Key

Taylor Series & Polynomials MC Review

Select the correct capital letter. NO CALCULATOR unless specified otherwise.

A 1. Let $T_5(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5$ be the fifth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f'''(0)$?

$f'(x) = 6x - 15x^2$
$f''(x) = 6 - 30x$
$f'''(x) = -30$ $\quad f'''(0) = -30$

(A) −30 (B) −15 (C) −5 (D) $-\frac{5}{6}$ (E) $-\frac{1}{6}$

D 2. For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

$\left(\frac{3}{4}\right)^n$ converges by Geometric Series Test

(A) 6 (B) 5 (C) 4 $\boxed{\text{(D) 3}}$ (E) 2

$\frac{(-1)^{3n}}{n}$ converges by AST.

C 3. (Calculator Permitted) The Taylor series for $\ln x$, centered at $x = 1$, is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$. Let f be the function given by the sum of the first three nonzero terms of this series. The maximum value of $|\ln x - f(x)|$ for $0.3 \le x \le 1.7$ is which of the following?

$f(x) = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$

(A) 0.030 (B) 0.039 $\boxed{\text{(C) 0.145}}$ (D) 0.153 (E) 0.529

For $x = 0.3$, $|\ln(0.3) - f(0.3)| \approx 0.14464$ *

For $x = 1.7$, $|\ln(1.7) - f(1.7)| \approx 0.03875$

Since graph is concave up from endpoint to endpoint, one of the endpts must be max value

B 4. What are the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

(A) $-3 < x < -1$ (B) $-3 \le x < -1$ (C) $-3 \le x \le -1$ (D) $-1 \le x < 1$ (E) $-1 \le x \le 1$

Fails p-series test

$\frac{(-3+2)^n}{\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$

Converges by AST $\lim_{n\to\infty} a_n = 0$ ✓

test $x=-1.1$

$\frac{(-1.1+2)^n}{\sqrt{n}} = \frac{(0.9)^n}{\sqrt{n}} = (0.9)^n \cdot \frac{1}{\sqrt{n}}$

converges by Direct Comparison with Geometric series $|r| = 0.9 < 1$

A 5. (Calculator Permitted) The graph of the function represented by the Maclaurin series $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots$ intersects the graph of $y = x^3$ at $x =$

(A) 0.773 (B) 0.865 (C) 0.929 (D) 1.000 (E) 1.857

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

* Look for intersection between $y = x^3$ and $y = e^{-x}$

Graph $y = x^3 - e^{-x}$ and find x-intercept

$\boxed{x = 0.773}$

D 6. Which of the following sequences converge?

I. $\left\{\frac{5n}{2n-1}\right\}$ II. $\left\{\frac{e^n}{n}\right\}$ III. $\left\{\frac{e^n}{1+e^n}\right\}$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

* sequence converges as long as $\lim_{n\to\infty} a_n = L$ ← Real number

✓ $\lim_{n\to\infty} \frac{5n}{2n-1} = \frac{5}{2}$

✗ $\lim_{n\to\infty} \frac{e^n}{n} = \infty$

✓ $\lim_{n\to\infty} \frac{e^n}{1+e^n} = 1$

E 7. What is the approximation of the value of $\sin 1$ obtained by using the fifth-degree Taylor polynomial about $x=0$ for $\sin x$?

(A) $1-\frac{1}{2}+\frac{1}{24}$ (B) $1-\frac{1}{2}+\frac{1}{4}$ (C) $1-\frac{1}{3}+\frac{1}{5}$ (D) $1-\frac{1}{4}+\frac{1}{8}$ $\boxed{\text{(E) } 1-\frac{1}{6}+\frac{1}{120}}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} = 1 - \frac{1}{6} + \frac{1}{120}$$

B 8. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$ II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n}$

(A) None $\boxed{\text{(B) II only}}$ (C) III only (D) I and II only (E) I and III only

x i) $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$ Diverges by n^{th} term test

✓ ii) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \frac{(-1)^n}{n}$ converges by AST (Alternating Harmonic Series)

x iii) $\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic series diverge

A 9. If $\lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

$\boxed{\text{(A) } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}}$ (B) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (C) $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges (E) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges

Integral Test (If $\int_k^{\infty} f(x)\,dx$ converges then $\sum_{n=k}^{\infty} a_n$ converge as well)

