CS365 Written Assignment 3
Khoa Cao
Due November 14, 2025

Question 1

1. Let $\mathbf{M}$ be the transition matrix of a Markov chain with $|V|$ states and let $\vec{pi}$ be the distribution:

$$\vec{\pi} = \left[\frac{1}{|V|}, \frac{1}{|V|}, \ldots, \frac{1}{|V|}\right]$$

First, we show that $\vec{\pi}$ is a probability vector:

$$\begin{aligned}
||\vec{\pi}||_1 &= \sum_{i=1}^{|V|} \frac{1}{|V|} && \text{Definition of } \vec{\pi} \\
&= \frac{|V|}{|V|} && \text{Sum of } |V| \text{ terms of } \frac{1}{|V|} \\
&= 1 && \text{(Simplification)} \\
&\Rightarrow \vec{\pi} \text{ is a probability vector}
\end{aligned}$$

Then, we show that $\vec{\pi}\mathbf{M} = \vec{\pi}$. Let $\vec{m}_i$ be the i-th column of $\mathbf{M}$:

$$\begin{aligned}
\vec{\pi}\mathbf{M} &= \begin{bmatrix} \vec{\pi} \cdot \vec{m}_1 & \vec{\pi} \cdot \vec{m}_2 & \cdots & \vec{\pi} \cdot \vec{m}_n \end{bmatrix} && \text{Definition of matrix multiplication} \\
&= \begin{bmatrix} \sum_{i=1}^{|V|} \frac{1}{|V|} m_{i1} & \sum_{i=1}^{|V|} \frac{1}{|V|} m_{i2} & \cdots & \sum_{i=1}^{|V|} \frac{1}{|V|} m_{in} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{|V|} \sum_{i=1}^{|V|} m_{i1} & \frac{1}{|V|} \sum_{i=1}^{|V|} m_{i2} & \cdots & \frac{1}{|V|} \sum_{i=1}^{|V|} m_{in} \end{bmatrix} && \text{Factoring out } \frac{1}{|V|} \\
&= \begin{bmatrix} \frac{1}{|V|} \cdot 1 & \frac{1}{|V|} \cdot 1 & \cdots & \frac{1}{|V|} \cdot 1 \end{bmatrix} && \mathbf{M} \text{ is symmetric and row-stochastic} \\
&= \vec{\pi} && \text{Definition of } \vec{\pi}
\end{aligned}$$

Since we have shown that $\vec{\pi}$ is a probability vector and that $\vec{\pi}\mathbf{M} = \vec{\pi}$, we conclude that $\vec{\pi}$ is a stationary distribution of the Markov chain with transition matrix $\mathbf{M}$.

2. Suppose we converted a connected, undirected graph into a Markov chain with. Let $\mathbf{M}$ be the transition matrix of this Markov chain and let $\vec{\pi}$ be the distribution where

$$\pi[v] = \frac{\deg(v)}{2|E|}$$

First, we will show that $\vec{\pi}$ is a probability vector:

$$\begin{aligned}
||\vec{\pi}||_1 &= \sum_{v \in V} \frac{\deg(v)}{2|E|} && \text{Definition of } \vec{\pi} \\
&= \frac{1}{2|E|} \sum_{v \in V} \deg(v) && \text{Factoring out } \frac{1}{2|E|} \\
&= \frac{1}{2|E|} \cdot 2|E| && \text{every vertex is counted twice in the sum of degrees} \\
&= 1 && \text{(Simplification)} \\
&\Rightarrow \vec{\pi} \text{ is a probability vector}
\end{aligned}$$

Next, we will show that $\vec{\pi}\mathbf{M} = \vec{\pi}$. Let $\vec{m}_i$ be the i-th column of $\mathbf{M}$:

