Math 341: Homework 9

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§1 A

Let W be a finite-dimensional subspace of an inner product space V

a. Prove that $V = W \oplus W^{\perp}$

   Proof.

   i. $W \cap W^{\perp} = \{0\}$

      Let $v \in W \cap W^{\perp}$. This means that $\langle v, v \rangle = 0$. This implies that $v = 0$ by Theorem 6.1.

   ii. $W + W^{\perp} = V$

      Theorem 6.6. states that for any $v \in V$, there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that $v = u + z$.

   Therefore, $V = W \oplus W^{\perp}$. □

b. Prove that there exists a projection T on W along $W^{\perp}$ that satisfies $N(T) = W^{\perp}$.

   Proof. From (a), we know that $V = W \oplus W^{\perp}$, so this means that there is a projection T on W along $W^{\perp}$ such that whenever $x = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in W^{\perp}$, $T(x) = x_1$.

   Let $n \in N(T)$. We can express n as $n = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in W^{\perp}$ such that $T(n) = 0$. This implies that $x_1 = 0$ so $n = x_2 \in W^{\perp}$. Thus, $N(T) \subseteq W^{\perp}$.

   Let $n \in W^{\perp}$. We can express n as $n = x_1 + x_2 = 0 + x_2$ where $x_1 \in W$ and $x_2 \in W^{\perp}$. It is clear that $T(n) = 0$, so $n \in N(T)$. Thus, $W^{\perp} \subseteq N(T)$. □

c. Prove that $||T(x)|| \leq ||x||$ for all $x \in V$.

   Proof. Let $x \in V$. We can express x as $x = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in W^{\perp}$.

   Notice that $||T(x)|| = ||x_1||$. Additionally, $||x|| = ||x_1 + x_2|| \leq ||x_1|| + ||x_2||$ by Theorem 6.2. Combining these two equations together, we get $||T(x)|| \leq ||x||$, as desired. □

§2 B

Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. Prove the following results

a. There is a unique adjoint T^* of T, and T^* is linear.

   Proof. Suppose we have two unique adjoints, T^* and U^*. Then, $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$ and $\langle T(x), y \rangle_2 = \langle x, U^*(y) \rangle_1$ Thus, $\langle x, T^*(y) \rangle_1 = \langle x, U^*(y) \rangle_1$ and it follows that $T^* = U^*$ by Theorem 6.1. Hence, the adjoint must be unique.

We prove that T^* is linear. Let $y_1, y_2 \in W$, $x \in V$, and $c \in F$. Then,

$$\begin{aligned}\langle x, T^*(cy_1 + y_2)\rangle_1 &= \langle T(x), cy_1 + y_2\rangle_2\\ &= \langle T(x), cy_1\rangle_2 + \langle T(x), y_2\rangle_2\\ &= c\langle T(x), y_1\rangle_2 + \langle T(x), y_2\rangle_2\\ &= c\langle x, T^*(y_1)\rangle_1 + \langle x, T^*(y_2)\rangle_1\end{aligned}$$

as desired. □

b. If β and γ are orthonormal bases for V and W, respectively, then $[\mathbf{T}^*]_\gamma^\beta = \left([\mathbf{T}]_\beta^\gamma\right)^*$

Proof. Let $\beta = \{v_1, \cdots v_n\}$, $\gamma = \{w_1, \cdots, w_m\}$, $A = [T^*]_\gamma^\beta$, $B = [T]_\gamma^\beta$.
Since $T^*(w_j) \in V = span(\beta) \Rightarrow T^*(w_j) = \sum_{i=1}^n \langle T^*(w_j), v_i\rangle_1 v_i$ by Corollary 1 of Theorem 6.3. Then using the corollary of Theorem 6.5, this implies that $A_{ij} = ([T^*]_\gamma^\beta)_{ij} = \langle T^*(w_j), v_i\rangle_1$
Similiarly, $T(v_j) \in W = span(\gamma) \Rightarrow T(v_j) = \sum_{i=1}^n \langle T(v_j), w_i\rangle_2 w_i$ and $B_{ij} = \langle T(v_j), w_i\rangle_2$. It follows that

$$\begin{aligned}(B_{ij})^* &= \overline{B_{ji}}\\ &= \overline{\langle T(v_i), w_j\rangle_2}\\ &= \langle w_j, T(v_i)\rangle_2\\ &= \langle T^*(w_j), v_i\rangle_1\\ &= A_{ij}\end{aligned}$$

