Math 341: Homework 8

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§1 A

Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$

Proof.

a. λ is an eigenvalue of $T \Rightarrow \lambda$ is an eigenvalue of $[T]_\beta$

   By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Using Theorem 2.14,

$$\begin{aligned} T(v) &= \lambda v \\ [T(v)]_\beta &= [\lambda v]_\beta \\ [T]_\beta [v]_\beta &= \lambda [v]_\beta \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of $[T]_\beta$.

b. λ is an eigenvalue of $[T]_\beta \Rightarrow \lambda$ is an eigenvalue of T

   By definition, there exists a eigenvector $v \in V$ such that $[T]_\beta [v]_\beta = \lambda [v]_\beta$. Using Theorem 2.14,

$$\begin{aligned} [T]_\beta [v]_\beta &= \lambda [v]_\beta \\ [T(v)]_\beta &= [\lambda v]_\beta \\ T(v) &= \lambda v \end{aligned}$$

   as desired. Thus, λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. $\square$

§2 B

a. Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   Proof.

   i. Linear operator T on a finite-dimensional vector space is invertible $\Rightarrow$ zero is not an eigenvalue of T.

By the corollary of Theorem 4.7, $\det(T) \neq 0$. Assume, for the sake of contradiction, suppose zero is an eigenvalue of T. It follows from Theorem 5.2 that

$$\begin{aligned} \det(T - \lambda I) &= 0 \\ \det(T - 0I) &= 0 \\ \det(T) &= 0 \end{aligned}$$

which is a contradiction. Thus, zero is not an eigenvalue of T.

ii. Zero is not an eigenvalue of $T \Rightarrow$ linear operator T on a finite-dimensional vector space is invertible.

By contrapositive, we will instead prove that if linear operator T on a finite-dimensional vector space is not invertible then zero is an eigenvalue of T. If T is not invertible then $\det(T) = 0$ by corollary of Theorem 4.7. It follows from Theorem 5.2 that

$$\det(T - \lambda I) = 0$$

It directly follows that zero is an eigenvalue of T.

Therefore, linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T. □

b. Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof.

i. A scalar λ is an eigenvalue of T $\Rightarrow \lambda^{-1}$ is an eigenvalue of T^{-1}.

By definition, there exists a eigenvector $v \in V$ such that $T(v) = \lambda v$. Given that T is invertible and by definition eigenvalues are non zero,

$$\begin{aligned} T(v) &= \lambda v \\ T^{-1}(T(v)) &= T^{-1}(\lambda v) \\ v &= T^{-1}(\lambda v) \\ v &= \lambda T^{-1}(v) \\ \lambda^{-1} v &= T^{-1}(v) \end{aligned}$$

as desired. Thus, λ^{-1} is an eigenvalue of T^{-1}.

ii. λ^{-1} is an eigenvalue of $T^{-1} \Rightarrow$ a scalar λ is an eigenvalue of T.

By definition, there exists a eigenvector $v \in V$ such that $T^{-1}(v) = \lambda^{-1} v$. Given that T^{-1} is invertible linear operator,

$$\begin{aligned} T^{-1}(v) &= \lambda^{-1} v \\ T(T^{-1}(v)) &= T(\lambda^{-1} v) \\ v &= \lambda^{-1} T(v) \\ \lambda v &= T(v) \end{aligned}$$

as desired. Thus, λ is an eigenvalue of T.

Therefore, a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. □

c. State and prove results analogous to (a) and (b) for matrices.

(a) A matrix A is invertible if and only if zero is not an eigenvalue of A.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (a) that "matrix A is invertible if and only if zero is not an eigenvalue of A." is true. $\square$

(b) Let A be an invertible matrix. λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Proof. Since A is an invertible matrix, the corresponding left multiplication transformation is also invertible by corollary 2 of Theorem 2.18. It directly follows from the proof from (b) that "λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}" is true. $\square$

§3 C

For any square matrix A, prove that A and A^t have the same characteristic polynomial, and hence the same eigenvalues.

