

CS365 Written Assignment 3

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Question 1

1. Let \mathbf{M} be the transition matrix of a Markov chain with $|V|$ states and let \vec{p} be the distribution:

$$\vec{p} = \left[\frac{1}{|V|}, \frac{1}{|V|}, \dots, \frac{1}{|V|} \right]$$

First, we show that \vec{p} is a probability vector:

$$\begin{aligned} \|\vec{p}\|_1 &= \sum_{i=1}^{|V|} \frac{1}{|V|} && \text{Definition of } \vec{p} \\ &= \frac{|V|}{|V|} && \text{Sum of } |V| \text{ terms of } \frac{1}{|V|} \\ &= 1 && \text{(Simplification)} \\ &\Rightarrow \vec{p} \text{ is a probability vector} \end{aligned}$$

Then, we show that $\vec{p}\mathbf{M} = \vec{p}$:

$$\begin{aligned} \vec{p}\mathbf{M} &= [\vec{p} \cdot \vec{m}_1 \quad \vec{p} \cdot \vec{m}_2 \quad \dots \quad \vec{p} \cdot \vec{m}_n] && \text{Definition of matrix multiplication} \\ &= \left[\sum_{i=1}^{|V|} \frac{1}{|V|} m_{i1} \quad \sum_{i=1}^{|V|} \frac{1}{|V|} m_{i2} \quad \dots \quad \sum_{i=1}^{|V|} \frac{1}{|V|} m_{in} \right] \\ &= \left[\frac{1}{|V|} \sum_{i=1}^{|V|} m_{i1} \quad \frac{1}{|V|} \sum_{i=1}^{|V|} m_{i2} \quad \dots \quad \frac{1}{|V|} \sum_{i=1}^{|V|} m_{in} \right] && \text{Factoring out } \frac{1}{|V|} \\ &= \left[\frac{1}{|V|} \cdot 1 \quad \frac{1}{|V|} \cdot 1 \quad \dots \quad \frac{1}{|V|} \cdot 1 \right] && \mathbf{M} \text{ is symmetric and row-stochastic} \\ &= \vec{p} && \text{Definition of } \vec{p} \end{aligned}$$

Since we have shown that \vec{p} is a probability vector and that $\vec{p}\mathbf{M} = \vec{p}$, we conclude that \vec{p} is a stationary distribution of the Markov chain with transition matrix \mathbf{M} .

2. Suppose we converted a connected, undirected graph into a Markov chain with. Let \mathbf{M} be the transition matrix of this Markov chain and let π be the distribution where

$$\pi[v] = \frac{\deg(v)}{2|E|}$$

First, we will show that $\vec{\pi}$ is a probability vector:

$$\begin{aligned}
\|\vec{\pi}\|_1 &= \sum_{v \in V} \frac{\deg(v)}{2|E|} && \text{Definition of } \vec{\pi} \\
&= \frac{1}{2|E|} \sum_{v \in V} \deg(v) && \text{Factoring out } \frac{1}{2|E|} \\
&= \frac{1}{2|E|} \cdot 2|E| && \text{every vertex is counted twice in the sum of degrees} \\
&= 1 && \text{(Simplification)} \\
&\Rightarrow \vec{\pi} \text{ is a probability vector}
\end{aligned}$$

Next, we will show that $\vec{\pi}\mathbf{M} = \vec{\pi}$:

$$\begin{aligned}
\vec{\pi}\mathbf{M} &= [\vec{\pi} \cdot \vec{m}_1 \quad \vec{\pi} \cdot \vec{m}_2 \quad \cdots \quad \vec{\pi} \cdot \vec{m}_n] && \text{Definition of matrix multiplication} \\
&= \left[\sum_{i=1}^{|V|} \pi[v_i] \cdot m_{i1} \quad \cdots \quad \sum_{i=1}^{|V|} \pi[v_i] \cdot m_{in} \right] && \text{Expanding the dot products} \\
&= \left[\sum_{i=1}^{|V|} \frac{\deg(v_i)}{2|E|} \cdot m_{i1} \quad \cdots \quad \sum_{i=1}^{|V|} \frac{\deg(v_i)}{2|E|} \cdot m_{in} \right] && \text{Definition of } \pi[v_i] \\
&= \left[\frac{1}{2|E|} \sum_{i=1}^{|V|} \deg(v_i) \cdot m_{i1} \quad \cdots \quad \frac{1}{2|E|} \sum_{i=1}^{|V|} \deg(v_i) \cdot m_{in} \right] && \text{Factoring out } \frac{1}{2|E|} \\
&= \left[\frac{1}{2|E|} \sum_{i=1}^{|V|} \frac{1}{\deg(v_i)} \frac{w(v_i \rightarrow v_1)}{|\mathcal{N}_1(v_i)|} \quad \cdots \quad \frac{1}{2|E|} \sum_{i=1}^{|V|} \frac{1}{\deg(v_i)} \frac{w(v_i \rightarrow v_n)}{|\mathcal{N}_1(v_i)|} \right] && \text{Definition of } m_{ij} \\
&= \left[\frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_1)} \frac{\deg(v_i)}{\deg(v_i)} \quad \cdots \quad \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_n)} \frac{\deg(v_i)}{\deg(v_i)} \right] && \text{Unweighted graph} \\
&= \left[\frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_1)} 1 \quad \cdots \quad \frac{1}{2|E|} \sum_{v_i \in \mathcal{N}(v_n)} 1 \right] && \text{Algebra} \\
&= \left[\frac{1}{2|E|} \cdot |\mathcal{N}(v_1)| \quad \frac{1}{2|E|} \cdot |\mathcal{N}(v_2)| \quad \cdots \quad \frac{1}{2|E|} \cdot |\mathcal{N}(v_n)| \right] && \text{Definition of summation} \\
&= \left[\frac{1}{2|E|} \cdot \deg(v_1) \quad \frac{1}{2|E|} \cdot \deg(v_2) \quad \cdots \quad \frac{1}{2|E|} \cdot \deg(v_n) \right] && \text{Count of neighbors equals degree} \\
&= \vec{\pi} && \text{Definition of } \vec{\pi}
\end{aligned}$$

Since we have shown that $\vec{\pi}$ is a probability vector and that $\vec{\pi}\mathbf{M} = \vec{\pi}$, we conclude that $\vec{\pi}$ is a stationary distribution of the Markov chain derived from the connected, undirected graph.

3. Suppose that we converted a connected, weighted, undirected graph into a Markov chain and let $\vec{\pi}$ be the distribution where:

$$\pi[v] = \frac{\sum_{u \in \mathcal{N}_1(v)} w(u \rightarrow v)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)}$$

First, we will show that $\vec{\pi}$ is a probability vector:

$$\begin{aligned}
\|\vec{\pi}\|_1 &= \sum_{v \in V} \pi[v] && \text{Definition of L1 norm} \\
&= \sum_{v \in V} \frac{\sum_{u \in \mathcal{N}_1(v)} w(u \rightarrow v)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Definition of } \pi[v] \\
&= \frac{\sum_{v \in V} \sum_{u \in \mathcal{N}_1(v)} w(u \rightarrow v)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Sum of fractions with same denominator} \\
&= \frac{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Every edge is counted twice} \\
&= 1 \\
&\Rightarrow \vec{\pi} \text{ is a probability vector}
\end{aligned}$$

Next, we will show that $\vec{\pi} \mathbf{M} = \vec{\pi}$. Let $\vec{p} = \vec{\pi} \mathbf{M}$:

$$\begin{aligned}
p[v_i] &= \vec{\pi} \cdot \vec{m}_i && \text{Definition of matrix multiplication} \\
&= \sum_{j=1}^{|V|} \pi[v_j] \cdot m_{ji} && \text{Expanding the dot product} \\
&= \sum_{j=1}^{|V|} \frac{\sum_{u \in \mathcal{N}_1(v_j)} w(u \rightarrow v_j)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} \cdot m_{ji} && \text{Definition of } \pi[v] \\
&= \sum_{j=1}^{|V|} \frac{\sum_{u \in \mathcal{N}_1(v_j)} w(u \rightarrow v_j)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} \cdot \frac{w(v_j \rightarrow v_i)}{\sum_{u \in \mathcal{N}_1(v_j)} w(v_j \rightarrow u)} && \text{Definition of } m_{ji} \\
&= \sum_{j=1}^{|V|} \frac{w(v_j \rightarrow v_i)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Dividing} \\
&= \frac{\sum_{j=1}^{|V|} w(v_j \rightarrow v_i)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Combining fractions} \\
&= \frac{\sum_{u \in \mathcal{N}_1(v_i)} w(u \rightarrow v_i)}{2 \sum_{(p \rightarrow q) \in E} w(p \rightarrow q)} && \text{Only neighbors of } v_i \text{ contribute} \\
&= \pi[v_i] && \text{Definition of } \pi[v_i] \\
&\Rightarrow \vec{p} = \vec{\pi} \mathbf{M} = \vec{\pi}
\end{aligned}$$

Since we have shown that $\vec{\pi}$ is a probability vector and that $\vec{\pi} \mathbf{M} = \vec{\pi}$, we conclude that $\vec{\pi}$ is a stationary distribution of the Markov chain derived from the connected, weighted, undirected graph.

Question 2

1. Let \mathbf{M} be row-stochastic. We will show that $\mathbf{M}' = \alpha\mathbf{M} + \frac{1-\alpha}{n}\mathbf{1}$ is also row-stochastic. First we will show that the ij -th entry of \mathbf{M}' is between 0 and 1:

$$\begin{aligned}
 m'_{ij} &= \alpha m_{ij} + \frac{1-\alpha}{n} \cdot 1 && \text{Matrix multiplication definition} \\
 &= \alpha m_{ij} + \frac{1-\alpha}{n} && \text{Simplification} \\
 \Rightarrow \min(m'_{ij}) &= \frac{1-\alpha}{n} \geq 0 && \text{Since } 1-\alpha \geq 0 \text{ and } n > 0 \text{ (1)} \\
 \max(m'_{ij}) &= \alpha + \frac{1-\alpha}{n} \\
 &= \frac{\alpha n + 1 - \alpha}{n} && \text{Finding common denominator} \\
 &\leq \frac{n}{n} && \text{Since } \alpha \leq 1 \\
 &\leq 1 && \text{Simplification (2)} \\
 (1), (2) &\Rightarrow 0 \leq m'_{ij} \leq 1
 \end{aligned}$$

Next, we will show that the sum of each row in \mathbf{M}' is equal to 1:

$$\begin{aligned}
 \sum_{j=1}^n m'_{ij} &= \sum_{j=1}^n \left(\alpha m_{ij} + \frac{1-\alpha}{n} \right) && \text{Using results from above} \\
 &= \alpha \sum_{j=1}^n m_{ij} + \sum_{j=1}^n \frac{1-\alpha}{n} && \text{Distributing the summation} \\
 &= \alpha \cdot 1 + (1-\alpha) && \text{Since } \mathbf{M} \text{ is row-stochastic} \\
 &= 1 && \text{Simplification}
 \end{aligned}$$

Since we have shown that each entry of \mathbf{M}' is between 0 and 1 and that the sum of each row in \mathbf{M}' is equal to 1, we conclude that \mathbf{M}' is row-stochastic.

2. We will show that the algorithm computes $\vec{y} = \vec{x}\mathbf{M}'$ where $\mathbf{M}' = \alpha\mathbf{M} + \frac{1-\alpha}{n}\mathbf{1}$.

