

# Math 341: Homework 1

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## §1 A

a.  $p \Rightarrow (p \vee q)$

$p$	$q$	$p \vee q$	$p \Rightarrow (p \vee q)$
$T$	$T$	$T$	$T$
$T$	$F$	$T$	$T$
$F$	$T$	$T$	$T$
$F$	$F$	$F$	$T$

b.  $p \vee F \Leftrightarrow F$

$p$	$F$	$p \vee F$	$p \vee F \Leftrightarrow F$
$T$	$F$	$T$	$T$
$F$	$F$	$F$	$T$

c.  $p \wedge \neg p \Leftrightarrow F$

$p$	$\neg p$	$p \wedge \neg p$	$p \wedge \neg p \Leftrightarrow F$
$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$

d.  $(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$

$p$	$q$	$p \Leftrightarrow q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$
$T$	$T$	$T$	$T$	$F$	$T$	$T$
$T$	$F$	$F$	$F$	$F$	$F$	$T$
$F$	$T$	$F$	$F$	$F$	$F$	$T$
$F$	$F$	$T$	$F$	$T$	$T$	$T$

e.  $[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$

$p$	$q$	$r$	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)$	$p \Leftrightarrow r$	$[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$
$T$	$T$	$T$	$T$	$T$	$T$	$T$	$T$
$T$	$T$	$F$	$T$	$F$	$F$	$F$	$T$
$T$	$F$	$T$	$F$	$F$	$F$	$T$	$T$
$T$	$F$	$F$	$F$	$T$	$F$	$F$	$T$
$F$	$T$	$T$	$F$	$T$	$F$	$T$	$T$
$F$	$T$	$F$	$F$	$F$	$F$	$T$	$T$
$F$	$F$	$T$	$T$	$F$	$F$	$F$	$T$
$F$	$F$	$F$	$T$	$T$	$T$	$T$	$T$

f.  $[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$

$p$	$q$	$p \wedge \neg q$	$(p \wedge \neg q) \Rightarrow \neg p$	$p \Rightarrow q$	$[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$
$T$	$T$	$F$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$	$T$	$T$
$F$	$F$	$F$	$T$	$T$	$T$

## §2 B

a.  $(p \vee q \Leftrightarrow p \wedge r) \Rightarrow ((p \Rightarrow p) \wedge (p \Rightarrow r))$

$$\begin{aligned}
 & (p \vee q \Leftrightarrow p \wedge r) \Rightarrow (p \Rightarrow r) && \text{(Transitivity)} \\
 & [(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \Rightarrow (p \Rightarrow r) && \text{(Def. of biconditional)} \\
 & \neg[(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \vee (\neg p \vee r) && \text{(Material Implication)} \\
 & \neg[\neg(\neg(p \vee q) \vee (p \wedge r)) \wedge (\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(Material Implication)} \\
 & [\neg(\neg(p \vee q) \vee (p \wedge r)) \vee \neg(\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(De Morgan's Law)} \\
 & (\neg\neg(p \vee q) \wedge \neg(p \wedge r)) \vee (\neg\neg(p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(De Morgan's Law)} \\
 & ((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(Double negation)} \\
 & (\neg p \vee r) \vee ((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Commutative)} \\
 & ((\neg p \vee r) \vee (p \vee q)) \wedge ((\neg p \vee r) \vee \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Distributive)} \\
 & (True \wedge True) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Excluded middle)} \\
 & True
 \end{aligned}$$

$$b. [(p \Rightarrow \neg q) \wedge (r \Rightarrow q)] \Rightarrow (p \Rightarrow \neg r)$$

$$\begin{aligned} & \neg[(\neg p \vee \neg q) \wedge (\neg r \wedge q)] \vee \neg p \vee \neg r && \text{(Material Implication)} \\ & \neg(\neg p \vee \neg q) \vee \neg(\neg r \wedge q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)} \\ & (p \wedge q) \vee (r \wedge \neg q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)} \\ & \neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)} \\ & [(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)} \\ & [(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)} \\ & \neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)} \\ & True \end{aligned}$$

