

Math 341: Homework 6

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Spring 2020

§1 A

Find the rank of the following matrices.

- a. Corollary 2a of Theorem 3.6 states that $\text{rank}(A^t) = \text{rank}(A)$. It is clear that the rank of A^t is 2 because the first and third columns are the same. So $\text{rank}(A)=2$.
- b. We will calculate the rank by performing row reduction on our matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$r_2 \leftarrow r_2 - 2r_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$r_3 \leftarrow r_3 - r_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

There are three linearly independent rows, so $\text{rank}(A)=3$.

- c. Corollary 2a of Theorem 3.6 states that $\text{rank}(A^t) = \text{rank}(A)$. It is clear that the rank of A^t is 2 because the second column has a zero in its second row. So $\text{rank}(A)=2$.
- d.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$
$$r_2 \leftarrow r_2 - 2r_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There is one linearly independent row, so $\text{rank}(A)=1$.

- e. Similarly to (a) and (b), by performing row reduction we get the matrix

$$\begin{bmatrix} 1 & 0 & 6 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three linearly independent columns, so $\text{rank}(A)=3$.

f. Similarly to (a) and (b), by performing row reduction we get the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three linearly independent columns, so $\text{rank}(A)=3$.

g. It is clear that the rank of A is 1 because the three columns are the same and the other column is just zero.

§2 B

Prove that any elementary row [column] operation of type 1 can be obtained by a succession of three elementary row [column] operations of type 3 followed by one elementary row [column] operation of type 2

Proof. Row operation type 1 on row i and row j can be done by the following:

1. Row operation type 3: Add -1 times row i to row j
2. Row operation type 3: Add row j to row i
3. Row operation type 3: Add -1 times row i to row j
4. Row operation type 2: Multiply row j by -1

Without loss of generality, same could be done for a elementary column operation of type 1. \square

§3 C

Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

Proof. Iterate through each column, let this variable be c . If $A_{c,c}$ equals 0, go through all the elements in that column below $A_{c,c}$ and find the first non zero element. Perform a type 1 row operation on row c and the row the non zero element was found. If there is no non zero element, do nothing and go to the next column.

If $A_{c,c}$ does not equal 0, perform a type 3 row operation on each row below $A_{c,c}$. Multiply $-\frac{A_{r,c}}{A_{c,c}}$ by the c th row to the r th row, where r is every row below c . \square

§4 D

Complete the proof of the corollary to Theorem 3.4 by showing that elementary column operations preserve rank.

Proof. If B is obtained from a matrix A by an elementary column operation, then there exists an elementary matrix E such that $B = AE$. By Theorem 3.2, E is invertible, and hence $\text{rank}(B) = \text{rank}(A)$ by Theorem 3.4. \square

§5 E

Let B and B' is an $m \times n$ matrix submatrix of B . Prove that $\text{rank}(B) = r$, then $\text{rank}(B') = r - 1$

Proof. Consider the matrix

$$M = \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & B' \end{array} \right]$$

M has the same number of linearly independent columns to that of B' , so $\text{rank}(M) = \text{rank}(B')$
Now consider the matrix below.

$$B = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & & & B' \end{array} \right]$$

B has one more linearly independent row to that of M .

Then $\text{rank}(B) = \text{rank}(M) + 1 = \text{rank}(B') + 1$.

Therefore, if $\text{rank}(B) = r$, then $\text{rank}(M) = \text{rank}(B') = r - 1$. □

§6 F

Let B' and D' be $m \times n$ matrices, and let B and D be $(m+1) \times (n+1)$ matrices respectively. Prove that if B' can be transformed into D' by an elementary row [column] operation, then B can be transformed into D by an elementary row [column] operation.

Proof. If B' can be transformed into D' by elementary row operations, there must exist an elementary matrix E such that $D' = EB'$ by theorem 3.1. Now consider the matrix below.

$$A = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & & & E \end{array} \right]$$

A is also an elementary matrix, We can observe that $D = AB$, thus B can be transformed to D by elementary row operations. Without loss of generality, there exist a matrix F such that $D' = B'F$ where F is the elementary column matrix. So, $D = BF$ where B is like the A matrix but with F instead of E . Therefore. B can be transformed to D by elementary column operations. □

§7 G

- a. Find a 5×5 matrix M with rank 2 such that $AM = O$ where O is the 4×5 zero matrix.

