Math 341: Midterm 2

Daniel Ko

Spring 2020

§1

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1}$$

a. Suppose that $a \neq 0$, compute the solution of $\mathbf{Ax} = \mathbf{b}$ using row reduction and provide the conditions on a, b, c, d such that your computations are valid. Express the result as a simplified expression. (**Hint:** recall that you can not divide by zero)

Proof. We perform reduced row echelon form (rref) on the augmented matrix

$$(A|b) = \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$$

$$R_2 \leftarrow R_2 - \frac{c}{a}R_1 \left[\begin{array}{cc|c} a & b & e \\ 0 & d - \frac{cb}{a} & f - \frac{ce}{a} \end{array}\right]$$

$$\left[\begin{array}{cc|c} a & b & e \\ 0 & \frac{ad-cb}{a} & \frac{af-ce}{a} \end{array}\right]$$

$$R_2 \leftarrow \frac{a}{ad-cb}R_2 \quad \text{Assuming that } ab - cd \neq 0 \quad \left[\begin{array}{cc|c} a & b & e \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]$$

$$R_1 \leftarrow R_1 - bR_2 \left[\begin{array}{cc|c} a & 0 & e - b\frac{af-ce}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]$$

$$R_1 \leftarrow \frac{R_1}{a} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}(e - b\frac{af-ce}{ad-cb}) \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]$$

$$x = \begin{bmatrix} \frac{de-bf}{ad-cb} \\ \frac{af-ce}{ad-cb} \end{bmatrix} \quad \text{where } ad - cb \neq 0$$

□

b. If $a = 0$, and $c \neq 0$, is your above computation still valid? How would you modify it? (explain briefly) (**Hint:** recall that you can swap the equations and the result is the same)

Proof. If $a = 0$, and $c \neq 0$, then the above computation will not be valid as we divided by a multiple times when we computed the rref. I would swap the first and second rows so that it would look like

$$\left[\begin{array}{cc|c} c & d & f \\ 0 & b & e \end{array}\right]$$

and compute the rref, assuming that $b \neq 0$. We obtain the rref,

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{bf-de}{bc} \\ 0 & 1 & \frac{e}{b} \end{array}\right]$$

$\square$

c. If $a = 0$, $c = 0$, but $b \neq 0$, $d \neq 0$, what are the conditions on e and f such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution? Is the solution unique? (**Hint:** recall that $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$)

Proof. If $a = 0$, $c = 0$, $b \neq 0$, $d \neq 0$, we get the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & e \\ 0 & d & f \end{array}\right]$$

Performing row reduction,

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{e}{b} \\ 0 & 1 & \frac{f}{d} \end{array}\right]$$

So the condition of the solution is,

$$x_2 = \frac{e}{b} = \frac{f}{d}$$

Thus, there exists a infinite amount of solution. $\square$

d. Solve the system

$$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}. \tag{2}$$

(**Hint:** You may want to use the formula you just deduced)

Proof.

$$\begin{aligned} x_1 &= \frac{de - bf}{ad - cb} \\ &= \frac{\sqrt{2}(5\sqrt{2}) - 3\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 30}{2 - 12} \\ &= \frac{-20}{-10} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\sqrt{2}(5\sqrt{2}) - 2\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 20}{-10} \\ &= 1 \end{aligned}$$

$\square$

§2

Let

$$\mathbf{A} = \begin{bmatrix} 0 & -\alpha & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 3 \\ -2 & -2 & 4 & 2\alpha \\ 0 & \alpha & -1 & 2\alpha + 1/2 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2+\alpha \\ 2\beta+\alpha-2 \end{bmatrix} \tag{3}$$

What are the conditions on α and β such that the system $\mathbf{Ax} = \mathbf{b}$:

a. Has no solution?

