

# Math 341: Homework 2

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## §1 A

Let  $D$  be the set of all differentiable functions defined on  $\mathbb{R}$ . Note that  $D$  is a subset of  $C$  because differentiable functions are continuous.

*Proof.*  $D$  is a subspace of  $C$

a.  $0 \in D$

Zero vector is defined as  $f(x) = 0$  where  $x \in \mathbb{R}$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{0 - 0}{h} \\&= 0\end{aligned}$$

Because the derivative of  $f(x) = 0$  exists,  $0 \in D$

b.  $f + g \in D$  where  $f, g \in D$

$$\begin{aligned}(f+g)'(x) &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\&= f'(x) + g'(x)\end{aligned}$$

Because the derivative of  $f + g$  exists,  $f + g \in D$

c.  $cf \in D$  where  $c \in \mathbb{R}$  and  $f \in D$

$$\begin{aligned}cf'(x) &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\&= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\end{aligned}$$

Because the derivative of  $cf$  exists,  $cf \in D$

$\therefore D$  is a subspace of  $C$

□

## §2 B

Prove the set of even functions in  $F(F_1, F_2)$  and odd functions in  $F(F_1, F_2)$  are subspaces of  $F(F_1, F_2)$

*Proof.* Let  $O$  be the set of all odd functions in  $F(F_1, F_2)$  and  $E$  be the set of all even functions in  $F(F_1, F_2)$ .

- a.  $0 \in O$  and  $0 \in E$

Zero function is defined as  $g(x) = 0$

$0 \in O$  is odd:

$$\begin{aligned} g(-x) &= 0 \\ -g(x) &= 0 \\ g(-x) &= -g(x) \end{aligned}$$

$0 \in E$  is even:

$$\begin{aligned} g(x) &= 0 \\ g(-x) &= 0 \\ g(x) &= g(-x) \end{aligned}$$

- b.  $X + Y \in O$  where  $X, Y \in O$  and  $t \in F_1$

$$\begin{aligned} (X + Y)(-t) &= X(-t) + Y(-t) \\ &= -X(t) + -Y(t) & (X, Y \in O) \\ &= -(X + Y)(t) \end{aligned}$$

$X + Y \in E$  where  $X, Y \in E$  and  $t \in F_1$

$$\begin{aligned} (X + Y)(t) &= X(t) + Y(t) \\ &= X(-t) + Y(-t) & (X, Y \in E) \\ &= (X + Y)(-t) \end{aligned}$$

- c.  $cX \in O$  where  $c \in F$  and  $X \in O$  and  $t \in F_1$

$$\begin{aligned} (cX)(-t) &= cX(-t) \\ &= -cX(t) \end{aligned}$$

$cY \in E$  where  $c \in F$  and  $Y \in E$  and  $t \in F_1$

$$\begin{aligned} (cY)(t) &= cY(t) \\ &= cY(-t) \end{aligned}$$

Therefore,  $O$  and  $E$  are subspaces of  $F(F_1, F_1)$

□

## §3 C

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \dots = a_{n-1} = 0\}$$

Show  $F^n = W_1 \oplus W_2$

*Proof.* Definition of direct sum is  $W_1 \cap W_2 = \{0\}$  and  $W_1 + W_2 = F^n$

a.  $W_1 \cap W_2 = \{0\}$

Let  $v \in W_1, W_2$

$$v = (a_1, a_2, \dots, a_n)$$

$$v \in W_1 \Rightarrow a_n = 0$$

$$v \in W_2 \Rightarrow a_1 = a_2 = \dots = a_{n-1} = 0$$

$$\therefore v = (0, 0, \dots, 0) \Rightarrow W_1 \cap W_2 = \{0\}$$

b.  $W_1 + W_2 = F^n$

Let  $v \in F^n$

$$v = (a_1, a_2, \dots, a_n)$$

Let  $w_1 \in W_1$  and  $w_2 \in W_2$

$$w_1 = (a_1, a_2, \dots, a_{n-1}, 0)$$

$$w_2 = (0, 0, \dots, a_n)$$

$$w_1 + w_2 = (a_1, a_2, \dots, a_n) = v$$

Thus, any vector in  $F^n$  can be expressed as a sum of vectors in  $W_1$  and  $W_2$

$$\therefore W_1 + W_2 = F^n$$

$$\therefore F^n = W_1 \oplus W_2$$

□

## §4 D

In  $M_{m \times n}(F)$

$$W_1 = \{A \in M_{m \times n}(F) : A_{i,j} = 0 \text{ whenever } i > j\}$$

$$W_2 = \{B \in M_{m \times n}(F) : B_{i,j} = 0 \text{ whenever } i \leq j\}$$

Show that  $M_{m \times n}(F) = W_1 \oplus W_2$

*Proof.*

a.  $W_1 \cap W_2 = \{0\}$

Let  $m \in W_1, W_2$

$$m \in W_1 \Rightarrow m_{i,j} = 0 \text{ whenever } i > j$$

$$m \in W_2 \Rightarrow m_{i,j} = 0 \text{ whenever } i \leq j$$

Thus,  $(\forall i, j)(m_{i,j} = 0)$  which is  $\{0\}$

$$\therefore W_1 \cap W_2 = \{0\}$$

b.  $W_1 + W_2 = M_{m \times n}(F)$

Let  $m \in M_{m \times n}(F)$

$$v = (a_1, a_2, \dots, a_n)$$

Let  $w_1 \in W_1$  and  $w_2 \in W_2$

$$w_1 = (a_1, a_2, \dots, a_{n-1}, 0)$$

$$w_2 = (0, 0, \dots, a_n)$$

$$w_1 + w_2 = (a_1, a_2, \dots, a_n) = v$$

T O BE CONTINUED

□