Math 341: Midterm 2

Daniel Ko

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§1

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1}$$

a. Suppose that $a \neq 0$, compute the solution of $\mathbf{Ax} = \mathbf{b}$ using row reduction and provide the conditions on a, b, c, d such that your computations are valid. Express the result as a simplified expression. (**Hint:** recall that you can not divide by zero)

Proof. We perform reduced row echelon form (rref) on the augmented matrix

$$\begin{aligned}
(A|b) &= \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \\
R_2 \leftarrow R_2 - \frac{c}{a}R_1 & \left[\begin{array}{cc|c} a & b & e \\ 0 & d - \frac{cb}{a} & f - \frac{ce}{a} \end{array}\right] \\
& \left[\begin{array}{cc|c} a & b & e \\ 0 & \frac{ad-cb}{a} & \frac{af-ce}{a} \end{array}\right] \\
R_2 \leftarrow \frac{a}{ad-cb}R_2 \quad \text{Assuming that } ab - cd \neq 0 & \left[\begin{array}{cc|c} a & b & e \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow R_1 - bR_2 & \left[\begin{array}{cc|c} a & 0 & e - b\frac{af-ce}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow \frac{R_1}{a} & \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}(e - b\frac{af-ce}{ad-cb}) \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
& \left[\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]
\end{aligned}$$

$$x = \begin{bmatrix} \frac{de-bf}{ad-cb} \\ \frac{af-ce}{ad-cb} \end{bmatrix} \qquad \text{where } ad - cb \neq 0$$

□

b. If $a = 0$, and $c \neq 0$, is your above computation still valid? How would you modify it? (explain briefly) (**Hint:** recall that you can swap the equations and the result is the same)

Proof. If $a = 0$, and $c \neq 0$, then the above computation will not be valid as we divided by a multiple times when we computed the rref. I would swap the first and second rows so that it would look like

$$\left[\begin{array}{cc|c} c & d & f \\ 0 & b & e \end{array}\right]$$

and compute the rref, assuming that $b \neq 0$. We obtain the rref,

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{bf-de}{bc} \\ 0 & 1 & \frac{e}{b} \end{array}\right]$$

$\square$

c. If $a = 0$, $c = 0$, but $b \neq 0$, $d \neq 0$, what are the conditions on e and f such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution? Is the solution unique? (**Hint:** recall that $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$)

Proof. If $a = 0$, $c = 0$, $b \neq 0$, $d \neq 0$, we get the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & e \\ 0 & d & f \end{array}\right]$$

Performing row reduction,

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{e}{b} \\ 0 & 1 & \frac{f}{d} \end{array}\right]$$

So the condition of the solution is,

$$x_2 = \frac{e}{b} = \frac{f}{d}$$

Thus, there exists a infinite amount of solution. $\square$

d. Solve the system

$$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}. \tag{2}$$

(**Hint:** You may want to use the formula you just deduced)

Proof.

$$\begin{aligned} x_1 &= \frac{de - bf}{ad - cb} \\ &= \frac{\sqrt{2}(5\sqrt{2}) - 3\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 30}{2 - 12} \\ &= \frac{-20}{-10} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\sqrt{2}(5\sqrt{2}) - 2\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 20}{-10} \\ &= 1 \end{aligned}$$

$\square$

§2

Let

$$\mathbf{A} = \begin{bmatrix} 0 & -\alpha & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 3 \\ -2 & -2 & 4 & 2\alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2+\alpha \\ 2\beta + \alpha - 2 \end{bmatrix} \tag{3}$$

What are the conditions on α and β such that the system $\mathbf{Ax} = \mathbf{b}$:

a. Has no solution?

Proof. We begin by putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ in its reduced form.

$$(\mathbf{A}|\mathbf{b}) = \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 & 2\beta + \alpha - 2 \end{array}\right]$$

$$R_5 \leftarrow R_5 + R_1 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array}\right]$$

$$R_3 \leftarrow R_3 - 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 + 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2\alpha + 2 & 4+\alpha \\ 0 & 0 & 1 & 2\alpha + 1/2 & 2\beta + \alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 - 2R_3, R_5 \leftarrow R_5 - \frac{1}{2}R_3 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 2\beta + \alpha + 1 \end{array}\right]$$

$$R_5 \leftarrow R_5 - R_4 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta - 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta - 1 \end{array}\right]$$

Thus this system will have no solution if $2\beta - 1 \neq 0$, which is when $\beta \neq \frac{1}{2}$. expain more... $\square$

b. Has an unique solution? Find the solution. (**Hint:** you will need to row reduce the augmented system to echelon form, and then use the theorems seen in class to impose the conditions on α and β).

