Math 341: Linear Alegbra

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Spring 2020

§1 Propositional Logic

System of figuring out if something is true or false.
Proposition $\rightarrow$ Statement $\rightarrow$ True or False
Examples:

- P: Today is sunny
- P: I'm 5'11

§1.1 We can compose them

P: Today is sunny
Q: Today is rainy
$P \vee Q \Rightarrow$ Today is sunny or Today is cloudy

§1.2 Connectors (functions on propositions)

§1.2.1 Negation ($\neg$)

P: Today is sunny
$\neg P$: Today is not sunny

Truth Table

P	$\neg Q$
T	F
F	T

§1.2.2 Or ($\vee$)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

§1.2.3 And ($\wedge$)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	T

§1.3 Implication

$P \Rightarrow Q$ means P implies Q
In other words: if P, then Q is true.
False can imply anything.
We will come back to this.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

§1.4 Equivalence

$P \Leftrightarrow Q$
Means they have same truth value on a truth table.
If you break this down to implications you get:
$[(P \Rightarrow Q) \wedge (Q \Rightarrow P)] \Leftrightarrow [P \Leftrightarrow Q]$

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

P	$\neg P$	Q	$P \Rightarrow Q$	$\neg P \vee Q$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

§1.5 Rules for Computing

§1.5.1 Distibutive Laws

$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$
$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$

§1.5.2 Associative Laws

$X \vee (Y \vee Z) \Leftrightarrow (X \vee Y) \vee Z$
$X \wedge (Y \wedge Z) \Leftrightarrow (X \wedge Y) \wedge Z$

§1.5.3 De Morgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$
The rules can be expressed in English as:

- the negation of a disjunction is the conjunction of the negations
- the negation of a conjunction is the disjunction of the negations

Proof of De Morgan's law using proof table

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$\neg P$	$\neg Q$
T	T	F	F	F	F
T	F	F	F	F	T
F	T	F	F	T	F
F	F	T	T	T	T

§1.5.4 Transitivity

$[(P \Rightarrow R) \wedge (R \Rightarrow Q)] \Rightarrow (P \Rightarrow Q)$

P	Q	R	$P \Rightarrow R$	$R \Rightarrow P$	$P \Rightarrow R \wedge R \Rightarrow P$	$P \Rightarrow Q$
T	T	T	T	T	T	
T	T	F	F	T	F	
T	F	T	T	T	T	
T	F	F	F	T	F	
F	T	T	T	F	F	
F	T	F	T	F	F	
F	F	T	T	F	F	
F	F	F	T	F	F	

I'll do this later lol

§1.6 Notation

$\forall$
$\exists \setminus \exists$

§1.7 Logical Concepts

§1.7.1 Logical Truth

Logical truth, sometimes called tautology, means a proposition is true in all possible cases.
For example: $A \vee \neg A$ is always true.

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

§1.7.2 Logical Contradiction

Similar to a logical truth, a logical condtradiction means a proposition is false in all possible cases For example: $A \wedge \neg A$ is always false.

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

§1.7.3 Law of Logically True Conjunct

If Y is a logical truth, then $X \wedge Y \Leftrightarrow X$

§1.7.4 Law of Contradictory Disjunct

If Y is a contradiction, then $X \vee Y \Leftrightarrow X$

§1.7.5 Disjunctive Normal Form

Forumla consisting of disjunction of conjunctions, described as an $\vee$ of $\wedge$
$A \vee \neg B \Leftrightarrow (A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$
Beyond the scope of this class.

§1.7.6 Expressive Completeness

A connective, or set of connectives is expressively complete iff every truth function can be represented just using the connective or connectives.
Example is sheffer stroke, also known as NAND
Simpler example is $\neg$ and $\wedge$.
Beyond the scope of this class.

§1.7.7 Logically Valid vs Logically Sound

Logically valid means if the premises are true, the conclusion must be true. In other words, an argument is logically valid iff it takes a form that makes it impossible for the premises to be true and the conclusion to be false. Doesn't mean the argument is actually true, just its structure.
An argument is logically sound if the premises are true and is logically valid.