Proof. Let $A \in M_{nxn}(F)$. The characteristic polynomial for A is $f(t) = \det(A - tI_n)$. Using Theorem 4.8, Theorem 2.12, and the trivial fact that the identity matrix is symmetric,

$$\begin{aligned} f(t) &= det(A - tI_n) \\ &= det((A - tI_n)^t) \\ &= det(A^t - (tI_n)^t)) \\ &= det(A^t - tI_n) \end{aligned}$$

which is exactly the characteristic polynomial for A^t. Therefore, A and A^t have the same characteristic polynomial, and hence the same eigenvalues. $\square$

§4 D

Let T be a linear operator on a finite-dimensional vector space V, and let c be any scalar.

a. Determine the relationship between the eigenvalues and eigenvectors of T (if any) and the eigenvalues and eigenvectors of $U = T - cI$ (where I is the identity transformation) Justify your answers.

Proof. Suppose $v \in V$ is an eigenvector of T where λ is its eigenvalue,

$$Tv = \lambda v$$

Applying the transformation U to v,

$$\begin{aligned} Uv &= (T - cI)v \\ &= Tv - cIv \\ &= \lambda v - cv \\ &= (\lambda - c)v \end{aligned}$$

Thus, if v is an eigenvector of T, then it is an eigenvector U with its corresponding eigenvalue being $\lambda - c$. $\square$

b. Prove that T is diagonalizable if and only if U is diagonalizable.

Proof. Since T is diagonalizable, there exists an ordered basis β for V consisting of eigenvectors of T by Theorem 5.1. From (a), we know that all eigenvectors of T are eigenvectors of U. Thus, β is an ordered basis consisting of eigenvectors of U. Thus, U is diagonalizable. Without loss of generality, if U is diagonalizable then T is diagonalizable. Therefore, T is diagonalizable if and only if U is diagonalizable. □

§5 E

For each of the following matrices $A \in \mathbf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$

a. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Proof. We follow "Test for Diagonalization" and example 5 in section 5.2 of our textbook. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has a single eigenvalue of $\lambda_1 = 1$. Because

$$A - \lambda_1 I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

has rank 1, we see that $2 - \text{rank}(A - \lambda_1 I) = 1$ which is not the multiplicity of λ_1. Thus condition 2 fails and therefore A is not diagonalizable. □

b. $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

Proof. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) - 9 = (\lambda+2)(\lambda-4)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = 4$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A - \lambda_1 I)x = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} x = 0$$

$$\begin{cases} 3x_1 + 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)x = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} x = 0$$

$$\begin{cases} -3x_1 + 3x_2 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has its columns the vectors in a basis of eigenvectors of A.

$$\begin{aligned}
Q &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
Q^{-1} &= \frac{1}{(1\cdot 1)-(1\cdot -1)}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
D &= Q^{-1}AQ \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}
\end{aligned}$$

as desired. □

c. $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

Proof. The characteristic polynomial of A is

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 12 = (\lambda+2)(\lambda-5)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = 5$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A-\lambda_1 I)x = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} x = 0$$

$$\begin{cases} 3x_1 + 4x_2 = 0 \\ 3x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_2 = 3 \Rightarrow v_{\lambda_1} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Similiarly,

$$v_{\lambda_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has as its columns the vectors in a basis of eigenvectors of A.

$$\begin{aligned}
Q &= \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix} \\
Q^{-1} &= \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix} \\
D &= Q^{-1}AQ \\
&= \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}
\end{aligned}$$

as desired. □

d. $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda+1)(\lambda-3)^2$ which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 3$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. So we only need to test condition 2 for λ_2. It is clear that

$$A - \lambda_2 I = \begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix}$$

has rank 1, we see that $3 - \text{rank}(A - \lambda_1 I) = 2$ which is the multiplicity of λ_2. Thus, condition 2 holds and A is diagonalizable. The eigenvector of λ_1 is

$$\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

and the eigenvectors of λ_2 is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus,

$$Q = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} D &= Q^{-1}AQ \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