Proof. Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(\mathbf{A}) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &\neq 0 \\ -4\alpha^2 &\neq 0 \\ \alpha &\neq 0 \end{aligned}$$

and from (a), $\beta = \frac{1}{2}$. Combining these two conditions we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By performing back substitution we compute the generic solution as

$$x = \begin{bmatrix} \frac{1}{2} + \frac{1}{2\alpha} \\ -\frac{1}{2\alpha} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

$\square$

c. Has infinite amount of solutions? Find the solution set in parametric form. (**Hint:** You may have one equations for α and one for β that have to be satisfied simultaneously).

Proof. Having an infinite amount of solutions is by definition another way of saying that a system that is consistent and that the solutions are not unique. A system is consistent if and only if $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ by Theorem 3.11. If $\det(\mathbf{A}) = 0$, the solution, if it exists, is not unique by Theorem 3.10 and the corollary to Theorem 4.7.

Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &= 0 \\ -4\alpha^2 &= 0 \\ \alpha &= 0 \end{aligned}$$

Given that $\alpha = 0$, and that $\beta = \frac{1}{2}$ from part (a) we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It is clear that $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ because $\mathbf{b}$ is a linear combination of the second and third column from $\mathbf{A}$. Hence, this system is consistent.

By performing back substitution we compute the generic solution as

$$x = \begin{bmatrix} \gamma \\ 1-\gamma \\ 1 \\ 0 \end{bmatrix}$$

where $\gamma \in \mathbb{R}$. □

§3

Let $A \in M_{n\times n}(F)$, for a field F. We want to prove that $rank(A^2)-rank(A^3) \leq rank(A)-rank(A^2)$. The solution to this exercise requires the notion of quotient spaces. Even though you should already be familiar with quotient spaces we will prove a few properties that will be useful.

Let W be a subspace of a vector space V over a field F. For any $v \in$ V the set $\{v\} +$ W $= \{v + w : w \in$ W$\}$ is called the coset of W containing v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$. Following this notation we write V/W $= \{v +$ W $: v \in$ V$\}$, which is usually called the quotient space V module W. Addition and scalar multiplication by scalars can be defined in the collection V/W as follows:

$$(v_1 + \mathrm{W}) + (v_2 + \mathrm{W}) = (v_1 + v_2) + \mathrm{W}$$

for all $v_1, v_2 \in$ V and

$$a(v + W) = av + W$$

for all $v \in$ V and $a \in F$

a. Prove that $v_1 +$ W $= v_2 +$ W if and only if $v_1 - v_2 \in$ W. (**Hint**: recall that $v_1 +$ W is a set, thus you need to prove equality between sets)

Proof.

i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$
If $v_1 + W = v_2 + W$, then $\exists w_1, w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$
$v_1 - v_2 = w_2 - w_1$
Since, $w_2 - w_1 \in W$ (clourse under addition)
Therefore, $v_1 - v_2 \in W$

ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$
This means $v_1 - v_2 = w$ where $w \in W$ $(*)$
Now let $x \in v_1 + W$
By definition, $\exists w_x \in W$ such that $x = v_1 + w_x$
By $(*)$ $v_1 = v_2 + w$
So, $x = v_2 + w + w_x$
Since, $w + w_x \in W$ (closure under addition)
We have $x \in v_2 + W$
So, $v_1 + W \subseteq v_2 + W$
Without loss of generality, we can show $v_2 + W \subseteq v_1 + W$
Therefore, $v_1 + W = v_2 + W$

Therefore, $v_1 +$ W $= v_2 +$ W if and only if $v_1 - v_2 \in$ W □

b. Show that V/W with the operations defined above is a linear vector space.

VS 1: For all x, y in V/W, $x + y = y + x$ (commutativity of addition)

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} x + y &= (v_x + W) + (v_y + W) \\ &= (v_x + v_y) + W \\ &= (v_y + v_x) + W \\ &= (v_y + W) + (v_x + W) \\ &= y + x \end{aligned}$$

$\square$

VS 2: For all x, y, z in V/W, $(x + y) + z = x + (y + z)$ (associativity of addition)

Proof. Let $x = v_x + W, y = v_y + W, z = v_z + W$, where $v_x, v_y, v_z \in V$.

$$\begin{aligned} (x + y) + z &= ((v_x + W) + (v_y + W)) + (v_z + W) \\ &= ((v_x + v_y) + W) + (v_z + W) \\ &= (v_y + v_x + v_z) + W \\ &= (v_x + W) + ((v_y + v_z) + W) \\ &= (v_x + W) + ((v_y + W) + (v_z + W)) \\ &= x + (y + z) \end{aligned}$$

$\square$

VS 3: There exists an element in V/W denoted by 0 such that $x + 0 = x$ for each x in V/W

Proof. Let $x = v_x + W$ where $v_x \in V$. Let the zero vector in V/W be defined as $0 + W$, where $0 \in V$.

$$\begin{aligned} x + (0 + W) &= (v_x + W) + (0 + W) \\ &= (v_x + 0) + W \\ &= v_x + W \\ &= x \end{aligned}$$

$\square$

§4