Math 341: Midterm 2

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§1

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1}$$

a. Suppose that $a \neq 0$, compute the solution of $\mathbf{Ax} = \mathbf{b}$ using row reduction and provide the conditions on a, b, c, d such that your computations are valid. Express the result as a simplified expression. (**Hint:** recall that you can not divide by zero)

Proof. We perform reduced row echelon form (rref) on the augmented matrix

$$\begin{aligned}
(A|b) &= \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \\
R_2 \leftarrow R_2 - \frac{c}{a}R_1 & \left[\begin{array}{cc|c} a & b & e \\ 0 & d - \frac{cb}{a} & f - \frac{ce}{a} \end{array}\right] \\
& \left[\begin{array}{cc|c} a & b & e \\ 0 & \frac{ad-cb}{a} & \frac{af-ce}{a} \end{array}\right] \\
R_2 \leftarrow \frac{a}{ad-cb}R_2 \quad \text{Assuming that } ab - cd \neq 0 & \left[\begin{array}{cc|c} a & b & e \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow R_1 - bR_2 & \left[\begin{array}{cc|c} a & 0 & e - b\frac{af-ce}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
R_1 \leftarrow \frac{R_1}{a} & \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}(e - b\frac{af-ce}{ad-cb}) \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right] \\
& \left[\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-cb} \\ 0 & 1 & \frac{af-ce}{ad-cb} \end{array}\right]
\end{aligned}$$

$$x = \begin{bmatrix} \frac{de-bf}{ad-cb} \\ \frac{af-ce}{ad-cb} \end{bmatrix} \qquad \text{where } ad - cb \neq 0$$

□

b. If $a = 0$, and $c \neq 0$, is your above computation still valid? How would you modify it? (explain briefly) (**Hint:** recall that you can swap the equations and the result is the same)

Proof. If $a = 0$, and $c \neq 0$, then the above computation will not be valid as we divided by a multiple times when we computed the rref. I would swap the first and second rows so that it would look like

$$\left[\begin{array}{cc|c} c & d & f \\ 0 & b & e \end{array}\right]$$

and compute the rref, assuming that $b \neq 0$. We obtain the rref,

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{bf-de}{bc} \\ 0 & 1 & \frac{e}{b} \end{array}\right]$$

$\square$

c. If $a = 0$, $c = 0$, but $b \neq 0$, $d \neq 0$, what are the conditions on e and f such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution? Is the solution unique? (**Hint:** recall that $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$)

Proof. If $a = 0$, $c = 0$, $b \neq 0$, $d \neq 0$, we get the augmented matrix

$$\left[\begin{array}{cc|c} 0 & b & e \\ 0 & d & f \end{array}\right]$$

Performing row reduction,

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{e}{b} \\ 0 & 1 & \frac{f}{d} \end{array}\right]$$

So the condition of the solution is,

$$x_2 = \frac{e}{b} = \frac{f}{d}$$

Thus, there exists a infinite amount of solution. $\square$

d. Solve the system

$$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}. \tag{2}$$

(**Hint:** You may want to use the formula you just deduced)

Proof.

$$\begin{aligned} x_1 &= \frac{de - bf}{ad - cb} \\ &= \frac{\sqrt{2}(5\sqrt{2}) - 3\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 30}{2 - 12} \\ &= \frac{-20}{-10} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\sqrt{2}(5\sqrt{2}) - 2\sqrt{2}(5\sqrt{2})}{\sqrt{2}\sqrt{2} - 3\sqrt{2}(2\sqrt{2})} \\ &= \frac{10 - 20}{-10} \\ &= 1 \end{aligned}$$

$\square$

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