

# Math 341: Homework 7

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## §1 A

For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution space.

a. 
$$\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 1$  because the two columns are a multiples of each other. If  $K$  is the solution set of this system, then  $\dim(K) = 2 - 1 = 1$ . Thus any nonzero solution constitutes a basis for  $K$ . For example, since  $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$  is a solution to the given system,  $\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$  is a basis for  $K$  by Corollary 2 of Theorem 1.10.  $\square$

b. 
$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 2$  because there are two linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If  $K$  is the solution set of this system, then  $\dim(K) = 3 - 2 = 1$ . Thus any nonzero solution constitutes a basis for  $K$ . For example, since  $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$  is a solution to the given system,  $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$  is a basis for  $K$  by Corollary 2 of Theorem 1.10.  $\square$

c. 
$$\{x_1 + 2x_2 - 3x_3 + x_4 = 0\}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 1$  because there are one linearly independent row (rank of a transposed matrix is the same as the rank of a matrix). If  $K$  is the solution set of this system,

then  $\dim(K) = 4 - 1 = 3$ . Note that,  $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$  are linearly independent vectors in

$K$ . Thus they form a basis by Corollary 2 of Theorem 1.10.  $\square$

d. 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 2$  because there are 2 linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If  $K$  is the solution set of this system, then

$\dim(K) = 4 - 2 = 2$ . Note that,  $\begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ -1 \end{pmatrix}$  are linearly independent vectors in  $K$ . Thus

they form a basis by Corollary 2 of Theorem 1.10.  $\square$

e. 
$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ 4x_1 + x_2 - 2x_3 = 0 \end{cases}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & -2 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 2$  because there are 2 linearly independent rows (rank of a transposed matrix is the same as the rank of a matrix). If  $K$  is the solution set of this system, then  $\dim(K) = 3 - 2 = 1$ . Thus any nonzero solution constitutes a basis for  $K$ . For example, since

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  is a solution for  $K$ , it forms a basis by Corollary 2 of Theorem 1.10.  $\square$

f. 
$$\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - 2x_3 = 0 \end{cases}$$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 2$  because there are 2 linearly independent columns (third column is second column multiplied by -1). If  $K$  is the solution set of this system, then  $\dim(K) = 3 -$

$2 = 1$ . Thus any nonzero solution constitutes a basis for  $K$ . For example, since  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  is a solution for  $K$ , it forms a basis by Corollary 2 of Theorem 1.10.  $\square$

g.  $\begin{cases} x_1 + 2x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$

*Proof.* Using Theorem 3.8 and following example 2 from the textbook, we define this system as an homogeneous system  $Ax = 0$ , where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

It is clear that  $\text{rank}(A) = 2$  because there are 2 linearly independent columns. If  $K$  is the solution set of this system, then  $\dim(K) = 2 - 2 = 0$ . This means the zero vector is the basis for  $K$ , i.e.  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .  $\square$

## §2 B

Using the results of Exercise 2, find all solutions to the following systems.

a.  $\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 + 6x_2 = 10 \end{cases}$

*Proof.* A solution to the above system is  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . Using Theorem 3.9 and following example 3 from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$\square$

b.  $\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + x_2 + x_3 = 6 \end{cases}$

*Proof.* A solution to the above system is  $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ . Using Theorem 3.9 and following example 3 from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$\square$

c.  $\{x_1 + 2x_2 - 3x_3 + x_4 = 1\}$

*Proof.* A solution to the above system is  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ . Using Theorem 3.9 and following example 3

from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

□

d. 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1 \\ x_2 - x_3 + x_4 = 1 \end{cases}$$

*Proof.* A solution to the above system is  $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ . Using Theorem 3.9 and following example 3

from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix} + t_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ -1 \end{pmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

□

e. 
$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 4x_1 + x_2 - 2x_3 = 3 \end{cases}$$

*Proof.* A solution to the above system is  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . Using Theorem 3.9 and following example 3

from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

f. 
$$\begin{cases} 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + x_3 = 1 \\ x_1 + 2x_2 - 2x_3 = 4 \end{cases}$$

*Proof.* A solution to the above system is  $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ . Using Theorem 3.9 and following example 3

from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

□

g. 
$$\begin{cases} x_1 + 2x_2 = 5 \\ x_1 - x_2 = -1 \end{cases}$$

*Proof.* A solution to the above system is  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Using Theorem 3.9 and following example 3 from the textbook, we know  $K = \{s\} + K_H = \{s + k : k \in K_H\}$ . So,

$$K = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

□

### §3 C

Prove that the system of linear equations  $Ax = b$  has a solution if and only if  $b \in R(L_A)$ .