$$\begin{aligned}
\vec{\pi}\mathbf{M} &= \begin{bmatrix} \vec{\pi} \cdot \vec{m}_1 & \vec{\pi} \cdot \vec{m}_2 & \cdots & \vec{\pi} \cdot \vec{m}_n \end{bmatrix} && \text{Definition of matrix multiplication} \\
&= \begin{bmatrix} \sum_{i=1}^{|V|} \pi[v_i] \cdot m_{i1} & \cdots & \sum_{i=1}^{|V|} \pi[v_i] \cdot m_{in} \end{bmatrix} && \text{Expanding the dot products} \\
&= \begin{bmatrix} \sum_{i=1}^{|V|} \frac{\deg(v_i)}{2|E|} \cdot m_{i1} & \cdots & \sum_{i=1}^{|V|} \frac{\deg(v_i)}{2|E|} \cdot m_{in} \end{bmatrix} && \text{Definition of } \pi[v_i] \\
&= \begin{bmatrix} \frac{1}{2|E|} \sum_{i=1}^{|V|} \deg(v_i) \cdot m_{i1} & \cdots & \frac{1}{2|E|} \sum_{i=1}^{|V|} \deg(v_i) \cdot m_{in} \end{bmatrix} && \text{Factoring out } \frac{1}{2|E|} \\
&= \begin{bmatrix} \frac{1}{2|E|} \sum_{i=1}^{|V|} \frac{1}{\deg(v_i)} \frac{w(v_i \to v_1)}{|\mathcal{N}_1(v_i)|} & \cdots & \frac{1}{2|E|} \sum_{i=1}^{|V|} \frac{1}{\deg(v_i)} \frac{w(v_i \to v_n)}{|\mathcal{N}_1(v_i)|} \end{bmatrix} && \text{Definition of } m_{ij} \\
&= \begin{bmatrix} \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_1)} \frac{\deg(v_i)}{\deg(v_i)} & \cdots & \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_n)} \frac{\deg(v_i)}{\deg(v_i)} \end{bmatrix} && \text{Unweighted graph} \\
&= \begin{bmatrix} \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_1)} 1 & \cdots & \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_n)} 1 \end{bmatrix} && \text{Algebra} \\
&= \begin{bmatrix} \frac{1}{2|E|} \cdot |\mathcal{N}(v_1)| & \frac{1}{2|E|} \cdot |\mathcal{N}(v_2)| & \cdots & \frac{1}{2|E|} \cdot |\mathcal{N}(v_n)| \end{bmatrix} && \text{Definition of summation} \\
&= \begin{bmatrix} \frac{1}{2|E|} \cdot \deg(v_1) & \frac{1}{2|E|} \cdot \deg(v_2) & \cdots & \frac{1}{2|E|} \cdot \deg(v_n) \end{bmatrix} && \text{Count of neighbors equals degree} \\
&= \vec{\pi} && \text{Definition of } \vec{\pi}
\end{aligned}$$

Since we have shown that $\vec{\pi}$ is a probability vector and that $\vec{\pi}\mathbf{M} = \vec{\pi}$, we conclude that $\vec{\pi}$ is a stationary distribution of the Markov chain derived from the connected, undirected graph.

3. Suppose that we converted a connected, weighted, undirected graph into a Markov chain and let $\vec{\pi}$ be the distribution where:

$$\pi[v] = \frac{\sum_{u \in \mathcal{N}_1(v)} w(u \to v)}{2 \sum_{(p \to q) \in E} w(p \to q)}$$

First, we will show that $\vec{\pi}$ is a probability vector:

$$\begin{aligned}
||\vec{\pi}||_1 &= \sum_{v \in V} \pi[v] && \text{Definition of L1 norm} \\
&= \sum_{v \in V} \frac{\sum_{u \in \mathcal{N}_1(v)} w(u \to v)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Definition of } \pi[v] \\
&= \frac{\sum_{v \in V} \sum_{u \in \mathcal{N}_1(v)} w(u \to v)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Sum of fractions with same denominator} \\
&= \frac{2 \sum_{(p \to q) \in E} w(p \to q)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Every edge is counted twice} \\
&= 1 \\
&\Rightarrow \vec{\pi} \text{ is a probability vector}
\end{aligned}$$

Next, we will show that $\vec{\pi}\mathbf{M} = \vec{\pi}$. Let $\vec{p} = \vec{\pi}\mathbf{M}$ and $\vec{m}_i$ be the i-th column of $\mathbf{M}$:

$$\begin{aligned}
p[v_i] &= \vec{\pi} \cdot \vec{m}_i && \text{Definition of matrix multiplication} \\
&= \sum_{j=1}^{|V|} \pi[v_j] \cdot m_{ji} && \text{Expanding the dot product} \\
&= \sum_{j=1}^{|V|} \frac{\sum_{u \in \mathcal{N}_1(v_j)} w(u \to v_j)}{2 \sum_{(p \to q) \in E} w(p \to q)} \cdot m_{ji} && \text{Definition of } \pi[v] \\
&= \sum_{j=1}^{|V|} \frac{\sum_{u \in \mathcal{N}_1(v_j)} w(u \to v_j)}{2 \sum_{(p \to q) \in E} w(p \to q)} \cdot \frac{w(v_j \to v_i)}{\sum_{u \in \mathcal{N}_1(v_j)} w(v_j \to u)} && \text{Definition of } m_{ji} \\
&= \sum_{j=1}^{|V|} \frac{w(v_j \to v_i)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Dividing} \\
&= \frac{\sum_{j=1}^{|V|} w(v_j \to v_i)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Combining fractions} \\
&= \frac{\sum_{u \in \mathcal{N}_1(v_i)} w(u \to v_i)}{2 \sum_{(p \to q) \in E} w(p \to q)} && \text{Only neighbors of } v_i \text{ contribute} \\
&= \pi[v_i] && \text{Definition of } \pi[v_i] \\
&\Rightarrow \vec{p} = \vec{\pi}\mathbf{M} = \vec{\pi}
\end{aligned}$$

