Math 341: Homework 4

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§1 A

Let V be a vector space having dimension n, and let S be a subset of V that generates V.

a. Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite).

   Proof.
   Since V is finite dimensional, there exists a basis for V.
   $B = \{v_1, v_2, \cdots, v_n\}$
   Any $v \in B$ can be expressed as a linear combination of S because span(S) = V.
   Let the subset of S that generates v_i be S_i
   $v_i = \sum_{j=1}^{m^k} a_j^k s_j^k$ where $a \in F$ and $s \in S_i$
   The span of the union of the sets that generates v, span$(\bigcup_{i=1}^{n} S_i) = V$
   Corollary 2(a) of Theorem 1.10 states that a generating set for V that contains exactly n vectors is a basis for V. The set above, which is a subset of S, contains exactly n vectors and generates V. Therefore, there is subset of S that is a basis for V. □

b. Prove that S contains at least n vectors.

   Proof.
   From (a) we know there is a subset of S that forms a basis. Since that subset contains n vectors, S must contain n or more vectors. □

§2 B

Let $f(x)$ be a polynomial of degree n in $P_n(R)$. Prove that for any $g(x) \in P_n(R)$ there exists scalars $c_0, c_1, \cdots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

Proof.
Let $B = \{f, f', f'', \cdots, f^{(n)}\}$.
If B forms a basis we can express any $g(x) \in P_n(R)$ in the format above (a linear combination).
We can determine if B is basis by seeing if it is linearly independent by using a matrix.
$\mu_0 f + \mu_1 f' + \mu_2 f'' + \cdots + \mu_n f^{(n)} = 0$

$$\begin{bmatrix} a_n & a_{n-1} & \cdots & \cdots & a_0 \\ & na_n & \cdots & \cdots & a_1 \\ & & \ddots & \cdots & \cdots & \vdots \\ & & & \ddots & \cdots & \vdots \\ & & & & n!a_n & (n-1)!a_{n-1} \\ & & & & & n!a_n \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \vdots \\ \mu_{n-1} \\ \mu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Solving this system of equations:
Looking at the bottom row, $n!a_n\mu_n = 0$
$\mu_n = \frac{0}{n!a_n} = 0$, a_n is non zero because f is a nth degree polynomial and a_n is its coefficient.
Looking at row n - 1,$n!a_n\mu_{n-1} + (n-1)!a_{n-1}\mu n = 0$
Because $\mu_n = 0$, $n!a_n\mu_{n-1} + 0 = 0$
$\mu_{n-1} = \frac{0}{n!a_n} = 0$
By back substitution, $\mu_n = \mu_{n-1} = \cdots = \mu_1 = \mu_0 = 0$
This means that B is linearly independent, which also means that B is a basis.
Therefore, any $g(x) \in P_n(R)$ can be a linear combination of B with the scalars $c_0, c_1, \cdots, c_n$ ☐

§3 C

a. prove blah blah

Proof.
W_1 and W_2 are finite dimensional subspaces of $V \Rightarrow$ subspace $W_1 + W_2$ is finite dimensional and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
Let $B_{1\cap 2}$ be a basis for $W_1 \cap W_2$
$B_{1\cap 2} = \{u_1, u_2, \cdots, u_k\}$
By using the replacement theorem, we can extend $B_{1\cap 2}$ to be a basis for W_1
So the basis for W_1 is B_1
$B_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$
Likewise, we can extend $B_{1\cap 2}$ to be a basis for W_2
$B_2 = \{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_p\}$
Basis for $W_1 + W_2$ will be $B_1 \cup B_2$, however they may contain the same vectors twice.
To prevent double counting, we must subtract $B_1 \cap B_2$ from $B_1 \cup B_2$
Thus the basis for $W_1 + W_2$ is
$B_{1+2} = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$
$W_1 + W_2$ is finite dimensional because its basis contains only a finite amount of vectors.
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
$k + m + p = k + m + k + p - k$
$k + m + p = k + m + p$ ☐

b. Let W 1 and W 2 be finite-dimensional subspaces of a vector space V, and let V = W 1 + W 2 . Deduce that V is the direct sum of W 1 and W 2 if and only if dim(V = dim)W 1)(+ dimW 2 ())
$V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$

Proof.
$V = W_1 \oplus W_2 \Rightarrow dim(V) = dim(W_1) + dim(W_2)$
From the definition of direct sum, $W_1 \cap W_2 = \{0\}$
This means $dim(W_1 \cap W_2) = 0$
From (a), we proved that $dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$
$= dim(W_1) + dim(W_2) - 0$

