Math 341: Linear Alegbra

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1 Propositional Logic

System of figuring out if something is true or false.

Proposition $\rightarrow$ Statement $\rightarrow$ True or False

Examples:

- P: Today is sunny
- P: I'm 5'11

1.1 We can compose them

P: Today is sunny

Q: Today is rainy

$P \lor Q \implies$ Today is sunny or Today is cloudy

1.2 Connectors (functions on propositions)

1.2.1 Negation ($\neg$)

P: Today is sunny

$\neg P$: Today is not sunny

Truth Table

P	$\neg Q$
T	F
F	T

1.2.2 Or ($\vee$)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1.2.3 And ($\wedge$)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	T

1.3 Implication

$P \implies Q$ means P implies Q

In other words: if P, then Q is true.

False can imply anything.

We will come back to this.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

1.4 Equivalence

$P \Leftrightarrow Q$

Means they have same truth value on a truth table.

If you break this down to implications you get:

$[(P \implies Q) \wedge (Q \implies P)] \Leftrightarrow [P \Leftrightarrow Q]$

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$(P \implies Q) \Leftrightarrow (\neg P \vee Q)$

P	$\neg P$	Q	$P \implies Q$	$\neg P \vee Q$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

1.5 Rules for Computing

1.5.1 Distibutive Laws

$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$

$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$

1.5.2 Associative Laws

$X \vee (Y \vee Z) \Leftrightarrow (X \vee Y) \vee Z$

$X \wedge (Y \wedge Z) \Leftrightarrow (X \wedge Y) \wedge Z$

1.5.3 De Morgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$

The rules can be expressed in English as:

- the negation of a disjunction is the conjunction of the negations
- the negation of a conjunction is the disjunction of the negations

Proof of De Morgan's law using proof table

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$\neg P$	$\neg Q$
T	T	F	F	F	F
T	F	F	F	F	T
F	T	F	F	T	F
F	F	T	T	T	T

1.5.4 Transitivity

$[(P \implies R) \wedge (R \implies Q)] \implies (P \implies Q)$

P	Q	R	$P \implies R$	$R \implies P$	$P \implies R \wedge R \implies P$	$P \implies Q$
T	T	T	T	T	T	
T	T	F	F	T	F	
T	F	T	T	T	T	
T	F	F	F	T	F	
F	T	T	T	F	F	
F	T	F	T	F	F	
F	F	T	T	F	F	
F	F	F	T	F	F	

I'll do this later lol

1.6 Notation

$\forall$

$\exists \setminus \exists$

1.7 Logical Concepts

1.7.1 Logical Truth

Logical truth, sometimes called tautology, means a proposition is true in all possible cases.

For example: $A \vee \neg A$ is always true.

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

1.7.2 Logical Contradiction

Similar to a logical truth, a logical condtradiction means a proposition is false in all possible cases For example: $A \wedge \neg A$ is always false.

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

1.7.3 Law of Logically True Conjunct

If Y is a logical truth, then $X \wedge Y \Leftrightarrow X$

1.7.4 Law of Contradictory Disjunct

If Y is a contradiction, then $X \vee Y \Leftrightarrow X$

1.7.5 Disjunctive Normal Form

Forumla consisting of disjunction of conjunctions, described as an $\vee$ of $\wedge$

asdasd iasd

1.7.6 Expressive Completeness

Example is sheffer stroke. also known as NOR

1.7.7 Logically Valid vs Logically Sound

1.7.8 Law of Conditional

1.7.9 Law of Contraposition

1.8 Proof Techniques (basic ones)

i $(P \Leftrightarrow Q) \Leftrightarrow (P \implies Q) \wedge (Q \implies P)$

ii $(P \implies R) \wedge (R \implies Q) \implies (P \implies Q)$

iii finish this

2 Vector space

2.1 Field

Def: A field F is a set with two operations (+, ·) satisfying

$\forall\ x, y \in F$

$$\exists!\ z \in F\ s.t.\ z = x + y$$

$$\exists!\ w \in F\ s.t.\ w = x \cdot y$$

This is called closure. Other properties:

$\forall\ a, b, c \in F$

1. a + b = b + a
2. (a + b) + c = a + (b + c)
3. $\exists\ 0 \in F,\ \exists\ 1 \in F$

   $0 + a = a,\ 1 \cdot a = a$
4. Additive and Multiplicative inverse

   $\forall\ a \in F,\ \forall\ b \in F \setminus \{0\}$

   $\exists\ c, d \in F\ s.t.\ a + b = 0,\ bd = 1$
5. $a \cdot (b + c) = ab + ac$

Examples of fields: $\mathbb{R},\ \mathbb{C}$

2.2 Vector space

Def: A vector space V over a field F is a set with two operations

- addition
- scalar multiplication

which satisfies

1. $\forall\ x, y \in V$

   $x + y = y + x$
2. $\forall\ x, y, z \in V$

   $(x + y) + z = x + (y + x)$

3. $\exists\ 0 \in V s.t.\ x + 0 = x$

4. $\forall\ x \in V\ \exists\ y \in V\ s.t.\ x + y = 0$

5. $\forall\ x \in V\ s.t.\ 1 \cdot x = x$ (1 from field F)

6. $\forall\ a, b \in F,\ \forall\ x \in V$

   $(a \cdot b) \cdot x = a \cdot (b \cdot x)$

7. $\forall\ a \in F,\ \exists\ x, y \in V$

   $a(x + y) = ax + ay$

8. $\forall\ a, b \in F,\ \forall\ x \in V$

   $(a + b)x = ax + bx$

9. $\forall\ x, y \in V\ \exists!\ z \in V\ s.t.\ x + y = z$

10. $\forall\ x \in F,\ \forall\ x \in V\ \exists!\ w\ s.t.\ w = x + y$

$\implies$ Elements of F are are called scalars

2.3 Tuples of scalars

An n-tuple is a sequence (or ordered list of n elements, aka order matters), where n is a non-negative integer. The set of all n-tuples with entries from a field F is denoted by F^n

$F^n = \{(a_1, a_2, a_3, \ldots, a_n)\ a_1 \in F\}$

2.3.1 Adding n-tuples

$u = (a_1, a_2, \ldots, a_n)\quad v = (b_1, b_2, \ldots, b_n)$

$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n)$

2.3.2 Multiplying n-tuples with a scalar

$c \in F$

$c \cdot u = (ca_1, ca_2, \ldots, ca_n)$