\det(\mathbf{D}_k\mathbf{A}_c)$ where,

$$\mathbf{D}_k = \begin{bmatrix}1 & 0 & 0\\ 0 & k & 0\\ 0 & 0 & 1\end{bmatrix}$$

Proof.

$$\begin{aligned}\det(\mathbf{D}_k) &= 1\cdot\det\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix} - 0\cdot\det\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + 0\cdot\det\begin{bmatrix}0 & k\\ 0 & 0\end{bmatrix}\\ &= k\end{aligned}$$

From Theorem 4.7, we know that,

$$\begin{aligned}\det(\mathbf{E}_k\mathbf{A}_c) &= \det(\mathbf{D}_k)\det(\mathbf{A}_c)\\ &= (k)(2c+2)\\ &= 2kc+2k\end{aligned}$$

$\square$

i. Compute $\det(\mathbf{A}_0^{-1})$

Proof. From (c) we know that,

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -\frac{1}{2} & 1 \\ -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

It follows that,

$$\begin{aligned} \det(\mathbf{A}_0^{-1}) &= 2 \cdot \det \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{2} & 2 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -2 & -\frac{1}{2} \\ -3 & -\frac{3}{2} \end{bmatrix} \\ &= 2(\frac{1}{2}) - 1(-1) - 1(\frac{3}{2}) \\ &= \frac{1}{2} \end{aligned}$$

$\square$

j. Compute the eigenvalues of $\mathbf{A}_0$. Can you diagonalize $\mathbf{A}_0$?

Proof. Using Theorem 5.2, we compute the eigenvalues of $\mathbf{A}_0$ by computing its characteristic polynomial.

$$\begin{aligned} \det(\mathbf{A}_0 - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -1 & 1 \\ 2 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{bmatrix} \\ &= -(\lambda - 2)(\lambda^2 - 3\lambda + 1) \end{aligned}$$

We compute when $-(\lambda-2)(\lambda^2 - 3\lambda + 1) = 0$. When $\lambda = 2$, the characteristic polynomial equals 0. Using the quadratic formula, when $\lambda = \frac{3\pm\sqrt{5}}{2}$, the characteristic polynomial equals 0. Hence the eigenvalues of $\mathbf{A}_0$ are

$$2, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

We can rewrite the characteristic polynomial as

$$f(\lambda) = -(\lambda - 2)(\lambda - \frac{3+\sqrt{5}}{2})(\lambda - \frac{3-\sqrt{5}}{2})$$

Using the "Test for Diagonalization" outlined in pg 269, we determine if $\mathbf{A}_0$ can be diagonalized. It is clear that the first condition, the characteristic polynomial of T splits, holds. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. $\square$

k. Compute the eigenvalues of $\mathbf{A}_0^{-1}$

Proof. Notice that since eigenvalues are non zero,

$$\begin{aligned}\mathbf{A}_0 x &= \lambda x\\ \mathbf{A}_0^{-1}\mathbf{A}_0 x &= \mathbf{A}_0^{-1}\lambda x\\ x &= \mathbf{A}_0^{-1}\lambda x\\ \frac{1}{\lambda}x &= \mathbf{A}_0^{-1}x\end{aligned}$$

Thus the eigenvalues of $\mathbf{A}_0^{-1}$ are

$$\frac{1}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

□

§2

Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T(\mathbf{x}) = \begin{pmatrix} x_2 - x_1 + \alpha\left(x_2^2 + x_3^3\right) \\ x_1 + x_2 + 2gx_3 + \alpha x_1^2 \\ x_1 + x_2 + 2x_3 \\ hx_3 + q \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

in which, h, g, q and α are real numbers.

a. What are the condition on α and q such that the transformation is linear? Explain briefly.

Proof. We know that if T is linear then $T(0) = 0$. Hence $q = 0$ for T to be linear. Notice that we must not have any terms higher than degree 1 because for example if we have a transformation $U(x) = x^2$ and let $c \in \mathbb{R}$ then,

$$\begin{aligned}U(cx) &= (cx)^2\\ &= c^2x^2\\ &\neq cU(x)\end{aligned}$$

which means U is not a linear transformation. Thus, $\alpha = 0$ so that there are no terms higher than degree 1. □

b. From now we suppose that $q = 0$ and $\alpha = 0$. Write the representation matrix $A = [T]_{\epsilon_3}^{\epsilon_4}$.

