Math 240: Homework 9

Daniel Ko

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§1

Let A be any nonempty set. (You do not get to decide what A is)

a. Let R be a relation on A. Explain what is wrong with the following "proof" that if R is symmetric and transitive, then it is reflexive.

   "Proof": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$

   Proof. This proof is wrong because it assumes a priori that $(a, b) \in R$ when no such definition of the relation R is defined. $\square$

b. Give an example of a relation R on A which is symmetric and transitive, but not reflexive. Justify your answer.

   Proof. The most trival example is a non empty set A and the empty set R. It is vacuously true that R is symmetric and transitive. However, R is not reflexive because there are no relations in it, i.e. the following statement does not hold: $\forall a \in A : (a, a) \in R$. $\square$

§2

Give an example of a relation R on $\mathbb{N}$ such that if S is a relation on $\mathbb{N}$ which contains R, then S is not antisymmetric. Justify your answer. (This implies that the antisymmetric closure of R does not exist.)

Proof. Consider the relation $R = \{(1, 2), (2, 1)\}$. It is clear that $1 \neq 2$, hence R is not antisymmetric. Therefore, if S contains R, then S is not antisymmetric. $\square$

§3

Come up with a topological ordering of the following DAG, i.e., write down the vertices in some order such that if (i, j) is an edge, then i is to the left of j.

Proof. Using Kahn's algorithm, we get the topological ordering of $2, 3, 4, 6, 1, 5$. $\square$

§4

Let R be the relation on $\mathbb{R}$ defined by $(x, y) \in R$ if and only if $x - y$ is rational. Is R symmetric? Antisymmetric? Reflexive? Transitive? Justify your answers. (You may use basic properties about rational numbers without proving them)

Proof.

a. R is symmetric

   This is because there is closure under addition/subtraction for rational numbers. If $x - y$ is a rational number then $y - x$ is a rational number, hence $(y, x) \in R$.

b. R is not antisymmetric

   Consider the following relation in R: $(1, 2)$ and $(2, 1)$. It is clear that $1 \neq 2$. Therefore, R is not antisymmetric.

c. R is reflexive

   Suppose $x \in \mathbb{R}$. Any number subtracted by itself will be zero. Zero is a rational number. Thus, $(x, x) \in R$. Therefore, R is reflexive.

d. R is transitive

   Suppose (x, y) and (y, z) are in R. This means that $x - y$ and $y - z$ are rational numbers. Now consider,

$$x - z = (x - y) + (y - z)$$

   Since there is closure under addition for rational numbers, $x - z$ must be a rational number. Hence, $(x, z) \in R$. Therefore, R is transitive.

□

§5

Let R be a relation on a set A. Prove that if R is reflexive, then $R \subseteq R^n$ for all $n \geq 1$.

Proof. We proceed by induction on $n \geq 1$. Let $P(n)$ be the predicate if R is reflexive, then $R \subseteq R^n$.
Base case: We prove that $P(1)$ holds. Since $R^1 = R$, $P(1)$ holds.
Inductive step: Suppose that $n \geq 1$ such that $P(n)$ holds. We prove that $P(n+1)$ holds. Suppose $(a, c) \in R$. By our inductive hypothesis, $(a, c) \in R^n$. Since $R^n = R^{n-1} \circ R$, there must be some some $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R^{n-1}$. Since R is reflexive, there exists $(c, c) \in R$. This directly implies that $(a, c) \in R^{n+1}$. Therefore, $P(n+1)$ holds and this concludes our inductive step.
We have proved by induction that if R is reflexive, then $R \subseteq R^n$ for all $n \geq 1$. □

§6

Let R be a relation on a set A. Prove that for any $m, n \geq 1$, $R^n \circ R^m = R^{n+m}$. (You may assume that composition of relations is associative, i.e., $(Q \circ R) \circ S = Q \circ (R \circ S)$)

Proof. Fix a positive nonzero m. We prove by induction on $n \in \mathbb{N}^+$ that $R^n \circ R^m = R^{n+m}$.
Base case: We prove that $P(1)$ holds. Since $n = 1$, $R^1 \circ R^m = R^{1+m}$, $P(1)$ holds.

Inductive step: Suppose that $n \in \mathbb{N}^+$ such that $P(n)$ holds. We prove that $P(n+1)$ holds. Since $P(n)$ holds, $R^n \circ R^m = R^{n+m}$. We apply R to both sides,

$$\begin{aligned} R^n \circ R^m &= R^{n+m} \\ R \circ R^n \circ R^m &= R \circ R^{n+m} \\ R^{n+1} \circ R^m &= R^{n+m+1} \\ R^{n+1} \circ R^m &= R^{(n+1)+m} \end{aligned}$$

as desired. Therefore, $P(n+1)$ holds and this concludes our inductive step.
We have proved by induction that for any $m, n \geq 1$, $R^n \circ R^m = R^{n+m}$. $\square$