Proof. We begin by putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ in its reduced form.

$$(\mathbf{A}|\mathbf{b}) = \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & \alpha & -1 & 2\alpha+1/2 & 2\beta+\alpha-2 \end{array}\right]$$

$$R_5 \leftarrow R_5 + R_1 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_3 \leftarrow R_3 - 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ -2 & -2 & 4 & 2\alpha & 2+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 + 2R_2 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2\alpha+2 & 4+\alpha \\ 0 & 0 & 1 & 2\alpha+1/2 & 2\beta+\alpha \end{array}\right]$$

$$R_4 \leftarrow R_4 - 2R_3, R_5 \leftarrow R_5 - \frac{1}{2}R_3 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 2\beta+\alpha+1 \end{array}\right]$$

$$R_5 \leftarrow R_5 - R_4 \left[\begin{array}{cccc|c} 0 & -\alpha & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta-1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 2\beta-1 \end{array}\right]$$

By Theorem 3.11, a system is consistent if and only if $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$. Thus this system will have no solution if $2\beta - 1 \neq 0$, which is when $\beta \neq \frac{1}{2}$. We observe that there will be no conditions on α. □

b. Has an unique solution? Find the solution. (**Hint:** you will need to row reduce the augmented system to echelon form, and then use the theorems seen in class to impose the conditions on α and β).

Proof. Combining Theorem 3.10 and the corollary to Theorem 4.7, we get that a system has a unique solution if and only if $\det(\mathbf{A}) \neq 0$. The determinant of an upper triangular matrix is the product of its diagonal entries by property 4 of determinants in section 4.4. Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &\neq 0 \\ -4\alpha^2 &\neq 0 \\ \alpha &\neq 0 \end{aligned}$$

and from (a), $\beta = \frac{1}{2}$. Combining these two conditions we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & -\alpha & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By performing back substitution we compute the unique solution as

$$x = \begin{bmatrix} \frac{1}{2} + \frac{1}{2\alpha} \\ -\frac{1}{2\alpha} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

□

c. Has infinite amount of solutions? Find the solution set in parametric form. (**Hint:** You may have one equations for α and one for β that have to be satisfied simultaneously).

Proof. Having an infinite amount of solutions is by definition another way of saying that a system that is consistent and that the solutions are not unique. A system is consistent if and only if $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ by Theorem 3.11. If $\det(\mathbf{A}) = 0$, the solution, if it exists, is not unique by Theorem 3.10 and the corollary to Theorem 4.7.

Using this fact we can compute the condition of α as such that

$$\begin{aligned} 1 * -\alpha * 2 * 2\alpha &= 0 \\ -4\alpha^2 &= 0 \\ \alpha &= 0 \end{aligned}$$

Given that $\alpha = 0$, and that $\beta = \frac{1}{2}$ from part (a) we get the following system,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It is clear that $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b})$ because $\mathbf{b}$ is a linear combination of the second and third column from $\mathbf{A}$. Hence, this system is consistent.
We can compute a solution space to $\mathbf{Ax} = \mathbf{b}$ as outlined in Theorem 3.9. We start by first computing the solution set to $\mathbf{Ax} = 0$ denoted by K_H. It is clear that $rank(\mathbf{A}) = 3$ because the first two columns are the same and the rest of the columns are linearly independent from each other. By Theorem 3.8, $dim(K_H) = 4 - 3 = 1$. Thus any nonzero solution constitutes a basis for K. For example, since

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

is a solution to the $\mathbf{Ax} = 0$, it is a basis for K_H by Corollary 2 of Theorem 1.10. So a solution set to K_H would be

$$K_H = \left\{ t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

A solution to $\mathbf{Ax} = \mathbf{b}$ is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, by Theorem 3.9 we compute solution space when this system has infinite amount of solutions, which is when $\alpha = 0$ and $\beta = \frac{1}{2}$ as

$$K = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$\square$

§3

Let $A \in M_{n\times n}(F)$, for a field F. We want to prove that $rank(A^2) - rank(A^3) \leq rank(A) - rank(A^2)$. The solution to this exercise requires the notion of quotient spaces. Even though you should already be familiar with quotient spaces we will prove a few properties that will be useful.

