

Math 341: Homework 2

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§1 A

Let D be the set of all differentiable functions defined on \mathbb{R} . Note that D is a subset of C because differentiable functions are continuous.

Proof. D is a subspace of C

a. $0 \in D$

Zero vector is defined as $f(x) = 0$ where $x \in \mathbb{R}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{0 - 0}{h} \\ &= 0 \end{aligned}$$

Because the derivative of $f(x) = 0$ exists, $0 \in D$

b. $f + g \in D$ where $f, g \in D$

$$\begin{aligned} (f+g)'(x) &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

Because the derivative of $f + g$ exists, $f + g \in D$

c. $cf \in D$ where $c \in \mathbb{R}$ and $f \in D$

$$\begin{aligned} cf'(x) &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Because the derivative of cf exists, $cf \in D$

$\therefore D$ is a subspace of C

□

§2 B

Prove the set of even functions in $F(F_1, F_2)$ and odd functions in $F(F_1, F_2)$ are subspaces of $F(F_1, F_2)$

Proof. Let O be the set of all odd functions in $F(F_1, F_2)$ and E be the set of all even functions in $F(F_1, F_2)$.

- a. $0 \in O$ and $0 \in E$

Zero function is defined as $g(x) = 0$

$0 \in O$ is odd:

$$\begin{aligned} g(-x) &= 0 \\ -g(x) &= 0 \\ g(-x) &= -g(x) \end{aligned}$$

$0 \in E$ is even:

$$\begin{aligned} g(x) &= 0 \\ g(-x) &= 0 \\ g(x) &= g(-x) \end{aligned}$$

- b. $X + Y \in O$ where $X, Y \in O$ and $t \in F_1$

$$\begin{aligned} (X + Y)(-t) &= X(-t) + Y(-t) \\ &= -X(t) + -Y(t) & (X, Y \in O) \\ &= -(X + Y)(t) \end{aligned}$$

$X + Y \in E$ where $X, Y \in E$ and $t \in F_1$

$$\begin{aligned} (X + Y)(t) &= X(t) + Y(t) \\ &= X(-t) + Y(-t) & (X, Y \in E) \\ &= (X + Y)(-t) \end{aligned}$$

- c. $cX \in O$ where $c \in F$ and $X \in O$ and $t \in F_1$

$$\begin{aligned} (cX)(-t) &= cX(-t) \\ &= -cX(t) \end{aligned}$$

$cY \in E$ where $c \in F$ and $Y \in E$ and $t \in F_1$

$$\begin{aligned} (cY)(t) &= cY(t) \\ &= cY(-t) \end{aligned}$$

Therefore, O and E are subspaces of $F(F_1, F_1)$

□

§3 C

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \dots = a_{n-1} = 0\}$$

Show $F^n = W_1 \oplus W_2$

Proof. Definition of direct sum is $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = F^n$

a. $W_1 \cap W_2 = \{0\}$

Let $v \in W_1, W_2$

$$v = (a_1, a_2, \dots, a_n)$$

$$v \in W_1 \Rightarrow a_n = 0$$

$$v \in W_2 \Rightarrow a_1 = a_2 = \dots = a_{n-1} = 0$$

$$\therefore v = (0, 0, \dots, 0) \Rightarrow W_1 \cap W_2 = \{0\}$$

b. $W_1 + W_2 = F^n$

Let $v \in F^n$

$$v = (a_1, a_2, \dots, a_n)$$

Let $w_1 \in W_1$ and $w_2 \in W_2$

$$w_1 = (a_1, a_2, \dots, a_{n-1}, 0)$$

$$w_2 = (0, 0, \dots, a_n)$$

$$w_1 + w_2 = (a_1, a_2, \dots, a_n) = v$$

Thus, any vector in F^n can be expressed as a sum of vectors in W_1 and W_2

$$\therefore W_1 + W_2 = F^n$$

$$\therefore F^n = W_1 \oplus W_2$$

□

§4 D

In $M_{m \times n}(F)$

$$W_1 = \{A \in M_{m \times n}(F) : A_{i,j} = 0 \text{ whenever } i > j\}$$

$$W_2 = \{B \in M_{m \times n}(F) : B_{i,j} = 0 \text{ whenever } i \leq j\}$$

Show that $M_{m \times n}(F) = W_1 \oplus W_2$

Proof.