$$\begin{aligned}
 \vec{y} &= \vec{y} + \beta \frac{1}{n} \vec{1} && \text{Line 3 of the algorithm} \\
 &= \alpha \vec{x}\mathbf{M} + (||\vec{x}||_1 - ||\alpha \vec{x}\mathbf{M}||_1) \frac{1}{n} \vec{1} && \text{Substituting } \vec{y} \text{ and } \beta \\
 &= \alpha \vec{x}\mathbf{M} + (1 - \alpha ||\vec{x}\mathbf{M}||_1) \frac{1}{n} \vec{1} && \vec{x} \text{ is a probability vector} \\
 &= \alpha \vec{x}\mathbf{M} + (1 - \alpha \cdot 1) \frac{1}{n} \vec{1} && \mathbf{M} \text{ is row-stochastic and } \vec{x} \text{ is a probability vector} \\
 &= \alpha \vec{x}\mathbf{M} + \frac{1-\alpha}{n} \vec{1} && \text{Simplification} \\
 &= \vec{x} \left(\alpha \mathbf{M} + \frac{1-\alpha}{n} \mathbf{1} \right) && \text{Factoring out } \vec{x} \\
 &= \vec{x}\mathbf{M}' && \text{Definition of } \mathbf{M}'
 \end{aligned}$$

This algorithm saves time by avoiding the need to compute \mathbf{M}' explicitly, which would require $O(n^2)$ time.

Question 3

Let $G \sim G_{n,p}$ be a graph generated from the $G_{n,p}$ model. Let v_1, v_2, \dots, v_n be the vertices of G and $E_{i,j}$ be the outcome of the edge between v_i and v_j (where $i < j$). Then, the probability of observing a specific graph G with a given edge set is determined by:

$$\begin{aligned}
 Pr[E_{i,j}] &= \begin{cases} p & \text{if } (v_i, v_j) \in E \\ 1 - p & \text{if } (v_i, v_j) \notin E \end{cases} && \text{Definition of } G_{n,p} \\
 \Rightarrow Pr[E] &= \prod_{1 \leq i < j \leq n} Pr[E_{i,j}] && \text{Independence of edges} \\
 &= \prod_{(v_i, v_j) \in E} p \cdot \prod_{(v_i, v_j) \notin E} (1 - p) && \text{Separating edges in } E \text{ and not in } E \\
 &= p^{|E|} (1 - p)^{\binom{n}{2} - |E|} && \text{Counting the number of edges}
 \end{aligned}$$

Therefore, the probability of observing a specific graph G with edge set E in the $G_{n,p}$ model is given by $Pr[E] = p^{|E|} (1 - p)^{\binom{n}{2} - |E|}$.

Question 4

Consider the expected distance between a fixed graph Q and a random graph $G \sim G_{n,p}$:

$$\begin{aligned}\mathbb{E}_{G \sim G_{n,p}}[d(G, Q)] &= \mathbb{E}_{G \sim G_{n,p}}[|E_G \setminus E_Q| + |E_Q \setminus E_G|] && \text{Definition of distance} \\ &= \mathbb{E}_{G \sim G_{n,p}}[|E_G \setminus E_Q|] + \mathbb{E}_{G \sim G_{n,p}}[|E_Q \setminus E_G|] && \text{Linearity of expectation}\end{aligned}$$

We compute each term separately. For the first term, note that $|E_G \setminus E_Q|$ counts the edges present in G but not in Q . The possible number of such edges ranges from 0 to $\binom{n}{2} - |E_Q|$. Each edge not in Q has a probability p of being present in G . Thus, the expected number of such edges is:

$$\begin{aligned}\mathbb{E}_{G \sim G_{n,p}}[|E_G \setminus E_Q|] &= \sum_{i=0}^{\binom{n}{2} - |E_Q|} i \cdot \binom{\binom{n}{2} - |E_Q|}{i} p^i (1-p)^{\binom{n}{2} - |E_Q| - i} && \text{Definition of expectation} \\ &= p \cdot \left[\binom{n}{2} - |E_Q| \right] && \text{Expected value of a Binomial distribution}\end{aligned}$$

For the second term, note that $|E_Q \setminus E_G|$ counts the edges present in Q but not in G . The possible number of such edges ranges from 0 to $|E_Q|$. Each edge in Q has a probability $1 - p$ of being absent in G . Thus, the expected number of such edges is:

$$\begin{aligned}\mathbb{E}_{G \sim G_{n,p}}[|E_Q \setminus E_G|] &= \sum_{i=0}^{|E_Q|} i \cdot \binom{|E_Q|}{i} (1-p)^i p^{|E_Q| - i} && \text{Definition of expectation} \\ &= (1-p) \cdot |E_Q| && \text{Expected value of a Binomial distribution}\end{aligned}$$