*Proof.* Let  $A$  be an  $m \times n$  matrix.

- i.  $Ax = b$  has a solution  $\Rightarrow b \in R(L_A)$ .

Let  $s$  be a solution to  $Ax = b$ . This means  $A_1s_1 + \cdots + A_ns_n = b$ . Notice that  $b$  is a linear combination of the columns of  $A$ , which is equivalent to  $R(L_A)$  by the proof given in Theorem 3.5. Therefore,  $b \in R(L_A)$ .

- ii.  $b \in R(L_A) \Rightarrow Ax = b$  has a solution.

$b \in R(L_A)$  means that  $b$  is a linear combination of the columns of  $A$ . So,  $b = A_1s_1 + \cdots + A_ns_n$ . This means there exists an  $x$ , composed of the  $s_1, \dots, s_n$  as seen in the previous equation, such that  $Ax = b$ . Therefore,  $Ax = b$  has a solution.

□

### §4 D

Prove or give a counterexample to the following statement: If the comatrix of a system of  $m$  linear equations in  $n$  unknowns has rank  $m$ , then the system has a solution.

*Proof.* Let  $A$  be a  $m \times n$  comatrix for the system  $Ax = b$ . By definition,  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ . Another way of saying that the system has a solution is to say that  $b \in R(L_A)$ , as we proved in problem C. Given that the  $\text{rank}(A) = m$ , it follows that  $\dim(R(L_A)) = m$  by Theorem 3.5. Since the range and the codomain of  $L_A$  are the same, it means that  $L_A$  is onto by definition. Thus,  $\forall b \in R(L_A) [\exists s \in \mathbb{F}^n \text{ s.t. } As = b]$ . Therefore, if the comatrix of a system of  $m$  linear equations in  $n$  unknowns has rank  $m$ , then system has a solution. □

### §5 E

Suppose that the augmented matrix of a system  $Ax = b$  is transformed into a matrix  $(A'|b')$  in reduced row echelon form by a finite sequence of elementary row operations.

- a. Prove that  $\text{rank}(A') \neq \text{rank}(A'|b')$  if and only if  $(A'|b')$  contains a row in which the only nonzero entry lies in the last column.

*Proof.*

- i.  $\text{rank}(A') \neq \text{rank}(A'|b') \Rightarrow (A'|b')$  contains a row in which the only nonzero entry lies in the last column.

If  $\text{rank}(A') \neq \text{rank}(A'|b')$ , then we know that  $b'$  is a linearly independent vector to  $A'$ . This must mean that  $(A'|b')$  contains a row in which the only nonzero entry lies in the last column, which is  $b'$ .

- ii.  $(A'|b')$  contains a row in which the only nonzero entry lies in the last column  $\Rightarrow \text{rank}(A') \neq \text{rank}(A'|b')$

$(A'|b')$  contains a row in which the only nonzero entry lies in the last column means that  $b'$  is a linearly independent vector to  $A'$ . Then this means that  $\text{rank}(A') \neq \text{rank}(A'|b')$ .  $\square$

- b. Deduce that  $Ax = b$  is consistent if and only if  $(A'|b')$  contains no row in which the only nonzero entry lies in the last column.

*Proof.*

- i.  $Ax = b$  is consistent  $\Rightarrow (A'|b')$  contains no row in which the only nonzero entry lies in the last column.

In problem C, we proved that  $Ax = b$  is consistent means that  $b$  is a linear combination of the columns in  $A$ , i.e. linearly dependent. This also must mean that  $b'$  is a linearly dependent of the columns in  $A'$  because row reduction is a rank and row space preserving operation. Thus,  $(A'|b')$  contains no row in which the only nonzero entry lies in the last column because  $b'$  is a linearly dependent.

- ii.  $(A'|b')$  contains no row in which the only nonzero entry lies in the last column  $\Rightarrow Ax = b$  is consistent.

From part (a), we know that the above premise is equivalent to  $\text{rank}(A') \neq \text{rank}(A'|b')$ . By Theorem 3.11, we know that  $A'x = b'$  must be consistent.  $Ax = b$  must also be consistent because row reduced echelon form preserves rank and row space.  $\square$

## §6 F

Let the reduced row echelon form of  $A$  be

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Determine  $A$  if the first, second, and fourth columns of  $A$  are

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

*Proof.* Notice that  $r_{j_3} = 2r_{j_1} - 5r_{j_2}$  and  $r_{j_5} = -2r_{j_1} - 3r_{j_2} + 6r_{j_4}$  where  $r_{j_i}$  is the  $i$ th column of  $R$ . Using Theorem 3.16(c) and (d) we can compute  $A$ .

