Math 240: Homework 7

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§1 1

Solve the following recurrence relations. Show your work. If you wish, you may use the formulas we stated in lecture, without proof

a. $a_0 = 1$ and $a_n = 3a_{n-1} + 2$ for $n \geq 1$

Proof. Given any recurrence relations in this form $a_n = ra_{n-1} + c$, we can rewrite the relation as $a_n = r^n a_0 + c\frac{r^n-1}{r-1}$ if $r \neq 1$.

$$\begin{aligned} a_n &= 3^n(1) + 2\frac{3^n - 1}{3 - 1} \\ &= 3^n + 3^n - 1 \\ &= 2(3^n) - 1 \end{aligned}$$

□

b. $a_0 = 3$, $a_1 = 6$ and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$

Proof. We can write the above recurrence relation as $r^2 - r - 6 = 0$. Solving for r gets us $r = 3, r = -2$. We use the equation given in the lecture for linear homogeneous recurrence relations of degree two.

$$\begin{aligned} a_n &= \alpha(3)^n + \beta(-2)^n \\ a_0 = 3 &\Rightarrow 3 = \alpha + \beta \\ a_1 = 6 &\Rightarrow 6 = \alpha(3) + \beta(-2) \\ \alpha = \frac{12}{5} &\qquad \beta = \frac{3}{5} \\ a_n &= \frac{12}{5}3^n + \frac{3}{5}(-2)^n \end{aligned}$$

□

c. $a_0 = 5$, $a_1 = -1$ and $a_n = a_{n-2}$ for $n \geq 2$

Proof. We can write the above recurrence relation as $r^2 - 1 = 0$. Solving for r gets us $r = \pm 1$. We use the equation given in the lecture for linear homogeneous recurrence relations of degree two.

$$\begin{aligned} a_n &= \alpha(-1)^n + \beta(1)^n \\ a_0 = 5 &\Rightarrow 5 = \alpha + \beta \\ a_1 = -1 &\Rightarrow -1 = -\alpha + \beta \\ \alpha = 3 &\qquad \beta = 2 \\ a_n &= 3(-1)^n + 2(1)^n \end{aligned}$$

□

§2 2

a. Prove that every string σ of length n with an even number of 0s has exactly one of the following forms:

   i $\sigma = \tau 1$ where τ is a string of length $n-1$ with an even number of 0s.

   When we concatenate a 1 to τ, the total number of 0s in σ is the same as τ. So σ will have an even number of 0s.

   ii $\sigma = \tau 0$ where τ is a string of length $n-1$ with an odd number of 0s.

   When we concatenate a 0 to τ, the total number of 0s in σ is one more than τ. An even number plus one is an odd number by definition. τ has an even number of 0's, so σ will have an odd number of 0s.

b. Use (a) to find the number of binary strings of length n with an even number of 0s, as a function of n.

We will assume that 0 is an even number, so for a number with no 0s will have an even number of 0s, including the empty string. From part (a) we can create a recurrence relation: $a_n = 2(a_{n-1})$. This because there are two cases when we concatenate a binary string to a_{n-1}. If a_{n-1} has an even number of 0s then we then we append a 1 to the end. If a_{n-1} has an odd number of 0s then we then we append a 0 to the end. We know that $a_0 = 1$ and $a_1 = 1$ because the empty string and string with no zeros will have an even amount of zeros by our assumption. We can write the above recurrence relation as $r^2 - 2r = 0$. Solving for r gets us $r = 0, r = 2$. We use the equation given in the lecture for linear homogeneous recurrence relations of degree two.

$$\begin{aligned} a_n &= \alpha(0)^n + \beta(2)^n \\ a_0 = 1 &\Rightarrow 1 = \alpha + \beta \\ a_1 = 1 &\Rightarrow 1 = -\alpha 0 + \beta 2 = \beta 2 \\ \alpha = \frac{1}{2} &\qquad \beta = \frac{1}{2} \\ a_n &= \frac{1}{2}(0)^n + \frac{1}{2}(2)^n \end{aligned}$$

§3 3

$$\begin{aligned} H'_{n+1} &= H'_n + (n+1)^2 + H'_n \\ &= 2H'_n + (n+1)^2 \end{aligned}$$

§4 4

a. Prove that $5x+4$ is $O(x^2)$

Proof. We need to choose $C, k > 0$ such that for every $x > k$

$$5x + 4 \leq Cx^2$$

Take $C = 9$ and $k = 2$. Then,

$$\begin{aligned} 9x^2 &= 5x^2 + 4x^2 \\ &> 5x + 4 && x > 1 \end{aligned}$$

as desired. □

b. Prove that x^2 is not $\Theta(5x+4)$

Proof. To prove the above statement, we can prove that x^2 is not $O(5x+4)$. We need to show that for every $C, k > 0$, there is some $x > k$ such that $x^2 > O(5x+4)$.
Suppose we are given $C, k > 0$. Consider $x = \max\{9C, k\} + 1$.
Then $x > k$ and

$$\begin{aligned} x^2 &> 9Cx && x > 9C \text{ and } x > 0 \\ &= 5Cx + 4Cx \\ &> 5Cx + 4C && x > 1 \text{ and } C > 0 \\ &= C(5x+4) \end{aligned}$$

as desired. Since x^2 is not $O(5x+4)$, by definition x^2 is not $\Theta(5x+4)$. □