Therfore, $[\mathbf{T}^*]_\gamma^\beta = \left([\mathbf{T}]_\beta^\gamma\right)^*$. □

c. rank(T^*) = rank(T)

Proof. We know that

$$dim(V) = dim(N(T)) + dim(R(T))$$

by the rank nullity theorem. By Theorem 6.7, we know that since $N(T)$ is a subspace of V,

$$dim(V) = dim(N(T)) + dim(N(T)^\perp)$$

Hence,

$$dim(R(T)) = dim(N(T)^\perp)$$

We claim that $N(T) = R(T^*)^\perp$. Suppose $x \in N(T)$. Let $y \in V$. Then,

$$T(x) = 0 = \langle 0, y\rangle = \langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

By definition, $x \in N(T)^\perp$, so $N(T) \subseteq R(T^*)^\perp$.
Suppose $x \in R(T^*)^\perp$. Then for all $y \in V$,

$$0 = \langle x, T^*(y)\rangle = \langle T(x), y\rangle$$

This means that $T(x) = 0$, hence $x \in N(T)$. So $R(T^*)^\perp \subseteq N(T)$
Thus, $N(T) = R(T^*)^\perp$. It directly follows that $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$. Therefore,

$$\begin{aligned}dim(R(T)) &= dim(N(T)^\perp)\\ &= dim(R(T^*))\\ rank(T) &= rank(T^*)\end{aligned}$$

$\square$

d. $\langle T^*(x), y\rangle_1 = \langle x, T(y)\rangle_2$ for all $x \in W$ and $y \in V$

Proof.

$$\begin{aligned}\langle T^*(x), y\rangle_1 &= \overline{\langle y, T^*(x)\rangle_1} \\ &= \overline{\langle T(y), x\rangle_2} \\ &= \langle x, T(y)\rangle_2\end{aligned}$$

as desired. $\square$

e. For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$

Proof.

i. $T^*T(x) = 0 \Rightarrow T(x) = 0$

This means that $\forall x \in V$,

$$\begin{aligned}0 &= \langle x, T^*T(x)\rangle_1 \\ &= \langle T(x), T(x)\rangle_2\end{aligned}$$

Therefore, $T(x) = 0$

ii. $T(x) = 0 \Rightarrow T^*T(x) = 0$

$$T^*T(x) = T^*(0) = 0$$

as desired.

Therefore, $T^*T(x) = 0$ if and only if $T(x) = 0$. $\square$

§3 C

a. Show that the equation $(A^*A)x_0 = A^*y$ of Theorem 6.12 takes the form of the normal equations:

$$\left(\sum_{i=1}^{m} t_i^2\right) c + \left(\sum_{i=1}^{m} t_i\right) d = \sum_{i=1}^{m} t_i y_i$$

and

$$\left(\sum_{i=1}^{m} t_i\right) c + md = \sum_{i=1}^{m} y_i$$

Proof. Suppose

$$A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{bmatrix}$$

Then,

$$A^* = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$A^*A = \begin{bmatrix} \sum_{i=1}^{m}(t_i)^2 & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & m \end{bmatrix}$$

