Math 341: Homework 4

Daniel Ko

Spring 2020

§1 A

Let V be a vector space having dimension n, and let S be a subset of V that generates V.

a. Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite).

   Proof.
   Since V is finite dimensional, there exists a basis for V.
   $B = \{v_1, v_2, \cdots, v_n\}$
   Any $v \in B$ can be expressed as a linear combination of S because span(S) = V.
   Let the subset of S that generates v_i be S_i
   $v_i = \sum_{j=1}^{m^k} a_j^k s_j^k$ where $a \in F$ and $s \in S_i$
   The span of the union of the sets that generates v, span$(\bigcup_{i=1}^{n} S_i) = V$
   Corollary 2(a) of Theorem 1.10 states that a generating set for V that contains exactly n vectors is a basis for V. The set above, which is a subset of S, contains exactly n vectors and generates V. Therefore, there is subset of S that is a basis for V. □

b. Prove that S contains at least n vectors.

   Proof.
   From (a) we know there is a subset of S that forms a basis. Since that subset contains n vectors, S must contain n or more vectors. □

§2 B

Let $f(x)$ be a polynomial of degree n in $P_n(R)$. Prove that for any $g(x) \in P_n(R)$ there exists scalars $c_0, c_1, \cdots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

Proof.
Let $B = \{f, f', f'', \cdots, f^{(n)}\}$.
If B forms a basis we can express any $g(x) \in P_n(R)$ in the format above (a linear combination).
We can determine if B is basis by seeing if it is linearly independent by using a matrix.
$\mu_0 f + \mu_1 f' + \mu_2 f'' + \cdots + \mu_n f^{(n)} = 0$

$$\begin{bmatrix} a_n & a_{n-1} & \cdots & \cdots & a_0 \\ & na_n & \cdots & \cdots & a_1 \\ & \ddots & \cdots & \cdots & \vdots \\ & & \ddots & \cdots & \vdots \\ & & & n!a_n & (n-1)!a_{n-1} \\ & & & & n!a_n \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \vdots \\ \mu_{n-1} \\ \mu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Solving this system of equations:
Looking at the bottom row, $n!a_n\mu_n = 0$
$\mu_n = \frac{0}{n!a_n} = 0$, a_n is non zero because f is a nth degree polynomial and a_n is its coefficient.
Looking at row n - 1,$n!a_n\mu_{n-1} + (n-1)!a_{n-1}\mu n = 0$
Because $\mu_n = 0$, $n!a_n\mu_{n-1} + 0 = 0$
$\mu_{n-1} = \frac{0}{n!a_n} = 0$
By back substitution, $\mu_n = \mu_{n-1} = \cdots = \mu_1 = \mu_0 = 0$
This means that B is linearly independent, which also means that B is a basis.
Therefore, any $g(x) \in P_n(R)$ can be a linear combination of B with the scalars $c_0, c_1, \cdots, c_n$ □

§3 C

a. Prove that if W 1 and W 2 are finite-dimensional subspaces of a vector space V,then the subspace W 1 +W 2 is finite-dimensional, and dim(W 1 + W 2 = di)m(W 1)+dimW 2)(dimW 1 W 2 (.))

Proof.
W_1 and W_2 are finite dimensional subspaces of $V \Rightarrow$ subspace $W_1 + W_2$ is finite dimensional and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
Let $B_{1\cap 2}$ be a basis for $W_1 \cap W_2$
$B_{1\cap 2} = \{u_1, u_2, \cdots, u_k\}$
By using the replacement theorem, we can extend $B_{1\cap 2}$ to be a basis for W_1
So the basis for W_1 is B_1
$B_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$
Likewise, we can extend $B_{1\cap 2}$ to be a basis for W_2
$B_2 = \{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_p\}$
Basis for $W_1 + W_2$ will be $B_1 \cup B_2$, however they may contain the same vectors twice.
To prevent double counting, we must subtract $B_1 \cap B_2$ from $B_1 \cup B_2$
Thus the basis for $W_1 + W_2$ is
$B_{1+2} = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$
$W_1 + W_2$ is finite dimensional because its basis contains only a finite amount of vectors.
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
$k + m + p = k + m + k + p - k$
$k + m + p = k + m + p$ □

b. Let W 1 and W 2 be finite-dimensional subspaces of a vector space V, and let V = W 1 + W 2 . Deduce that V is the direct sum of W 1 and W 2 if and only if dim(V = dim)W 1)(+ dimW 2 ())
$V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$