Since we have shown that $\vec{\pi}$ is a probability vector and that $\vec{\pi}\mathbf{M} = \vec{\pi}$, we conclude that $\vec{\pi}$ is a stationary distribution of the Markov chain derived from the connected, weighted, undirected graph.

Question 2

1. Let $\mathbf{M}$ be row-stochastic. We will show that $\mathbf{M'} = \alpha\mathbf{M} + \frac{1-\alpha}{n}\mathbf{1}$ is also row-stochastic. First we will show that the ij-th entry of $\mathbf{M'}$ is between 0 and 1:

$$\begin{aligned}
m'_{ij} &= \alpha m_{ij} + \frac{1-\alpha}{n} \cdot 1 && \text{Matrix multiplication definition} \\
&= \alpha m_{ij} + \frac{1-\alpha}{n} && \text{Simplification} \\
\Rightarrow min(m'_{ij}) &= \frac{1-\alpha}{n} \geq 0 && \text{Since } 1-\alpha \geq 0 \text{ and } n > 0 \text{ (1)} \\
max(m'_{ij}) &= \alpha + \frac{1-\alpha}{n} \\
&= \frac{\alpha n + 1 - \alpha}{n} && \text{Finding common denominator} \\
&\leq \frac{n}{n} && \text{Since } \alpha \leq 1 \\
&\leq 1 && \text{Simplification (2)} \\
(1),(2) &\Rightarrow 0 \leq m'_{ij} \leq 1
\end{aligned}$$

Next, we will show that the sum of each row in $\mathbf{M'}$ is equal to 1:

$$\begin{aligned}
\sum_{j=1}^{n} m'_{ij} &= \sum_{j=1}^{n} \left(\alpha m_{ij} + \frac{1-\alpha}{n}\right) && \text{Using results from above} \\
&= \alpha \sum_{j=1}^{n} m_{ij} + \sum_{j=1}^{n} \frac{1-\alpha}{n} && \text{Distributing the summation} \\
&= \alpha \cdot 1 + (1-\alpha) && \text{Since } \mathbf{M} \text{ is row-stochastic} \\
&= 1 && \text{Simplification}
\end{aligned}$$

Since we have shown that each entry of $\mathbf{M'}$ is between 0 and 1 and that the sum of each row in $\mathbf{M'}$ is equal to 1, we conclude that $\mathbf{M'}$ is row-stochastic.

2. We will show that the algorithm computes $\vec{y} = \vec{x}\mathbf{M'}$ where $\mathbf{M'} = \alpha\mathbf{M} + \frac{1-\alpha}{n}\mathbf{1}$.

$$\begin{aligned}
\vec{y} &= \vec{y} + \beta\frac{1}{n}\vec{1} && \text{Line 3 of the algorithm} \\
&= \alpha\vec{x}\mathbf{M} + (||\vec{x}||_1 - ||\alpha\vec{x}\mathbf{M}||_1)\,\frac{1}{n}\vec{1} && \text{Substituting } \vec{y} \text{ and } \beta \\
&= \alpha\vec{x}\mathbf{M} + (1 - \alpha||\vec{x}\mathbf{M}||_1)\,\frac{1}{n}\vec{1} && \vec{x} \text{ is a probability vector} \\
&= \alpha\vec{x}\mathbf{M} + (1 - \alpha \cdot 1)\,\frac{1}{n}\vec{1} && \mathbf{M} \text{ is row-stochastic and } \vec{x} \text{ is a probability vector} \\
&= \alpha\vec{x}\mathbf{M} + \frac{1-\alpha}{n}\vec{1} && \text{Simplification} \\
&= \vec{x}\left(\alpha\mathbf{M} + \frac{1-\alpha}{n}\mathbf{1}\right) && \text{Factoring out } \vec{x} \\
&= \vec{x}\mathbf{M'} && \text{Definition of } \mathbf{M'}
\end{aligned}$$

This algorithm saves time by avoiding the need to compute **M'** explicitly, which would require $O(n^2)$ time.