a. $W_1 \cap W_2 = \{0\}$

Let $m \in W_1, W_2$

$$m \in W_1 \Rightarrow m_{i,j} = 0 \text{ whenever } i > j$$

$$m \in W_2 \Rightarrow m_{i,j} = 0 \text{ whenever } i \leq j$$

Thus, $(\forall i, j)(m_{i,j} = 0)$ which is $\{0\}$

$$\therefore W_1 \cap W_2 = \{0\}$$

b. $W_1 + W_2 = M_{m \times n}(F)$

Let $q \in M_{m \times n}(F)$

Let $w_1 \in W_1$ and $w_2 \in W_2$

$$w_1 = \{(w_1)_{i,j} = 0 \text{ whenever } i > j\}$$

$$w_2 = \{(w_2)_{i,j} = 0 \text{ whenever } i \leq j\}$$

$$w_1 + w_2 = \{(w_1)_{ij} \text{ wherever } i \leq j \text{ and } (w_2)_{ij} \text{ wherever } i > j\} = q$$

Thus, any matrix in $M_{m \times n}(F)$ can be expressed as a sum of matrices in W_1 and W_2

$$\therefore W_1 + W_2 = M_{m \times n}(F)$$

$$\therefore M_{m \times n}(F) = W_1 \oplus W_2$$

□

§5 E

Let W be a subspace of a vector space V over a field F .

For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is the coset W containing v .

- a. Prove that $v + W$ is in the subspace of V if and only if $v \in W$.

Proof.

$v + W$ is in the subspace of $V \Rightarrow v \in W$.

$0 \in v + W$ because $v + W$ is a subspace.

$$0 = v + w, w \in W$$

$$v = -w$$

$$v \in W$$

$v \in W \Rightarrow v + W$ is in the subspace of V .

- i. $0 \in v + W$

$$w \in W \text{ and let } v = -w$$

$$v + w = 0$$

Thus, $0 \in v + W$

- ii. $a + b \in v + W$ where $a, b \in v + W$

$$\text{Let } a = v + w_a, w_a \in W \text{ and } b = v + w_b, w_b \in W$$

$$a + b = v + w_a + v + w_b$$

Because $v \in W$, $w_a + v + w_b \in W$.

Thus, $a + b \in v + W$

- iii. $ca \in v + w, a \in v + W, c \in F$

$$\text{Let } a = v + w_a, w_a \in W$$

$$ca = c(v + w_a)$$

$$= cv + cw_a$$

$$= v + cv + cw_a - v$$

$cv + cw_a - v \in W$ by closure under scalar multiplication and vector addition.

Thus, $ca \in v + w$

□

- b. Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

Proof.

- i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$

$$\text{Let } w_1, w_2 \in W$$

$$v_1 + w_1 = v_2 + w_2$$

$$v_1 - v_2 = w_2 - w_1$$

Since, $w_2 - w_1 \in W$ (closure under addition)

Therefore, $v_1 - v_2 \in W$

- ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$

This means $v_1 - v_2 = w$ where $w \in W$ (*)

Now let $x \in v_1 + W$

By definition, $\exists w_x \in W : x = v_1 + w_x$
 By (*) $v_1 = v_2 + w$
 So, $x = v_2 + w + w_x$
 Since, $w + w_x \in W$ (closure under addition)
 We have $x \in v_2 + W$
 So, $v_1 + W \subseteq v_2 + W$
 Similarly, we can show $v_2 + W \subseteq v_1 + W$
 Therefore, $v_1 + W = v_2 + W$

□

- c. Show that if $v_1 + W = v'_1 + W$ and $v_2 + W = v'_2 + W$, then
 $(v_1 + W) + (v_2 + W) = (v'_1 + W) + (v'_2 + W)$ and
 $a(v_1 + W) = a(v'_1 + W)$ for all $a \in F$

Proof.

