Math 341: Homework 6

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§1 A

Find the rank of the following matrices.

a. 2 show work later

b. 3

c. 2

d. 1

e. 3

f. 3

g. 1

§2 B

Prove that any elementary row [column] operation of type 1 can be obtained by a succession of three elementary row [column] operations of type 3 followed by one elementary row [column] operation of type 2

Proof. Row operation type 1 on row i and row j can be done by the following:

1. Row operation type 3: Add -1 times row i to row j
2. Row operation type 3: Add row j to row i
3. Row operation type 3: Add -1 times row i to row j
4. Row operation type 2: Multiply row j by -1

Without loss of generality, same could be done for a elementary column operation of type 1. $\square$

§3 C

Let A be an mxn matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

Proof. Iterate through each column, let this variable be c. If $A_{c,c}$ equals 0, go through all the elements in that column below $A_{c,c}$ and find the first non zero element. Perform a type 1 row operation on row c and the row the non zero element was found. If there is no non zero element, do nothing and go to the next column.

If $A_{c,c}$ does not equal 0, perform a type 3 row operation on each row below $A_{c,c}$. Multiply $-\frac{A_{r,c}}{A_{c,c}}$ by the cth row to the rth row, where r is every row below c. □

§4 D

Complete the proof of the corollary to Theorem 3.4 by showing that elementary column operations preserve rank.

Proof. If B is obtained from a matrix A by an elementary column operation, then there exists an elementary matrix E such that B = AE. By Theorem 3.2 (p. 150), E is invertible, and hence rank(B) = rank(A) by Theorem 3.4. □

§5 E

Let B and B' is an mxn matrix submatrix of B. Prove that rank(B) = r, then rank(B') = $r-1$

Proof. By Theorem 3.5, we know that rank(B) = dim($R(L_B)$), where $R(L_B) = \text{span}(B_1, B_2, \cdots, B_{n+1})$ and B_i is the ith column of B. In other words, is the rank is the number of linearly independent rows/columns in a matrix. Notice that B' has one less linear independent row/column than B. It follows that rank of B' would be 1 less than the rank of B. Therefore, rank(B')= $r-1$. Consider the matrix

$$M = \left[\begin{array}{c|c} 0 & \\ \vdots & B' \\ 0 & \end{array}\right]$$

M has the same number of linearly independent columns to that of B', so rank(M) = rank(B') Now consider the matrix below.

$$B = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \vdots & & B' & \\ 0 & & & \end{array}\right]$$

B has one more linearly independent row to that of M.
Then rank(B) = rank(M) + 1 = rank(B') + 1.
Therefore, if rank(B) = r, then rank(M) = rank(B') = r - 1. □

§6 F

Let S be the subspace of $M_{nxn}(R)$ generated by all matrices of the form $AB - BA$ with A and B in $M_{nxn}(R)$. Prove that $\dim(S) = n^2 - 1$. (You may want to use the trace together with the rank-nullity theorem)

Proof.
Trace is a linear transformation.
Tr : $M_{nxn}(R) \to R$
The subspace S is defined as $\{AB - BA : A, B \in M_{nxn}(R)\}$
Tr$(AB - BA) =$ Tr$(AB) -$ Tr(BA)
$=$ Tr$(AB) -$ Tr(AB)
$= 0$
All matrices that can be expressed as $AB - BA$ is in the null space of Tr. This means that N(Tr)$= S$.
The rank-nullity theorem states:
dim(N(Tr)) + dim(R(Tr)) = dim($M_{nxn}(R)$)
N(Tr)$= S$, so dim(S) + dim(R(Tr)) = dim($M_{nxn}(R)$)
dim(S) = dim($M_{nxn}(R)$) - dim(R(Tr))
= n^2- dim(R)
= n^2- 1

□

§7 G

Let T be a linear transformation of a vector space V into itself. Suppose that $x \in V$ is such that $T^m(x) = 0$, and $T^{m-1}(x) \neq 0$ for some positive m. Show that $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent.