§1.8 Proof Techniques (basic ones)

i $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

ii $(P \Rightarrow R) \wedge (R \Rightarrow Q) \Rightarrow (P \Rightarrow Q)$

iii $(P \vee Q) \Rightarrow R \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$
Note: false can imply anything

iv $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

v $[(P \Rightarrow Q) \Leftrightarrow True] \Leftrightarrow [\neg(P \Rightarrow Q) \Leftrightarrow False]$

§2 Vector space

§2.1 Field

Def: A field F is a set with two operations (+, ·) satisfying
$\forall\ x, y \in F$

$$\exists!\ z \in F\ s.t.\ z = x + y$$
$$\exists!\ w \in F\ s.t.\ w = x \cdot y$$

This is called closure. Other properties:
$\forall\ a, b, c \in F$

1. a + b = b + a
2. (a + b) + c = a + (b + c)
3. $\exists\ 0 \in F,\ \exists\ 1 \in F$
   $0 + a = a,\ 1 \cdot a = a$
4. Additive and Multiplicative inverse
   $\forall\ a \in F,\ \forall\ b \in F \setminus \{0\}$
   $\exists\ c, d \in F\ s.t.\ a + b = 0,\ bd = 1$
5. $a \cdot (b + c) = ab + ac$

Examples of fields: $\mathbb{R},\ \mathbb{C}$

§2.2 Vector space

Def: A vector space V over a field F is a set with two operations

- addition
- scalar multiplication

which satisfies

1. $\forall\ x, y \in V$
   $x + y = y + x$
2. $\forall\ x, y, z \in V$
   $(x + y) + z = x + (y + x)$
3. $\exists\ 0 \in V s.t.\ x + 0 = x$
4. $\forall\ x \in V\ \exists\ y \in V\ s.t.\ x + y = 0$
5. $\forall\ x \in V\ s.t.\ 1 \cdot x = x$ (1 from field F)
6. $\forall\ a, b \in F,\ \forall\ x \in V$
   $(a \cdot b) \cdot x = a \cdot (b \cdot x)$
7. $\forall\ a \in F,\ \exists\ x, y \in V$
   $a(x + y) = ax + ay$

8. $\forall\ a, b \in F,\ \forall\ x \in V$
   $(a + b)x = ax + bx$

9. $\forall\ x, y \in V\ \exists!\ z \in V\ s.t.\ x + y = z$

10. $\forall\ x \in F,\ \forall\ x \in V\ \exists!\ w\ s.t.\ w = x + y$

$\Rightarrow$ Elements of F are are called scalars

§2.3 Example of a vector space: tuples of scalars

An n-tuple is a sequence (or ordered list of n elements, aka order matters), where n is a non-negative integer. The set of all n-tuples with entries from a field F is denoted by F^n
$F^n = \{(a_1, a_2, a_3, \ldots, a_n)\ a_1 \in F\}$

§2.3.1 Adding n-tuples

$u = (a_1, a_2, \ldots, a_n)\quad v = (b_1, b_2, \ldots, b_n)$
$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n)$

§2.3.2 Multiplying n-tuples with a scalar

$c \in F$
$c \cdot u = (ca_1, ca_2, \ldots, ca_n)$

§2.4 Another example of vector space: Matrices

$M_{mxn}(F)$ is the set of matrices with element in F of dimensions m x n
Generic matrix in M where $a \in F$:

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

§2.4.1 Caveat: How do we define $A = B$?

$$A = B \Leftrightarrow A_{i,j} = B_{i,j} \text{ for } 1 \leq i \leq m \\ 1 \leq j \leq n$$

§2.4.2 Rules for matrices

Addition: $(A + B)_{i,j} = A_{i,j} + B_{i,j}$
Multiplication by scalar: $(cA)_{i,j} = cA_{i,j}$

An example: $F = \mathbb{R}\quad M_{2x2}(\mathbb{R})$
An element of $M_{2x2}(\mathbb{R})$ is for example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} \pi & 1 \\ \sqrt{2} & 3 \end{pmatrix}$$

$$(A + B) = \begin{pmatrix} 1 + \pi & 2 + 1 \\ 3 + \sqrt{2} & 4 + 3 \end{pmatrix} = \begin{pmatrix} 1 + \pi & 3 \\ 3 + \sqrt{2} & 7 \end{pmatrix}$$

$$c = 5 \qquad cA = \begin{pmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot 3 & 5 \cdot 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

§2.5 Hey, functions can be vector spaces too.

