CS365 Written Assignment 1

Khoa Cao

Due

Question 1

Proof. First, since we pick the largest number from the k numbers we sampled, if at least one of the k numbers is larger than the median, then the largest number we picked must be larger than the median.
Since we are sampling without replacement, the number of ways to sample k numbers from n numbers is:

$$|S| = \binom{n}{k}$$

The number of numbers, out of the n numbers, that are smaller than or equal to the median is:

$$|E| = \lfloor n/2 \rfloor$$

Note that if $k > \lceil n/2 \rceil$, then we are guaranteed to select a number larger than the median, since there are only $\lfloor n/2 \rfloor$ numbers that are smaller than or equal to the median.
Let E be the event that all k numbers we examined are smaller than or equal to the median of the n numbers. Then, the probability of us selecting a number larger than the median is:

$$\begin{aligned}
Pr(E^c) &= 1 - Pr(E) && \text{complement rule} \\
&= \begin{cases} 1 - \frac{|E|}{|S|} & \text{if } k \leq \lfloor n/2 \rfloor \\ 1 & \text{if } k > \lceil n/2 \rceil \end{cases} && \text{definition of probability} \\
&= \begin{cases} 1 - \frac{\binom{\lceil n/2 \rceil}{k}}{\binom{n}{k}} & \text{if } k \leq \lceil n/2 \rceil \\ 1 & \text{if } k > \lceil n/2 \rceil \end{cases} && \text{substituting in } |E| \text{ and } |S|
\end{aligned}$$

$\square$

Problem 2

Proof. Let X be the RV representing the number of successes we get. Then,

$$\begin{aligned} X &= \sum_{i=1}^{n} X_i && \text{definition of } X \\ X &\sim Binomial(n, p) && \text{since each coin flip is i.i.d. Bernoulli trial} \end{aligned}$$

Examining $\bar{X}$, the average number of successes, we have:

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i && \text{definition of } \bar{X} \\ &= \frac{1}{n}X && \text{definition of } X \end{aligned}$$

Using the linearity of expectation, we can compute the expected value of $\bar{X}$:

$$\begin{aligned} E(\bar{X}) &= E(\frac{1}{n}X) && \text{definition of } \bar{X} \\ &= \frac{1}{n}E(X) && \text{linearity of expectation} \\ &= \frac{1}{n}(np) && X \sim Binomial(n, p) \\ &= p && \text{simplifying} \end{aligned}$$

Next, we can compute the variance of $\bar{X}$:

$$\begin{aligned} Var(\bar{X}) &= Var(\frac{1}{n}X) && \text{definition of } \bar{X} \\ &= \left(\frac{1}{n}\right)^2 Var(X) && \text{property of variance} \\ &= \frac{1}{n^2}(np(1-p)) && X \sim Binomial(n, p) \\ &= \frac{p(1-p)}{n} && \text{simplifying} \end{aligned}$$

Using Chebyshev's inequality, we can give an upper bound on how likely the value of $\bar{X}$ differs from the bias of the coin p by at least $\frac{p}{10}$:

$$\begin{aligned}
Pr(|\bar{X} - E(\bar{X})| \geq \frac{p}{10}) &\leq \frac{Var(\bar{X})}{(p/10)^2} && \text{Chebyshev's inequality} \\
\Rightarrow Pr(|\bar{X} - p| \geq \frac{p}{10}) &\leq \frac{Var(\bar{X})}{(p/10)^2} && \text{substituting in } E(\bar{X}) \\
&= \frac{\frac{p(1-p)}{n}}{(p/10)^2} && \text{substituting in } Var(\bar{X}) \\
&= \frac{100(1-p)}{np} && \text{simplifying}
\end{aligned}$$

Therefore:

$$Pr(|\bar{X} - p| \geq \frac{p}{10}) \leq \frac{100(1-p)}{np}$$

□

Problem 3

Claim. *Let f and g be valid probability distributions defined over the same domain, S. Then, the convex combination of f and g*

$$h := \lambda f + (1 - \lambda)g$$

for some lambda $\in [0, 1]$ is also a valid probability distribution.

Proof. To show that h is a valid probability distribution, we need to show that:

(i) $h(x) \geq 0 \forall x \in S$

(ii) $\sum_{x \in S} h(x) = 1$

Since the domains of f and g are the same, h is also defined over the same domain, S.
For (i), we have:

$$\begin{aligned}
h(x) &= \lambda f(x) + (1 - \lambda)g(x) && \text{definition of } h \\
\sum_{x \in S} h(x) &= \sum_{x \in S} \lambda f(x) + (1 - \lambda)g(x) && \text{applying summation} \\
&= \sum_{x \in S} \lambda f(x) + \sum_{x \in S} (1 - \lambda)g(x) && \text{linearity of summation} \\
&= \lambda \sum_{x \in S} f(x) + (1 - \lambda) \sum_{x \in S} g(x) && \text{linearity of summation} \\
&= \lambda(1) + (1 - \lambda)(1) && \text{since } f \text{ and } g \text{ are valid probability distributions} \\
&= \lambda + 1 - \lambda && \text{simplifying} \\
&= 1 && \text{simplifying}
\end{aligned}$$

Thus, h satisfies (i).
For (ii), we have:

$$\begin{aligned}
\lambda f(x) &\geq 0 && \text{since } \lambda \in [0, 1] \text{ and } f(x) \geq 0 \\
(1 - \lambda)g(x) &\geq 0 && \text{since } (1 - \lambda) \in [0, 1] \text{ and } g(x) \geq 0 \\
\Rightarrow h(x) = \lambda f(x) + (1 - \lambda)g(x) &\geq 0 + 0 && \text{adding the two inequalities} \\
&= 0 && \text{simplifying}
\end{aligned}$$

Thus, h satisfies (ii).
Since h satisfies both (i) and (ii), h is a valid probability distribution. $\square$

Problem 4