Math 341: Homework 8

Daniel Ko

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§1 A

Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$

Proof.

    a. λ is an eigenvalue of $T \Rightarrow \lambda$ is an eigenvalue of $[T]_\beta$

       By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Using Theorem 2.14,

$$T(v) = \lambda v$$
$$[T(v)]_\beta = [\lambda v]_\beta$$
$$[T]_\beta [v]_\beta = \lambda [v]_\beta$$

       as desired. Thus, λ is an eigenvalue of $[T]_\beta$.

    b. λ is an eigenvalue of $[T]_\beta \Rightarrow \lambda$ is an eigenvalue of T

       By definition, there exists a eigenvector $v \in V$ such that $[T]_\beta [v]_\beta = \lambda [v]_\beta$. Using Theorem 2.14,

$$[T]_\beta [v]_\beta = \lambda [v]_\beta$$
$$[T(v)]_\beta = [\lambda v]_\beta$$
$$T(v) = \lambda v$$

       as desired. Thus, λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. $\qquad\square$

§2 B

    a. Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

      Proof.

        i. Linear operator T on a finite-dimensional vector space is invertible $\Rightarrow$ zero is not an eigenvalue of T.

By the corollary of Theorem 4.7, $\det(T) \neq 0$. Assume, for the sake of contradiction, suppose zero is an eigenvalue of T. It follows from Theorem 5.2 that

$$\det(T - \lambda I) = 0$$
$$\det(T - 0I) = 0$$
$$\det(T) = 0$$

which is a contradiction. Thus, zero is not an eigenvalue of T.

  ii. Zero is not an eigenvalue of $T \Rightarrow$ linear operator T on a finite-dimensional vector space is invertible.

  By contrapositive, we will instead prove that if linear operator T on a finite-dimensional vector space is not invertible then zero is an eigenvalue of T. If T is not invertible then $\det(T) = 0$ by corollary of Theorem 4.7. It follows from Theorem 5.2 that

$$\det(T - \lambda I) = 0$$

  It directly follows that zero is an eigenvalue of T.

Therefore, linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.   □

b. Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

  Proof.

  i. A scalar λ is an eigenvalue of T $\Rightarrow \lambda^{-1}$ is an eigenvalue of T^{-1}.

  By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Given that T is invertible and by definition eigenvalues are non zero,

$$T(v) = \lambda v$$
$$T^{-1}(T(v)) = T^{-1}(\lambda v)$$
$$v = T^{-1}(\lambda v)$$
$$v = \lambda T^{-1}(v)$$
$$\lambda^{-1} v = T^{-1}(v)$$

  as desired. Thus, λ^{-1} is an eigenvalue of T^{-1}.

  ii. λ^{-1} is an eigenvalue of $T^{-1} \Rightarrow$ a scalar λ is an eigenvalue of T.

  By definition, there exists a eigenvector $v \in V$ such that $T^{-1}(v) = \lambda^{-1} v$. Given that T^{-1} is invertible linear operator,

$$T^{-1}(v) = \lambda^{-1} v$$
$$T(T^{-1}(v)) = T(\lambda^{-1} v)$$
$$v = \lambda^{-1} T(v)$$
$$\lambda v = T(v)$$

  as desired. Thus, λ is an eigenvalue of T.

Therefore, a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.   □

c. State and prove results analogous to (a) and (b) for matrices.

  (a) A matrix A is invertible if and only if zero is not an eigenvalue of A.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (a) that "matrix A is invertible if and only if zero is not an eigenvalue of A." is true. $\qquad\square$

(b) Let A be an invertible matrix. λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (b) that "λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}" is true. $\qquad\square$

§3 C

For any square matrix A, prove that A and A^t have the same characteristic polynomial, and hence the same eigenvalues.

Proof. Let $A \in M_{n \times n}(F)$. The characteristic polynomial for A is $f(t) = \det(A - tI_n)$. Using Theorem 4.8, Theorem 2.12, and the trivial fact that the identity matrix is symmetric,

$$
\begin{aligned}
f(t) &= det(A - tI_n) \\
&= det((A - tI_n)^t) \\
&= det(A^t - (tI_n)^t)) \\
&= det(A^t - tI_n)
\end{aligned}
$$

which is exactly the characteristic polynomial for A^t. Therefore, A and A^t have the same characteristic polynomial, and hence the same eigenvalues. $\qquad\square$

§4 D

Let T be a linear operator on a finite-dimensional vector space V, and let c be any scalar.

a. Determine the relationship between the eigenvalues and eigenvectors of T (if any) and the eigenvalues and eigenvectors of $U = T - cI$ (where I is the identity transformation) Justify your answers.

Proof. Suppose $v \in V$ is an eigenvector of T where λ is its eigenvalue,

$$Tv = \lambda v$$

Applying the transformation U to v,

$$
\begin{aligned}
Uv &= (T - cI)v \\
&= Tv - cIv \\
&= \lambda v - cv \\
&= (\lambda - c)v
\end{aligned}
$$

Thus, if v is an eigenvector of T, then it is an eigenvector U with its corresponding eigenvalue being c. $\qquad\square$

b. Prove that T is diagonalizable if and only if U is diagonalizable.