Let W be a subspace of a vector space V over a field F. For any $v \in$ V the set $\{v\} +$ W $= \{v + w : w \in$ W$\}$ is called the coset of W containing v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$. Following this notation we write V/W $= \{v +$ W $: v \in$ V$\}$, which is usually called the quotient space V module W. Addition and scalar multiplication by scalars can be defined in the collection V/W as follows:

$$(v_1 + \mathrm{W}) + (v_2 + \mathrm{W}) = (v_1 + v_2) + \mathrm{W}$$

for all $v_1, v_2 \in$ V and

$$a(v + W) = av + W$$

for all $v \in$ V and $a \in F$

a. Prove that $v_1 +$ W $= v_2 +$ W if and only if $v_1 - v_2 \in$ W. (**Hint**: recall that $v_1 +$ W is a set, thus you need to prove equality between sets)

Proof.

i. $v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W$
If $v_1 + W = v_2 + W$, then $\exists w_1, w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$
$v_1 - v_2 = w_2 - w_1$
Since, $w_2 - w_1 \in W$ (clourse under addition)
Therefore, $v_1 - v_2 \in W$

ii. $v_1 - v_2 \in W \Rightarrow v_1 + W = v_2 + W$
This means $v_1 - v_2 = w$ where $w \in W$ $(*)$
Now let $x \in v_1 + W$
By definition, $\exists w_x \in W$ such that $x = v_1 + w_x$
By $(*)$ $v_1 = v_2 + w$
So, $x = v_2 + w + w_x$
Since, $w + w_x \in W$ (closure under addition)
We have $x \in v_2 + W$
So, $v_1 + W \subseteq v_2 + W$
Without loss of generality, we can show $v_2 + W \subseteq v_1 + W$
Therefore, $v_1 + W = v_2 + W$

Therefore, $v_1 + \mathrm{W} = v_2 + \mathrm{W}$ if and only if $v_1 - v_2 \in \mathrm{W}$ □

b. Show that V/W with the operations defined above is a linear vector space.

VS 1: For all x, y in V/W, $x + y = y + x$ (commutativity of addition)

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} x + y &= (v_x + W) + (v_y + W) \\ &= (v_x + v_y) + W \\ &= (v_y + v_x) + W \\ &= (v_y + W) + (v_x + W) \\ &= y + x \end{aligned}$$

□

VS 2: For all x, y, z in V/W, $(x + y) + z = x + (y + z)$ (associativity of addition)

Proof. Let $x = v_x + W, y = v_y + W, z = v_z + W$, where $v_x, v_y, v_z \in V$.

$$\begin{aligned} (x + y) + z &= ((v_x + W) + (v_y + W)) + (v_z + W) \\ &= ((v_x + v_y) + W) + (v_z + W) \\ &= (v_y + v_x + v_z) + W \\ &= (v_x + W) + ((v_y + v_z) + W) \\ &= (v_x + W) + ((v_y + W) + (v_z + W)) \\ &= x + (y + z) \end{aligned}$$

□

VS 3: There exists an element in V/W denoted by $\mathbf{0}$ such that $x + \mathbf{0} = x$ for each x in V/W

Proof. Let $x = v_x + W$ where $v_x \in V$. Let $\mathbf{0}$ in V/W be defined as $0 + W$, where $0 \in V$.

$$\begin{aligned} x + \mathbf{0} &= x + (0 + W) \\ &= (v_x + W) + (0 + W) \\ &= (v_x + 0) + W \\ &= v_x + W \\ &= x \end{aligned}$$

□

VS 4: For each element x in V/W there exists an element y in V/W such that $x + y = \mathbf{0}$

Proof. Let $x = v_x + W$ where $v_x \in V$. Fix y such that $y = -v_x + W$.

$$\begin{aligned} x + y &= (v_x + W) + (-v_x + W) \\ &= (v_x - v_x) + W \\ &= 0 + W \\ &= \mathbf{0} \end{aligned}$$

□

VS 5: For each element x in V/W, $1x = x$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} 1x &= 1(v_x + W) \\ &= (1v_x + W) \\ &= (v_x + W) \\ &= x \end{aligned}$$

□

VS 6: For each pair of elements a, b in $\mathbb{F}$ and each element x in V/W, $(ab)x = a(bx)$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned} (ab)x &= abv_x + W \\ &= a(bv_x + W) \\ &= a(bx) \end{aligned}$$

□

VS 7: For each element a in $\mathbb{F}$ and each pair of elements x, y in V/W, $a(x + y) = ax + ay$

Proof. Let $x = v_x + W, y = v_y + W$ where $v_x, v_y \in V$.