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{bmatrix}$$

□

## §7 G

Let the reduced row echelon form of  $A$  be

$$R = \begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine  $A$  if the first, third, and sixth columns of  $A$  are

$$\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix}$$

*Proof.* Notice that  $r_{j_2} = -3r_{j_1}$ ,  $r_{j_4} = 4r_{j_1} + 3r_{j_3}$ , and  $r_{j_5} = 5r_{j_1} + 2r_{j_3} - r_{j_6}$  where  $r_{j_i}$  is the  $i$ th column of  $R$ . Using Theorem 3.16(c) and (d) we can compute  $A$ .

$$A = \begin{bmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{bmatrix}$$

□

## §8 H

Let  $W$  be the subspace of  $M_{2 \times 2}(R)$  consisting of the symmetric  $2 \times 2$  matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates  $W$ . Find a subset of  $S$  that is a basis for  $W$ .

*Proof.* We put  $S$  in an augmented matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix}$$

Then we compute reduced row echelon form of this matrix.

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the first, second, and fourth columns of  $R$  is linearly independent. Then, the coore-

sponding columns in  $A$  would be the linearly independent subset by Theorem 3.16.

$$S' = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}$$

forms a basis for  $W$ . □

## §9 I

Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique.

### Lemma 9.1

The pivot columns of matrix  $A$ 's reduced row echelon form is unique

*Proof.* We proceed by induction on  $n \in \mathbb{N}$ . Let  $P(n)$  be the following predicate "pivot columns of matrix  $A$ 's reduced row echelon form is unique, where matrix  $A$  has  $n$  columns".

a. Base case:

We prove  $P(1)$ . There is only one column so it must be unique.

b. Inductive step:

Suppose  $P(k)$  holds. We show that  $P(k+1)$  holds.

Let  $A = [A' | u_{k+1}]$  where  $A'$  is a matrix which has  $k$  columns and has unique pivot columns in its reduced row echelon form. Consider two cases: where  $u_{k+1}$  is linearly dependent to  $A'$  and where it is linearly independent to  $A'$ .

Case 1:  $u_{k+1}$  is linearly dependent to  $A'$

If  $u_{k+1}$  is linearly dependent it means that  $\text{rank}(A) = \text{rank}(A')$ . So the reduced row echelon form of  $A$  is  $[R' | b]$  where  $R'$  are the pivot columns from  $A'$ . That must mean that  $A$  has a unique amount of pivot columns because  $A'$  has a unique amount of pivot columns by our induction hypothesis.

Case 2:  $u_{k+1}$  is linearly independent to  $A'$

If  $u_{k+1}$  is linearly independent it means that  $\text{rank}(A) = \text{rank}(A') + 1 = r$ , where  $r$  is just some fixed integer. This must mean that  $A$  has one more pivot column than  $A'$ . The reduced row echelon form of  $A$  is  $[R' | b]$ , and  $b$  must be  $e_r$  by Theorem 3.16(b) and the fact that it is the right most column. That must mean that  $A$  has a unique amount of pivot columns.

Therefore, by induction we have proved this lemma. □

Let's assume for the sake of contradiction that matrix  $A$  has two unique reduced row echelon forms,  $B$  and  $B'$ . Theorem 3.16(d) states that "For each  $k = 1, 2, \dots, n$ , if column  $k$  of  $B$  is  $d_1 e_1 + d_2 e_2 + \dots + d_r e_r$ , then column  $k$  of  $A$  is  $d_1 a_{j_1} + d_2 a_{j_2} + \dots + d_r a_{j_r}$ . Let's the  $k$ th column of  $B$  be  $d_1 e_1 + d_2 e_2 + \dots + d_r e_r$  and the  $k$ th column of  $B'$  be  $d'_1 e_1 + d'_2 e_2 + \dots + d'_r e_r$  where  $k$ th column of  $B$  does not equal the  $k$ th column of  $B'$  because they are both unique. We know that the  $e_1, e_2, \dots, e_r$  are unique by our lemma. However, this would imply that the  $k$ th column of the original matrix  $A$  are different depending if you compute using  $B$  or  $B'$ . Thus, this is a contradiction and  $A$  must have a unique reduced row echelon form. □