$= dim(W_1) + dim(W_2)$

$dim(V) = dim(W_1) + dim(W_2) \Rightarrow V = W_1 \oplus W_2$
$V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$
$V = W_1 + W_2$ is true by the definition of the problem.
From part (a), $dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$
Our antecedent is $dim(V) = dim(W_1) + dim(W_2)$
Setting the two equations equal to each other:
$dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = dim(W_1) + dim(W_2)$
$dim(W_1 \cap W_2) = 0$
This means $W_1 \cap W_2 = \{0\}$
Thus, $dim(V) = dim(W_1) + dim(W_2)$

Therefore, $V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$ □

§4 D

§5 E

Let V be the vector space of sequences. Define the functions T,U : V → V by T(a 1 ,a 2 ,.. . = ()a 2 ,a 3 ,.. .) and U(a 1 ,a 2 ,.. .) = (0,a 1 ,a 2 ,.. . T and U are called the left shift and right shift operators o)n V respectively.

a. Prove that T and U are linear.

Proof.
T is linear if and only if $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$
Let $x, y \in V \quad c \in F$
$x = (x_1, x_2, \cdots) \quad y = (y_1, y_2, \cdots)$
$x + y = (x_1 + y_1, x_2 + y_2, \cdots)$
$T(x + y) = (x_2 + y_2, x_3 + y_3, \cdots)$
$T(x) = (x_2, x_3, \cdots)$
$T(y) = (y_2, y_3, \cdots)$
$T(x) + T(y) = (x_2 + y_2, x_3 + y_3, \cdots)$
Thus, $T(x + y) = T(x) + T(y)$

$x = (x_1, x_2, \cdots)$
$cx = (cx_1, cx_2, \cdots)$
$T(cx) = (cx_2, cx_3, \cdots)$
$T(x) = (x_2, x_3, \cdots)$
$cT(x) = (cx_2, cx_3, \cdots)$
Thus, $T(cx) = cT(x)$

Therefore, T is linear. The proof for U being linear is similiar. □

b. T is onto but not one to one

Proof.
T is onto if $\forall y \in V$ $\exists x \in V$ such that $f(x) = y$
Let $y = (a_1, a_2, \cdots)$ be arbitrary
$f(x) = y = (a_1, a_2, \cdots)$
$x = (a_0, a_1, a_2, \cdots)$

Since y was chosen arbitrarily, there exists an x for any y.
Therefore, T is onto.

T is one to one if $\forall a, b \in V, T(a) = T(b) \Rightarrow a = b$
Let $a = (u_\alpha, u_2, u_3, \cdots)$ $b = (u_\gamma, u_2, u_3, \cdots)$ where $u_\alpha \neq u_\gamma \Leftrightarrow a \neq b$
$T(a) = (u_2, u_3, \cdots)$ $T(b) = (u_2, u_3, \cdots)$
Therefore, T is not one to one because $T(a) = T(b)$ and $a \neq b$. □

c. U is one to one but not onto.

Proof.
U is one to one if $\forall a, b \in V, U(a) = U(b) \Rightarrow a = b$
Let $U(a) = (0, u_1, u_2, \cdots) = U(b) = (0, v_1, v_2, \cdots)$
This means $u_1 = v_1, u_2 = v_2, \cdots$
So, $a = (u_1, u_2, u_3, \cdots)$ $b = (v_1, v_2, v_3, \cdots)$
Hence, $a = b$
Therefore, U is one to one because $U(a) = U(b)$ and $a = b$.

U is onto if $\forall y \in V\ \exists x \in V$ such that $f(x) = y$
Let $y = (a_1, a_2, \cdots)$ where $a_1 \neq 0$
There is no x such that $U(x) = y$ because the linear transformation always makes the first term always zero. Therefore, T is not onto. □

§6 F

Let S be the subspace of $M_{nxn}(R)$ generated by all matrices of the form $AB - BA$ with A and B in $M_{nxn}(R)$. Prove that $\dim(S) = n^2 - 1$. (You may want to use the trace together with the rank-nullity theorem)

Proof.
Trace is a linear transformation.
$\text{Tr} : M_{nxn}(R) \to R$
The subspace S is defined as $\{AB - BA : A, B \in M_{nxn}(R)\}$
$\text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA)$
$= \text{Tr}(AB) - \text{Tr}(AB)$
$= 0$
All matrices that can be expressed as $AB - BA$ is in the null space of Tr. This means that N(Tr)$= S$.
The rank-nullity theorem states:
dim(N(Tr)) + dim(R(Tr)) = dim($M_{nxn}(R)$)
N(Tr)$= S$, so dim(S) + dim(R(Tr)) = dim($M_{nxn}(R)$)
dim(S) = dim($M_{nxn}(R)$) - dim(R(Tr))
= $n^2 -$ dim(R)
= $n^2 -$ 1

□

§7 G

Let T be a linear transformation of a vector space V into itself. Suppose that $x \in V$ is such that $T^m(x) = 0$, and $T^{m-1}(x) \neq 0$ for some positive m. Show that $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent.