Proof.
$V = W_1 \oplus W_2 \Rightarrow dim(V) = dim(W_1) + dim(W_2)$
From the definition of direct sum, $W_1 \cap W_2 = \{0\}$
This means $dim(W_1 \cap W_2) = 0$

From (a), we proved that $dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$
$= dim(W_1) + dim(W_2) - 0$
$= dim(W_1) + dim(W_2)$

$dim(V) = dim(W_1) + dim(W_2) \Rightarrow V = W_1 \oplus W_2$
$V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$
$V = W_1 + W_2$ is true by the definition of the problem.
From part (a), $dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$
Our antecedent is $dim(V) = dim(W_1) + dim(W_2)$
Setting the two equations equal to each other:
$dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = dim(W_1) + dim(W_2)$
$dim(W_1 \cap W_2) = 0$
This means $W_1 \cap W_2 = \{0\}$
Thus, $dim(V) = dim(W_1) + dim(W_2)$

Therefore, $V = W_1 \oplus W_2 \Leftrightarrow dim(V) = dim(W_1) + dim(W_2)$ □

§4 D

a. Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$

Proof.
Since W_1 is a subspace, let B_1 be its basis.

$$B_1 = \{u_1, u_2, \cdots, u_k\}$$

Using the replacement theorem, we can extend B_1 to be a basis for V. Let this basis for V be B_v.

$$B_v = \{u_1, u_2, \cdots, u_k, u_{k+1}, \cdots, u_n\}$$

Let the set of the vectors we added to B_1 to create B_v be called B_2.

$$B_2 = \{u_{k+1}, \cdots, u_n\}$$

Let W_2 be the subspace where its span is B_2.
To prove $W_1 \oplus W_2 = V$ we need to show that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$.

i. $W_1 \cap W_2 = \{0\}$

Let $v \in W_1$ and $v \in W_2$

$$v = a_1u_1 + a_2u_2 + \cdots a_ku_k = b_{k+1}u_{k+1}b_{k+2}u_{k+2} + \cdots + b_nu_n \quad \text{where } a, b \in F$$

$$v = a_1u_1 + a_2u_2 + \cdots a_ku_k - (b_{k+1}u_{k+1}b_{k+2}u_{k+2} + \cdots + b_nu_n) = 0$$

Notice that v is written as a linear combination of B_v, which means that all the constants equal 0: $a_1 = a_2 = \cdots = a_k = b_{k+1} = b_{k+2} + \cdots = b_n = 0$
So, $v = 0$.
Therefore, $W_1 \cap W_2 = \{0\}$

ii. $W_1 + W_2 = V$

Let $v \in V$

$$v = \sum_{i=1}^{n} a_i u_i \quad \text{where } a \in F$$

v can also be expressed as the sum of B_1 and B_2

$$v = \sum_{i=1}^{k} a_i u_i + \sum_{k+1}^{n} a_i u_i$$

Thus, any vector in V and be expressed as a sum of vectors in W_1 and W_2.
Therefore, $W_1 + W_2 = V$

Therefore, $W_1 \oplus W_2 = V$ □

b. Let $V = R^2$ and $W_1 = \{(a_1, 0) : a_1 \in R\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$