Proof. We now define

$$T(\mathbf{x}) = \begin{pmatrix} x_2 - x_1 \\ x_1 + x_2 + 2gx_3 \\ x_1 + x_2 + 2x_3 \\ hx_3 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We now compute the representation matrix

$$T(e_1) = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 2g \\ 2 \\ h \end{pmatrix}$$

$$\begin{aligned} A &= [[T(e_1)]_{\epsilon_4}[T(e_2)]_{\epsilon_4}[T(e_3)]_{\epsilon_4}] \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2g \\ 1 & 1 & 2 \\ 0 & 0 & h \end{bmatrix} \end{aligned}$$

□

c. What are the conditions on h and g such that the transformation T maps $\mathbb{R}^3$ onto $\mathbb{R}^4$? Explain briefly.

Proof. Since $T : \mathbb{R}^3 \to \mathbb{R}^4$, there are no h and g such that T is onto. This is because T is onto if and only if the range of T equals the codomain, but in this linear transformation, the dimension of codomain is greater than the domain so by rank nullity, T cannot be onto. □

d. What are the conditions on h and g such that the transformation T is one to one? Explain briefly.

Proof. By Theorem 2.4, T is one to one if and only if $N(T) = \{0\}$. Using the rank nullity theorem, this is equivalent to saying that $rank(T) = 3$, i.e.

$$\begin{aligned} dim(\mathbb{R}^3) &= rank(T) + nullity(T) \\ 3 &= rank(T) \end{aligned}$$

We know that $rank(T) = rank([T]_{\epsilon_3}^{\epsilon_4})$ by Theorem 3.3. Additionally by Theorem 3.5., the rank of a matrix is the number of linearly independent columns. It is clear that A will have rank 3 if $h \neq 0$. There are no conditions for g. □

e. Suppose that $h = 0$, then

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2g \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and let } b = \begin{bmatrix} 0 \\ 2 \\ r \\ 0 \end{bmatrix}$$

What are the conditions on r and g such that the system $Ax = b$ has a solution? When is the solution unique?

Proof. We begin by putting the augmented matrix $(A|b)$ in its reduced form.

$$(A'|b') = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 2 & 2g & 2 \\ 0 & 0 & 2-2g & r-2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By Theorem 3.11 and 3.13, a system is consistent if and only if $rank(A') = rank(A'|b')$. So for the system to have as solution, $2 - 2g \neq 0$ if $r - 2 \neq 0$. If $r - 2 = 0$, there is no conditions on g.

In summary, for the system to have as solution if $r \neq 2$ then $g \neq 1$. If $r = 2$, there are no conditions on g.

Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(A) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of g as such that

$$\begin{aligned} -1 * 2 * (2 - 2g) &\neq 0 \\ 4g - 4 &\neq 0 \\ g &\neq 1 \end{aligned}$$

There is no conditions for r when the solution is unique. □

f. Suppose that $h = 0, g = 1, r = 2$. Solve $\mathbf{Ax} = \mathbf{b}$ and give the answer in parametric form.

Proof. We define

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

We can compute a solution space to $\mathbf{Ax} = \mathbf{b}$ as outlined in Theorem 3.9. We start by first computing the solution set to $\mathbf{Ax} = 0$ denoted by K_H. It is clear that $rank(\mathbf{A}) = 2$ because the first two columns are linearly independent and the third column is the sum of the first two columns. By Theorem 3.8, $dim(K_H) = 3 - 2 = 1$. Thus any nonzero solution constitutes a basis for K. For example, since

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

is a solution to the $\mathbf{Ax} = 0$, it is a basis for K_H by Corollary 2 of Theorem 1.10. So a solution set to K_H would be

$$K_H = \left\{ t \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

A solution to $\mathbf{Ax} = \mathbf{b}$ is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, by Theorem 3.9 we compute the solution space as

$$K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

§3

Let T and U be positive semidefinite operators on an inner product space V. Prove the following results

a. $T + U$ is positive semidefinite.

Proof. By definition, $\langle T(x), x\rangle \geq 0$, $\langle U(x), x\rangle > 0$ for all $x \neq 0$

$$\begin{aligned}\langle (T+U)(x), x\rangle &= \langle T(x) + U(x), x\rangle \\ &= \langle T(x), x\rangle + \langle U(x), x\rangle \\ &\geq 0\end{aligned}$$

$T + U$ is self adjoint because

$$\begin{aligned}(T+U)^* &= T^* + U^* \\ &= T + U\end{aligned}$$

Therefore, $T + U$ is positive semidefinite. ☐

b. If $c > 0$, then $cI + T$ is positive definite, where I is the identity transformation.

