Math 341: Homework 2

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§1 A

Let D be the set of all differentiable functions defined on $\mathbb{R}$. Note that D is a subset of C because differentiable functions are continuous.

Proof. D is a subspace of C

a. $0 \in D$
Zero vector is defined as $f(x) = 0$ where $x \in \mathbb{R}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{0} \\ &= 0 \end{aligned}$$

Because the derivative of $f(x) = 0$ exists, $0 \in D$

b. $f + g \in D$ where $f, g \in D$

$$\begin{aligned} (f+g)'(x) &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

Because the derivative of $f + g$ exists, $f + g \in D$

c. $cf \in D$ where $c \in \mathbb{R}$ and $f \in D$

$$\begin{aligned} cf'(x) &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Because the derivative of cf exists, $cf \in D$

$$\therefore D \text{ is a subspace of } C$$

□

§2 B

Prove the set of even functions in $F(F_1, F_2)$ and odd functions in $F(F_1, F_2)$ are subspaces of $F(F_1, F_2)$

Proof. Let O be the set of all odd functions in $F(F_1, F_2)$ and E be the set of all even functions in $F(F_1, F_2)$.

a. $0 \in O$ and $0 \in E$
   Zero function is defined as $g(x) = 0$

   $0 \in O$ is odd:

$$\begin{aligned} g(-x) &= 0 \\ -g(x) &= 0 \\ g(-x) &= -g(x) \end{aligned}$$

   $0 \in E$ is even:

$$\begin{aligned} g(x) &= 0 \\ g(-x) &= 0 \\ g(x) &= g(-x) \end{aligned}$$

b. $X + Y \in O$ where $X, Y \in O$ and $t \in F_1$

$$\begin{aligned} (X+Y)(-t) &= X(-t) + Y(-t) \\ &= -X(t) + -Y(t) && (X, Y \in O) \\ &= -(X+Y)(t) \end{aligned}$$

   $X + Y \in E$ where $X, Y \in E$ and $t \in F_1$

$$\begin{aligned} (X+Y)(t) &= X(t) + Y(t) \\ &= X(-t) + Y(-t) && (X, Y \in E) \\ &= (X+Y)(-t) \end{aligned}$$

c. $cX \in O$ where $c \in F$ and $X \in O$ and $t \in F_1$

$$\begin{aligned} (cX)(-t) &= cX(-t) \\ &= -cX(t) \end{aligned}$$

   $cY \in E$ where $c \in F$ and $Y \in E$ and $t \in F_1$

$$\begin{aligned} (cY)(t) &= cY(t) \\ &= cY(-t) \end{aligned}$$

Therefore, O and E are subspaces of $F(F_1, F_1)$ $\square$

§3 C

$W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_n = 0\}$
$W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}$
Show $F^n = W_1 \oplus W_2$

Proof. Definition of direct sum is $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = F^n$

a. $W_1 \cap W_2 = \{0\}$

   Let $v \in W_1, W_2$
   $v = (a_1, a_2, \cdots, a_n)$
   $v \in W_1 \Rightarrow a_n = 0$
   $v \in W_2 \Rightarrow a_1 = a_2 = \cdots = a_{n-1} = 0$
   $\therefore v = (0, 0, \cdots, 0) \Rightarrow W_1 \cap W_2 = \{0\}$

b. $W_1 + W_2 = F^n$

   Let $v \in F^n$
   $v = (a_1, a_2, \cdots, a_n)$
   Let $w_1 \in W_1$ and $w_2 \in W_2$
   $w_1 = (a_1, a_2, \cdots, a_{n-1}, 0)$
   $w_2 = (0, 0, \cdots, a_n)$
   $w_1 + w_2 = (a_1, a_2, \cdots, a_n) = v$
   Thus, any vector in F^n can be expressed as a sum of vectors in W_1 and W_2
   $\therefore W_1 + W_2 = F^n$

$\therefore F^n = W_1 \oplus W_2$ □

§4 D

In $M_{mxn}(F)$
$W_1 = \{A \in M_{mxn}(F) : A_{i,j} = 0 \text{ whenever } i > j\}$
$W_2 = \{B \in M_{mxn}(F) : B_{i,j} = 0 \text{ whenever } i \leq j\}$
Show that $M_{mxn}(F) = W_1 \oplus W_2$