$$\begin{aligned} a(x + y) &= a((v_x + v_y) + W) \\ &= ((av_x + av_y) + W) \\ &= (av_x + W) + (av_y + W) \\ &= a(v_x + W) + a(v_y + W) \\ &= ax + ay \end{aligned}$$

□

VS 8: For each pair of elements a, b in $\mathbb{F}$ and each element x in V/W, $(a+b)x = ax + bx$

Proof. Let $x = v_x + W$ where $v_x \in V$.

$$\begin{aligned}(a+b)x &= (a+b)(v_x + W)\\ &= ((a+b)v_x) + W\\ &= ((av_x + bv_x) + W\\ &= (av_x + W) + (bv_x + W)\\ &= a(v_x + W) + b(v_x + W)\\ &= ax + bx\end{aligned}$$

□

Therefore, V/W is a vector space because it holds all the properties above.

c. Prove that if $dim(\text{V}) < \infty$ then $dim(\text{V/W}) = dim(\text{V}) - dim(\text{W})$. (Hint: Define a linear map $T : \text{V} \to \text{V/W}$ such that the range of T is V/W, and then use the rank-nullity theorem)

Proof. We define the linear map $T : \text{V} \to \text{V/W}$ by

$$T(v) = v + W$$

We first prove that T is in fact linear, where $v_1, v_2 \in V$ and $c \in F$.

$$\begin{aligned}T(cv_1 + v_2) &= (cv_1 + v_2) + W\\ &= (cv_1 + W) + ((v_2) + W)\\ &= c(v_1 + W) + ((v_2) + W)\\ &= cT(v_1) + T(v_2)\end{aligned}$$

We claim that $N(T) = W$ and $R(T) = \text{V/W}$

1. $R(T) = \text{V/W}$
   - i. $R(T) \subseteq \text{V/W}$
     By Theorem 2.1
   - ii. $\text{V/W} \subseteq R(T)$
     Let $y = v_y + W$ where $y \in \text{V/W}$ and $v_y \in V$.
     For prove that $y \in N(T)$ we need to show that $\exists x$ such that $T(x) = y$.
     Notice that $x = v_y$. Therfore, $y \in R(T)$ so $\text{V/W} \subseteq R(T)$.
2. $N(T) = W$
   - i. $N(T) \subseteq W$
     Let $x \in N(T)$.
     By definition,

$$\begin{aligned}T(x) &= \mathbf{0}\\ &= 0 + W\\ &= W\\ &= w + W\end{aligned}$$

where $w \in W$ and $\mathbf{0} \in \text{V/W}$.
This must mean that $x = w$ and since $w \in W$, $x \in W$. Therefore, $N(T) \subseteq W$

ii. $W \subseteq N(T)$
Let $w \in W$.

$$\begin{aligned} T(w) &= w + W \\ &= W \\ &= 0 + W \\ &= \mathbf{0} \end{aligned}$$

So $w \in N(T)$. Therefore, $W \subseteq N(T)$.

Notice that V/W $\subseteq$ V so V/W is finite dimensional. Since T is a linear map and that V and V/W are indeed finite dimensional vector spaces (by part b) we can use the rank-nullity theorem (Theorem 2.3).

$$\begin{aligned} dim(N(T)) + dim(R(T)) &= dim(V) \\ dim(W) + dim(\text{V/W}) &= dim(V) \\ dim(\text{V/W}) &= dim(V) - dim(W) \end{aligned}$$

as desired. □

d. Let $K = F^n$, define $AK = R(L_A)$, and $A^2K = R(L_{A^2})$. Show that AK/A^2K is a vector space of dimension $rank(A) - rank(A^2)$.