Proof.
Let $W_2 = \{(0, a_2) : a_2 \in R\}$

i. $W_1 \cap W_2 = \{0\}$

Let $v \in W_1$ and $v \in W_2$

$$v = (a_1, 0) = (0, a_2)$$

$$a_1 = 0, a_2 = 0$$

So, $v = (0, 0)$.
Therefore, $W_1 \cap W_2 = \{0\}$

ii. $W_1 + W_2 = V$

Let $v \in V$

$$v = (u_1, u_2)$$

v can also be expressed as the sum of vectors in W_1 and W_2
Let $x = (a_1, 0) \in W_1$ and $y = (0, a_2) \in W_2$.

$$v = c_1 x + c_2 y = (c_1 a_1, c_2 a_2), \text{ where } c \in F \text{ and } c_1 = \frac{v_1}{a_1}, c_2 = \frac{v_2}{a_2}$$

Thus, any vector in V and be expressed as a sum of vectors in W_1 and W_2.
Therefore, $W_1 + W_2 = V$
Therefore, $V = W_1 \oplus W_2$

Let $W_2' = \{(d, d) : d \in R\}$

i. $W_1 \cap W_2' = \{0\}$

Let $v \in W_1$ and $v \in W_2'$

$$v = (a_1, 0) = (d, d)$$

$$a_1 = d, d = 0$$

So, $v = (0, 0)$.
Therefore, $W_1 \cap W_2' = \{0\}$

ii. $W_1 + W_2' = V$

Let $v \in V$

$$v = (u_1, u_2)$$

v can also be expressed as the sum of vectors in W_1 and W_2'
Let $x = (u_1 - u_2, 0) \in W_1$ and $y = (u_2, u_2) \in W_2'$.

$$x + y = (u_1 - u_2 + u_2, 0 + u_2) = (u_1, u_2)$$

Thus, any vector in V and be expressed as a sum of vectors in W_1 and W_2'.
Therefore, $W_1 + W_2' = V$
Therefore, $V = W_1 \oplus W_2'$ □

§5 E

Let V be the vector space of sequences. Define the functions T,U : V → V by T(a 1 ,a 2 ,.. . = ()a 2 ,a 3 ,.. .) and U(a 1 ,a 2 ,.. .) = (0,a 1 ,a 2 ,.. . T and U are called the left shift and right shift operators o)n V respectively.

a. Prove that T and U are linear.

Proof.
T is linear if and only if $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$
Let $x, y \in V \quad c \in F$
$x = (x_1, x_2, \cdots) \quad y = (y_1, y_2, \cdots)$
$x + y = (x_1 + y_1, x_2 + y_2, \cdots)$
$T(x + y) = (x_2 + y_2, x_3 + y_3, \cdots)$
$T(x) = (x_2, x_3, \cdots)$
$T(y) = (y_2, y_3, \cdots)$
$T(x) + T(y) = (x_2 + y_2, x_3 + y_3, \cdots)$
Thus, $T(x + y) = T(x) + T(y)$

$x = (x_1, x_2, \cdots)$
$cx = (cx_1, cx_2, \cdots)$
$T(cx) = (cx_2, cx_3, \cdots)$
$T(x) = (x_2, x_3, \cdots)$
$cT(x) = (cx_2, cx_3, \cdots)$
Thus, $T(cx) = cT(x)$

Therefore, T is linear. The proof for U being linear is similiar. □

b. T is onto but not one to one

Proof.
T is onto if $\forall y \in V \ \exists x \in V$ such that $f(x) = y$
Let $y = (a_1, a_2, \cdots)$ be arbitrary
$f(x) = y = (a_1, a_2, \cdots)$
$x = (a_0, a_1, a_2, \cdots)$
Since y was chosen arbitrarily, there exists an x for any y.
Therefore, T is onto.

T is one to one if $\forall a, b \in V, T(a) = T(b) \Rightarrow a = b$
Let $a = (u_\alpha, u_2, u_3, \cdots) \quad b = (u_\gamma, u_2, u_3, \cdots)$ where $u_\alpha \neq u_\gamma \Leftrightarrow a \neq b$
$T(a) = (u_2, u_3, \cdots) \quad T(b) = (u_2, u_3, \cdots)$
Therefore, T is not one to one because $T(a) = T(b)$ and $a \neq b$. □

c. U is one to one but not onto.