D 10. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1) =$

(A) 0 (B) a_1 (C) $\sum_{n=0}^{\infty} a_n$ $\boxed{\text{(D) } \sum_{n=1}^{\infty} na_n}$ (E) $\sum_{n=1}^{\infty} na_n^{n-1}$

$$f'(x) = \sum a_n \cdot n \cdot x^{n-1}$$

$$f'(1) = \sum a_n \cdot n \cdot 1^{n-1} = \underline{\underline{\sum a_n \cdot n}}$$

C 11. What is the value of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$? $= \frac{2^n \cdot 2}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n$

(A) 1 (B) 2 $\boxed{\text{(C) } 4}$ (D) 6 (E) The series diverges

$$S = \frac{a_1}{1-r} = \frac{\frac{4}{3}}{1-\frac{2}{3}} = \frac{\frac{4}{3}}{\frac{1}{3}} = \boxed{4}$$

D 12. The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$? $\longrightarrow x^2\left(\frac{1}{1-x^2}\right) = x^2\left[1 + x^2 + x^4 + x^6 + x^8 + \ldots\right]$

(A) $1 + x^2 + x^4 + x^6 + x^8 + \cdots$ (B) $x^2 + x^3 + x^4 + x^5 + \cdots$ (C) $x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$

$\boxed{\text{(D) } x^2 + x^4 + x^6 + x^8 + \cdots}$ (E) $x^2 - x^4 + x^6 - x^8 + \cdots$

D 13. A function f has a Maclaurin series given by $\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}+\cdots+\frac{x^{n+3}}{(n+1)!}+\cdots$. Which of the following is an expression for $f(x)$?

resembles pattern of e^x: $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^n}{n!}$

(A) $-3x\sin x+3x^2$ (B) $-\cos\left(x^2\right)+1$ (C) $-x^2\cos x+x^2$

(D) $x^2e^x-x^3-x^2$ (E) $e^{x^2}-x^2-1$

$$x^2\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right]-x^3-x^2$$

$$\cancel{x^2}+\cancel{x^3}+\boxed{\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}}-\cancel{x^3}-\cancel{x^2}$$

matches series

$$e^{x^2}=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots$$

$$\cancel{1}+\cancel{x^2}+\boxed{\frac{x^4}{2!}+\frac{x^6}{3!}}-\cancel{x^2}-\cancel{1}$$

Does not match!

D 14. Which of the following series diverge?

I. $\sum_{n=0}^{\infty}\left(\frac{\sin 2}{\pi}\right)^n$ II. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$ III. $\sum_{n=1}^{\infty}\left(\frac{e^n}{e^n+1}\right)$

(A) III only (B) I and II only (C) I and III only $\boxed{\text{(D) II and III only}}$ (E) I, II, and III

i) $\frac{\sin 2}{\pi}<1$, converges by GST

ii) $\sum\frac{1}{n^{1/3}}$ diverges by p-series test $p<1$

iii) $\sum\left(\frac{e^n}{e^n+1}\right)$ diverges by n^{th} term test $\lim_{n\to\infty}\frac{e^n}{e^n+1}=1\neq 0$

D 15. What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x=0$?

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) 1 $\boxed{\text{(D) 3}}$ (E) 6

$\boxed{\frac{f''(0)}{2!}}(x-0)^2$

Coefficient

$f(x)=(1+x)^{-2}$

$f'(x)=-2(1+x)^{-3}$

$f''(x)=6(1+x)^{-4}$

$f''(0)=6(1)^{-4}=6$

$\frac{f''(0)}{2!}(x-0)^2$

$\boxed{\frac{6}{2!}}=\boxed{3}$

C 16. The sum of the infinite geometric series $\overset{\times 3\ \ \times 3}{\underset{\times 8\ \ \times 8}{\frac{3}{2}+\frac{9}{16}+\frac{27}{128}}}+\frac{81}{1024}+\cdots$ is

(A) 1.60 (B) 2.35 $\boxed{\text{(C) } 2.40}$ (D) 2.45 (E) 2.50

$r=\frac{3}{8}$ $S=\frac{a_1}{1-r}=\frac{\frac{3}{2}}{1-\frac{3}{8}}=\frac{\frac{3}{2}}{\frac{5}{8}}=\frac{3}{2}\cdot\frac{8}{5}=\frac{24}{10}=\boxed{2.4}$

E 17. What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n3^n}$ converges?

