Math 341: Linear Alegbra

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1 Propositional Logic

System of figuring out if something is true or false.

Proposition $\rightarrow$ Statement $\rightarrow$ True or False

Examples:

- P: Today is sunny
- P: I'm 5'11

1.1 We can compose them

P: Today is sunny

Q: Today is rainy

$P \vee Q \implies$ Today is sunny or Today is cloudy

1.2 Connectors (functions on propositions)

1.2.1 Negation ($\neg$)

P: Today is sunny

$\neg P$: Today is not sunny

Truth Table

P	$\neg Q$
T	F
F	T

1.2.2 Or ($\vee$)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1.2.3 And ($\wedge$)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	T

1.3 Implication

$P \implies Q$ means P implies Q

In other words: if P, then Q is true.

False can imply anything.

We will come back to this.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

1.4 Equivalence

$P \Leftrightarrow Q$

Means they have same truth value on a truth table.

If you break this down to implications you get:

$[(P \implies Q) \wedge (Q \implies P)] \Leftrightarrow [P \Leftrightarrow Q]$

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$(P \implies Q) \Leftrightarrow (\neg P \vee Q)$

P	$\neg P$	Q	$P \implies Q$	$\neg P \vee Q$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

1.5 Rules for Computing

1.5.1 Distibutive Laws

$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$

$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$

1.5.2 Associative Laws

$X \vee (Y \vee Z) \Leftrightarrow (X \vee Y) \vee Z$

$X \wedge (Y \wedge Z) \Leftrightarrow (X \wedge Y) \wedge Z$

1.5.3 De Morgan's Laws

$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$

The rules can be expressed in English as:

- the negation of a disjunction is the conjunction of the negations
- the negation of a conjunction is the disjunction of the negations

Proof of De Morgan's law using proof table

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$\neg P$	$\neg Q$
T	T	F	F	F	F
T	F	F	F	F	T
F	T	F	F	T	F
F	F	T	T	T	T

1.5.4 Transitivity

$[(P \implies R) \wedge (R \implies Q)] \implies (P \implies Q)$

P	Q	R	$P \implies R$	$R \implies P$	$P \implies R \wedge R \implies P$	$P \implies Q$
T	T	T	T	T	T	
T	T	F	F	T	F	
T	F	T	T	T	T	
T	F	F	F	T	F	
F	T	T	T	F	F	
F	T	F	T	F	F	
F	F	T	T	F	F	
F	F	F	T	F	F	

I'll do this later lol

1.6 Notation

$\forall$

$\exists \setminus \exists$

1.7 Logical Concepts

1.7.1 Logical Truth

Logical truth, sometimes called tautology, means a proposition is true in all possible cases.

For example: $A \vee \neg A$ is always true.

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

1.7.2 Logical Contradiction

Similar to a logical truth, a logical condtradiction means a proposition is false in all possible cases For example: $A \wedge \neg A$ is always true.

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

1.7.3 Law of Logically True Conjunct

If Y is a logical truth, then $X \wedge Y \Leftrightarrow X$

1.7.4 Law of Contradictory Disjunct

If Y is a contradiction, then $X \vee Y \Leftrightarrow X$

1.8 Proof Techniques (basic ones)

i $(P \Leftrightarrow Q) \Leftrightarrow (P \implies Q) \wedge (Q \implies P)$

ii $(P \implies R) \wedge (R \implies Q) \implies (P \implies Q)$

iii finish this

2 Vector space

2.1 Field

Def: A field F is a set with two operations $(+, \cdot\ )$ satisfying

$\forall\ x, y \in F$

$$\exists!\ z \in F\ s.t.\ z = x + y$$

$$\exists!\ w \in F\ s.t.\ w = x \cdot y$$

This is called closure. Other properties:

$\forall\ a, b, c \in F$

1. a + b = b + a
2. (a + b) + c = a + (b + c)
3. $\exists\ 0 \in F,\ \exists\ 1 \in F$

   $0 + a = a,\ 1 \cdot a = a$
4. Additive and Multiplicative inverse

   $\forall\ a \in F,\ \forall\ b \in F \setminus \{0\}$

   $\exists\ c, d \in F\ s.t.\ a + b = 0,\ bd = 1$
5. $a \cdot (b + c) = ab + ac$

Examples of fields: $\mathbb{R},\ \mathbb{C}$

2.2 Vector space

Def: A vector space V over a field F is a set with two operations

- addition
- scalar multiplication

which satisfies

1. $\forall\ x, y \in V$

   $x + y = y + x$
2. $\forall\ x, y, z \in V$

   $(x + y) + z = x + (y + x)$
3. $\exists\ 0 \in V s.t.\ x + 0 = x$
4. $\forall\ x \in V\ \exists\ y \in V\ s.t.\ x + y = 0$
5. $\forall\ x \in V\ s.t.\ 1 \cdot x = x$ (1 from field F)
6. $\forall\ a, b \in F,\ \forall\ x \in V$

   $(a \cdot b) \cdot x = a \cdot (b \cdot x)$
7. $\forall\ a \in F,\ \exists\ x, y \in V$

   $a(x + y) = ax + ay$
8. $\forall\ a, b \in F,\ \forall\ x \in V$

   $(a + b)x = ax + bx$
9. $\forall\ x, y \in V\ \exists!\ z \in V\ s.t.\ x + y = z$

10. $\forall\ x \in F,\ \forall\ x \in V\ \exists!\ w\ s.t.\ w = x + y$

$\implies$ Elements of F are are called scalars

2.3 Tuples of scalars

An n-tuple is a sequence (or ordered list of n elements, aka order matters), where n is a non-negative integer. The set of all n-tuples with entries from a field F is denoted by F^n

$F^n = \{(a_1, a_2, a_3, \ldots, a_n)\ a_1 \in F\}$

2.3.1 Adding n-tuples

$u = (a_1, a_2, \ldots, a_n)\ \ v = (b_1, b_2, \ldots, b_n)$

$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n)$

2.3.2 Multiplying n-tuples with a scalar

$c \in F$

$c \cdot u = (ca_1, ca_2, \ldots, ca_n)$

2.3.3 other stuff