□

e. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda-1)(\lambda^2+1)$ which does not split over $\mathbb{R}$, so condition 1 of the test for diagonalization is not satisfied. Therefore, A is not diagonalizable. □

f. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Proof. The characteristic polynomial of A is $-(\lambda-1)^2(\lambda-3)$ which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = 3$. We test condition 2 for λ_1. It is clear that

$$A - \lambda_1 I = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

has rank 2, we see that $3 - \text{rank}(A - \lambda_1 I) = 1$ which is not the multiplicity of λ_2. Thus, condition 2 fails and A not diagonalizable. ☐

g. $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ The characteristic polynomial of A is $-(\lambda-2)^2(\lambda-4)$. which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = 2$ and $\lambda_2 = 4$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. So we only need to test condition 2 for λ_1. It is clear that

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

has rank 1, we see that $3 - \text{rank}(A - \lambda_1 I) = 2$ which is the multiplicity of λ_2. Thus, condition 2 holds and A is diagonalizable. The eigenvectors of λ_1 is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and the eigenvector of λ_2 is

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Thus,

$$Q = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} D &= Q^{-1}AQ \\ &= \begin{bmatrix} -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

§6 F

For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R)$$

find an expression for A^n, where n is an arbitrary positive integer.

Proof. We claim that A is diagonalizable and we can find invertible matrices that will give us the expression A^n. Consider the characteristic polynomial of A

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = (\lambda+1)(\lambda-5)$$

which splits, so condition 1 of the test for diagonalization is satisfied. A has eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 5$. By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1. Therefore, A is diagonalizable. To find a invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$, we first calculate the eigenvectors for λ_1 and λ_2 using Theorem 5.4.

$$(A - \lambda_1 I)x = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} x = 0$$

$$\begin{cases} 2x_1 + 4x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = -2x_2 \Rightarrow \text{fix } x_2 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)x = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} x = 0$$

$$\begin{cases} -4x_1 + 4x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \text{fix } x_1 = 1 \Rightarrow v_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

v_{λ_1} and v_{λ_2} are eigenvectors of λ_1 and λ_2 respectively. We can use the corollary to Theorem 2.23 to find an invertible matrix Q. The matrix Q has its columns the vectors in a basis of eigenvectors of A.

$$\begin{aligned} Q &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \\ Q^{-1} &= \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ D &= Q^{-1}AQ \\ &= \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

Rewriting $D = Q^{-1}AQ$ gets us $A = QDQ^{-1}$. Since A and D are both diagonal matrices, $A^n = QD^nQ^{-1}$. So,

$$\begin{aligned} A^n &= QD^nQ^{-1} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}^n \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned}$$

as desired. □

§7 G

Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

a. Prove that the characteristic polynomial for T splits.

> **Lemma 7.1**
> Characteristic polynomial of a linear operator is indepedent of the choice of basis used.
>
> *Proof.* as □

Using the above lemma and the fact that the determinant of a upper triangular matrix is the product of the diagonal entries by property 4 of determinants in section 4.4, the characteristic polynomial of T is

$$\det([T]_\beta - \lambda I) = \prod_{i=1}^{n}(([T]_\beta)_{i,i} - \lambda)$$

Therefore, the characteristic polynomial for T splits. □

b. State and prove an analogous result for matrices.

Proof. The analogous result for matrices is as follows: if $A \in M_{nxn}(F)$ is similar to an upper triangular matrix A' (i.e. $A = Q^{-1}A'Q$), then the characteristic polynomial of A splits. Notice that the characteristic polynomial of A is the same as the characteristic polynomial of A'.

$$\begin{aligned}
\det(A - \lambda I) &= \det(Q^{-1}A'Q - \lambda I_n) \\
&= \det(Q^{-1}A'Q - Q^{-1}\lambda I_n Q) \\
&= \det(Q^{-1}(A' - \lambda I_n)Q) \\
&= \det(Q^{-1})\det(A' - \lambda I_n)\det(Q) \\
&= \det(A' - \lambda I_n)
\end{aligned}$$

We calculate the characteristic polynomial of A by using the fact that A' is an upper triangular matrix,

$$\begin{aligned}
\det(A - \lambda I) &= \det(A' - \lambda I_n) \\
&= \prod_{i=1}^{n}(A'_{i,i} - \lambda)
\end{aligned}$$

and A splits as desired. Therefore, analogous result of (a) for matrices holds. □

c. Prove that if $A \in M_{nxn}(F)$ and the characteristic polynomial of A splits, then A is similar to an upper triangular matrix.

