CS 577: HW 4

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§1 Given the random process described above, what is the expected number of incoming links to node v_j in the resulting network? Give an exact formula in terms of n and j, and also try to express this quantity asymptotically (via an expression without large summations) using Θ notation

We know that v_j can be only linked to nodes that come after it. For every node v_i, where $i > j$, the probability of v_i connecting with v_j is $\frac{1}{i-1}$, as there are $i-1$ nodes to choose from and v_i connects to only one node. Hence, the expected number of incoming links to node v_j in the resulting network is the following:

$$\sum_{i=j+1}^{n} \frac{1}{i-1}$$

By Theorem 13.10 in the textbook we know that $H(\alpha) = \Theta(\log(\alpha))$,

$$\begin{aligned}
\sum_{i=j+1}^{n} \frac{1}{i-1} &= \sum_{i=j}^{n-1} \frac{1}{i} \\
&= \sum_{i=1}^{n-1} \frac{1}{i} - \sum_{i=1}^{j-1} \frac{1}{i} \\
&= H(n-1) - H(j-1) \\
&= \Theta(\log(n-1)) - \Theta(\log(j-1)) \\
&= \Theta(\log(n)) - \Theta(\log(j)) \\
&= \Theta(\log\left(\frac{n}{j}\right))
\end{aligned}$$

§2 Part (1) makes precise sense in which the nodes that arrive early carry an "unfair" share of the connections in the network. Another way to quantify the imbalance is to observe that, in a run of this random process, we expect many nodes to end up with no incoming links. Give a formula for the expected number of nodes with no incoming links in a network grown randomly according to this model.

Let X_j be a random variable such that

$$X_j = \begin{cases} 1 & \textbf{if } v_j \text{ has no incoming links} \\ 0 & \textbf{otherwise} \end{cases}$$

From part 1, we know that the probability for some v_i to connect with v_j, where $i > j$, is $\frac{1}{i-1}$. It follows then that the probability that v_i does not connect with v_j is $1 - \frac{1}{i-1}$. To find the probability that no nodes connect to v_j, we multiply the probability of all v_i not connecting to v_j. Hence the expected value of X_j is

$$\begin{aligned} E[X_j] &= \prod_{i=j+1}^{n} 1 - \frac{1}{i-1} \\ &= \left(1 - \frac{1}{j}\right)\left(1 - \frac{1}{j+1}\right)\cdots\left(1 - \frac{1}{n-1}\right) \\ &= \left(\frac{j-1}{j}\right)\left(\frac{j}{j+1}\right)\cdots\left(\frac{n-2}{n-1}\right) \\ &= \frac{j-1}{n-1} \end{aligned}$$

assuming $n > 1$.

To find the expected number of all nodes with no incoming links is to sum up the expected value of no incoming links for all nodes. Using the formula for triangle numbers,

$$\begin{aligned} \sum_{j=1}^{n} E[X_j] &= \sum_{j=1}^{n} \frac{j-1}{n-1} \\ &= \frac{1}{n-1}\left(\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1\right) \\ &= \frac{1}{n-1}\left(\frac{n(n+1)}{2} - n\right) \\ &= \frac{1}{n-1}\left(\frac{n(n-1)}{2}\right) \\ &= \frac{n}{2} \end{aligned}$$