Math 341: Final

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§1

For $c \in \mathbb{R}$, we define the matrix $\mathbf{A}_c \in \mathbb{R}^{3\times 3}$ by

$$\mathbf{A}_c = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & c & 2 \end{bmatrix}$$

a. Compute $\det(\mathbf{A}_c)$. Does it depend of c?

Proof.

$$\begin{aligned} \det(\mathbf{A}_c) &= 1 \cdot \det \begin{bmatrix} 2 & 0 \\ c & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & 2 \\ 3 & c \end{bmatrix} \\ &= (4) + (4) + (2c - 6) \\ &= 2c + 2 \end{aligned}$$

Yes, it depends on c. □

b. For which c is the matrix $\mathbf{A}_c$ invertible?

Proof. Corollary to Theorem 4.7 states that a matrix is invertible if and only its determinant does not equal 0. From (a),

$$\begin{aligned} \det(\mathbf{A}_c) = 2c + 2 &= 0 \\ c &= -1 \end{aligned}$$

$\det(\mathbf{A}_c) = 0$ when $c = -1$. Hence, $\mathbf{A}_c$ is invertible for all c except $c = -1$. □

c. Compute $\mathbf{A}_0^{-1}$ (i.e. when $c = 0$).

Proof. Using the proof to Theorem 3.2, we compute the inverse by constructing an agumented matrix $(\mathbf{A}_0 | I_3)$ and applying elementary row operations to transform it into the form of

$(I_3|\mathbf{A}_0^{-1})$.

$$(\mathbf{A}_0|I_3) = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \leftarrow R_2 - 2R_1 \quad R_3 \leftarrow R_3 - 3R_1 = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 3 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 + \frac{1}{4}R_2 \quad R_3 \leftarrow R_3 - \frac{3}{4}R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_3 \quad R_2 \leftarrow R_2 + 4R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 4 & 0 & -8 & -2 & 4 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right]$$

$$R_2 \leftarrow \frac{1}{4}R_2 \quad R_3 \leftarrow 2R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -3 & -\frac{3}{2} & 2 \end{array}\right]$$

Therefore,

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -\frac{1}{2} & 1 \\ -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

□

d. Let $b = (1, -4, 2)^t$, find the solution of $\mathbf{A}_0 x = b$

Proof. Let $x = (x_1, x_2, x_3)^t$.

$$\mathbf{A}_0 x = b$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + 2x_2 = -4 \\ 3x_1 + 2x_3 = 2 \end{cases}$$

$$x_1 = -4$$
$$x_2 = 2$$
$$x_3 = 7$$

□

e. Compute $\det(\mathbf{A}_c^2)$

Proof. From Theorem 4.7, we know that

$$\begin{aligned}\det(\mathbf{A}_c^2) &= \det(\mathbf{A}_c\mathbf{A}_c)\\ &= \det(\mathbf{A}_c)\cdot\det(\mathbf{A}_c)\\ &= (2c+2)(2c+2)\\ &= 4c^2+8c+4\end{aligned}$$

□

f. Compute $\det(5\mathbf{A}_c)$

Proof. Using the second "Properties of the Determinant" on pg 234,

$$\begin{aligned}\det(5\mathbf{A}_c) &= 5^3\det(\mathbf{A}_c)\\ &= 5^3(2c+2)\\ &= 250c+250\end{aligned}$$

□

g. Compute $\det(\mathbf{E}_k\mathbf{A}_c)$ where,

$$\mathbf{E}_k = \begin{bmatrix}1 & 0 & 0\\ k & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Proof.

$$\begin{aligned}\det(\mathbf{E}_k) &= 1\cdot\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 0\cdot\det\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix} + 0\cdot\det\begin{bmatrix}k & 1\\ 0 & 0\end{bmatrix}\\ &= 1\end{aligned}$$

From Theorem 4.7, we know that,

$$\begin{aligned}\det(\mathbf{E}_k\mathbf{A}_c) &= \det(\mathbf{E}_k)\det(\mathbf{A}_c)\\ &= (1)(2c+2)\\ &= 2c+2\end{aligned}$$