Proof. We proceed by induction on $n \in \mathbb{N}^+$. Let $P(n)$ be the predicate that if $A \in M_{nxn}(F)$ and the characteristic polynomial of A splits, then A is similar to an upper triangular matrix.

Base case: It is trival that $P(1)$ holds as all matrix $1x1$ are upper triangular.

Inductive step: We prove that if $P(n)$ holds, then $P(n+1)$ holds. □

§8 H

a. Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis β

Proof. Since T and U are simultaneously diagonalizable, there exists an ordered basis γ such that $[T]_\gamma$ and $[U]_\gamma$ are diagonal matrices. Let β be an arbitrary ordered basis. Consider the change of basis matrix $[I]^{\beta}_{\gamma}$. Notice that

$$[T]_{\beta} = [I]^{\beta}_{\gamma}[T]_{\gamma}[I]^{\gamma}_{\beta}$$

Similiarly,

$$[U]_{\beta} = [I]^{\beta}_{\gamma}[U]_{\gamma}[I]^{\gamma}_{\beta}$$

Therefore, the matrices $[T]_{\beta}$ and $[U]_{\beta}$ are simultaneously diagonalizable for any ordered basis β. $\square$

b. Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operator

Proof. Since A and B are simultaneously diagonalizable matrices, there exists an invertible matrix Q such that that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. Let α be the basis from the columns vectors of Q and β be the standard basis for the domain of L_A and L_B. Notice,

$$[L_A]_{\alpha} = [I]^{\alpha}_{\beta}[L_A]_{\beta}[I]^{\beta}_{\alpha} = Q^{-1}AQ$$
$$[L_B]_{\alpha} = [I]^{\alpha}_{\beta}[L_B]_{\beta}[I]^{\beta}_{\alpha} = Q^{-1}BQ$$

$\square$

Therefore, L_A and L_B are simultaneously diagonalizable linear operators.

c. Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $TU = UT$).

Proof. Since T and U are simultaneously diagonalizable operators, there exists an ordered basis γ such that $[T]_{\gamma}$ and $[U]_{\gamma}$ are diagonal matrices. Notice that

$$[T]_{\gamma}[U]_{\gamma} = [U]_{\gamma}[T]_{\gamma}$$

because diagonal matrices commute. This directly implies that T and U commute. $\square$

d. Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

Proof. Since A and B are simultaneously diagonalizable matrices, there exists an invertible matrix Q such that that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

$$\begin{aligned} AB &= (Q^{-1}AQ)(Q^{-1}BQ) \\ &= (Q^{-1}BQ)(Q^{-1}AQ) \\ &= Q^{-1}BAQ \\ &= BA \end{aligned}$$

as desired. Therfore, if A and B are simultaneously diagonalizable matrices, A and B commute $\square$

§9 I

Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k\right)$$

Proof. We proceed by induction on $k \in \mathbb{N}^+$. Let $P(n)$ be the predicate that given $A \in M_{kxk}(F)$, the characteristic polynomial is $(-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k\right)$
Base case: We prove that $P(1)$ holds. The $1x1$ matrix is $[-a_0]$. The characteristic polynomial is

$$\det(A - tI) = -a_0 - t$$

Using the formula,

$$\begin{aligned}(-1)^k(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k) &= (-1)^1(a_0 + t^1) \\ &= -a_0 - t^1\end{aligned}$$

Thus, $P(1)$ holds.
Inductive step: Suppose $P(k)$ holds. We prove that $P(k+1)$ holds. We compute,

$$\begin{aligned}\det(A - tI) &= \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k \end{pmatrix} \\ &= -t \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k \end{pmatrix} + (-1)^{k+2}(-a_0) \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k \end{pmatrix} \\ &= (-t)(-1)^k(a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k) + (-1)^{k+2}(-a_0)(1) \\ &= (-1)^{k+1}(a_1 t + a_2 t^2 + \cdots + a_k t^k + t^{k+1}) + (-1)^{k+1}(a_0) \\ &= (-1)^{k+1}(a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + t^{k+1})\end{aligned}$$

as desried. Thus we have proven the inductive step. Therefore, $P(k)$ is true for all $k \in \mathbb{N}^+$. $\square$