CS 577: HW 7

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Summer 2020

§1 Prove that Strategic Advertising is NP-complete.

Define a set S to fulfill Strategic Advertising Problem (SAP) on G with paths $\{P_i\}$, if it is possible to place advertisements on at most k of the nodes in G, so that each path P_i includes at least one node containing an advertisement.

§1.1 Strategic Advertising is NP.

Proof. Suppose we have a set S that claims to fulfill SAP. We can iterate through all nodes in each t paths, checking to see if a path contains a node that is also in the set S. Since, a path is at most n, where n is the amount of nodes in the graph G, the computing complexity to check if a set S fulfills SAP is $O(nkt)$ which is clearly polynomial.

$\square$

§1.2 Reduce Strategic Advertising to a known NP-complete problem.

Proof. We claim that we can reduce Strategic Advertising to the Vertex Cover problem. We show that Vertex Cover $\leq_p$ Strategic Advertising.

Suppose we have a graph $G = (V, E)$, and we want to find a vertex cover of size of at most k. We can convert this problem into a Strategic Advertising problem in polynomial time by doing the following. Since Strategic Advertising algorithm requires a directed graph, let $G^* = (V, E^*)$ where E^* is where we arbitrary set a direction to the edges in E. Generate paths, P_i, where each edge in E^* is a path. We show that G has a vertex cover of size of at most $k \Leftrightarrow$ there exists a set S such that it fulfills SAP on G^* with paths $\{P_i\}$.

i. G has a vertex cover of size of at most $k \Rightarrow$ There exists a set S such that it fulfills SAP on G^* with paths $\{P_i\}$

   Let S be a set of nodes for a vertex cover for G. S will fulfill SAP because all edges will at least one endpoint in S by the fact that S is a vertex cover and since $\{P_i\}$ consists of all edges in E.

ii. There exists a set S such that it fulfills SAP on G^* with paths $\{P_i\} \Rightarrow G$ has a vertex cover of size of at most k

   This means that we have most k nodes in S such that each path P_i includes such nodes. Notice that we since we defined $\{P_i\}$ to be the set of all edges in E, S will be a vertex cover for G.

Thus, Vertex Cover $\leq_p$ Strategic Advertising as desired. Using Theorem 8.14, we conclude that Strategic Advertising is NP-complete. $\square$

§2 Prove that Scheduling is NP-complete.

We define the Scheduling problem to be the following: If you're given a set of n jobs with each specified by a set of time intervals. Is it possible to accept at least k of them so that none of the accepted jobs overlap in time?

§2.1 Scheduling is NP.

Proof. Suppose we have a set S that claims to fulfill the Scheduling problem. We can clearly check if none of the jobs overlap by first sorting the intervals by start time and iterate through S, checking if the end time of the ith job is equal or earlier than the start time of the $i+1$th job. We also make sure that $|S| \geq k$. Using an efficient sorting algorithm like merge sort, will take $O(n \log n)$, where n is the number of intervals given in the set, and checking if any job overlap will take, $O(n)$. Hence, checking if S fulfills Scheduling problem will take $O(n \log n)$, which clearly takes polynomial time. □

§2.2 Reduce Scheduling to a known NP-complete problem.

Proof. We claim that we can reduce Scheduling problem to the Independent Set problem. We show that Independent Set $\leq_p$ Scheduling.

Suppose we have a graph $G = (V, E)$, and we want to find an independent set of at least k. We can convert this problem into a Scheduling problem in polynomial time by doing the following. Let where e is the number of edges in our graph. Let the available period of time that our jobs be from $(00 : 00)$ to $(\delta : 00)$. Divide this period of time into e intervals where each period represent an edge in E. Let n be the number of vertices in G and create a set with n jobs with each job corresponding to a vertex in G. For each job, set its intervals to the ones that represent its edges.

We show that G has an independent set of at least k if and only if there is at least k jobs that do not overlap. If there is independent set of size k, these nodes do not share any edges, thus the corresponding time intervals for such jobs do not overlap. Conversely, if there is at least k jobs that do not overlap, the corresponding graph contains at least k jobs that do not share edges with each other, which is an independent set. □

Thus, Vertex Cover $\leq_p$ Strategic Advertising as desired. Using Theorem 8.14, we conclude that Scheduling is NP-complete.