(A) $-3\le x\le 3$ (B) $-3<x<3$ (C) $-1<x\le 3$ (D) $-1\le x\le 5$ $\boxed{\text{(E) } -1\le x<5}$

*Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-2)^n}(x-2)\cdot n\cancel{3^n}}{(n+1)\cdot\cancel{3^n}\cdot 3\,\cancel{(x-2)^n}}\right| = \left|\frac{x-2}{3}\right|<1 \to |x-2|<3$$

$-3<x-2<3$

$-1<x<5$

*Test endpts:

$x=-1$ $\frac{(-1-2)^n}{n3^n}=\frac{-1}{n}$ converges | $x=5$ $\sum\frac{(5-2)^n}{n\cdot 3^n}=\frac{1}{n}$ diverges

D 18. The Taylor series for $\sin x$ about $x=0$ is $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$. If f is a function such that $f'(x)=\sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x=0$ os

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 $\boxed{\text{(D) } -\frac{1}{42}}$ (E) $-\frac{1}{7!}$

$$f'(x)=\sin(x^2)=x^2-\frac{(x^2)^3}{3!}+\frac{(x^2)^5}{5!}=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}$$

$$\int x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}=\frac{x^3}{3}-\frac{x^7}{3!\cdot 7}+\frac{x^{11}}{5!\cdot 11}$$

$$\hookrightarrow \frac{-1}{3!\,7}x^7=\frac{-1}{6\cdot 7}x^7=\boxed{-\frac{1}{42}x^7}$$

D 19. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ is the Taylor series about zero for which of the following functions?

(A) $\sin x$ (B) $\cos x$ (C) e^x $\boxed{\text{(D) } e^{-x}}$ (E) $\ln(1+x)$

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!}$

then $e^{(-x)} = \frac{(-1)^n x^n}{n!}$

B 20. For what values of x does the series $1+2^x+3^x+4^x+\cdots+n^x+\cdots$ converge?

(A) No values of x (B) $x<-1$ (C) $x \geq -1$ (D) $x>-1$ (E) All values of x

$n^{-1.1} = \frac{1}{n^{1.1}}$ → converges by p-series test, $p > 1$

E 21. The complete interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2}$ is $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2}$

(A) $0<x<2$ (B) $0 \leq x \leq 2$ (C) $-2<x \leq 0$ (D) $-2 \leq x<0$ $\boxed{\text{(E) } -2 \leq x \leq 0}$

$\lim_{n \to \infty} \left| \frac{(x+1)^{n+1}}{(n+1)^2} \cdot \frac{(n)^2}{(x+1)^n} \right| = |x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

↳ Test endpts

If $x = -2$, $\frac{(-2+1)^n}{n^2} = \frac{(-1)^n}{n^2}$ converges p-series

If $x = 0$ $\frac{(0+1)^n}{n^2} = \frac{1}{n^2}$ converges p-series

A 21. For $-1<x<1$, if $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{2n-1}$, then $f'(x)=$

$\boxed{\text{(A) } \sum_{n=1}^{\infty}\boxed{(-1)^{n+1}}x^{2n-2}}$ (B) $\sum_{n=1}^{\infty}(-1)^n x^{2n-2}$ (C) $\sum_{n=1}^{\infty}(-1)^{2n}x^{2n}$ (D) $\sum_{n=1}^{\infty}(-1)^n x^{2n}$ (E) $\sum_{n=1}^{\infty}(-1)^{n+1}x^{2n}$

$$f'(x)=\frac{(2\cancel{n-1})\cdot x^{2n-2}}{(2\cancel{n-1})} \quad (-1)^{n+1}x^{2n-2}$$