Proof. By solving $Ax = 0$, we get this system of equation:

$$\begin{cases} x_1 - x_3 + 2x_4 + x_5 = 0 \\ -x_1 + x_2 + 3x_3 - x_4 = 0 \\ -2x_1 + x_2 + 4x_3 - x_4 + 3x_5 = 0 \\ 3x_1 - x_2 - 5x_3 + x_4 - 6x_5 = 0 \end{cases}$$

Solving this system of equations by computing reduced row echelon form, we get that x_1, x_2, x_4 are the pivot variables and x_3, x_5 are the free variables. So solutions are in the form $(x_3 + 3x_5, -2x_3 + x_5, x_3, -2x_5, x_5)$. From this, we are able to construct a basis for $Ax=0$,

$\{(1, -2, 1, 0, 0), (3, 1, 0, -2, 1)\}$. Define

$$M = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It is obvious that this matrix has rank 2 and is 5×5 . Because the column is a basis for $Ax = 0$, the resulting matrix will be O . \square

- b. Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $\text{rank}(B) \leq 2$

Proof. Since $AB = O$, we know that the columns of B is a solution to $Ax = 0$, which is a subset of the nullspace of L_A . From the rank nullity theorem, we know that $\dim(\mathbb{F}^5) = \text{rank}(L_A) + \text{nullity}(L_A)$.
 $\text{nullity}(L_A) = \dim(\mathbb{F}^5) - \text{rank}(L_A) = 5 - 3 = 2$. So, $\text{rank}(B)$ cannot be greater than 2. Therefore, $\text{rank}(B) \leq 2$. \square

§8 H

For each of the following linear transformations T , determine whether T is invertible, and compute T^{-1} if it exists.

Let α be the standard basis for the domain and γ be the standard basis for the codomain for each of these transformations below. For each of these problems, we can use theorem 2.18 and its corollaries to determine if the transformations are invertible, namely we need to show that $[T]_{\alpha}^{\gamma}$ is invertible.

- a. $T : P_2(R) \rightarrow P_2(R)$ defined by $T(f(x)) = f''(x) + 2f'(x) - f(x)$

Proof.

$$\begin{aligned} [T]_{\alpha}^{\gamma} &= [[T(1)]_{\gamma}, [T(x)]_{\gamma}, [T(x^2)]_{\gamma}] \\ &= [[-1]_{\gamma}, [-x + 2]_{\gamma}, [-x^2 + 4x + 2]_{\gamma}] \\ &= \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

To see if the matrix above is invertible, we can use an augmented matrix $([T]_{\alpha}^{\gamma} | I_3)$ and apply

elementary row operations to transform it into the form of $(I_3|[T^{-1}]_\gamma^\alpha)$.

$$\begin{aligned}
 ([T]_\alpha^\gamma | I_3) &= \left[\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \\
 r_1 &\leftarrow -r_1, \quad r_2 \leftarrow -r_2, \quad r_3 \leftarrow -r_3 \\
 &\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & -1 & 0 & 0 \\ 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right] \\
 r_1 &\leftarrow 2r_2 + r_1 \\
 &\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & -1 & -2 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right] \\
 r_1 &\leftarrow 10r_3 + r_1, \quad r_2 \leftarrow 4r_3 + r_2 \\
 (I_3|[T^{-1}]_\gamma^\alpha) &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right]
 \end{aligned}$$

Therefore, the inverse of T exists and is

$$T^{-1}(a_0 + a_1x + a_2x^2) = -a_0 - 2a_2 - 10a_2 + (-a_1 - 4a_2)x - a_2x^2$$

□

- b. $T : P_2(R) \rightarrow P_2(R)$ defined by $T(f(x)) = (x+1)f'(x)$

Proof.

Similar to (a), we will calculate $[T]_\alpha^\gamma$.

$$[T]_\alpha^\gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Notice that the rank of this matrix is 2, which means $\text{rank}([T]_\alpha^\gamma) \neq 3$. Therefore, there is no inverse for this matrix exists by the remark in chapter 3.2, "an $n \times n$ matrix is invertible if and only if its rank is n ".

□

- c. $T : R^3 \rightarrow R^3$ defined by $T(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3, -a_1 + a_2 + 2a_3, a_1 + a_3)$

Proof.

