Math 341: Homework 1

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§1 A

a. $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

b. $p \vee F \Leftrightarrow F$

p	F	$p \vee F$	$p \vee F \Leftrightarrow F$
T	F	T	T
F	F	F	T

c. $p \wedge \neg p \Leftrightarrow F$

p	$\neg p$	$p \wedge \neg p$	$p \wedge \neg p \Leftrightarrow F$
T	F	F	T
F	T	F	T

d. $(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$

p	q	$p \Leftrightarrow q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$
T	T	T	T	F	T	T
T	F	F	F	F	F	T
F	T	F	F	F	F	T
F	F	T	F	T	T	T

e. $[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$

p	q	r	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)$	$p \Leftrightarrow r$	$[(p \Leftrightarrow q) \wedge (q \Leftrightarrow r)] \Rightarrow (p \Leftrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	T	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

f. $[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$

p	q	$p \wedge \neg q$	$(p \wedge \neg q) \Rightarrow \neg p$	$p \Rightarrow q$	$[p \wedge \neg q \Rightarrow \neg p] \Rightarrow (p \Rightarrow q)$
T	T	F	T	T	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	F	T	T	T

§2 B

a. $(p \vee q \Leftrightarrow p \wedge r) \Rightarrow ((p \Rightarrow p) \wedge (p \Rightarrow r))$

$$
\begin{aligned}
&(p \vee q \Leftrightarrow p \wedge r) \Rightarrow (p \Rightarrow r) && \text{(Transitivity)}\\
&[(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \Rightarrow (p \Rightarrow r) && \text{(Def. of bicondtional)}\\
&\neg[(p \vee q \Rightarrow p \wedge r) \wedge (p \wedge r \Rightarrow p \vee q)] \vee (\neg p \vee r) && \text{(Material Implication)}\\
&\neg[(\neg(p \vee q) \vee (p \wedge r)) \wedge (\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(Material Implication)}\\
&[\neg(\neg(p \vee q) \vee (p \wedge r)) \vee \neg(\neg(p \vee r) \vee (p \vee q))] \vee (\neg p \vee r) && \text{(De Morgan's Law)}\\
&(\neg\neg(p \vee q) \wedge \neg(p \wedge r)) \vee (\neg\neg(p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(De Morgan's Law)}\\
&((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) \vee (\neg p \vee r) && \text{(Double negation)}\\
&(\neg p \vee r) \vee ((p \vee q) \wedge \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Commutative)}\\
&((\neg p \vee r) \vee (p \vee q)) \wedge ((\neg p \vee r) \vee \neg(p \wedge r)) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Distributive)}\\
&(True \wedge True) \vee ((p \vee r) \wedge \neg(p \vee q)) && \text{(Excluded middle)}\\
&True
\end{aligned}
$$

b. $[(p \Rightarrow \neg q) \wedge (r \Rightarrow q)] \Rightarrow (p \Rightarrow \neg r)$

$$
\begin{aligned}
\neg[(\neg p \vee \neg q) \wedge (\neg r \wedge q)] \vee \neg p \vee \neg r && \text{(Material Implication)}\\
\neg(\neg p \vee \neg q) \vee \neg(\neg r \wedge q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)}\\
(p \wedge q) \vee (r \wedge \neg q) \vee \neg p \vee \neg r && \text{(De Morgan's Law)}\\
\neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)}\\
\neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)}\\
True &&
\end{aligned}
$$

c. $(p \Rightarrow q) \Rightarrow [\neg(q \wedge r) \Rightarrow \neg(p \wedge r)]$

$$
\begin{aligned}
\neg(\neg p \vee q) \vee [\neg\neg(q \wedge r) \vee \neg(p \wedge r)] && \text{(De Morgan's Law)}\\
(p \wedge \neg q) \vee (q \wedge r) \vee (\neg p \vee \neg r) && \text{(De Morgan's Law + Negation)}\\
\neg p \vee (p \wedge \neg q) \vee \neg r \vee (q \wedge r) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (\neg r \vee r)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee \neg q)] \vee [(\neg r \vee q) \wedge (True)] && \text{(Excluded middle)}\\
\neg p \vee \neg q \vee \neg r \vee q && \text{(Excluded middle)}\\
True &&
\end{aligned}
$$