Proof. Since T is diagonalizable, there exists an ordered basis β for V consisting of eigenvectors of T by Theorem 5.1. From (a), we know that all eigenvectors of T are eigenvectors of U. Thus, β is an ordered basis consisting of eigenvectors of U. Thus, U is diagonalizable. Without loss of generality, if U is diagonalizable then T is diagonalizable. Therefore, T is diagonalizable if and only if U is diagonalizable.

$\square$

§5 **E**

For each of the following matrices $A \in \mathbf{M}_{n \times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$

a. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Proof. We follow "Test for Diagonalization" and example 5 in section 5.2 of our textbook. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 2 \\ 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has a single eigenvalue of $\lambda_1 = 1$. Because

$$A - \lambda_1 I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

has rank 1, we see that $2 - \text{rank}(A - \lambda_1 I) = 1$ which is not the multiplicity of λ_1. Thus condition 2 fails and therefore A is not diagonalizable. $\square$

b. $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

Proof. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{pmatrix} = (1 - \lambda)(1 - \lambda) - 9 = (\lambda + 2)(\lambda - 4)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = 4$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A - \lambda_1 I)x = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} x = 0$$

$$\begin{cases} 3x_1 + 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)x = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} x = 0$$

$$\begin{cases} -3x_1 + 3x_2 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has its columns the vectors in a basis of

eigenvectors of A.

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$Q^{-1} = \frac{1}{(1 \cdot 1) - (1 \cdot -1)} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$D = Q^{-1}AQ$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$

as desired. $\square$

c. $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

Proof. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 4 \\ 3 & 2 - \lambda \end{pmatrix} = (1 - \lambda)(2 - \lambda) - 12 = (\lambda + 2)(\lambda - 5)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = 5$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A - \lambda_1 I)x = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} x = 0$$

$$\begin{cases} 3x_1 + 4x_2 = 0 \\ 3x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_2 = 3 \Rightarrow v_{\lambda_1} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Similiarly,

$$v_{\lambda_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has as its columns the vectors in a basis of eigenvectors of A.

$$Q = \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix}$$

$$D = Q^{-1}AQ$$

$$= \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

as desired. $\square$

d. $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda + 1)(\lambda - 3)^2$ which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 3$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. So we only need to test condition 2 for λ_2. It is clear that

$$A - \lambda_2 I = \begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix}$$

has rank 1, we see that $3 - \text{rank}(A - \lambda_1 I) = 2$ which is the multiplicity of λ_2. Thus, condition 2 holds and A is diagonalizable. The eigenvector of λ_1 is

$$\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

and the eigenvectors of λ_2 is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus,

$$Q = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

and

$$D = Q^{-1}AQ$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\square$

e. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda - 1)(\lambda^2 + 1)$ which does not split over $\mathbb{R}$, so condition 1 of the test for diagonalization is not satisfied. Therefore, A is not diagonalizable. $\square$

f. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda - 1)^2(\lambda - 3)$ which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = 3$. We test

condition 2 for λ_1. It is clear that

$$A - \lambda_1 I = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

has rank 2, we see that $3 - \text{rank}(A - \lambda_1 I) = 1$ which is not the multiplicity of λ_2. Thus, condition 2 fails and A not diagonalizable. $\qquad\square$

g. $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ The characteristic polynomial of A is $-(\lambda - 2)^2(\lambda - 4)$. which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = 2$ and $\lambda_2 = 4$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. So we only need to test condition 2 for λ_1. It is clear that

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

has rank 1, we see that $3 - \text{rank}(A - \lambda_1 I) = 2$ which is the multiplicity of λ_2. Thus, condition 2 holds and A is diagonalizable. The eigenvectors of λ_1 is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and the eigenvector of λ_2 is

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Thus,

$$Q = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$D = Q^{-1}AQ$$
$$= \begin{bmatrix} -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

§6 F

For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(R)$$

find an expression for A^n, where n is an arbitrary positive integer.

Proof. We claim that A is diagonalizable and we can find invertible matrices that will give us the expression A^n. Consider the characteristic polynomial of A

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{pmatrix} = (\lambda + 1)(\lambda - 5)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 5$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A - \lambda_1 I)x = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} x = 0$$

$$\begin{cases} 2x_1 + 4x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = -2x_2 \Rightarrow \text{fix } x_2 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)x = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} x = 0$$

$$\begin{cases} -4x_1 + 4x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has its columns the vectors in a basis of eigenvectors of A.

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$D = Q^{-1}AQ$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Rewriting $D = Q^{-1}AQ$ gets us $A = QDQ^{-1}$. Since A and D are both diagonal matrices, $A^n = QD^nQ^{-1}$. So,

$$A^n = QD^nQ^{-1}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}^n \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

as desired. $\square$

§7   G

Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.