Math 341: Homework 8

Daniel Ko

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§1 A

Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$

Proof.

a. λ is an eigenvalue of $T \Rightarrow \lambda$ is an eigenvalue of $[T]_\beta$

   By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Using Theorem 2.14,

$$\begin{aligned} T(v) &= \lambda v \\ [T(v)]_\beta &= [\lambda v]_\beta \\ [T]_\beta [v]_\beta &= \lambda [v]_\beta \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of $[T]_\beta$.

b. λ is an eigenvalue of $[T]_\beta \Rightarrow \lambda$ is an eigenvalue of T

   By definition, there exists a eigenvector $v \in V$ such that $[T]_\beta [v]_\beta = \lambda [v]_\beta$. Using Theorem 2.14,

$$\begin{aligned} [T]_\beta [v]_\beta &= \lambda [v]_\beta \\ [T(v)]_\beta &= [\lambda v]_\beta \\ T(v) &= \lambda v \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. $\square$

§2 B

a. Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   Proof.

   i. Linear operator T on a finite-dimensional vector space is invertible $\Rightarrow$ zero is not an eigenvalue of T.

By the corollary of Theorem 4.7, $\det(T) \neq 0$. Assume, for the sake of contradiction, suppose zero is an eigenvalue of T. It follows from Theorem 5.2 that

$$\begin{aligned} \det(T - \lambda I) &= 0 \\ \det(T - 0I) &= 0 \\ \det(T) &= 0 \end{aligned}$$

which is a contradiction. Thus, zero is not an eigenvalue of T.

ii. Zero is not an eigenvalue of $T \Rightarrow$ linear operator T on a finite-dimensional vector space is invertible.

By contrapositive, we will instead prove that if linear operator T on a finite-dimensional vector space is not invertible then zero is an eigenvalue of T. If T is not invertible then $\det(T) = 0$ by corollary of Theorem 4.7. It follows from Theorem 5.2 that

$$\det(T - \lambda I) = 0$$

It directly follows that zero is an eigenvalue of T.

Therefore, linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T. □

b. Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof.

i. A scalar λ is an eigenvalue of T $\Rightarrow \lambda^{-1}$ is an eigenvalue of T^{-1}.

By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Given that T is invertible and by definition eigenvalues are non zero,

$$\begin{aligned} T(v) &= \lambda v \\ T^{-1}(T(v)) &= T^{-1}(\lambda v) \\ v &= T^{-1}(\lambda v) \\ v &= \lambda T^{-1}(v) \\ \lambda^{-1} v &= T^{-1}(v) \end{aligned}$$

as desired. Thus, λ^{-1} is an eigenvalue of T^{-1}.

ii. λ^{-1} is an eigenvalue of $T^{-1} \Rightarrow$ a scalar λ is an eigenvalue of T.

By definition, there exists a eigenvector $v \in V$ such that $T^{-1}(v) = \lambda^{-1} v$. Given that T^{-1} is invertible linear operator,

$$\begin{aligned} T^{-1}(v) &= \lambda^{-1} v \\ T(T^{-1}(v)) &= T(\lambda^{-1} v) \\ v &= \lambda^{-1} T(v) \\ \lambda v &= T(v) \end{aligned}$$

as desired. Thus, λ is an eigenvalue of T.

Therefore, a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. □

c. State and prove results analogous to (a) and (b) for matrices.

(a) A matrix A is invertible if and only if zero is not an eigenvalue of A.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (a) that "matrix A is invertible if and only if zero is not an eigenvalue of A." is true. □

(b) Let A be an invertible matrix. λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (b) that "λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}" is true. □

§3 C

For any square matrix A, prove that A and A^t have the same characteristic polynomial, and hence the same eigenvalues.

Proof. Let $A \in M_{nxn}(F)$. The characteristic polynomial for A is $f(t) = \det(A - tI_n)$. Using Theorem 4.8, Theorem 2.12, and the trivial fact that the identity matrix is symmetric,

$$\begin{aligned} f(t) &= det(A - tI_n) \\ &= det((A - tI_n)^t) \\ &= det(A^t - (tI_n)^t)) \\ &= det(A^t - tI_n) \end{aligned}$$

which is exactly the characteristic polynomial for A^t. Therefore, A and A^t have the same characteristic polynomial, and hence the same eigenvalues. □

§4 D

Let T be a linear operator on a finite-dimensional vector space V, and let c be any scalar.

a. Determine the relationship between the eigenvalues and eigenvectors of T (if any) and the eigenvalues and eigenvectors of $U = T - cI$ (where I is the identity transformation) Justify your answers.

   Proof. Suppose $v \in V$ is an eigenvector of T where λ is its eigenvalue,

$$Tv = \lambda v$$

   Applying the transformation U to v,

$$\begin{aligned} Uv &= (T - cI)v \\ &= Tv - cIv \\ &= \lambda v - cv \\ &= (\lambda - c)v \end{aligned}$$

   Thus, if v is an eigenvector of T, then it is an eigenvector U with its corresponding eigenvalue being c. □

b. Prove that T is diagonalizable if and only if U is diagonalizable.

Proof. Since T is diagonalizable, there exists an ordered basis β for V consisting of eigenvectors of T by Theorem 5.1. From (a), we know that all eigenvectors of T are eigenvectors of U. Thus, β is an ordered basis consisting of eigenvectors of U. Thus, U is diagonalizable. Without loss of generality, if U is diagonalizable then T is diagonalizable. Therefore, T is diagonalizable if and only if U is diagonalizable.

$\square$

§5 E