□

h. Compute $\det(\mathbf{D}_k\mathbf{A}_c)$ where,

$$\mathbf{D}_k = \begin{bmatrix}1 & 0 & 0\\ 0 & k & 0\\ 0 & 0 & 1\end{bmatrix}$$

Proof.

$$\begin{aligned}\det(\mathbf{D}_k) &= 1\cdot\det\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix} - 0\cdot\det\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + 0\cdot\det\begin{bmatrix}0 & k\\ 0 & 0\end{bmatrix}\\ &= k\end{aligned}$$

From Theorem 4.7, we know that,

$$\begin{aligned}\det(\mathbf{E}_k\mathbf{A}_c) &= \det(\mathbf{D}_k)\det(\mathbf{A}_c)\\ &= (k)(2c+2)\\ &= 2kc+2k\end{aligned}$$

□

i. Compute $\det(\mathbf{A}_0^{-1})$

Proof. From (c) we know that,

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -\frac{1}{2} & 1 \\ -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

It follows that,

$$\begin{aligned}\det(\mathbf{A}_0^{-1}) &= 2 \cdot \det \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{2} & 2 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -2 & -\frac{1}{2} \\ -3 & -\frac{3}{2} \end{bmatrix} \\ &= 2(\frac{1}{2}) - 1(-1) - 1(\frac{3}{2}) \\ &= \frac{1}{2}\end{aligned}$$

□

j. Compute the eigenvalues of $\mathbf{A}_0$. Can you diagonalize $\mathbf{A}_0$?

Proof. Using Theorem 5.2, we compute the eigenvalues of $\mathbf{A}_0$ by computing its characteristic polynomial.

$$\begin{aligned}\det(\mathbf{A}_0 - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -1 & 1 \\ 2 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{bmatrix} \\ &= -(\lambda - 2)(\lambda^2 - 3\lambda + 1)\end{aligned}$$

We compute when $-(\lambda-2)(\lambda^2 - 3\lambda + 1) = 0$. When $\lambda = 2$, the characteristic polynomial equals 0. Using the quadratic formula, when $\lambda = \frac{3 \pm \sqrt{5}}{2}$, the characteristic polynomial equals 0. Hence the eigenvalues of $\mathbf{A}_0$ are

$$2, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

We can rewrite the characteristic polynomial as

$$f(\lambda) = -(\lambda - 2)(\lambda - \frac{3+\sqrt{5}}{2})(\lambda - \frac{3-\sqrt{5}}{2})$$

Using the "Test for Diagonalization" outlined in pg 269, we determine if $\mathbf{A}_0$ can be diagonalized. It is clear that the first condition, the characteristic polynomial of T splits, holds. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. □

k. Compute the eigenvalues of $\mathbf{A}_0^{-1}$

Proof. Notice that since eigenvalues are non zero,

$$\begin{aligned} \mathbf{A}_0 x &= \lambda x \\ \mathbf{A}_0^{-1}\mathbf{A}_0 x &= \mathbf{A}_0^{-1}\lambda x \\ x &= \mathbf{A}_0^{-1}\lambda x \\ \frac{1}{\lambda}x &= \mathbf{A}_0^{-1}x \end{aligned}$$

Thus the eigenvalues of $\mathbf{A}_0^{-1}$ are

$$\frac{1}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

□

§2

Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T(\mathbf{x}) = \begin{pmatrix} x_2 - x_1 + \alpha\left(x_2^2 + x_3^3\right) \\ x_1 + x_2 + 2gx_3 + \alpha x_1^2 \\ x_1 + x_2 + 2x_3 \\ hx_3 + q \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

in which, h, g, q and α are real numbers.

a. What are the condition on α and q such that the transformation is linear? Explain briefly.

Proof. We know that if T is linear then $T(0) = 0$. Hence $q = 0$ for T to be linear. Notice that we must not have any terms higher than degree 1 because for example if we have a transformation $U(x) = x^2$ and let $c \in \mathbb{R}$ then,

$$\begin{aligned} U(cx) &= (cx)^2 \\ &= c^2x^2 \\ &\neq cU(x) \end{aligned}$$

which means U is not a linear transformation. Thus, $\alpha = 0$ so that there are no terms higher than degree 1. □

b. From now we suppose that $q = 0$ and $\alpha = 0$. Write the representation matrix $A = [T]_{\epsilon_3}^{\epsilon_4}$.