- i. $(v_1 + W) + (v_2 + W) = (v'_1 + W) + (v'_2 + W)$
 Let $q \in (v_1 + W) + (v_2 + W)$
 $q \in (v_1 + v_2) + W$ by definition of vector addition
 So, $q = v_1 + v_2 + w_q$ where $w_q \in W$
 $= v_1 + v_2 + w_q + v'_1 - v'_1 + v'_2 - v'_2$
 $= v'_1 + v'_2 + w_q + v_1 - v'_1 + v_2 - v'_2$
 From b. i, $v_1 - v'_1$ and $v_2 - v'_2 \in W$
 Which means, $(v_1 - v'_1) + (v_2 - v'_2) \in W$
 Thus, $w_q + v_1 - v'_1 + v_2 - v'_2 \in W$
 So, $q \in (v'_1 + v'_2) + W$
 So, $(v_1 + W) + (v_2 + W) \subseteq (v'_1 + W) + (v'_2 + W)$
 Similarly, we can show $(v'_1 + W) + (v'_2 + W) \subseteq (v_1 + W) + (v_2 + W)$
 Therefore, $(v_1 + W) + (v_2 + W) = (v'_1 + W) + (v'_2 + W)$

- ii. $a(v_1 + W) = a(v'_1 + W)$
 Let $q \in a(v_1 + W)$
 $q \in av_1 + W$ by definition of scalar multiplication.
 So, $q = av_1 + w_q$ where $w_q \in W$
 $= av_1 + w_q + av'_1 - av'_1$
 $= av'_1 + w_q + av_1 - av'_1$
 $= av'_1 + a(v_1 - v'_1) + w_q$
 From b. i, $a(v_1 - v'_1) \in W$
 $a(v_1 - v'_1) + w_q \in W$ because closure under vector addition.
 So, $q \in av'_1 + W$
 So, $a(v_1 + W) \subseteq a(v'_1 + W)$
 Similarly, we can show $a(v'_1 + W) \subseteq a(v_1 + W)$
 Therefore, $a(v_1 + W) = a(v'_1 + W)$

□

- d. Prove that the set S is a vector space with the operations defined in (c).

- i. $0 \in S$
 The zero vector in S is $0 = v_0 + W$
 Let $s \in S$
 So $s = v_s + W$
 If the zero vector exists we should be able to show, $s + 0 = s$
 $s + 0 = s \Leftrightarrow (v_s + W) + (v_0 + W) = v_s + W$

$(v_s + v_0) + W = v_s + W$ by definition of addition

Thus $v_0 = 0$ and the zero vector is $0 + W$ which is just W

Therefore, the zero vector is W .

ii. $X + Y \in S$ where $X, Y \in S$

This means $X = v_x + W$ $Y = v_y + W$

$X + Y = (v_x + W) + (v_y + W) = (v_x + v_y) + W$ by definition of addition.

$(v_x + v_y) \in V$ by closure under vector addition.

Therefore $X + Y \in S$

iii. $aX \in S$ $a \in F$

$aX = a(v_x + W)$

$= av_x + W$

$av_x \in V$ by closure under vector addition.

Therefore, $aX \in S$

§6 F

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Proof.

$$\text{Sym}(M_{2 \times 2}(F)) = \{m \in M_{2 \times 2}(F) : m = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Leftrightarrow m = m^t\}$$

$$m \in \text{span}(\{M_1, M_2, M_3\}) \text{ if } m = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ where } c_1, c_2, c_3 \in F$$

$$= \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}$$

$$m^t = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}$$

$$\therefore \text{Sym}(M_{2 \times 2}(F)) = \text{span}(\{M_1, M_2, M_3\})$$

□

§7 G

§8 H

§9 I

§10 J