Similar to (a), we will calculate $[T]_\alpha^\gamma$.

$$[T]_\alpha^\gamma = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

We can calculate the inverse of T by following the same steps we did in (a).

$$[T^{-1}]_\gamma^\alpha = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

Therefore, the inverse of T exists and is

$$T^{-1}(a_1, a_2, a_3) = (\frac{1}{6}a_1 - \frac{1}{3}a_2 + \frac{1}{2}a_3, \frac{1}{2}a_1 - \frac{1}{2}a_3, -\frac{1}{6}a_1 + \frac{1}{3}a_2 + \frac{1}{2}a_3)$$

□

- d. $T : R^3 \rightarrow P_2(R)$ defined by $T(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2$

Proof.

Similar to (a), we will calculate $[T]_{\alpha}^{\gamma}$.

$$[T]_{\alpha}^{\gamma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We can calculate the inverse of T by following the same steps we did in (a).

$$[T^{-1}]_{\gamma}^{\alpha} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$$

Therefore, the inverse of T exists and is

$$T^{-1}(a_1 + a_2x + a_3x^2) = (a_3, \frac{1}{2}a_1 - \frac{1}{2}a_2, \frac{1}{2}a_1 + \frac{1}{2}a_2 - a_3)$$

□

- e. $T : P_2(R) \rightarrow R^3$ defined by $T(f(x)) = (f(-1), f(0), f(1))$

Proof.

Similar to (a), we will calculate $[T]_{\alpha}^{\gamma}$.

$$[T]_{\alpha}^{\gamma} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We can calculate the inverse of T by following the same steps we did in (a).

$$[T^{-1}]_{\gamma}^{\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

Therefore, the inverse of T exists and is

$$T^{-1}(a_1, a_2, a_3) = a_2 + (-\frac{1}{2}a_1 + \frac{1}{2}a_3)x + (\frac{1}{2}a_1 - a_2 + \frac{1}{2}a_3)x^2$$

□

- f. $T : M_{2 \times 2}(R) \rightarrow R^4$ defined by $T(A) = (tr(A), tr(A^t), tr(EA), tr(AE))$ where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Proof.

Similar to (a), we will calculate $[T]_{\alpha}^{\gamma}$.

$$[T]_{\alpha}^{\gamma} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Notice that the rank of this matrix is 2, which means $\text{rank}([T]_{\alpha}^{\gamma}) \neq 4$. Therefore, there is no inverse for this matrix exists by the remark in chapter 3.2, "an $n \times n$ matrix is invertible if and only if its rank is n ".

□

§9 I

Express the invertible matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ as a product of elementary matrices

Proof. Corollary 1 of Theorem 3.6 states that for any invertible matrix A

$I_n = A^{-1}A = E_p E_{p-1} \cdots E_1 A$, where E_i are invertible elementary matrices. So, $A = E_1^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$.

To calculate the values of E_i , we can use an augmented matrix $(A|I_3)$ and apply elementary row operations to transform it into the form of $(I_3|A^{-1})$. Each transformation to the right matrix will be our E .

$$\begin{aligned}
 (A|I_3) &= \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \\
 r_3 &\leftarrow r_3 - r_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] \\
 r_2 &\leftarrow r_2 - r_1 \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] \\
 r_1 &\leftarrow r_1 + r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] \\
 r_2 &\leftarrow -\frac{1}{2}r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] \\
 r_3 &\leftarrow r_3 - r_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right] \\
 r_1 &\leftarrow r_1 - r_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & -1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right]
 \end{aligned}$$

By construction, A is the product of the inverse of the elementary matrices on the right of the augmented matrix starting the first matrix shown above. \square

§10 J

Suppose that A and B are matrices having n rows. Prove that $M(A|B) = (MA|MB)$ for any $m \times n$ matrix

Proof. Let A have a number of columns and B have b number of columns.

Notice for $1 \leq j \leq a$: $(M(A|B))_{ij} = \sum_{k=1}^n M_{i,k} A_{k,j} = (MA)_{ij} = (MA|MB)_{ij}$

Similarly for $a < j \leq a + b$: $(M(A|B))_{ij} = \sum_{k=1}^n M_{i,k} B_{k,j} = (MB)_{ij} = (MA|MB)_{ij}$

Therefore, $M(A|B) = (MA|MB)$ for any $m \times n$ matrix. \square

§11 1

Let $V = P_n(F)$, and let c_0, c_1, \dots, c_n be distinct scalars in F

For $0 \leq i \leq n$, define $f_i \in V^*$ by $f_i(p(x)) = p(c_i)$. Prove that $\{f_0, f_1, \dots, f_n\}$ is a basis for V^* .