Proof. We now define

$$T(\mathbf{x}) = \begin{pmatrix} x_2 - x_1 \\ x_1 + x_2 + 2gx_3 \\ x_1 + x_2 + 2x_3 \\ hx_3 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We now compute the representation matrix

$$T(e_1) = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 2g \\ 2 \\ h \end{pmatrix}$$

$$\begin{aligned} A &= [[T(e_1)]_{\epsilon_4}[T(e_2)]_{\epsilon_4}[T(e_3)]_{\epsilon_4}] \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2g \\ 1 & 1 & 2 \\ 0 & 0 & h \end{bmatrix} \end{aligned}$$

□

c. What are the conditions on h and g such that the transformation T maps $\mathbb{R}^3$ onto $\mathbb{R}^4$? Explain briefly.

Proof. Since $T : \mathbb{R}^3 \to \mathbb{R}^4$, there are no h and g such that T is onto. This is because T is onto if and only if the range of T equals the codomain, but in this linear transformation, the dimension of codomain is greater than the domain so by rank nullity, T cannot be onto. □

d. What are the conditions on h and g such that the transformation T is one to one? Explain briefly.

Proof. By Theorem 2.4, T is one to one if and only if $N(T) = \{0\}$. Using the rank nullity theorem, this is equivalent to saying that $rank(T) = 3$, i.e.

$$\begin{aligned} dim(\mathbb{R}^3) &= rank(T) + nullity(T) \\ 3 &= rank(T) \end{aligned}$$

We know that $rank(T) = rank([T]_{\epsilon_3}^{\epsilon_4})$ by Theorem 3.3. Additionally by Theorem 3.5., the rank of a matrix is the number of linearly independent columns. It is clear that A will have rank 3 if $h \neq 0$. There are no conditions for g. □

e. Suppose that $h = 0$, then

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2g \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and let } b = \begin{bmatrix} 0 \\ 2 \\ r \\ 0 \end{bmatrix}$$

What are the conditions on r and g such that the system $Ax = b$ has a solution? When is the solution unique?

Proof. We begin by putting the augmented matrix $(A|b)$ in its reduced form.

$$(A'|b') = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 2 & 2g & 2 \\ 0 & 0 & 2-2g & r-2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By Theorem 3.11 and 3.13, a system is consistent if and only if $rank(A') = rank(A'|b')$. So for the system to have as solution, $2 - 2g \neq 0$ if $r - 2 \neq 0$. If $r - 2 = 0$, there is no conditions on g.

In summary, for the system to have as solution if $r \neq 2$ then $g \neq 1$. If $r = 2$, there are no conditions on g.

Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(A) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of g as such that

$$\begin{aligned} -1 * 2 * (2 - 2g) &\neq 0 \\ 4g - 4 &\neq 0 \\ g &\neq 1 \end{aligned}$$

There is no conditions for r when the solution is unique. □

f. Suppose that $h = 0, g = 1, r = 2$. Solve $\mathbf{Ax} = \mathbf{b}$ and give the answer in parametric form.

Proof. We define

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

We can compute a solution space to $\mathbf{Ax} = \mathbf{b}$ as outlined in Theorem 3.9. We start by first computing the solution set to $\mathbf{Ax} = 0$ denoted by K_H. It is clear that $rank(\mathbf{A}) = 2$ because the first two columns are linearly independent and the third column is the sum of the first two columns. By Theorem 3.8, $dim(K_H) = 3 - 2 = 1$. Thus any nonzero solution constitutes a basis for K. For example, since

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

is a solution to the $\mathbf{Ax} = 0$, it is a basis for K_H by Corollary 2 of Theorem 1.10. So a solution set to K_H would be

$$K_H = \left\{ t \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

A solution to $\mathbf{Ax} = \mathbf{b}$ is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, by Theorem 3.9 we compute the solution space as

$$K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

§3