Hence,

$$A^*Ax_0 = A^*y$$

$$\begin{bmatrix} \sum_{i=1}^m (t_i)^2 & \sum_{i=1}^m t_i \\ \sum_{i=1}^m t_i & m \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t_1 & t_2 & \cdots & t_m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\left(\sum_{i=1}^m t_i^2\right) c + \left(\sum_{i=1}^m t_i\right) d = \sum_{i=1}^m t_i y_i$$

and

$$\left(\sum_{i=1}^m t_i\right) c + md = \sum_{i=1}^m y_i$$

as desired. □

b. Use the second normal equation of (a) to show that the least squares line must pass through the center of mass, $(\bar{t}, \bar{y})$, where

$$\bar{t} = \frac{1}{m}\sum_{i=1}^m t_i \quad \text{and} \quad \bar{y} = \frac{1}{m}\sum_{i=1}^m y_i$$

Proof. Consider what happens when we divide the following equation by m.

$$\begin{aligned} \left(\sum_{i=1}^m t_i\right) c + md &= \sum_{i=1}^m y_i \\ \frac{1}{m}\left(\sum_{i=1}^m t_i\right) c + d &= \frac{1}{m}\sum_{i=1}^m y_i \\ \bar{t}c + d &= \bar{y} \end{aligned}$$

as desired. □

§4 D

An nxn real matrix A is said to be a Gramian matrix if there exists a real (square) matrix B such that $A = B^t B$. Prove that A is a Gramian matrix if and only if A is symmetric and all of its eigenvalues are nonnegative. Hint: Apply Theorem 6.17 to $T = L_A$ to obtain an orthonormal basis $\{v_1, v_2, \cdots, v_n\}$ of eigenvectors with the associated eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Define the linear operator U by $U(v_i) = \sqrt{\lambda_i} v_i$.

Proof.

i. A is a Gramian matrix $\Rightarrow$ A is symmetric and all of its eigenvalues are nonnegative.

Since A is a Gramian matrix, there exists a real (square) matrix B such that $A = B^t B$. This implies that A is symmetric

$$\begin{aligned} A^t &= (B^t B)^t \\ &= B^t B \\ &= A \end{aligned}$$

as desired.

Suppose λ is an eigenvalue of A with a normal/unit eigenvector of x, i.e. $Ax = \lambda x$. It follows that

$$\begin{aligned}\lambda &= \lambda\langle x, x\rangle \\ &= \langle Ax, x\rangle \\ &= \langle B^t Bx, x\rangle \\ &= \langle Bx, Bx\rangle \\ &\geq 0\end{aligned}$$

because $Bx \neq 0$ and Theorem 6.1.

Therefore, A is symmetric and all of its eigenvalues are nonnegative.

ii. A is symmetric and all of its eigenvalues are nonnegative $\Rightarrow$ A is a Gramian matrix

Since A is symmetric it follows that A and $T = L_A$ must be self-adjoint because A is a real matrix, by definition. By Theorem 6.17, there exists an orthonormal basis $\{v_1, v_2, \cdots, v_n\}$ of eigenvectors with the associated eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. By Theorem 6.16., T is normal and there exists an digaonal matrix $[T]_\beta$ such that the ith digaonal entries are λ_i. Suppose we construct a digaonal matrix C such that its ith digaonal entries are $\sqrt{\lambda_i}$, i.e. $C^2 = [T]_\beta$. This will still be a real matrix because the eigenvalues are nonnegative. Let α be the standard orded basis. Hence,

$$A = [I]_\beta^\alpha [T]_\beta [I]_\alpha^\beta = [I]_\beta^\alpha C^2 [I]_\alpha^\beta$$

Since β is orthonormal, $([I]_\beta^\alpha)^t [I]_\beta^\alpha = [I]$. Thus, $([I]_\beta^\alpha)^t = ([I]_\beta^\alpha)^{-1} = ([I]_\alpha^\beta)$. We can fix B to be $C[I]_\alpha^\beta$ such that

$$\begin{aligned}A &= [I]_\beta^\alpha C^2 [I]_\alpha^\beta \\ &= [I]_\beta^\alpha CC[I]_\alpha^\beta \\ &= B^t B\end{aligned}$$

as desired. Therefore, A is a Gramian matrix.