Proof.
U is one to one if $\forall a, b \in V, U(a) = U(b) \Rightarrow a = b$
Let $U(a) = (0, u_1, u_2, \cdots) = U(b) = (0, v_1, v_2, \cdots)$
This means $u_1 = v_1, u_2 = v_2, \cdots$
So, $a = (u_1, u_2, u_3, \cdots) \quad b = (v_1, v_2, v_3, \cdots)$
Hence, $a = b$
Therefore, U is one to one because $U(a) = U(b)$ and $a = b$.

U is onto if $\forall y \in V\ \exists x \in V$ such that $f(x) = y$
Let $y = (a_1, a_2, \cdots)$ where $a_1 \neq 0$
There is no x such that $U(x) = y$ because the linear transformation always makes the first term always zero. Therefore, T is not onto. □

§6 F

Let S be the subspace of $M_{nxn}(R)$ generated by all matrices of the form $AB - BA$ with A and B in $M_{nxn}(R)$. Prove that $\dim(S) = n^2 - 1$. (You may want to use the trace together with the rank-nullity theorem)

Proof.
Trace is a linear transformation.
$\text{Tr} : M_{nxn}(R) \to R$
The subspace S is defined as $\{AB - BA : A, B \in M_{nxn}(R)\}$
$\text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA)$
$= \text{Tr}(AB) - \text{Tr}(AB)$
$= 0$
All matrices that can be expressed as $AB - BA$ is in the null space of Tr. This means that N(Tr)$= S$.
The rank-nullity theorem states:
dim(N(Tr)) + dim(R(Tr)) = dim($M_{nxn}(R)$)
N(Tr)$= S$, so dim(S) + dim(R(Tr)) = dim($M_{nxn}(R)$)
dim(S) = dim($M_{nxn}(R)$) - dim(R(Tr))
= $n^2 -$ dim(R)
= $n^2 - 1$

□

§7 G

Let T be a linear transformation of a vector space V into itself. Suppose that $x \in V$ is such that $T^m(x) = 0$, and $T^{m-1}(x) \neq 0$ for some positive m. Show that $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent.

Proof.
The linear combination of the above set is

$$a_0 x + a_1 T(x) + a_2 T^2(x) + \cdots + a_{n-1} T^{m-1}(x) = 0$$

Notice that $T^n(x) = 0$ for all $n \geq m$.
$T^{m+1}(x) = T(T^m(x)) = T(0) = 0$

Let's take T^{m-1} on both sides of the linear combination.

$$\begin{aligned} T^{m-1}(a_0x + a_1T(x) + a_2T^2(x) + \cdots + a_{n-1}T^{m-1}(x)) &= T^{m-1}(0) \\ T^{m-1}(a_0x) + T^{m-1}(a_1T(x)) + T^{m-1}(a_2T^2(x)) + \cdots + T^{m-1}(a_{n-1}T^{m-1}(x)) &= 0 \\ T^{m-1}(a_0x) + 0 + 0 + \cdots + 0 &= 0 \\ T^{m-1}(a_0x) &= 0 \\ a_0 = \frac{0}{T^{m-1}(x)} &= 0 \end{aligned}$$

By back substitution we know that $a_0 = a_1 = \cdots = a_{n-1} = 0$
Therefore, $x, T(x), T^2(x), \cdots, T^{m-1}(x)$ are linearly independent. □

§8 H

Let $T : R^3 \to R^3$

a. If T(a,b,c) = (a,b,0), show that T is the projection on the xy-plane along the z-axis.