Combining both terms, we have:

$$\mathbb{E}_{G \sim G_{n,p}}[d(G, Q)] = p \cdot \left[\binom{n}{2} - |E_Q| \right] + (1-p) \cdot |E_Q| \quad \text{Combining both terms}$$

Therefore, to compute the expected distance between a fixed graph Q and a random graph $G \sim G_{n,p}$, we can count the number of edges in Q and use the formula above. This algorithm runs in $O(n^2)$ time since counting the edges in Q takes $O(n^2)$ time.

Extra Credit

1. Let $\mathbf{M} \in \mathcal{R}^{n \times n}$ be a row-stochastic transition matrix. Consider an arbitrary eigenvector \vec{e} of \mathbf{M} with corresponding eigenvalue λ . We will show that \vec{e} is also an eigenvector of \mathbf{M}^t with eigenvalue λ^t for any integer $t \geq 1$ using induction on t .

- **Base Case:** Let $t = 1$. Then, we have:

$$\begin{aligned}\vec{e}\mathbf{M}^1 &= \vec{e}\mathbf{M} && \text{Matrix exponentiation} \\ &= \lambda\vec{e} && \text{Definition of eigenvector} \\ &\Rightarrow \vec{e} \text{ is an eigenvector of } \mathbf{M}^1 \text{ with eigenvalue } \lambda^1\end{aligned}$$

Thus, the base case holds.

- **Inductive Step:** Assume that for some integer $k \geq 1$, \vec{e} is an eigenvector of \mathbf{M}^k with eigenvalue λ^k . We will show that \vec{e} is also an eigenvector of \mathbf{M}^{k+1} with eigenvalue λ^{k+1} .

$$\begin{aligned}\vec{e}\mathbf{M}^{k+1} &= \vec{e}\mathbf{M}^k\mathbf{M} && \text{Matrix exponentiation} \\ &= \lambda^k\vec{e}\mathbf{M} && \text{Inductive Hypothesis} \\ &= \lambda^k\lambda\vec{e} && \text{Definition of eigenvector} \\ &= \lambda^{k+1}\vec{e} && \text{Simplification} \\ &\Rightarrow \vec{e} \text{ is an eigenvector of } \mathbf{M}^{k+1} \text{ with eigenvalue } \lambda^{k+1}\end{aligned}$$

Thus, the inductive step holds.

2. Let $\vec{\pi}$ be a stationary distribution of \mathbf{M} . We will show that $\|\vec{x}\mathbf{M}^t - \vec{\pi}\|_2 \leq \sqrt{n}\lambda_2^t$ for any probability vector \vec{x} , where λ_2 is the second largest eigenvalue (in absolute value) of \mathbf{M} . Consider the difference $\|\vec{x}\mathbf{M}^t - \vec{\pi}\|_2$:

$$\begin{aligned}\|\vec{x}\mathbf{M}^t - \vec{\pi}\|_2 &= \left\| \left(\sum_{i=1}^n \vec{x}e_i^T \vec{e}_i \right) \mathbf{M}^t - \vec{\pi} \right\|_2 && \text{Expanding } \vec{x} \text{ in terms of eigenvectors} \\ &= \left\| \left(\sum_{i=1}^n \vec{x}e_i^T \vec{e}_i \right) \mathbf{M}^t - \vec{\pi}\mathbf{M}^t \right\|_2 && \text{Property of stationary distribution} \\ &= \left\| \sum_{i=1}^n \vec{x}e_i^T \vec{e}_i \mathbf{M}^t - \vec{\pi}\mathbf{M}^t \right\|_2 && \text{Distributing } \mathbf{M}^t \\ &= \left\| \sum_{i=1}^n \vec{x}e_i^T \lambda_i^t \vec{e}_i - \vec{\pi} \right\|_2 && \text{From Extra Credit 1}\end{aligned}$$

TODO: Unfinished