$\square$

§5 E

Let T and U be self-adjoint linear operators on an n-dimensional inner product space V,and let $A = [T]_\beta$,where β is an orthonormal basis for V. Prove the following results.

a. T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [nonnegative].

Proof.

i. T is positive definite $\Rightarrow$ all of its eigenvalues are positive

By definition of positive definite, $\langle T(x), x\rangle > 0$ for all $x \neq 0$. Let λ be a eigenvalue of

T with v being its corresponding orthonormal eigenvector. Consider,

$$\begin{aligned}\lambda &= \lambda\langle v, v\rangle \\ &= \langle \lambda v, v\rangle \\ &= \langle T(v), v\rangle \\ &> 0\end{aligned}$$

as desired.

ii. All of T's eigenvalues are positive $\Rightarrow$ T is positive definite

By Theorem 6.17, we know that there exists an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$ for V consisting of eigenvectors of T. Let $x \in V$. We can express x in terms of β,

$$x = \sum_{i=1}^{n} c_i v_i$$

Now consider the following inner product,

$$\begin{aligned}\langle T(x), x\rangle &= \langle T(\sum_{i=1}^{n} c_i v_i), \sum_{i=1}^{n} c_i v_i\rangle \\ &= \langle \sum_{i=1}^{n} c_i \lambda_i v_i, \sum_{i=1}^{n} c_i v_i\rangle \\ &= \sum_{i=1}^{n} |c_i|^2 \lambda_i \langle v_i, v_i\rangle \\ &= \sum_{i=1}^{n} |c_i|^2 \lambda_i \\ &> 0\end{aligned}$$

as desired.

Without loss of generality, the above statements hold for the semidefinite case. Therefore, T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [nonnegative]. □

b. T is positive definite if and only if $\sum_{i,j} A_{ij} a_j \overline{a_i} > 0$ for all nonzero n-tuples $(a_1, \cdots, a_n)$

Proof.

i. T is positive definite $\Rightarrow \sum_{i,j} A_{ij} a_j \overline{a_i} > 0$ for all nonzero n-tuples $(a_1, \cdots, a_n)$

Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V. Let $a = (a_1, \cdots, a_n)$ be a nonzero n-tuple. We can define $x = \sum_{i=0}^{n} a_i v_i \in V$. It follows that by corollary of Theorem 6.5

and the definition of positive definite,

$$\begin{aligned}
\langle T(x), x\rangle &= \langle T(\sum_{j=0}^{n} a_j v_j), \sum_{i=0}^{n} a_i v_i\rangle \\
&= \sum_{j=0}^{n} a_j \langle T(v_j), \sum_{i=0}^{n} a_i v_i\rangle \\
&= \sum_{i,j=0}^{n} a_j \overline{a_i} \langle T(v_j), v_i\rangle \\
&= \sum_{i,j=0}^{n} A_{ij} \overline{a_i} a_j \\
&> 0
\end{aligned}$$

as desired.

ii. $\sum_{i,j} A_{ij} a_j \overline{a_i} > 0$ for all nonzero n-tuples $(a_1, \cdots, a_n) \Rightarrow$ T is positive definite

Let $x = \sum_{i=0}^{n} a_i v_i \in V$. It follows from corollary of Theorem 6.5,

$$\begin{aligned}
\sum_{i,j}^{n} A_{ij} a_j \overline{a_i} &= \sum_{i,j}^{n} a_j \overline{a_i} \langle T(v_j), v_i\rangle \\
&= \langle T(\sum_{j}^{n} a_j v_j), \sum_{i}^{n} a_i v_i\rangle \\
&= \langle T(x), x\rangle \\
&> 0
\end{aligned}$$

as desired.

Therefore, T is positive definite if and only if $\sum_{i,j} A_{ij} a_j \overline{a_i} > 0$ for all nonzero n-tuples $(a_1, \cdots, a_n)$ $\square$

c. T is positive semidefinite if and only if $A = B^*B$ for some square matrix B.