Proof. To prove that $\{f_0, f_1, \dots, f_n\}$ is a basis for V^* , we must show f_i is linear, the cardinality of the set is equal to the dimension of V^* , and that the set is linearly independent by Theorem 1.10 corollary 2.

i. f_i is linear

Show that $f_i(g(x) + ch(x)) = f_i(g(x)) + cf_i(h(x))$ where $g(x), h(x) \in P_n(F)$ and $c \in F$.

$$\begin{aligned} f_i(g(x) + ch(x)) &= g(c_i) + ch(c_i) \\ &= f_i(g(x)) + cf_i(h(x)) \end{aligned}$$

ii. Cardinality of the set is equal to the dimension of V^*

By the corollary to Theorem 2.20, we know $\dim(V^*) = \dim(L(V, F)) = \dim(V) \cdot \dim(F) = \dim(V)$. So, $\dim(V^*) = \dim(V) = n + 1$ because $V = P_n(F)$. So cardinality of the set is equal to the dimension of V^* .

iii. Set is linearly independent

Let $c_0 f_0(p(x)) + c_1 f_1(p(x)) + \dots + c_n f_n(p(x)) = 0$ for some c_i .

Let $p_i(x) = \prod_{j \neq i} (x - c_j)$, where $p_i(c_j) = 0$ and $p_i(c_i) \neq 0$.

Thus, $c_0 f_0(p(x)) + c_1 f_1(p(x)) + \dots + c_n f_n(p(x)) = c_0 f_0(c_0) + c_1 f_1(c_1) + \dots + c_n f_n(c_1)$

Because $f_0(c_0), f_1(c_1), \dots, f_n(c_n) \neq 0$, c_0 must equal zero and all c_i must be zero as well.

Therefore, this set is linearly independent.

□

§12 2

Use the corollary to Theorem 2.26 and (a) to show that there exist unique polynomials $p_0(x), p_1(x), \dots, p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. These polynomials are the Lagrange polynomials defined in Section 1.6

Proof. Corollary to Theorem 2.26 states that every ordered basis for V^* , $\{f_0, f_1, \dots, f_n\}$, is the dual basis for some basis, $\beta = \{p_0, p_1, \dots, p_m\}$, for V such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. Define polynomial $z \in V$ such that $z(c_j) = \delta_{kj}$. Since β is a basis for V we can write z as a linear combination of V . So $z = \sum_{i=0}^n a_i p_i$. Notice that when $z(c_0) = 1 = \sum_{i=0}^n a_i p_i(c_0) = a_1$ and $z(c_j) = 0 = \sum_{i=0}^n a_i p_i(c_j) = a_j$ when $j \neq 1$. Thus, by definition, $z = p_0$ and is unique because it was constructed from basis β . Without loss of generality, we can define p_1, p_2, \dots, p_n in the same way.

□

§13 3

For any scalars a_0, a_1, \dots, a_n (not necessarily distinct), deduce that there exists a unique polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $0 \leq i \leq n$. In fact, $q(x) = \sum_{i=0}^n a_i p_i(x)$

Proof. Let $\beta = \{p_0, p_1, \dots, p_m\}$. Let $q_0(c_i), q_1(c_i) \in V$. We define them as:

$$q_0(x) = \sum_{i=0}^n a_i^0 p_i(x)$$

$$q_1(x) = \sum_{i=0}^n a_i^1 p_i(x)$$

Notice that $a_i^0 = q_1(c_i) = \sum_{i=0}^n a_i^1 p_i(c_i) = a_i^1$. Therefore, $q_0 = q_1$ and q_0 is unique.

□

§14 4

Proof. Substitute a_i^0 with $p_1(c_i)$ from the equation in question 3. □

§15 5

Prove that

$$\int_a^b p(t) dt = \sum_{i=0}^n p(c_i) d_i$$

where

$$d_i = \int_a^b p_i(t) dt$$

Suppose now that

$$c_i = a + \frac{i(b-a)}{n} \text{ for } i = 0, 1, \dots, n$$

For $n = 1$, the preceding result yields the trapezoidal rule for evaluating the definite integral of a polynomial. For $n = 2$, this result yields Simpson's rule for evaluating the definite integral of a polynomial.

Proof.

$$\begin{aligned} \int_a^b p(t) dt &= \int_a^b \sum_{i=0}^n p(c_i) p_i(x) dt \\ &= \sum_{i=0}^n \int_a^b p(c_i) p_i(x) dt \\ &= \sum_{i=0}^n \int_a^b p(c_i) d_i \\ &= \sum_{i=0}^n p(c_i) d_i \end{aligned}$$

□