Proof.
The linear combination of the above set is

$$a_0x + a_1T(x) + a_2T^2(x) + \cdots + a_{n-1}T^{m-1}(x) = 0$$

Notice that $T^n(x) = 0$ for all $n \geq m$.
$T^{m+1}(x) = T(T^m(x)) = T(0) = 0$
Let's take T^{m-1} on both sides of the linear combination.

$$\begin{aligned}
T^{m-1}(a_0x + a_1T(x) + a_2T^2(x) + \cdots + a_{n-1}T^{m-1}(x)) &= T^{m-1}(0) \\
T^{m-1}(a_0x) + T^{m-1}(a_1T(x)) + T^{m-1}(a_2T^2(x)) + \cdots + T^{m-1}(a_{n-1}T^{m-1}(x)) &= 0 \\
T^{m-1}(a_0x) + 0 + 0 + \cdots + 0 &= 0 \\
T^{m-1}(a_0x) &= 0 \\
a_0 = \frac{0}{T^{m-1}(x)} &= 0
\end{aligned}$$

By back substitution we know that $a_0 = a_1 = \cdots = a_{n-1} = 0$
Therefore, $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent. □

§8 H

Let $T : R^3 \to R^3$

a. If T(a,b,c) = (a,b,0), show that T is the projection on the xy-plane along the z-axis.

   Proof.
   We want to projection to be on the xy-plane along the z-axis. Let the projection be (x,y,0). To minimize the distance, we must choose x and y such that

   $$(a - x)^2 + (b - y)^2 + (c - 0)^2$$

   is minimum. Since the equation above is a difference of squares, x = a and b = y will give us the minimum value. Therefore, the projection on the xy-plane will be (a,b,0), which is T. □

b. Find a formula for T(a,b,c), where T represents the projection on the z-axis along the xy-plane.

   Proof.
   We want to projection to be on the z-axis along the xy-plane. Let the projection be (0,0,z). To minimize the distance, we must choose z such that

   $$(a - 0)^2 + (b - 0)^2 + (c - z)^2$$

   is minimum. z = c will give us the minimum value. Therefore, the equation for T will be T(a,b,c)=(0,0,c). □

c. If T(a,b,c) = (a-c,b,0), show that T is the projection on the xy-plane along the line L $= \{(a, 0, a) : a \in R\}$

   Proof.
   We want to projection to be on the xy-plane along the line L. Let the projection be $(x, y, 0)$. A vector that is on L is $(1, 0, 1)$. To minimize the distance, we must choose λ such that

   $$(a, b, c) + \lambda(1, 0, 1) = (x, y, 0)$$

is minimum. Writing the equation above as a system:

$$a + \lambda = x$$
$$b = y$$
$$c + \lambda = 0$$

Solving this system gives us, $x = a - c, y = b$
Therefore, the projection on the xy-plane along the line L will be $(a - c, b, 0)$.

□

§9 I

In $M_{mxn}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Proof.
If E^{ij} is linearly independent then $a_{1,1}E^{1,1} + \cdots + a_{m,n}E^{m,n} \neq 0$
This sum can only equal the 0 matrix if all a are 0.
Therefore, E^{ij} is linearly independent. □

§10 J

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof.
Let's first show that if u or v is a multiple of the other then $\{u, v\}$ is linearly dependent.
Being a mutiple means $u = nv$ or $v = nu$ where $n \in F$
If $\{u, v\}$ is linearly dependent then $a_1u + a_2v = 0$ where $a \in F$
Using definition of mutiple $a_1u + a_2nu = 0$
Factoring, $u(a_1 + a_2n) = 0$
This means $(a_1 + a_2n) = 0$
So, $n = \frac{-a_1}{a_2}$ which is a solution for linearly dependency.
Without loss of generality, we can prove the case where $v = nu$
Therefore, $\{u, v\}$ is linearly dependent.

Now let's show that if $\{u, v\}$ is linearly dependent then u or v is a multiple of the other.
If $\{u, v\}$ is linearly dependent then $a_1u + a_2v = 0$ where $a \in F$
We can rewrite the equation above as $a_1u = -a_2v$
$u = \frac{-a_2}{a_1}v$
Thus, u is a multiple of v.
Without loss of generality, we can prove v is a multiple of u.
Therefore, u or v is a mutiple of the other. □