

Math 240: Homework 9

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§1 1

Let A be any nonempty set. (You do not get to decide what A is)

- a. Let R be a relation on A . Explain what is wrong with the following “proof” that if R is symmetric and transitive, then it is reflexive.

"Proof": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$

Proof. This proof is wrong because it assumes a priori that $(a, b) \in R$ when no such definition of the relation R is defined. \square

- b. Give an example of a relation R on A which is symmetric and transitive, but not reflexive. Justify your answer.

Proof. The most trivial example is a non empty set A and the empty set R . It is vacuously true that R is symmetric and transitive. However, R is not reflexive because there are no relations in it, i.e. the following statement does not hold: $\forall a \in A : (a, a) \in R$. \square

§2 2

Give an example of a relation R on \mathbb{N} such that if S is a relation on \mathbb{N} which contains R , then S is not antisymmetric. Justify your answer. (This implies that the antisymmetric closure of R does not exist.)

Proof. Consider the relation $R = \{(1, 2), (2, 1)\}$. It is clear that $1 \neq 2$, hence R is not antisymmetric. Therefore, if S contains R , then S is not antisymmetric. \square

§3 3

Come up with a topological ordering of the following DAG, i.e., write down the vertices in some order such that if (i, j) is an edge, then i is to the left of j .

Proof. Using Kahn's algorithm, we get the topological ordering of 2, 3, 4, 6, 1, 5. \square

§4 4

Let R be the relation on \mathbb{R} defined by $(x, y) \in R$ if and only if $x - y$ is rational. Is R symmetric? Antisymmetric? Reflexive? Transitive? Justify your answers. (You may use basic properties about rational numbers without proving them)

Proof.

a. R is symmetric

This is because there is closure under addition/subtraction for rational numbers. If $x - y$ is a rational number then $y - x$ is a rational number, hence $(y, x) \in R$.

b. R is not antisymmetric

Consider the following relation in R : $(1, 2)$ and $(2, 1)$. It is clear that $1 \neq 2$. Therefore, R is not antisymmetric.

c. R is reflexive

Suppose $x \in \mathbb{R}$. Any number subtracted by itself will be zero. Zero is a rational number. Thus, $(x, x) \in R$. Therefore, R is reflexive.

d. R is transitive

Suppose (x, y) and (y, z) are in R . This means that $x - y$ and $y - z$ are rational numbers. Now consider,

$$x - z = (x - y) + (y - z)$$

Since there is closure under addition for rational numbers, $x - z$ must be a rational number. Hence, $(x, z) \in R$. Therefore, R is transitive.

□

§5 5

Let R be a relation on a set A . Prove that if R is reflexive, then $R \subseteq R^n$ for all $n \geq 1$.

Proof. yeet

□