$$c. (p \Rightarrow q) \Rightarrow [\neg(q \wedge r) \Rightarrow \neg(p \wedge r)]$$

$$\begin{aligned} & \neg(\neg p \vee q) \vee [\neg\neg(q \wedge r) \vee \neg(p \wedge r)] && \text{(De Morgan's Law)} \\ & (p \wedge \neg q) \vee (q \wedge r) \vee (\neg p \vee \neg r) && \text{(De Morgan's Law + Negation)} \\ & \neg p \vee (p \wedge \neg q) \vee \neg r \vee (q \wedge r) && \text{(Commutative)} \\ & [(\neg p \vee p) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (\neg r \vee r)] && \text{(Distributive)} \\ & [(True) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (True)] && \text{(Excluded middle)} \\ & \neg p \vee \neg q \vee \neg r \vee q && \text{(Excluded middle)} \\ & True \end{aligned}$$

$$d. [(p \Rightarrow \neg q) \wedge (\neg r \vee q) \wedge r] \Rightarrow \neg p$$

$$\begin{aligned} & \neg[(\neg p \vee \neg q) \wedge (\neg r \vee q) \wedge r] \vee \neg p && \text{(Material Implication)} \\ & \neg(\neg p \vee \neg q) \vee \neg(\neg r \vee q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)} \\ & (p \wedge q) \vee (r \wedge \neg q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)} \\ & \neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)} \\ & [(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)} \\ & [(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)} \\ & \neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)} \\ & True && \text{(Excluded middle)} \end{aligned}$$

### §3 C

- a. Proposition r means q is true if p(x) is true for one x.  
 Proposition s means q is true if p(x) is true for all x.

b.

$$\begin{aligned}
r &\Leftrightarrow (\forall x)(p(x) \Rightarrow q) \\
\neg r &\Leftrightarrow \neg[(\forall x)(p(x) \Rightarrow q)] \\
&\Leftrightarrow \exists x \neg(p(x) \Rightarrow q) && \text{(Quantifier Negation)} \\
&\Leftrightarrow \exists x \neg(\neg p(x) \vee q) && \text{(Material Implication)} \\
&\Leftrightarrow \exists x (\neg \neg p(x) \wedge \neg q) && \text{(De Morgan's Law)} \\
&\Leftrightarrow \exists x (p(x) \wedge \neg q) && \text{(Double Negation)}
\end{aligned}$$

$$\begin{aligned}
s &\Leftrightarrow ((\forall x)p(x)) \Rightarrow q \\
\neg s &\Leftrightarrow \neg[((\forall x)p(x)) \Rightarrow q] \\
&\Leftrightarrow \neg[\neg((\forall x)p(x)) \vee q] && \text{(Material Implication)} \\
&\Leftrightarrow ((\forall x)p(x)) \wedge \neg q && \text{(De Morgan's Law)}
\end{aligned}$$

c.  $s \Rightarrow r$  is a tautology. If  $s$  is true, then  $r$  would be true because  $s$  requires  $p(x)$  to be true for all  $x$  while  $r$  only requires  $p(x)$  for one  $x$  to be true. If  $s$  is false, then the whole statement would be vacuously true.

## §4 D

### Corollary

The additive inverse is unique.

*Proof.* Suppose  $u, v$  are the additive inverse of  $x$ .

$$x + u = 0 \quad x + v = 0$$

$$\begin{aligned}
x + u &= x + v && \text{(Transitive property)} \\
u + x &= v + x && \text{(Commutative property)} \\
u &= v && \text{(Theorem 1.1)}
\end{aligned}$$

□

### Corollary

The vector  $0$  is unique.

