Math 341: Homework 4

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§1 A

Let V be a vector space having dimension n, and let S be a subset of V that generates V.

a. Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite).

   Proof.
   Since V is finite dimensional, there exists a basis for V.
   $B = \{v_1, v_2, \cdots, v_n\}$
   Any $v \in B$ can be expressed as a linear combination of S because span(S) = V.
   Let the subset of S that generates v_i be S_i
   $v_i = \sum_{j=1}^{m^k} a_j^k s_j^k$ where $a \in F$ and $s \in S_i$
   The span of the union of the sets that generates v, span$(\bigcup_{i=1}^{n} S_i) = V$
   Corollary 2(a) of Theorem 1.10 states that a generating set for V that contains exactly n vectors is a basis for V. The set above, which is a subset of S, contains exactly n vectors and generates V. Therefore, there is subset of S that is a basis for V. □

b. Prove that S contains at least n vectors.

   Proof.
   From (a) we know there is a subset of S that forms a basis. Since that subset contains n vectors, S must contain n or more vectors. □

§2 B

Let $f(x)$ be a polynomial of degree n in $P_n(R)$. Prove that for any $g(x) \in P_n(R)$ there exists scalars $c_0, c_1, \cdots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

Proof.
If $\{f, f', f'', \cdots, f^{(n)}\}$ form a basis we can express any $g(x) \in P_n(R)$ as seen above (a linear combination).

□

§3 C

$W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_n = 0\}$
$W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}$
Show $F^n = W_1 \oplus W_2$

Proof. Definition of direct sum is $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = F^n$

a. $W_1 \cap W_2 = \{0\}$

   Let $v \in W_1, W_2$
   $v = (a_1, a_2, \cdots, a_n)$
   $v \in W_1 \Rightarrow a_n = 0$
   $v \in W_2 \Rightarrow a_1 = a_2 = \cdots = a_{n-1} = 0$
   $\therefore v = (0, 0, \cdots, 0) \Rightarrow W_1 \cap W_2 = \{0\}$

b. $W_1 + W_2 = F^n$

   Let $v \in F^n$
   $v = (a_1, a_2, \cdots, a_n)$
   Let $w_1 \in W_1$ and $w_2 \in W_2$
   $w_1 = (a_1, a_2, \cdots, a_{n-1}, 0)$
   $w_2 = (0, 0, \cdots, a_n)$
   $w_1 + w_2 = (a_1, a_2, \cdots, a_n) = v$
   Thus, any vector in F^n can be expressed as a sum of vectors in W_1 and W_2
   $\therefore W_1 + W_2 = F^n$

$\therefore F^n = W_1 \oplus W_2$ □

§4 D

In $M_{mxn}(F)$
$W_1 = \{A \in M_{mxn}(F) : A_{i,j} = 0 \text{ whenever } i > j\}$
$W_2 = \{B \in M_{mxn}(F) : B_{i,j} = 0 \text{ whenever } i \leq j\}$
Show that $M_{mxn}(F) = W_1 \oplus W_2$

Proof.

a. $W_1 \cap W_2 = \{0\}$

   Let $m \in W_1, W_2$
   $m \in W_1 \Rightarrow m_{i,j} = 0$ whenever $i > j$
   $m \in W_2 \Rightarrow m_{i,j} = 0$ whenever $i \leq j$
   Thus, $(\forall i, j)(m_{i,j} = 0)$ which is $\{0\}$
   $\therefore W_1 \cap W_2 = \{0\}$

b. $W_1 + W_2 = M_{mxn}(F)$

   Let $q \in M_{mxn}(F)$
   Let $w_1 \in W_1$ and $w_2 \in W_2$
   $w_1 = \{(w_1)_{i,j} = 0 \text{ whenever } i > j\}$
   $w_2 = \{(w_2)_{i,j} = 0 \text{ whenever } i \leq j\}$

$w_1 + w_2 = \{(w_1)_{i,j}$ wherever $i \leq j$ and $(w_2)_{i,j}$ wherever $i > j\} = q$
Thus, any matrix in $M_{mxn}(F)$ can be expressed as a sum of matrices in W_1 and W_2
$\therefore W_1 + W_2 = M_{mxn}(F)$

$\therefore M_{mxn}(F) = W_1 \oplus W_2$ □

§5 E

Let W be a subspace of a vector space V over a field F.
For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is the coset W containing v.

a. Prove that $v + W$ is in the subspace of V if and only if $v \in W$.