Proof. We begin by proving that AK/A^2K is a vector space. It is enough to show that A^2K is a subspace of AK to prove that AK/A^2K is a vector space, by part (b). It is clear that AK is a subspace because $AK = R(L_A)$ and Theorem 2.3. Moreover, $A^2K \subseteq AK$ because $R(L_{A^2}) \subseteq R(L_A)$. (Theorem 3.7?) **Not sure if $R(L_{A^2}) \subseteq R(L_A)$ holds**
Because $A^2K = R(L_{A^2})$, A^2K has the properties of a vector space because of Theorem 2.1. It follows from part (c) that

$$\begin{aligned} dim(AK/A^2K) &= dim(AK) - dim(A^2K)) \\ &= dim(R(L_A)) - dim(R(L_{A^2})) \\ &= rank(A) - rank(A^2) \end{aligned}$$

as desired. □

e. Show that A^2K/A^3K is a vector space of dimension $rank(A^2) - rank(A^3)$, where $A^3K = R(L_{A^3})$.

Proof. Similiar to what we did in part (d), we can prove that A^2K/A^3K is a vector space. Then it follows that,

$$\begin{aligned} dim(A^2K/A^3K) &= dim(A^2K) - dim(A^3K)) \\ &= dim(R(L_{A^2})) - dim(R(L_{A^3})) \\ &= rank(A^2) - rank(A^3) \end{aligned}$$

as desired. □

f. Define $T : AK/A^2K \to A^2K/A^3K$, by $T(v) = L_A(v)$, i.e, we left multiply each element of v by the matrix A. Show that $R(T) = A^2K/A^3K$.

Proof. To show that $R(T) = A^2K/A^3K$, is another way of saying show that T is onto. We must show that $\forall x \in AK/A^2K[\exists y \in A^2K/A^3K : T(x) = y]$. Let $x = Ak' + A^2K$ where

$k' \in \mathbb{F}^n$.

$$\begin{aligned} T(x) &= T(Ak' + A^2K) \\ &= L_A(Ak' + A^2K) \\ &= L_A(Ak') + L_A(A^2K) \\ &= A^2k' + A^3K \in A^2K/A^3K \end{aligned}$$

Therefore, T is onto and $R(T) = A^2K/A^3K$. □

g. Use the rank-nullity theorem on T to conclude that $rank(A^2) - rank(A^3) \leq rank(A) - rank(A^2)$.

Proof. We begin by proving that T is linear. We define $x_1 = Ak_1' + A^2K$ and $x_2 = Ak_2' + A^2K$ such that $x_1, x_2 \in AK/A^2K$ and $k_1', k_2' \in F^n$ and $c \in F$. We use properties of matrices as outlined in Theorem 2.12.

$$\begin{aligned} T(ck_1' + k_2') &= L_A(ck_1' + k_2') \\ &= A(ck_1' + k_2') \\ &= A(c(Ak_1' + A^2K) + (Ak_2' + A^2K)) \\ &= A(c(Ak_1' + A^2K)) + A(Ak_2' + A^2K) \\ &= c(A(Ak_1' + A^2K)) + A(Ak_2' + A^2K) \\ &= cL_A(k_1') + L_A(k_2') \\ &= cT(k_1') + T(k_2') \end{aligned}$$

Since T is linear, we can use rank nullity theorem. From part (d), (e), (f), and assuming that $nullity(T) \geq 0$ because the number of elements of a set cannot be negative

$$\begin{aligned} rank(T) + nullity(T) &= dim(AK/A^2K) \\ &= rank(A) - rank(A^2) \\ rank(T) &\leq rank(A) - rank(A^2) \\ rank(A^2) - rank(A^3) &\leq rank(A) - rank(A^2) \end{aligned}$$

as desired. □

§4

Let V be a finite-dimensional vector space. Let T and P be two linear transformations from V to itself, such that $T^2 = P^2 = 0$, and $T \circ P + P \circ T = I$, where I is the identity in V.

a. Denote $N_T = N(T)$ and $N_P = N(P)$, the null spaces of T and P, respectively. Show that $N_P = P(N_T)$, and $N_T = T(N_P)$, where $T(N_P) = \{T(v) : v \in N_P\}$ and $P(N_T) = \{P(v) : v \in N_T\}$.

Proof.

i. Show $N_P = P(N_T)$

1. $N_P \subseteq P(N_T)$

Let $x \in N_P$. By definition, $P(x) = 0$.

$$\begin{aligned} P \circ T(x) + T \circ P(x) &= x \\ P \circ T(x) + T(0) &= x \\ P \circ T(x) &= x \end{aligned}$$

Notice that $T(x) \in N_T$, so $x \in P(N_T)$. Thus, $N_p \subseteq P(N_T)$.