Proof.

a. $W_1 \cap W_2 = \{0\}$

   Let $m \in W_1, W_2$
   $m \in W_1 \Rightarrow m_{i,j} = 0$ whenever $i > j$
   $m \in W_2 \Rightarrow m_{i,j} = 0$ whenever $i \leq j$
   Thus, $(\forall i, j)(m_{i,j} = 0)$ which is $\{0\}$
   $\therefore W_1 \cap W_2 = \{0\}$

b. $W_1 + W_2 = M_{mxn}(F)$

   Let $q \in M_{mxn}(F)$
   Let $w_1 \in W_1$ and $w_2 \in W_2$
   $w_1 = \{(w_1)_{i,j} = 0 \text{ whenever } i > j\}$
   $w_2 = \{(w_2)_{i,j} = 0 \text{ whenever } i \leq j\}$

$w_1 + w_2 = \{(w_1)_{i,j}$ wherever $i \leq j$ and $(w_2)_{i,j}$ wherever $i > j\} = q$
Thus, any matrix in $M_{mxn}(F)$ can be expressed as a sum of matrices in W_1 and W_2
$\therefore W_1 + W_2 = M_{mxn}(F)$

$\therefore M_{mxn}(F) = W_1 \oplus W_2$ □

§5 E

Let W be a subspace of a vector space V over a field F.
For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is the coset W containing v.

a. Prove that $v + W$ is in the subspace of V if and only if $v \in W$.

Proof.
$v + W$ is in the subspace of $V \Rightarrow v \in W$.
$0 \in v + W$ because $v + W$ is a subspace.
$0 = v + w, w \in W$
$v = -w$
$v \in W$

$v \in W \Rightarrow v + W$ is in the subspace of V.

i. $0 \in v + W$
$w \in W$ and let $v = -w$
$v + w = 0$
Thus, $0 \in v + W$

ii. $a + b \in v + W$ where $a, b \in v + W$
Let $a = v + w_a, w_a \in W$ and $b = v + w_b, w_b \in W$
$a + b = v + w_a + v + w_b$
Because $v \in W$, $w_a + v + w_b \in W$.
Thus, $a + b \in v + W$

iii. $ca \in v + w, a \in v + W, c \in F$
Let $a = v + w_a, w_a \in W$
$ca = c(v + w_a)$
$= cv + cw_a$
$= v + cv + cw_a - v$
$cv + c_w a - v \in W$ by closure under scalar multplication and vector addition.
Thus, $ca \in v + w$

□

b. Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

Proof.

i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$
Let $w_1, w_2 \in W$
$v_1 + w_1 = v_2 + w_2$
$v_1 - v_2 = w_2 - w_1$
Since, $w_2 - w_1 \in W$ (clourse under addition)
Therefore, $v_1 - v_2 \in W$

ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$
This means $v_1 - v_2 = w$ where $w \in W$ $(*)$
Now let $x \in v_1 + W$

By definition, $\exists w_x \in W : x = v_1 + w_x$
By $(*)$ $v_1 = v_2 + w$
So, $x = v_2 + w + w_x$
Since, $w + w_x \in W$ (closure under addition)
We have $x \in v_2 + W$
So, $v_1 + W \subseteq v_2 + W$
Similarly, we can show $v_2 + W \subseteq v_1 + W$
Therefore, $v_1 + W = v_2 + W$

$\square$

§6 F

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric $2x2$ matrices.

Proof.
$Sym(M_{2x2}(F)) = \{m \in M_{2x2}(F) : m = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Leftrightarrow m = m^t\}$

$$\begin{aligned} m \in span(\{M_1, M_2, M_3\}) \text{ if } m &= c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{where } c_1, c_2, c_3 \in F \\ &= \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} \\ m^t &= \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} \end{aligned}$$

$\therefore Sym(M_{2x2}(F)) = span(\{M_1, M_2, M_3\})$ $\square$