Proof.
We want to projection to be on the xy-plane along the z-axis. Let the projection be (x,y,0).
To minimize the distance, we must choose x and y such that

$$(a-x)^2 + (b-y)^2 + (c-0)^2$$

is minimum. Since the equation above is a difference of squares, x = a and b = y will give us the minimum value. Therefore, the projection on the xy-plane will be (a,b,0), which is T. □

b. Find a formula for T(a,b,c), where T represents the projection on the z-axis along the xy-plane.

Proof.
We want to projection to be on the z-axis along the xy-plane. Let the projection be (0,0,z).
To minimize the distance, we must choose z such that

$$(a-0)^2 + (b-0)^2 + (c-z)^2$$

is minimum. z = c will give us the minimum value. Therefore, the equation for T will be T(a,b,c)=(0,0,c). □

c. If T(a,b,c) = (a-c,b,0), show that T is the projection on the xy-plane along the line L = $\{(a, 0, a) : a \in R\}$

Proof.
We want to projection to be on the xy-plane along the line L. Let the projection be $(x, y, 0)$.
A vector that is on L is $(1, 0, 1)$. To minimize the distance, we must choose λ such that

$$(a, b, c) + \lambda(1, 0, 1) = (x, y, 0)$$

is minimum. Writing the equation above as a system:

$$\begin{aligned} a + \lambda &= x \\ b &= y \\ c + \lambda &= 0 \end{aligned}$$

Solving this system gives us, $x = a - c, y = b$
Therefore, the projection on the xy-plane along the line L will be $(a - c, b, 0)$.

□

§9 I

Suppose that the linear transformation $T : V \to V$ is the projection on $W \subset V$ along some subspace $W' \subset V$. Prove that W is T-invariant and that $T_W = I_W$

Proof.

a. W is T-invariant.

   We need to show that $T(x) \in W$ for every $x \in W$.

   $$x = x_1 + x_2, T(x) = T(x_1 + x_2) = x_1 \text{ where } x_1 \in W, x_2 \in W'$$

   It is trivial to see that T restricts to the identity on W.
   Therefore, W is T-invariant.

b. $T_W = I_W$

   For any $w \in W$, we can express w as $w = w + 0$ where $0 \in W'$ because W' is a subspace. Because W_1 and W_2 are a direct sum, there is no way to express w as the sum of a vector in one and a vector in the other.
   $T(w + 0) = w$, shows that T_w is the same as the indentity tranformation I_W.
   Therefore, $T_W = I_W$.

□

§10 J

Let V be a finite-dimensional vector space and $T : V \to V$ be linear.

a. Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$

   Proof.
   Recall the properties of dimensions we proved in problem C.

   $$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$$

   $\dim(R(T))$ and $\dim(N(T))$ must be finite because $\dim(V)$ is finite. Because we are supposing that $V = R(T) + N(T)$ we can rewrite the equation above as

   $$\begin{aligned} \dim(V) &= \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) \\ \dim((R(T) \cap N(T)) &= \dim(R(T)) + \dim(N(T)) - \dim(V) \\ \dim((R(T) \cap N(T)) &= 0 \end{aligned}$$

$\dim(R(T)) + \dim(N(T)) - \dim(V)$ is equal to zero because of the rank nullity theorem and that V is finite dimensional. $\dim((R(T) \cap N(T)) = 0$ means that $R(T) \cap N(T) = \{0\}$ Therefore, $V = R(T) \oplus N(T)$ because $R(T) \cap N(T) = \{0\}$ and $V = R(T) + N(T)$. $\square$

b. Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

Proof.
$\dim(R(T))$ and $\dim(N(T))$ must be finite because $\dim(V)$ is finite.

$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$$
$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(\{0\})$$
$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - 0$$
$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T))$$

Given that V is finite dimensional and using the rank nullity theorem:

$$\dim(V) = \dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T))$$

This means that $V = R(T) + N(T)$.
Therefore, $V = R(T) \oplus N(T)$ because $V = R(T) + N(T)$ and $R(T) \cap N(T) = \{0\}$. $\square$