Question 3

Let $G \sim G_{n,p}$ be a graph generated from the $G_{n,p}$ model. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $E_{i,j}$ be the outcome of the edge between v_i and v_j (where $i < j$). Then, the probability of observing a specific graph G with a given edge set is determined by:

$$
\begin{aligned}
Pr[E_{i,j}] &= \begin{cases} p & \text{if } (v_i, v_j) \in E \\ 1-p & \text{if } (v_i, v_j) \notin E \end{cases} && \text{Definition of } G_{n,p} \\
\Rightarrow Pr[E] &= \prod_{1 \le i < j \le n} Pr[E_{i,j}] && \text{Independence of edges} \\
&= \prod_{(v_i,v_j) \in E} p \cdot \prod_{(v_i,v_j) \notin E} (1-p) && \text{Separating edges in } E \text{ and not in } E \\
&= p^{|E|}(1-p)^{\binom{n}{2} - |E|} && \text{Counting the number of edges}
\end{aligned}
$$

Therefore, the probability of observing a specific graph G with edge set E in the $G_{n,p}$ model is given by $Pr[E] = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$.

Question 4

Consider the expected distance between a fixed graph Q and a random graph $G \sim G_{n,p}$:

$$\begin{aligned}
\mathbb{E}_{G\sim G_{n,p}}[d(G,Q)] &= \mathbb{E}_{G\sim G_{n,p}}\left[|E_G \setminus E_Q| + |E_Q \setminus E_G|\right] && \text{Definition of distance} \\
&= \mathbb{E}_{G\sim G_{n,p}}\left[|E_G \setminus E_Q|\right] + \mathbb{E}_{G\sim G_{n,p}}\left[|E_Q \setminus E_G|\right] && \text{Linearity of expectation}
\end{aligned}$$

We compute each term separately. For the first term, note that $|E_G \setminus E_Q|$ counts the edges present in G but not in Q. The possible number of such edges ranges from 0 to $\binom{n}{2} - |E_Q|$. Each edge not in Q has a probability p of being present in G. Thus, the expected number of such edges is:

$$\begin{aligned}
\mathbb{E}_{G\sim G_{n,p}}\left[|E_G \setminus E_Q|\right] &= \sum_{i=0}^{\binom{n}{2}-|E_Q|} i \cdot \binom{\binom{n}{2} - |E_Q|}{i} p^i (1-p)^{\binom{n}{2}-|E_Q|-i} && \text{Definition of expectation} \\
&= p \cdot \left[\binom{n}{2} - |E_Q|\right] && \text{Expected value of a Binomial distribution}
\end{aligned}$$

For the second term, note that $|E_Q \setminus E_G|$ counts the edges present in Q but not in G. The possible number of such edges ranges from 0 to $|E_Q|$. Each edge in Q has a probability $1-p$ of being absent in G. Thus, the expected number of such edges is:

$$\begin{aligned}
\mathbb{E}_{G\sim G_{n,p}}\left[|E_Q \setminus E_G|\right] &= \sum_{i=0}^{|E_Q|} i \cdot \binom{|E_Q|}{i} (1-p)^i p^{|E_Q|-i} && \text{Definition of expectation} \\
&= (1-p) \cdot |E_Q| && \text{Expected value of a Binomial distribution}
\end{aligned}$$

Combining both terms, we have:

$$\mathbb{E}_{G\sim G_{n,p}}[d(G,Q)] = p \cdot \left[\binom{n}{2} - |E_Q|\right] + (1-p) \cdot |E_Q| \qquad \text{Combining both terms}$$

Therefore, to compute the expected distance between a fixed graph Q and a random graph $G \sim G_{n,p}$, we can count the number of edges in Q and use the formula above. This algorithm runs in $O(n^2)$ time since counting the edges in Q takes $O(n^2)$ time.

Extra Credit

1. Let $\mathbf{M} \in \mathcal{R}^{n\times n}$ be a row-stochastic transition matrix. Consider an arbitrary eigenvector $\vec{e}$ of $\mathbf{M}$ with corresponding eigenvalue λ. We will show that $\vec{e}$ is also an eigenvector of $\mathbf{M}^t$ with eigenvalue λ^t for any integer $t \geq 1$ using induction on t.

   - **Base Case**: Let $t = 1$. Then, we have:

     $$\begin{aligned} \vec{e}\mathbf{M}^1 &= \vec{e}\mathbf{M} && \text{Matrix exponentation} \\ &= \lambda\vec{e} && \text{Definition of eigenvector} \\ &\Rightarrow \vec{e} \text{ is an eigenvector of } \mathbf{M}^1 \text{ with eigenvalue } \lambda^1 \end{aligned}$$

     Thus, the base case holds.