Let's call F a vector space. In this vector/function space, there contains the set of all functions from $\mathbb{S} \to \mathbb{F}$. $\mathbb{S}$ is just a nonempty set that represents our domain/input for our function and $\mathbb{F}$ is our field where the outputs are.
$f, g \in F$

§2.5.1 Equality

$f = g \Leftrightarrow f(s) = g(s)\ \forall\ s \in \mathbb{F}$

§2.5.2 Operations

$(f + g)(s) = f(s) + g(s)$
$(cf)(s) = c[f(s)]$

§2.6 How can be abstract and generalize this idea of a vector space?

We want things like this to be true in all vector spaces:
$x + \cancel{z} = y + \cancel{z} \Rightarrow x = y$
We need to prove this.

Theorem (1.1 Cancellation Law for Vector Addition)
If x, y, x are vectors in vector space V such that $x + z = y + z$. Then,

$$x = y$$

Proof. Given that $z \in V$

$$\text{VS3:} \quad \exists v \in V \text{ such that } z + v = 0$$
$$\text{VS1:} \quad \exists 0 \in V \text{ such that } x + 0 = x$$

$$\begin{aligned} x = x + 0 &= x + (z + x) \\ &= (x + z) + v \\ &= (y + z) + v \\ &= y + 0 \\ &= y \\ \therefore x &= y \end{aligned}$$

□

Corollary
The additive inverse is unique.

Proof. Suppose u, v are the additive inverse of x.

$$x + u = 0 \quad x + v = 0$$

$$\begin{aligned} x + u &= x + v && \text{(Transitive property)} \\ u + x &= v + x && \text{(Commutative property)} \\ u &= v && \text{(Theorem 1.1)} \end{aligned}$$

□

Corollary
The vector 0 is unique.

Proof. Suppose $u, v \in V$ satisfies the "zero property", which is defined as:

$$\begin{aligned} \forall x \in V \quad & x + u = x \Rightarrow v + u = v \\ \forall x \in V \quad & x + v = x \Rightarrow u + v = u \\ u = u + v &= v + u = v && \text{(Transitive property)} \\ u &= v && \text{(Theorem 1.1)} \end{aligned}$$

□

Theorem (1.2 In any vector space the following statements are true.)

a. $0\mathbf{x} = \mathbf{0}$
   $\forall \mathbf{x} \in \mathbf{V} \quad 0 \in F$
   Where the left 0 is a scalar, $\mathbf{x}$ is a vector, and right $\mathbf{0}$ is a vector. The equality is a vector equality.
   Any vector multiplied by the 0 scalar will result in the 0 vector.

b. $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$
   $\forall x \in \mathbf{V} \quad a \in F$
   Where $-x$ is the additive inverse of x.
   Whether the negative is in the scalar, vector, or after the scalar vector multiplication, the result will be the same.

c. $a\mathbf{0} = \mathbf{0}$
   $\forall a \in F \quad \mathbf{0} \in \mathbf{V}$
   Any scalar multiplied by the 0 vector will result in the 0 vector.

§2.7 Subspaces

$\mathbf{W} \subset \mathbf{V}$
Is $\mathbf{W}$ a subspace; does it behaves like a vector space on its own?

Definition (1.3 Subspaces)**.** A subset W of V(vector space over F) is called a subspace of $V \Leftrightarrow W$ is a vector space over F with the operations inherited from V.

Trival subspace: $V, \{0\}$

§2.7.1 Conditions to be subspace (ie vector space)

a. $\mathbf{x} + \mathbf{y} \in \mathbf{W}$

b. $c\mathbf{x} \in \mathbf{W}$

c. $\exists \mathbf{0} \in \mathbf{W}, \mathbf{x} + \mathbf{0} = \mathbf{x}$

d. $\forall \mathbf{x} \in \mathbf{W} \quad \exists \mathbf{y} \in \mathbf{W}$
$\mathbf{x} + \mathbf{y} = \mathbf{0}$

Theorem (1.3 Conditions for subspace)
$\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if the three conditions hold in $\mathbf{W}$ defined in $\mathbf{V}$.

a. $\mathbf{0} \in \mathbf{W}$
$\mathbf{0}$ vector exists in W.

b. $\mathbf{x} + \mathbf{y} \in \mathbf{W}$
Closure under vector addition.

c. $c\mathbf{x} \in \mathbf{W}$
Closure under scalar multiplication.