Math 240: Midterm 2 Q5

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Spring 2020

§1 Q5

Consider the following function $f : \{0,1\}^* \to \{0,1\}^*$ defined inductively as follows:
Foundation rule: $f(\lambda) = \lambda$
Constructor rule: $f(x0) = f(x)1$ and $f(x1) = f(x)0$
Prove by structural induction that for all $x \in \{0,1\}^*$, $f(f(x)) = x$

Proof.

i. Base case: Prove $f(f(\lambda)) = \lambda$

$$\begin{aligned} f(f(\lambda)) &= f(\lambda) \\ &= \lambda \end{aligned}$$

by the foundation rule as desired.

ii. Inductive step: Suppose $f(f(n)) = n$ where $n \in \{0,1\}^*$. We prove that $f(f(ni)) = ni$ where $i \in \{0,1\}$. Consider two cases

a. $i = 0$

Then, $f(f(ni)) = f(f(n0))$

$$\begin{aligned} f(f(n0)) &= f(f(n)1) && \text{constructor rule} \\ &= f(n1) && \text{inductive hypothesis} \\ &= f(n)0 && \text{constructor rule} \\ &= n0 && \text{inductive hypothesis} \end{aligned}$$

as desired.

b. $i = 1$

Then, $f(f(ni)) = f(f(n1))$

$$\begin{aligned} f(f(n1)) &= f(f(n)0) && \text{constructor rule} \\ &= f(n0) && \text{inductive hypothesis} \\ &= f(n)1 && \text{constructor rule} \\ &= n1 && \text{inductive hypothesis} \end{aligned}$$

as desired.

Therefore, we have proved by structural induction that for all $x \in \{0,1\}^*$, $f(f(x)) = x$ ☐