Proof. By definition, $\langle T(x), x\rangle \geq 0$, $\langle x, x\rangle > 0$ for all $x \neq 0$

$$\begin{aligned}\langle (cI+T)(x), x\rangle &= \langle cI(x) + T(x), x\rangle \\ &= \langle cI(x), x\rangle + \langle T(x), x\rangle \\ &= c\langle x, x\rangle + \langle T(x), x\rangle \\ &> 0\end{aligned}$$

$cI + T$ is self adjoint because

$$\begin{aligned}(cI+T)^* &= (cI)^* + T^* \\ &= \overline{c}I^* + T^* \\ &= cI + T\end{aligned}$$

Therefore, $cI + T$ is positive definite. ☐

c. $(cI + T)^{-1}$ is positive definite. From (b) we know that $cI + T$ is self adjoint because. It follows that

Proof. Let $y = (cI + T)^{-1}(x)$.

$$\begin{aligned}\langle (cI+T)^{-1}(x), x\rangle &= \langle y, (cI+T)(y)\rangle \\ &= \langle (cI+T)^*(y), y\rangle \\ &= \langle (cI+T)(y), y\rangle \\ &> 0\end{aligned}$$

$(cI + T)^{-1}$ is self adjoint because

$$\begin{aligned}((cI+T)^{-1})^* &= ((cI+T)^*)^{-1} \\ &= (cI+T)^{-1}\end{aligned}$$

Therefore, $(cI + T)^{-1}$ is positive definite. ☐

§4

Let V be a finite-dimensional inner product space. Suppose that U is a partial isometry of W on V, where W is a subspace of V, and let $\{v_1, v_2, \cdots, v_k\}$ be an orthonormal basis for W.

a. Show that $\{U(v_1), U(v_2), \cdots, U(v_k)\}$ is an orthonormal basis for $R(U)$

Proof. By definition $||U(w)|| = ||w||$ for all $w \in W$. By Theorem 6.18 (b), we know that $\langle U(w_1), U(w_2)\rangle = \langle w_1, w_2\rangle$ where $w_1, w_2 \in W$.

Notice that $\langle U(v_i), U(v_j)\rangle = \langle v_i, v_j\rangle$ which means that $\{U(v_1), U(v_2), \cdots, U(v_k)\}$ is an orthonormal basis for $U(W)$ because $\{v_1, v_2, \cdots, v_k\}$ is an orthonormal basis and the fact that U is a partial isometry.

Additionally, suppose $w \in W$ and $w' \in W^{\perp}$. Then by definition of partial isometry we know that $U(w + w') = w$. This implies that $U(W) = R(U)$.

Therefore, since $\{U(v_1), U(v_2), \cdots, U(v_k)\}$ is an orthonormal basis for $U(W)$, it is an orthonormal basis for $R(U)$, □

b. Show that there exists an orthonormal basis γ for V such that the first k columns of $[U]_\gamma$ form an orthonormal set and the remaining columns are zero.

Proof. Suppose the dimension of V is n. Using Corollary 2 of Theorem 1.10 (Replacement Theorem), we can extend the orthonormal basis for W from (a) $\{v_1, v_2, \cdots, v_k\}$ to orthonormal basis for V, denoted as $\gamma = \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_{n-1}, v_n\}$. Notice that $U(v_i) \neq 0$ where $1 \leq i \leq k$ by (a). Since γ is an orthonormal basis, $v_i \in W^{\perp} \Rightarrow U(v_i) = 0$ where $k + 1 \leq i \leq n$.

Therefore, there exists an orthonormal basis γ for V such that the first k columns of $[U]_\gamma$ form an orthonormal set and the remaining columns are zero. □

c. Let $\{w_1, w_2, \cdots, w_j\}$ be an orthonormal basis for $\mathrm{R(U)}^{\perp}$ and

$$\beta = \{\mathrm{U}(v_1), \mathrm{U}(v_2), \cdots, \mathrm{U}(v_k), w_1, \cdots, w_j\}$$

Show that β is an orthonormal basis for V

Proof.

Lemma 4.1

Let V be an inner product space and W be a finite-dimensional subspace of V. Then, $V = W \oplus W^{\perp}$

Proof. From Theorem 6.6, $V = W + W^{\perp}$.