*Proof.* Suppose  $u, v \in V$  satisfies the "zero property", which is defined as:

$$\begin{aligned}
\forall x \in V \quad x + u &= x \Rightarrow v + u = v \\
\forall x \in V \quad x + v &= x \Rightarrow u + v = u \\
u &= u + v = v + u = v && \text{(Transitive property)} \\
u &= v && \text{(Theorem 1.1)}
\end{aligned}$$

□

## §5 E

**Theorem** (1.2(c)) In any vector space the following statements are true.)

$$a\mathbf{0} = \mathbf{0}$$

$$\forall a \in F \quad \mathbf{0} \in V$$

Any scalar multiplied by the 0 vector will result in the 0 vector.

*Proof.*

$$a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) \quad (\text{Identity element of addition})$$

$$a\mathbf{0} = a\mathbf{0} + a\mathbf{0} \quad (\text{Distributive})$$

$$a\mathbf{0} - a\mathbf{0} = a\mathbf{0} + a\mathbf{0} - a\mathbf{0} \quad (\text{Inverse element of addition})$$

$$\mathbf{0} = a\mathbf{0}$$

□

## §6 F

Prove that diagonal matrices (as defined in your book in Example 3, Section 1.3) are symmetric.

*Proof.*

Let  $D$  equal a diagonal matrix

$$D \text{ is symmetric} \Leftrightarrow (\forall i, j)(D_{i,j} = D_{j,i})$$

By definition of diagonal matrix:

$$\text{When } i \neq j, D_{i,j} = D_{j,i} = 0$$

$$\text{When } i = j, D_{i,j} = D_{j,i}$$

$$\therefore (\forall i, j)(D_{i,j} = D_{j,i}) \text{ so diagonal matrix is symmetric}$$

□

## §7 G

Prove that

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 0\}$$

is a subspace of  $F^n$ , but

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 1\}$$

is not.

*Proof.* Proof that  $W_1$  is a subspace of  $F^n$

a.  $0 \in W_1$

$$\text{Let } a_1, a_2, \dots, a_n = 0$$

$$0 + 0 + \dots + 0 = 0$$

$$\text{So the } 0 \text{ vector: } (0, 0, \dots, 0) \in W_1$$

b.  $X, Y \in W_1 \Rightarrow X + Y \in W_1$

$$\begin{aligned}
 X &= (x_1, x_2, \dots, x_n) \quad Y = (y_1, y_2, \dots, y_n) \\
 X + Y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\
 \sum_{i=1}^n x_i + y_i &= x_1 + y_1 + x_2 + y_2 + \dots + x_n + y_n \\
 &= (x_1 + x_2 + \dots + x_n) + (y_1 + y_2 + \dots + y_n) \\
 &= 0 + 0 \\
 &= 0 \in W_1
 \end{aligned}$$

c.  $c \in F, X \in W_1 \Rightarrow cX \in W_1$

$$\begin{aligned}
 X &= (x_1, x_2, \dots, x_n) \\
 cX &= (cx_1, cx_2, \dots, cx_n) \\
 \sum_{i=1}^n cx_i &= cx_1 + cx_2 + \dots + cx_n \\
 &= (x_1 + x_2 + \dots + x_n) + (y_1 + y_2 + \dots + y_n) \\
 &= c(x_1 + x_2 + \dots + x_n) \\
 &= c(0) \\
 &= 0 \in W_1
 \end{aligned}$$

□

*Proof.*  $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 1\}$  is not a subspace of  $F^n$ .  
If  $W_2$  is a subspace, there must be closure under vector addition.

$$\begin{aligned}
 X &= (x_1, x_2, \dots, x_n) \quad Y = (y_1, y_2, \dots, y_n) \\
 X + Y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\
 \sum_{i=1}^n x_i + y_i &= x_1 + y_1 + x_2 + y_2 + \dots + x_n + y_n \\
 &= (x_1 + x_2 + \dots + x_n) + (y_1 + y_2 + \dots + y_n) \\
 &= 1 + 1 \\
 &= 2 \notin W_2
 \end{aligned}$$

$\therefore W_2$  is not a subspace of  $F^n$ .

Additionally, there is no 0 vector in  $W_2$ . 0 vector for polynomial space only exists if each component in a vector is 0. This is not possible in  $W_2$  because the sum of the components must equal 1. □

## §8 H

$$W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$$