Math 341: Homework 6

Daniel Ko

Spring 2020

§1 A

Find the rank of the following matrices.

a. 2 show work later

b. 3

c. 2

d. 1

e. 3

f. 3

g. 1

§2 B

Prove that any elementary row [column] operation of type 1 can be obtained by a succession of three elementary row [column] operations of type 3 followed by one elementary row [column] operation of type 2

Proof. Row operation type 1 on row i and row j can be done by the following:

1. Row operation type 3: Add -1 times row i to row j
2. Row operation type 3: Add row j to row i
3. Row operation type 3: Add -1 times row i to row j
4. Row operation type 2: Multiply row j by -1

Without loss of generality, same could be done for a elementary column operation of type 1. $\square$

§3 C

Let A be an mxn matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

Proof. Iterate through each column, let this variable be c. If $A_{c,c}$ equals 0, go through all the elements in that column below $A_{c,c}$ and find the first non zero element. Perform a type 1 row operation on row c and the row the non zero element was found. If there is no non zero element, do nothing and go to the next column.

If $A_{c,c}$ does not equal 0, perform a type 3 row operation on each row below $A_{c,c}$. Multiply $-\frac{A_{r,c}}{A_{c,c}}$ by the cth row to the rth row, where r is every row below c. □

§4 D

Complete the proof of the corollary to Theorem 3.4 by showing that elementary column operations preserve rank.

Proof. If B is obtained from a matrix A by an elementary column operation, then there exists an elementary matrix E such that B = AE. By Theorem 3.2 (p. 150), E is invertible, and hence rank(B) = rank(A) by Theorem 3.4. □

§5 E

Let B and B' is an mxn matrix submatrix of B. Prove that rank(B) = r, then rank(B') = $r-1$

Proof. By Theorem 3.5, we know that $\text{rank}(B) = \dim(R(L_B))$, where $R(L_B) = \text{span}(B_1, B_2, \cdots, B_{n+1})$ and B_i is the ith column of B. In other words, is the rank is the number of linearly independent rows/columns in a matrix. Notice that B' has one less linear independent row/column than B. It follows that rank of B' would be 1 less than the rank of B. Therefore, rank(B')= $r-1$. Consider the matrix

$$M = \left[\begin{array}{c|c} 0 & \\ \vdots & B' \\ 0 & \end{array}\right]$$

M has the same number of linearly independent columns to that of B', so rank(M) = rank(B') Now consider the matrix below.

$$B = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \vdots & & B' & \\ 0 & & & \end{array}\right]$$

B has one more linearly independent row to that of M.
Then rank(B) = rank(M) + 1 = rank(B') + 1.
Therefore, if rank(B) = r, then rank(M) = rank(B') = r - 1. □

§6 F

Let B' and D' be mxn matrices, and let B and D be $(m+1)x(n+1)$ matrices respectively. Prove that if B' can be transformed into D' by an elementary row [column] operation, then B can be transformed into D by an elementary row [column] operation.

Proof. If B' can be tranformed into D' by elementary row operations, there must exist an elementary matrix E such that $D' = EB'$ by theorem 3.1. Now consider the matrix below.

$$A = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \vdots & & E & \\ 0 & & & \end{array}\right]$$

A is also an elementary matrix, We can observe that $D = AB$, thus B can be tranformed to D by elementary row operations. Without loss of generality, there exist a matrix F such that $D' = B'F$ where F is the elementary column matrix. So, $D = BF$ where B is like the A matrix but with F instead of E. Therefore. B can be tranformed to D by elementary column operations. ☐

§7 G

a. Find a 5x5 matrix M with rank 2 such that $AM = O$ where O is the 4x5 zero matrix.

Proof. By solving $Ax = 0$, we get this system of equation:

$$\begin{cases} x_1 \quad - x_3 + 2x_4 + x_5 = 0 \\ -x_1 + x_2 + 3x_3 - x_4 \quad = 0 \\ -2x_1 + x_2 + 4x_3 - x_4 + 3x_5 = 0 \\ 3x_1 - x_2 - 5x_3 + x_4 - 6x_5 = 0 \end{cases}$$

Solving this system of equations by computing reduced row echelon form, we get that x_1, x_2, x_4 are the pivot variables and x_3, x_5 are the free variables. So solutions are in the form $(x_3 + 3x_5, -2x_3 + x_5, x_3, -2x_5, x_5)$. From this, we are able to construct a basis for Ax=0, $(1, -2, 1, 0, 0), (3, 1, 0, -2, 1)$. Define

$$M = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It is obvious that this matrix has rank 2 and is 5x5. Because the column is a basis for $Ax = 0$, the resulting matrix will be O. ☐

b. Suppose that B is a 5x5 matrix such that $AB = O$. Prove that $\text{rank}(B) \leq 2$

Proof. Since $AB = O$, we know that the columns of B is a solution to $Ax = 0$, which is a subset of the nullspace of L_A. From the rank nullity theorem, we know that $\dim(\mathbb{F}^5) = \text{rank}(L_A) + \text{nullity}(L_A)$.
$\text{nullity}(L_A) = \dim(\mathbb{F}^5) - \text{rank}(L_A) = 5 - 3 = 2$. So, $\text{rank}(B)$ cannot be greater than 2. Therefore, $\text{rank}(B) \leq 2$. ☐

§8 H

For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

a. $T : P_2(R) \to P_2(R)$ defined by $T(f(x)) = f''(x) + 2f'(x) - f(x)$

Proof.
We want to projection to be on the xy-plane along the z-axis. Let the projection be (x,y,0).

To minimize the distance, we must choose x and y such that

$$(a-x)^2+(b-y)^2+(c-0)^2$$

is minimum. Since the equation above is a difference of squares, x = a and b = y will give us the minimum value. Therefore, the projection on the xy-plane will be (a,b,0), which is T. $\square$

b. Find a formula for T(a,b,c), where T represents the projection on the z-axis along the xy-plane.

Proof.
We want to projection to be on the z-axis along the xy-plane. Let the projection be (0,0,z). To minimize the distance, we must choose z such that

$$(a-0)^2+(b-0)^2+(c-z)^2$$

is minimum. z = c will give us the minimum value. Therefore, the equation for T will be T(a,b,c)=(0,0,c). $\square$

c. If T(a,b,c) = (a-c,b,0), show that T is the projection on the xy-plane along the line L $= \{(a,0,a) : a \in R\}$

Proof.
We want to projection to be on the xy-plane along the line L. Let the projection be $(x, y, 0)$. A vector that is on L is $(1, 0, 1)$. To minimize the distance, we must choose λ such that

$$(a,b,c)+\lambda(1,0,1)=(x,y,0)$$

is minimum. Writing the equation above as a system:

$$\begin{aligned} a+\lambda &= x \\ b &= y \\ c+\lambda &= 0 \end{aligned}$$

Solving this system gives us, $x = a - c, y = b$
Therefore, the projection on the xy-plane along the line L will be $(a - c, b, 0)$.

$\square$

§9 I

Express the invertible matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ as a product of elementary matrices

Proof. $\square$

§10 J

Suppose that A and B are matrices having n rows. Prove that $M(A|B) = (MA|MB)$ for any mxn matrix