Let $x \in W \cap W^{\perp}$. Since $x \in W^{\perp}$, $\langle x, g\rangle = 0$ for any $g \in W$. Since $x \in W$, this means that $\langle x, x\rangle = 0$. Thus, $x = 0$ and $W \cap W^{\perp} = \{0\}$.

Therefore, $V = W \oplus W^{\perp}$. □

Since $R(U) \subseteq V$, by by our lemma we know that $V = R(U) \oplus R(U)^{\perp}$. From (a), we know that $\{\mathrm{U}(v_1), \mathrm{U}(v_2)\}$ is a basis for $R(U)$. It directly follows that β is an orthonormal basis for V. □

d. Show that U^* is a partial isometry.

Proof. We want to show that there exists a subspace X of V such that $||U^*(x)|| = ||x||$ for all $x \in X$ and $U(x) = 0$ for all $x \in X^{\perp}$.

We claim that $X = R(U)$. Let $x \in R(U)$. We can express x as $\sum_{i=1}^{k} c_i U(v_i)$. It follows that,

$$\begin{aligned}
||U^*(x)|| &= ||U^*(\sum_{i=1}^{k} c_i U(v_i))|| \\
&= ||\sum_{i=1}^{k} c_i U^*(U(v_i))|| \\
&= ||\sum_{i=1}^{k} c_i v_i|| \\
&= ||\sum_{i=1}^{k} c_i U(v_i)|| \\
&= ||x||
\end{aligned}$$

by Theorem 6.18 and the fact that U is a partial isometry, i.e. $||U(v_i)|| = ||v_i||$.

Let $x \in R(U)^\perp$. We can express x as $\sum_{i=1}^{j} \langle x, w_i \rangle w_i$ from (c) and Theorem 6.5.

$$\begin{aligned}
U^*(x) &= U^*(\sum_{i=1}^{j} \langle x, w_i \rangle w_i) \\
&= \sum_{i=1}^{j} \langle U^*(x), w_i \rangle w_i \\
&= \sum_{i=1}^{j} \langle x, U(w_i) \rangle w_i \\
&= \sum_{i=1}^{j} \langle x, 0 \rangle w_i \\
&= 0
\end{aligned}$$

Therefore, U^* is a partial isometry. □

e. Show that U^*U is an orthogonal projection on W.

Proof. We first show that U^*U is a projection on W. From the lemma in (c), $x \in V$ can be expressed as $x = w + w'$ where $w \in W, w' \in W^\perp$. Then by definition of partial isometry of U,

$$\begin{aligned}
U^*U(x) &= U^*U(w + w') \\
&= U^*U(w) + U^*U(w') \\
&= U^*(w) + U^*U(0) \\
&= U^*(w)
\end{aligned}$$

Thus, U^*U is a projection on W.

Lemma 4.2

Let V be a finite dimensional inner product space and W be a subspace of V. Then, $(W^{\perp})^{\perp} = W$.

Proof. Let $w \in W$. For all $w' \in W^{\perp}$, $\langle w, w' \rangle = 0$. This implies that $w \in (W^{\perp})^{\perp}$.

Let $w' \in (W^{\perp})^{\perp}$. That means for some $w \in W$, $\langle w', w \rangle = 0$.This implies that $w \in W$.

Therefore, $(W^{\perp})^{\perp} = W$. □

By definition, U^*U is an orthogonal projection if $R(U^*U)^{\perp} = N(U^*U)$ and $N(U^*U)^{\perp} = R(U^*U)$. We only need to prove one conditions because from our lemma,

$$\begin{aligned} R(U^*U)^{\perp} &= N(U^*U) \\ (R(U^*U)^{\perp})^{\perp} &= N(U^*U)^{\perp} \\ R(U^*U) &= N(U^*U)^{\perp} \end{aligned}$$

We claim that $R(U^*U)^{\perp} = N(U^*U)$. Let $x \in R(U^*U)^{\perp}$.

$$\begin{aligned} \langle U^*U(x), x \rangle &= \langle U(x), U(x) \rangle \\ &= 0 \end{aligned}$$

This implies that $U(x) = 0$, so $x \in N(U^*U)$.

Let $x \in N(U^*U)$. By definition, $U^*U(x) = 0$. Let $y \in R(U^*U)$,

$$\begin{aligned} \langle x, U^*U(y) \rangle &= \langle U(x), U(y) \rangle \\ &= \langle U^*U(x), y \rangle \\ &= \langle 0, y \rangle \\ &= 0 \end{aligned}$$

So $x \in R(U^*U)^{\perp}$.

Therefore, U^*U is an orthogonal projection on W. □