2. $P(N_T) \subseteq N_P$
   Let $P(x) \in P(N_T)$ where $x \in N_T$.
   $P \circ P(x) = 0$ so $P(x) \in N_P$. Thus, $P(N_T) \subseteq N_P$.

Therfore, $N_P = P(N_T)$.

ii. Show $N_T = T(N_P)$

1. $N_T \subseteq T(N_P)$
   Let $x \in N_T$. By definition, $T(x) = 0$.

$$\begin{aligned} P \circ T(x) + T \circ P(x) &= x \\ P(0) + T \circ P(x) &= x \\ T \circ P(x) &= x \end{aligned}$$

Notice that $P(x) \in N_P$, so $x \in T(N_P)$. Thus, $N_T \subseteq T(N_P)$.

2. $T(N_P) \subseteq N_T$
   Let $T(x) \in T(N_P)$ where $x \in N_P$.
   $T \circ T(x) = 0$ so $T(x) \in N_T$. Thus, $T(N_P) \subseteq N_T$.

Therfore, $N_P = P(N_T)$.

□

b. Show that $\mathrm{V} = N_T \oplus N_P$.

Proof. Need to prove the following two conditions.

i. $N_T \cap N_P = \{0\}$.

Let $x \in N_T$ and $x \in N_P$. This means that $T(x) = 0$ and $P(x) = 0$.

$$\begin{aligned} x &= P \circ T(x) + T \circ P(x) \\ &= P(0) + T(0) \\ &= 0 \end{aligned}$$

Thus, $N_T \cap N_P = \{0\}$.

ii. $N_T + N_P = \mathrm{V}$.

1. $N_T + N_P \subseteq \mathrm{V}$
   Let $x \in N_T + N_P$. Define $x = n_t + n_p$ where $n_t \in N_T \subseteq \mathrm{V}$ and $n_p \in N_P \subseteq \mathrm{V}$ by Theorem 2.11. Since V is a vector space $n_t + n_p \in V$ by closure under addition. Thus, $x \in \mathrm{V}$ and it follows that $N_T + N_P \subseteq \mathrm{V}$.
2. $\mathrm{V} \subseteq N_T + N_P$
   Let $x \in V$. We know that $x = P \circ T(x) + T \circ P(x)$.
   Notice that $P \circ T(x) \in N_P$ because $P(P \circ T(x)) = 0$.
   Similarly, $T \circ P(x) \in N_T$ because $T(T \circ P(x)) = 0$.
   Thus, $N_T + N_P \subseteq \mathrm{V}$.

Therefore, $N_T + N_P = V$.

Therefore, by definition $V = N_T \oplus N_P$. ☐

c. Prove that the dimension of V is even.

Proof.

Lemma 4.1

$V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$

Proof. 1.6.29(b) in the textbook **Actually remember to type this in** ☐

By the lemma above, $dim(V) = dim(N_T) + dim(N_P)$.
Let's construct a basis β_{N_T} for N_T and β_{N_P} for N_P.
$\beta_{N_T} = \{u_1, u_2, \cdots, u_m\}$
$\beta_{N_P} = \{w_1, w_2, \cdots, w_n\}$
From (a), since $N_T = T(N_P)$, this must mean that

$$span(\beta_{N_T}) = T(span(\beta_{N_P}))$$

Thus, $dim(N_P) \leq dim(N_T)$ because $dim(T(N_P)) \leq dim(N_P)$.
Likewise, since $N_P = P(N_T)$, this must mean that

$$span(\beta_{N_P}) = P(span(\beta_{N_T}))$$

Thus, $dim(N_T) \leq dim(N_P)$ because $dim(P(N_T)) \leq dim(N_T)$.

Thus, $dim(N_P) = dim(N_T) = n = m$.

$$\begin{aligned} dim(V) &= dim(N_T) + dim(N_P) \\ &= n + m \\ &= n + n = m + m \\ &= 2n = 2m \end{aligned}$$

Therefore, $dim(V)$ is even. ☐

d. Suppose that the dimension of V is two. Prove that V has a basis β, such that

$$[T]_\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad [P]_\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Proof. ☐