E 22. The coefficient of x^3 in the Taylor series fo e^{3x} about $x=0$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$ $\boxed{\text{(E) } \frac{9}{2}}$

$$e^x = 1+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$e^{3x} = 1+(3x)+\frac{(3x)^2}{2!}+\boxed{\frac{(3x)^3}{3!}}$$

$$\frac{27}{6}x^3 = \boxed{\frac{9}{2}x^3}$$

C 23. $\sum_{i=n}^{\infty}\left(\frac{1}{3}\right)^i=$

$r=\frac{1}{3}$

$a_1=\left(\frac{1}{3}\right)^n$

(A) $\frac{3}{2}-\left(\frac{1}{3}\right)^n$ (B) $\frac{3}{2}\left[1-\left(\frac{1}{3}\right)^n\right]$ $\boxed{\text{(C) } \frac{3}{2}\left(\frac{1}{3}\right)^n}$ (D) $\frac{2}{3}\left(\frac{1}{3}\right)^n$ (E) $\frac{2}{3}\left(\frac{1}{3}\right)^{n+1}$

$$S=\frac{a_1}{1-r}=\frac{\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}=\frac{\left(\frac{1}{3}\right)^n}{\frac{2}{3}}=\frac{3}{2}\left(\frac{1}{3}\right)^n$$

A 24. Which of the followign series converge?

I. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$ II. $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{2}\right)^{n}$ III. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

$\boxed{\text{(A) I only}}$ (B) II only (C) III only (D) I and III only (E) I, II, and III

√ i) Converges by AST $\lim_{n\to\infty}\frac{1}{2n+1}=0$

ii) diverges since $\left|\frac{3}{2}\right|>1$

iii) $\lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}dx \rightarrow u=\ln x$, $\frac{du}{dx}=\frac{1}{x}$ $du=xdu$ $\int\frac{1}{x\cdot u}\cdot x\,du$ $\int\frac{1}{u}du$

$\lim_{b\to\infty}\ln|\ln x|\Big]_2^b$

$\lim_{b\to\infty}\ln|\ln b|-\ln|\ln 2|=\infty$

Diverges, so Series diverge

A 25. If $s_n=\left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right)$, to what number does the sequence $\{s_n\}$ converge?

$\boxed{\text{(A) } \frac{1}{5}}$ (B) 1 (C) $\frac{5}{4}$ (D) $\left(\frac{5}{4}\right)^{100}$ (E) The sequence does not converge

$\lim_{n\to\infty}\left[\frac{(5+n)^{100}}{(4+n)^{100}}\cdot\frac{\cancel{5^n}}{\cancel{5^n}\cdot 5^1}\right]=\frac{1}{5}$

D 26. (Calculator Permitted) If $f(x)=\sum_{k=1}^{\infty}\left(\sin^2 x\right)^k$, then $f(1)$ is

(A) 0.369 (B) 0.585 (C) 2.400 $\boxed{\text{(D) 2.426}}$ (E) 3.426

$\sum_{n=1}^{\infty}\left[(\sin x)^2\right]^n$

$[\sin(1)]^2<1$ Converges

$S=\frac{a_1}{1-r}=\frac{(\sin 1)^2}{1-(\sin 1)^2}=\frac{0.708}{1-0.708}=\boxed{2.426}$

B
____ 27. Let the function given by $f(x)=\ln(3-x)$. The third-degree Taylor polynomial for f about $x=2$ is

(A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$ (B) $\boxed{-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}}$ (C) $(x-2)+(x-2)^2+(x-2)^3$

(D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$ (E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

$f(x)=\ln(3-x) \rightarrow f(2)=\ln 1=0$

$f'(x)=\frac{-1}{3-x}=-(3-x)^{-1} \rightarrow f'(2)=-1$

$f''(x)=(3-x)^{-2}(-1) \rightarrow f''(2)=-1$

$f'''(x)=2(3-x)^{-3}(-1) \rightarrow f'''(2)=-2$

$\frac{f^n(2)}{n!}(x-2)^n$

$0-1(x-2)-1\frac{(x-2)^2}{2}-\frac{2}{3!}(x-2)^3$

$\boxed{-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}}$

E
____ 28. (Calculator Permitted) Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x=1$. If the maximum value of the fifth derivative between $x=1$ and $x=3$ is 0.01, that is, $\left|f^{(5)}(x)\right|<0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

$R_n(x)=\left|\frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}\right|$

$x=3$
$c=1$

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 $\boxed{\text{(E) } 0.00267}$

$R_4(x)=\left|\frac{f^5(z)}{5!}(x-c)^5\right|$

$R_4(3)=\left|\frac{f^5(z)}{5!}(3-1)^5\right|$

$R_4(3)\leq\left|\frac{0.01}{5!}(2)^5\right|$

$R_4(3)\leq 0.00267$