Math 341: Homework 1

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§1 A

a. $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

b. $p \vee F \Leftrightarrow F$

p	F	$p \vee F$	$p \vee F \Leftrightarrow F$
T	F	T	T
F	F	F	T

c. $p \wedge \neg p \Leftrightarrow F$

p	$\neg p$	$p \wedge \neg p$	$p \wedge \neg p \Leftrightarrow F$
T	F	F	T
F	T	F	T

d. $(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$

p	q	$p \Leftrightarrow q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$
T	T	T	T	F	T	T
T	F	F	F	F	F	T
F	T	F	F	F	F	T
F	F	T	F	T	T	T

e. $[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$

p	q	r	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)$	$p \Leftrightarrow r$	$[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	T	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

f. $[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$

p	q	$p \wedge \neg q$	$(p \wedge \neg q) \Rightarrow \neg p$	$p \Rightarrow q$	$[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$
T	T	F	T	T	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	F	T	T	T

§2 B

a. $(p \vee q \Leftrightarrow p \wedge r) \Rightarrow ((p \Rightarrow p) \wedge (p \Rightarrow r))$

$$
\begin{aligned}
(p \vee q \Leftrightarrow p \wedge r) \Rightarrow (p \Rightarrow r) && \text{(Transitivity)}\\
[(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \Rightarrow (p \Rightarrow r) && \text{(Def. of bicondtional)}\\
\neg[(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \vee (\neg p \vee r) && \text{(Material Implication)}\\
\neg[(\neg(p \vee q) \vee (p \wedge r)) \wedge (\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(Material Implication)}\\
[\neg(\neg(p \vee q) \vee (p \wedge r)) \vee \neg(\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(De Morgan's Law)}\\
(\neg\neg(p \vee q) \wedge \neg(p \wedge r)) \vee (\neg\neg(p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(De Morgan's Law)}\\
((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(Double negation)}\\
(\neg p \vee r) \vee ((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Commutative)}\\
((\neg p \vee r) \vee (p \vee q)) \wedge ((\neg p \vee r) \vee \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Distributive)}\\
(True \wedge True) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Excluded middle)}\\
True &&
\end{aligned}
$$

b. $[(p \Rightarrow \neg q) \wedge (r \Rightarrow q)] \Rightarrow (p \Rightarrow \neg r)$

$$\begin{aligned}
\neg[(\neg p \vee \neg q) \wedge (\neg r \wedge q)] \vee \neg p \vee \neg r && \text{(Material Implication)}\\
\neg(\neg p \vee \neg q) \vee \neg(\neg r \wedge q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)}\\
(p \wedge q) \vee (r \wedge \neg q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)}\\
\neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)}\\
\neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)}\\
True &&
\end{aligned}$$

c. $(p \Rightarrow q) \Rightarrow [\neg(q \wedge r) \Rightarrow \neg(p \wedge r)]$

$$\begin{aligned}
\neg(\neg p \vee q) \vee [\neg\neg(q \wedge r) \vee \neg(p \wedge r)] && \text{(De Morgan's Law)}\\
(p \wedge \neg q) \vee (q \wedge r) \vee (\neg p \vee \neg r) && \text{(De Morgan's Law + Negation)}\\
\neg p \vee (p \wedge \neg q) \vee \neg r \vee (q \wedge r) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (\neg r \vee r)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (True)] && \text{(Excluded middle)}\\
\neg p \vee \neg q \vee \neg r \vee q && \text{(Excluded middle)}\\
True &&
\end{aligned}$$

d. $[(p \Rightarrow \neg q) \wedge (\neg r \vee q) \wedge r] \Rightarrow \neg p$

$$\begin{aligned}
\neg[(\neg p \vee \neg q) \wedge (\neg r \vee q) \wedge r] \vee \neg p && \text{(Material Implication)}\\
\neg(\neg p \vee \neg q) \vee \neg(\neg r \vee q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)}\\
(p \wedge q) \vee (r \wedge \neg q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)}\\
\neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)}\\
\neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)}\\
True && \text{(Excluded middle)}
\end{aligned}$$

§3 C

loren ipsum or something

§4 D

Corollary
The additive inverse is unique.

Proof. Suppose u, v are the additive inverse of x.

$$x + u = 0 \quad x + v = 0$$

$$\begin{aligned} x + u &= x + v && \text{(Transitive property)} \\ u + x &= v + x && \text{(Commutative property)} \\ u &= v && \text{(Theorem 1.1)} \end{aligned}$$

□

Corollary
The vector 0 is unique.

Proof. Suppose $u, v \in V$ satisfies the "zero property", which is defined as:

$$\begin{aligned} &\forall x \in V \quad x + u = x \Rightarrow v + u = v \\ &\forall x \in V \quad x + v = x \Rightarrow u + v = u \\ &u = u + v = v + u = v && \text{(Transitive property)} \\ &u = v && \text{(Theorem 1.1)} \end{aligned}$$

□

§5 E

Theorem (1.2(c) In any vector space the following statements are true.)
$a\mathbf{0} = \mathbf{0}$
$\forall a \in F \quad \mathbf{0} \in \mathbf{V}$
Any scalar multiplied by the 0 vector will result in the 0 vector.

Proof.

$$\begin{aligned} a\mathbf{0} &= a(\mathbf{0} + \mathbf{0}) && \text{(Identity element of addition)} \\ a\mathbf{0} &= a\mathbf{0} + a\mathbf{0} && \text{(Distributive)} \\ a\mathbf{0} - a\mathbf{0} &= a\mathbf{0} + a\mathbf{0} - a\mathbf{0} && \text{(Inverse element of addition)} \\ \mathbf{0} &= a\mathbf{0} \end{aligned}$$

□

§6 F

Prove that diagonal matrices (as defined in your book in Example 3, Section 1.3) are symmetric.

Proof.

Let D equal a diagonal matrix
D is symmetric $\Leftrightarrow (\forall i, j)(D_{i,j} = D_{j,i})$
By definition of diagonal matrix:
When $i \neq j$, $D_{i,j} = D_{j,i} = 0$
When $i = j$, $D_{i,j} = D_{j,i}$
$\therefore (\forall i, j)(D_{i,j} = D_{j,i})$ so diagonal matrix is symmetric

□

§7 G

Prove that
$W_1 = (a_1, a_2, \cdots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0$
is a subspace of F_n, but
$W_2 = (a_1, a_2, \cdots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1$
is not.

Proof. Proof that W_1 is a subspace of F^n

a. $0 \in W_1$

$$\begin{aligned} &\text{Let } a_1, a_2, \cdots, a_n = 0 \\ &0 + 0 + \cdots + 0 = 0 \in W_1 \end{aligned}$$

b. $X, Y \in W_1 \Rightarrow X + Y \in W_1$

$$\begin{aligned} X = (x_1, x_2, \cdots, x_n) \quad Y &= (y_1, y_2, \cdots, y_n) \\ X + Y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ \sum_{i=1}^{n} x_i + y_i &= x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \\ &= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\ &= 0 + 0 \\ &= 0 \in W_1 \end{aligned}$$

c. $c \in F, X \in W_1 \Rightarrow cX \in W_1$

$$\begin{aligned} X &= (x_1, x_2, \cdots, x_n) \\ cX &= (cx_1, cx_2, \cdots, cx_n) \\ \sum_{i=1}^{n} cx_i &= cx_1 + cx_2 + \cdots + cx_n \\ &= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\ &= c(x_1, x_2, \cdots, x_n) \\ &= c(0) \\ &= 0 \in W_1 \end{aligned}$$

$\square$