   - **Inductive Step**: Assume that for some integer $k \geq 1$, $\vec{e}$ is an eigenvector of $\mathbf{M}^k$ with eigenvalue λ^k. We will show that $\vec{e}$ is also an eigenvector of $\mathbf{M}^{k+1}$ with eigenvalue λ^{k+1}.

     $$\begin{aligned} \vec{e}\mathbf{M}^{k+1} &= \vec{e}\mathbf{M}^k\mathbf{M} && \text{Matrix exponentation} \\ &= \lambda^k\vec{e}\mathbf{M} && \text{Inductive Hypothesis} \\ &= \lambda^k\lambda\vec{e} && \text{Definition of eigenvector} \\ &= \lambda^{k+1}\vec{e} && \text{Simplification} \\ &\Rightarrow \vec{e}\text{is an eigenvector of } \mathbf{M}^{k+1} \text{ with eigenvalue } \lambda^{k+1} \end{aligned}$$

     Thus, the inductive step holds.

2. Let $\vec{\pi}$ be a stationary distribution of $\mathbf{M}$. We will show that $||\vec{x}\mathbf{M}^t - \vec{\pi}||_2 \leq \sqrt{n}\lambda_2^t$ for any probability vector $\vec{x}$, where λ_2 is the second largest eigenvalue of $\mathbf{M}$. Consider the difference

$\vec{x}\mathbf{M}^t - \vec{\pi}$:

$$\begin{aligned}
\vec{x}\mathbf{M}^t - \vec{\pi} &= \left(\sum_{i=1}^{n} \alpha_i \vec{e_i}\right)\mathbf{M}^t - \vec{\pi} && \text{Expanding } \vec{x} \text{ in terms of eigenvectors} \\
&= \sum_{i=1}^{n} \alpha_i \lambda_i^t \vec{e_i} - \vec{\pi} && \text{Using result from part (a)} \\
&= \sum_{i=2}^{n} \alpha_i \lambda_i^t \vec{e_i} && \vec{\pi} = \alpha_1 \vec{e_1} \text{ and } \lambda_1 = 1 \\
\Rightarrow \left\|\vec{x}\mathbf{M}^t - \vec{\pi}\right\|_2^2 &= \left\|\sum_{i=2}^{n} \alpha_i \lambda_i^t \vec{e_i}\right\|_2^2 \\
&= \sum_{i=2}^{n} \left\|\alpha_i \lambda_i^t \vec{e_i}\right\|_2^2 && \text{Pythagorean theorem} \\
&= \sum_{i=2}^{n} \alpha_i^2 \lambda_i^{2t} && \vec{e_i} \text{ are unit vectors} \\
&\leq \lambda_2^{2t} \sum_{i=2}^{n} \alpha_i^2 && \text{Since } |\lambda_i| \leq \lambda_2 \text{ for } i \geq 2
\end{aligned}$$

Now, we will bound the value of α_i. Note that:

$$\begin{aligned}
\alpha_i &= \vec{x}\vec{e_i}^T && \text{Definition of } \alpha_i \\
&\leq ||\vec{x}||_2 \cdot ||\vec{e_i}||_2 && \text{Cauchy-Schwarz inequality} \\
&\leq ||\vec{x}||_2 && \vec{e_i} \text{ are unit vectors} \\
&\leq \sum_{j=1}^{n} (x_j)^2 && \text{Definition of L2 norm} \\
&\leq \sum_{j=1}^{n} (x_j) && \text{Since } 0 \leq x_j \leq 1 \text{ for probability vectors} \\
&\leq 1
\end{aligned}$$

Then:

$$\begin{aligned}
\left\|\vec{x}\mathbf{M}^t - \vec{\pi}\right\|_2^2 &\leq \lambda_2^{2t} \sum_{i=2}^{n} ||\vec{x}||_2^2 && \text{Substituting the bound on } \alpha_i \\
&\leq \lambda_2^{2t}(n-1)||\vec{x}||_2^2 && \text{There are } n-1 \text{ terms in the sum} \\
&\leq \lambda_2^{2t} n ||\vec{x}||_2^2 && \text{Simplification} \\
&\leq \lambda_2^{2t} n && \text{Since } ||\vec{x}||_2^2 \leq 1 \text{ for probability vectors} \\
\Rightarrow \left\|\vec{x}\mathbf{M}^t - \vec{\pi}\right\|_2 &\leq \sqrt{n}\lambda_2^t && \text{Taking the square root of both sides}
\end{aligned}$$

Therefore, we have shown that $||\vec{x}\mathbf{M}^t - \vec{\pi}||_2 \leq \sqrt{n}\lambda_2^t$ for any probability vector $\vec{x}$.