Math 240: Midterm 2 Q7

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§1 7

Given $a, b \in \mathbb{N}$, we want to compute $a \cdot b$.
Consider the following recursive algorithm $PROD(a, b)$:

If $b = 0$, return 0
If b is odd, return $PROD(a, b - 1) + a$
If $b > 0$ and b is even, return $PROD(a, b/2) + PROD(a, b/2)$

Prove that $PROD$ is correct.

Proof. Fix a nonzero integer a. We prove by strong induction on $b \in \mathbb{N}$ that $PROD(a, b)$ halts and returns $a \cdot b$. Suppose that $PROD(a, s)$ holds for all $s < b$. We prove that $POW(a, b)$ holds.

i. Case 1: $b = 0$ (base case)
If $b = 0$, then $PROD(a, 0)$ halts and returns $0 = a \cdot 0$ as desired.

ii. Case 2: b is odd
If b is odd, $PROD(a, b)$ halts and returns $PROD(a, b - 1) + a$. Since $b - 1 < b$, $PROD(a, b - 1)$ halts and returns $a \cdot (b - 1)$ by our strong induction hypothesis.

$$\begin{aligned} PROD(a, b) &= PROD(a, b - 1) + a \\ &= a \cdot (b - 1) + a \\ &= ab - a + a \\ &= ab \end{aligned}$$

as desired.

iii. Case 3: $b > 0$ and is even
If $b > 0$ and is even, $PROD(a, b)$ halts and returns $PROD(a, b/2) + PROD(a, b/2)$. Since $b/2 < b$, $PROD(a, b/2)$ halts and returns $a \cdot (b/2)$ by our strong induction hypothesis.

$$\begin{aligned} PROD(a, b) &= PROD(a, b/2) + PROD(a, b/2) \\ &= a \cdot (b/2) + a \cdot (b/2) \\ &= \frac{ab}{2} + \frac{ab}{2} \\ &= ab \end{aligned}$$

as desired. This proves the inductive step. By induction, we conclude that for all $a, b \in \mathbb{N}$ $PROD(a, b)$ halts and returns $a \cdot b$. Therefore, $PROD$ is correct.

$\square$