Proof.
$v + W$ is in the subspace of $V \Rightarrow v \in W$.
$0 \in v + W$ because $v + W$ is a subspace.
$0 = v + w, w \in W$
$v = -w$
$v \in W$

$v \in W \Rightarrow v + W$ is in the subspace of V.

i. $0 \in v + W$
$w \in W$ and let $v = -w$
$v + w = 0$
Thus, $0 \in v + W$

ii. $a + b \in v + W$ where $a, b \in v + W$
Let $a = v + w_a, w_a \in W$ and $b = v + w_b, w_b \in W$
$a + b = v + w_a + v + w_b$
Because $v \in W$, $w_a + v + w_b \in W$.
Thus, $a + b \in v + W$

iii. $ca \in v + w, a \in v + W, c \in F$
Let $a = v + w_a, w_a \in W$
$ca = c(v + w_a)$
$= cv + cw_a$
$= v + cv + cw_a - v$
$cv + c_w a - v \in W$ by closure under scalar multplication and vector addition.
Thus, $ca \in v + w$

□

b. Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

Proof.

i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$
Let $w_1, w_2 \in W$
$v_1 + w_1 = v_2 + w_2$
$v_1 - v_2 = w_2 - w_1$
Since, $w_2 - w_1 \in W$ (clourse under addition)
Therefore, $v_1 - v_2 \in W$

ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$
This means $v_1 - v_2 = w$ where $w \in W$ $(*)$
Now let $x \in v_1 + W$

By definition, $\exists w_x \in W : x = v_1 + w_x$
By $(*)$ $v_1 = v_2 + w$
So, $x = v_2 + w + w_x$
Since, $w + w_x \in W$ (closure under addition)
We have $x \in v_2 + W$
So, $v_1 + W \subseteq v_2 + W$
Similarly, we can show $v_2 + W \subseteq v_1 + W$
Therefore, $v_1 + W = v_2 + W$

$\square$

c. Show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then
$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$ and
$a(v_1 + W) = a(v_1' + W)$ for all $a \in F$

Proof.

i. $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$
Let $q \in (v_1 + W) + (v_2 + W)$
$q \in (v_1 + v_2) + W$ by definition of vector addition
So, $q = v_1 + v_2 + w_q$ where $w_q \in W$
$= v_1 + v_2 + w_q + v_1' - v_1' + v_2' - v_2'$
$= v_1' + v_2' + w_q + v_1 - v_1' + v_2 - v_2'$
From b. i, $v_1 - v_1'$ and $v_2 - v_2' \in W$
Which means, $(v_1 - v_1') + (v_2 - v_2') \in W$
Thus, $w_q + v_1 - v_1' + v_2 - v_2' \in W$
So, $q \in (v_1' + v_2') + W$
So, $(v_1 + W) + (v_2 + W) \subseteq (v_1' + W) + (v_2' + W)$
Similarly, we can show $(v_1' + W) + (v_2' + W) \subseteq (v_1 + W) + (v_2 + W)$
Therefore, $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$

ii. $a(v_1 + W) = a(v_1' + W)$
Let $q \in a(v_1 + W)$
$q \in av_1 + W$ by definition of scalar multplication.
So, $q = av_1 + w_q$ where $w_q \in W$
$= av_1 + w_q + av_1' - av_1'$
$= av_1' + w_q + av_1 - av_1'$
$= av'1 + a(v_1 - v_1') + w_q$
From b. i, $a(v_1 - v_1') \in W$
$a(v_1 - v_1') + w_q \in W$ because closure under vector addition.
So, $q \in av_1' + W$
So, $a(v_1 + W) \subseteq a(v_1' + W)$
Similarly, we can show $a(v_1' + W) \subseteq a(v_1 + W)$
Therefore, $a(v_1 + W) = (v_1' + W)$

$\square$

d. Prove that the set S is a vector space with the operations defined in (c).

i. $0 \in S$
The zero vector in S is $0 = v_0 + W$
Let $s \in S$
So $s = v_s + W$
If the zero vector exists we should be able to show, $s + 0 = s$
$s + 0 = s \Leftrightarrow (v_s + W) + (v_0 + W) = v_s + W$

$(v_s + v_0) + W = v_s + W$ by definition of addition
Thus $v_0 = 0$ and the zero vector is $0 + W$ which is just W
Therefore, the zero vector is W.

ii. $X + Y \in S$ where $X, Y \in S$
This means $X = v_x + W \quad Y = v_y + W$
$X + Y = (v_x + W) + (v_y + W) = (v_x + v_y) + W$ by defintion of addition.
$(v_x + v_y) \in V$ by closure under vector addition.
Therefore $X + Y \in S$

iii. $aX \in S \quad a \in F$
$aX = a(v_x + W)$
$= av_x + W$
$av_x \in V$ by closure under vector addition.
Therefore, $aX \in S$

§6 F

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric $2x2$ matrices.