Proof.
The linear combination of the above set is

$$a_0x + a_1T(x) + a_2T^2(x) + \cdots + a_{n-1}T^{m-1}(x) = 0$$

Notice that $T^n(x) = 0$ for all $n \geq m$.
$T^{m+1}(x) = T(T^m(x)) = T(0) = 0$
Let's take T^{m-1} on both sides of the linear combination.

$$\begin{aligned} T^{m-1}(a_0x + a_1T(x) + a_2T^2(x) + \cdots + a_{n-1}T^{m-1}(x)) &= T^{m-1}(0) \\ T^{m-1}(a_0x) + T^{m-1}(a_1T(x)) + T^{m-1}(a_2T^2(x)) + \cdots + T^{m-1}(a_{n-1}T^{m-1}(x)) &= 0 \\ T^{m-1}(a_0x) + 0 + 0 + \cdots + 0 &= 0 \\ T^{m-1}(a_0x) &= 0 \\ a_0 = \frac{0}{T^{m-1}(x)} &= 0 \end{aligned}$$

By back substitution we know that $a_0 = a_1 = \cdots = a_{n-1} = 0$
Therefore, $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent. □

§8 H

Let $T : R^3 \to R^3$

a. If T(a,b,c) = (a,b,0), show that T is the projection on the xy-plane along the z-axis.

   Proof.
   We want to projection to be on the xy-plane along the z-axis. Let the projection be (x,y,0).

To minimize the distance, we must choose x and y such that

$$(a-x)^2+(b-y)^2+(c-0)^2$$

is minimum. Since the equation above is a difference of squares, x = a and b = y will give us the minimum value. Therefore, the projection on the xy-plane will be (a,b,0), which is T. $\square$

b. Find a formula for T(a,b,c), where T represents the projection on the z-axis along the xy-plane.

Proof.
We want to projection to be on the z-axis along the xy-plane. Let the projection be (0,0,z). To minimize the distance, we must choose z such that

$$(a-0)^2+(b-0)^2+(c-z)^2$$

is minimum. z = c will give us the minimum value. Therefore, the equation for T will be T(a,b,c)=(0,0,c). $\square$

c. If T(a,b,c) = (a-c,b,0), show that T is the projection on the xy-plane along the line L $= \{(a, 0, a) : a \in R\}$

Proof.
We want to projection to be on the xy-plane along the line L. Let the projection be $(x, y, 0)$. A vector that is on L is $(1, 0, 1)$. To minimize the distance, we must choose λ such that

$$(a, b, c) + \lambda(1, 0, 1) = (x, y, 0)$$

is minimum. Writing the equation above as a system:

$$\begin{aligned} a+\lambda &= x \\ b &= y \\ c+\lambda &= 0 \end{aligned}$$

Solving this system gives us, $x = a - c, y = b$
Therefore, the projection on the xy-plane along the line L will be $(a - c, b, 0)$.

$\square$

§9 I

Suppose that the linear transformation $T : V \to V$ is the projection on $W \subset V$ along some subspace $W' \subset V$. Prove that W is T-invariant and that $T_W = I_W$

Proof.

a. W is T-invariant.

We need to show that $T(x) \in W$ for every $x \in W$.

$$x = x_1 + x_2, T(x) = T(x_1 + x_2) = x_1 \text{ where } x_1 \in W, x_2 \in W'$$

It is trivial to see that T restricts to the identity on W.
Therefore, W is T-invariant.

b. $T_W = I_W$

For any $w \in W$, we can express w as $w = w + 0$ where $0 \in W'$ because W' is a subspace. Because W_1 and W_2 are a direct sum, there is no way to express w as the sum of a vector in one and a vector in the other.
$T(w + 0) = w$, shows that T_w is the same as the indentity tranformation I_W.
Therefore, $T_W = I_W$.

$\square$

§10 J