Proof.

i. T is positive semidefinite $\Rightarrow A = B^*B$ for some square matrix B.

Since T is a self adjoint operator, there exists an orthonormal basis $\gamma = \{v_1, \cdots, v_n\}$ for V consisting of eigenvectors of T by Theorem 6.17. We also know from (a) that all of the corresponding eigenvalues are positive. Additionally, since T is a self adjoint operator we know that $A = P^*DP$ where D is diagonal. It follows that on the diagonals on D are positive. Let's define a matrix $E_{ii} = \sqrt{D_{ii}}$, i.e. $E^2 = D$. Then we have $A = (P^*E)(EP)$ where we can fix $B = EP$ and we know that $B^* = (EP)^* = P * E^* = P^*E$ because E is a self-adjoint matrix. Therefore, $A = B^*B$.

ii. $A = B^*B$ for some square matrix $B \Rightarrow T$ is positive semidefinite.

Consider $v \in V$ and the following inner product,

$$\begin{aligned}
\langle L_A(v), v\rangle &= \langle Av, v\rangle \\
&= \langle B^*Bv, v\rangle \\
&= B^*\langle Bv, v\rangle \\
&= \langle Bv, Bv\rangle \\
&\geq 0
\end{aligned}$$

Since $A = [T]_\beta$, T is positive semidefinite.

□

d. If T and U are positive semidefinite operators such that $T^2 = U^2$ then $T = U$.

Proof. Since T is a self adjoint operator, there exists an orthonormal basis $\gamma = \{v_1, \cdots, v_n\}$ for V consisting of eigenvectors of T by Theorem 6.17. We also know from (a) that all of the corresponding eigenvalues are nonnegative. Then, $T^2(v_i) = \lambda^2 v_i = U^2(v_i)$. It follows that

$$\begin{aligned}
0 &= U^2 v_i - \lambda^2 v_i \\
&= (U^2 - \lambda^2 I)v_i \\
&= (U + \lambda I)(U - \lambda I)v_i
\end{aligned}$$

This means that $(U - \lambda I)v_i = 0$ because eigenvalues are nonnegative for T. Thus

$$\begin{aligned}
(U - \lambda I)v_i &= 0 \\
U(v_i) &= \lambda v_i \\
&= T(v_i)
\end{aligned}$$

Therefore, $T = U$. □

e. If T and U are positive definite operators such that $TU = UT$ then TU is positive definite.

Proof. Since T and U are positive definite operators, TU is self adjoint because

$$(TU)^* = U^*T^* = UT = TU$$

Let $Q^2 = T$ such that Q is positive definite. We know that $Q^2U = QQU = QUQ$ because of part (d).

$$\begin{aligned}
\langle TU(x), x\rangle &= \langle Q^2U(x), x\rangle \\
&= \langle QUQ(x), x\rangle \\
&= \langle UQ(x), Q(x)\rangle \\
&> 0
\end{aligned}$$

Therefore, TU is positive definite. □

f. T is positive definite [semidefinite] if and only if A is positive definite [semidefinite]. Because of (f), results analogous to items (a) through (d) hold for matrices as well as operators.

Proof. Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V. Let $a = (a_1, \cdots, a_n)$ be a nonzero n-tuple. We can define $x = \sum_{i=0}^{n} a_i v_i \in V$. It follows that by corollary of Theorem

6.5 and the definition of positive definite,

$$\begin{aligned}
\langle T(x), x\rangle &= \langle T(\sum_{j=0}^{n} a_j v_j), \sum_{i=0}^{n} a_i v_i\rangle \\
&= \sum_{j=0}^{n} a_j \langle T(v_j), \sum_{i=0}^{n} a_i v_i\rangle \\
&= \sum_{i,j=0}^{n} a_j \overline{a_i} \langle T(v_j), v_i\rangle \\
&= \sum_{i,j=0}^{n} A_{ij} \overline{a_i} a_j \\
&= \langle Ax, x\rangle
\end{aligned}$$

as desired. □

§6 F

Let T and U be positive definite operators on an inner product space V. Prove the following results.