Proof.
$Sym(M_{2x2}(F)) = \{m \in M_{2x2}(F) : m = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Leftrightarrow m = m^t\}$

$$m \in span(\{M_1, M_2, M_3\}) \text{ if } m = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{where } c_1, c_2, c_3 \in F$$

$$= \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}$$

$$m^t = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}$$

$\therefore Sym(M_{2x2}(F)) = span(\{M_1, M_2, M_3\})$ □

§7 G

Show that if S_1 and S_2 are subsets of the vector space V such that $S_1 \subseteq S_2$ then span$(S_1) \subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span(S_1) = V, deduce that span(S_2) = V

Proof.
Let $z_1 \in \text{span}(S_1)$
So $z_1 = \sum_{i=1}^{n} a_i x_i$ where $a \in F$ and $x \in S_1$
If $S_1 \subseteq S_2$, then $x \in S_2$
So $z_1 \in \text{span}(S_2)$ because we can write z_1 as a linear combination of S_2
Therefore, if $S_1 \subseteq S_2$ then $\text{span}(S_1) \subseteq \text{span}(S_2) \quad (*)$
Defined in the problem, span(S_1) = V
By $(*)$, $\text{span}(S_1) = V \subseteq \text{span}(S_2)$

Using theorem 1.5, $\text{span}(S_2) \subseteq V$
Therefore, $\text{span}(S_2) \subseteq V \subseteq \text{span}(S_2) \Leftrightarrow V = \text{span}(S_2)$ $\square$

§8 H

Show that $P_n(F)$ is generated by $\{1, x, \cdots, x^n\}$

Proof.
Definition of generates is $\text{span}(\{1, x, \cdots, x^n\}) = P_n(F)$

First let's show that $\text{span}(\{1, x, \cdots, x^n\}) \subseteq P_n(F)$
By theorem 1.5, this is true because $\{1, x, \cdots, x^n\} \subset P_n(F)$

Now let's show that $P_n(F) \subseteq \text{span}(\{1, x, \cdots, x^n\})$
Let $w \in P_n(F)$
$w = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0$ where $a \in F$
Let $v \in \text{span}(\{1, x, \cdots, x^n\})$ where $b \in F$
$v = b_0 1 + b_2 x + \cdots + b_n x^n$
Any w can be expressed as a v, if we fix $a_0 = b_0, \cdots, a_n = b_n$.
Thus, $P_n(F) \subseteq \text{span}(\{1, x, \cdots, x^n\})$

Therefore, $P_n(F)$ is generated by $\{1, x, \cdots, x^n\}$ $\square$

§9 I

In $M_{mxn}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is linearly independent.

Proof.
If E^{ij} is linearly independent then $a_{1,1}E^{1,1} + \cdots + a_{m,n}E^{m,n} \neq 0$
This sum can only equal the 0 matrix if all a are 0.
Therefore, E^{ij} is linearly independent. $\square$

§10 J

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof.
Let's first show that if u or v is a multiple of the other then $\{u, v\}$ is linearly dependent.
Being a mutiple means $u = nv$ or $v = nu$ where $n \in F$
If $\{u, v\}$ is linearly dependent then $a_1 u + a_2 v = 0$ where $a \in F$
Using definition of mutiple $a_1 u + a_2 nu = 0$
Factoring, $u(a_1 + a_2 n) = 0$
This means $(a_1 + a_2 n) = 0$
So, $n = \frac{-a_1}{a_2}$ which is a solution for linearly dependency.
Without loss of generality, we can prove the case where $v = nu$
Therefore, $\{u, v\}$ is linearly dependent.

Now let's show that if $\{u, v\}$ is linearly dependent then u or v is a multiple of the other.

If $\{u, v\}$ is linearly dependent then $a_1u + a_2v = 0$ where $a \in F$
We can rewrite the equation above as $a_1u = -a_2v$
$u = \frac{-a_2}{a_1}v$
Thus, u is a multiple of v.
Without loss of generality, we can prove v is a multiple of u.
Therefore, u or v is a mutiple of the other. $\square$