d. $[(p \Rightarrow \neg q) \wedge (\neg r \vee q) \wedge r] \Rightarrow \neg p$

$$
\begin{aligned}
\neg[(\neg p \vee \neg q) \wedge (\neg r \vee q) \wedge r] \vee \neg p && \text{(Material Implication)}\\
\neg(\neg p \vee \neg q) \vee \neg(\neg r \vee q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)}\\
(p \wedge q) \vee (r \wedge \neg q) \vee \neg r \vee \neg p && \text{(De Morgan's Law)}\\
\neg p \vee (p \wedge q) \vee \neg r \vee (r \wedge \neg q) && \text{(Commutative)}\\
[(\neg p \vee p) \wedge (\neg p \vee q)] \vee [(\neg r \vee r) \wedge (\neg r \vee \neg q)] && \text{(Distributive)}\\
[(True) \wedge (\neg p \vee q)] \vee [(True) \wedge (\neg r \vee \neg q)] && \text{(Excluded middle)}\\
\neg p \vee q \vee \neg r \vee \neg q && \text{(Excluded middle)}\\
True && \text{(Excluded middle)}
\end{aligned}
$$

§3 C

a. Proposition r means q is true if p(x) is true for one x.
Proposition s means q is true if p(x) is true for all x.

b.

$$\begin{aligned}
r &\Leftrightarrow (\forall x)(p(x) \Rightarrow q) \\
\neg r &\Leftrightarrow \neg[(\forall x)(p(x) \Rightarrow q)] \\
&\Leftrightarrow \exists x \neg(p(x) \Rightarrow q) && \text{(Quantifer Negation)} \\
&\Leftrightarrow \exists x \neg(\neg p(x) \vee q) && \text{(Material Implication)} \\
&\Leftrightarrow \exists x(\neg\neg p(x) \wedge \neg q) && \text{(De Morgan's Law)} \\
&\Leftrightarrow \exists x(p(x) \wedge \neg q) && \text{(Double Negation)}
\end{aligned}$$

$$\begin{aligned}
s &\Leftrightarrow ((\forall x)p(x)) \Rightarrow q \\
\neg s &\Leftrightarrow \neg[((\forall x)p(x)) \Rightarrow q] \\
&\Leftrightarrow \neg[\neg((\forall x)p(x)) \vee q] && \text{(Material Implication)} \\
&\Leftrightarrow ((\forall x)p(x)) \wedge \neg q && \text{(De Morgan's Law)}
\end{aligned}$$

c. s ⇒ r is a tautology. If s is true, then r would be true because s requires p(x) to be true for all x while r only requires p(x) for one x to be true. If s is false, then the whole statement would be vacuously true.

§4 D

Corollary

The additive inverse is unique.

Proof. Suppose u, v are the additive inverse of x.

$$x + u = 0 \quad x + v = 0$$

$$\begin{aligned}
x + u &= x + v && \text{(Transitive property)} \\
u + x &= v + x && \text{(Commutative property)} \\
u &= v && \text{(Theorem 1.1)}
\end{aligned}$$

□

Corollary

The vector 0 is unique.

Proof. Suppose $u, v \in V$ satisfies the "zero property", which is defined as:

$$\begin{aligned}
\forall x \in V \quad x + u = x &\Rightarrow v + u = v \\
\forall x \in V \quad x + v = x &\Rightarrow u + v = u \\
u = u + v = v + u &= v && \text{(Transitive property)} \\
u &= v && \text{(Theorem 1.1)}
\end{aligned}$$

□

§5 E

Theorem (1.2(c) In any vector space the following statements are true.)
$a\mathbf{0} = \mathbf{0}$
$\forall a \in F \quad \mathbf{0} \in \mathbf{V}$
Any scalar multiplied by the 0 vector will result in the 0 vector.

Proof.

$$\begin{aligned} a\mathbf{0} &= a(\mathbf{0}+\mathbf{0}) && \text{(Identity element of addition)} \\ a\mathbf{0} &= a\mathbf{0}+a\mathbf{0} && \text{(Distributive)} \\ a\mathbf{0}-a\mathbf{0} &= a\mathbf{0}+a\mathbf{0}-a\mathbf{0} && \text{(Inverse element of addition)} \\ \mathbf{0} &= a\mathbf{0} \end{aligned}$$

□

§6 F

Prove that diagonal matrices (as defined in your book in Example 3, Section 1.3) are symmetric.