a. $T + U$ is positive definite

Proof. By definition, $\langle T(x), x\rangle > 0$, $\langle U(x), x\rangle > 0$ for all $x \neq 0$

$$\begin{aligned}
\langle (T+U)(x), x\rangle &= \langle T(x) + U(x), x\rangle \\
&= \langle T(x), x\rangle + \langle U(x), x\rangle \\
&> 0
\end{aligned}$$

Therefore, $T + U$ is positive definite. □

b. If $c > 0$, then cT is positive definite.

Proof. By definition, $\langle T(x), x\rangle > 0$. It follows that

$$\begin{aligned}
\langle cT(x), x\rangle &= c\langle T(x), x\rangle \\
&> 0
\end{aligned}$$

Therefore, if $c > 0$, then cT is positive definite. □

c. T^{-1} is positive definite.

Proof. Let $y = T^{-1}(x)$. We know that $T^* = T$ because T is self adjoint.

$$\begin{aligned}
\langle T^{-1}(x), x\rangle &= \langle y, T(y)\rangle \\
&= \langle T^*(y), y\rangle \\
&= \langle T(y), y\rangle \\
&> 0
\end{aligned}$$

□

§7 G

Prove that if T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary square root; that is, there exists a unitary operator U such that $T = U^2$.

Proof. By Theorem 6.18, there exists an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$ consisting of eigenvectors of T with corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of absolute value 1. It follows by Theorem 5.1 that

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

We can fix $[U]_\beta$ as $\sqrt{[T]_\beta}$

$$[U]_\beta = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$$

U is a unitary operator by Theorem 6.18 because.

$$|\sqrt{\lambda_i}| = \sqrt{|\lambda_i|} = \sqrt{1} = 1$$

Therefore, there exists a unitary operator U such that $T = U^2$. □

§8 H

Let A and B be nxn matrices that are unitarily equivalent.

a. Prove that $tr(A^*A) = tr(B^*B)$

   Proof. By definition, there exists a unitary matrix P such that $A = P^*BP$. It follows that

$$\begin{aligned} tr(A^*A) &= tr((P^*BP)^*(P^*BP)) \\ &= tr(P^*B^*PP^*BP) \\ &= tr((P^*B^*)(BP)) \\ &= tr((BP)(P^*B^*)) \\ &= tr(BB^*) \\ &= tr(B^*B) \end{aligned}$$

   as desired. □

b. Use (a) to prove that

$$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2$$

Proof. Notice that

$$\begin{aligned}
\sum_{i,j=1}^{n} \left|A_{ij}\right|^2 &= \sum_{i,j=1}^{n} |A_{ij}|^2 \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \overline{A_{ij}}A_{ij} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} (A^*)_{ji}A_{ij} \\
&= \sum_{i=1}^{n} (A^*A)_{ii} \\
&= tr(A^*A)
\end{aligned}$$

Without loss of generality,

$$\sum_{i,j=1}^{n} \left|B_{ij}\right|^2 = tr(B^*B)$$

Therefore, by (a)

$$\sum_{i,j=1}^{n} \left|A_{ij}\right|^2 = \sum_{i,j=1}^{n} \left|B_{ij}\right|^2$$

□

c. Use (b) to show that the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 4 \\ 1 & 1 \end{bmatrix}$$

are not unitarily equivalent.

Proof. Consider,

$$\begin{aligned}
\sum_{i,j=1}^{n} |A_{ij}|^2 &= |1|^2 + |2|^2 + |2|^2 + |i|^2 \\
&= 10
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j=1}^{n} |B_{ij}|^2 &= |i|^2 + |4|^2 + |1|^2 + |1|^2 \\
&= 19
\end{aligned}$$

Therefore, A and B are not unitarily equivalent. □