Proof.

$$\begin{gathered} \text{Let } D \text{ equal a diagonal matrix} \\ D \text{ is symmetric} \Leftrightarrow (\forall i,j)(D_{i,j} = D_{j,i}) \\ \text{By definition of diagonal matrix:} \\ \text{When } i \neq j, D_{i,j} = D_{j,i} = 0 \\ \text{When } i = j, D_{i,j} = D_{j,i} \\ \therefore (\forall i,j)(D_{i,j} = D_{j,i}) \text{ so diagonal matrix is symmetric} \end{gathered}$$

□

§7 G

Prove that
$W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0\}$
is a subspace of F_n, but
$W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1\}$
is not.

Proof. Proof that W_1 is a subspace of F^n

a. $0 \in W_1$

$$\begin{aligned} \text{Let } a_1, a_2, \cdots, a_n &= 0 \\ 0 + 0 + \cdots + 0 &= 0 \\ \text{So the 0 vector: } (0, 0, \cdots, 0) &\in W_1 \end{aligned}$$

b. $X, Y \in W_1 \Rightarrow X + Y \in W_1$

$$\begin{aligned} X = (x_1, x_2, \cdots, x_n) \quad Y = (y_1, y_2, \cdots, y_n) \\ X + Y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ \sum_{i=1}^{n} x_i + y_i &= x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \\ &= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$\therefore X + Y \in W_1$$

c. $c \in F, X \in W_1 \Rightarrow cX \in W_1$

$$\begin{aligned} X &= (x_1, x_2, \cdots, x_n) \\ cX &= (cx_1, cx_2, \cdots, cx_n) \\ \sum_{i=1}^{n} cx_i &= cx_1 + cx_2 + \cdots + cx_n \\ &= c(x_1 + x_2 + \cdots + x_n) \\ &= c(0) \\ &= 0 \end{aligned}$$

$$\therefore cX \in W_1$$

□

Proof. $W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1\}$ is not a subspace of F^n
If W_2 is a subspace, there must be closure under vector addition.

$$\begin{aligned} X = (x_1, x_2, \cdots, x_n) \quad Y = (y_1, y_2, \cdots, y_n) \\ X + Y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ \sum_{i=1}^{n} x_i + y_i &= x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \\ &= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\ &= 1 + 1 \\ &= 2 \neq 1 \end{aligned}$$

$$X + Y \notin W_2$$
$$\therefore W_2 \text{ is not a subspace of } F^n.$$

Additionally, there is no 0 vector in W_2. 0 vector for polynomial space only exists if each component in a vector is 0. This is not possible in W_2 because the sum of the components must equal 1. □

§8 H

Let $P(F)$ be the set of all polynomials in F.
$W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ is not a subspace of $P(F)$ if $n \geq 1$

Proof. If W is a subspace, there must be closure under vector addition.
Let $f(x) = a_n x^n + a_{n-1}x^{n-1}$ and $g(x) = -a_n x^n + (a_{n-1}+1)x^{n-1}$
$f(x) + g(x) = (a_n - a_n)x^n + (a_{n-1} + a_{n-1} + 1)x^{n-1}$
$= (0)x^n + (a_{n-1} + a_{n-1} + 1)x^{n-1}$
$= (a_{n-1} + a_{n-1} + 1)x^{n-1}$
This polynomial is not the 0 vector nor has degree n. Therefore, W is not a subspace of $P(F)$ when $n \geq 1$ □

§9 I

Prove that $A^T + A$ is symmetric for any square matrix A

Proof. We first need to prove $A^{TT} = A$ and $(A+B)^T = A^T + B^T$

The definition of the transpose of a matrix is for any value of row i and column j in A, transpose of A will have that value in row j and column i. The transpose of the tranpose of A will have that value in row i and column j which is the same as the original matrix. $A_{i,j} = (A^T)_{j,i} = (A^{TT})_{i,j} = A_{i,j}$
Thus, the matrix A^{TT} is equivalent to A

The transpose of A + B for any value of at row i and column j is the sum of A + B at j,i. $((A+B)^T)_{i,j} = A_{j,i} + B_{j,i}$. By definition of transpose, $A_{i,j} = (A^T)_{j,i} \Leftrightarrow A_{j,i} = (A^T)_{i,j}$. So we can rewrite $((A+B)^T)_{i,j} = A_{j,i} + B_{j,i}$ as $((A+B)^T)_{i,j} = (A^T)_{i,j} + (B^T)_{i,j}$.
Thus, $(A+B)^T = A^T + B^T$

For a matrix to be symmetric, the matrix and its transpose must be the same.
$(A^T + A)^T = A^{TT} + A^T = A + A^T = A^T + A$
Thus, $A^T + A$ is symmetric for any square matrix A. □