Question 1

Proof. First, since we pick the largest number from the k numbers we sampled, if at least one of the k numbers is larger than the median, then the largest number we picked must be larger than the median.

Since we are sampling without replacement, the number of ways to sample k numbers from n numbers is:

$$|S| = \binom{n}{k}$$

The number of numbers, out of the n numbers, that are smaller than or equal to the median is:

$$|E| = \lfloor n/2 \rfloor$$

Note that if $k > \lceil n/2 \rceil$, then we are guaranteed to select a number larger than the median, since there are only $\lfloor n/2 \rfloor$ numbers that are smaller than or equal to the median.

Let E be the event that all k numbers we examined are smaller than or equal to the median of the n numbers. Then, the probability of us selecting a number larger than the median is:

$$
\begin{aligned}
Pr(E^c) &= 1 - Pr(E) && \text{complement rule} \\
&= \begin{cases} 1 - \frac{|E|}{|S|} & \text{if } k \le \lfloor n/2 \rfloor \\ 1 & \text{if } k > \lceil n/2 \rceil \end{cases} && \text{definition of probability} \\
&= \begin{cases} 1 - \frac{\binom{\lceil n/2 \rceil}{k}}{\binom{n}{k}} & \text{if } k \le \lceil n/2 \rceil \\ 1 & \text{if } k > \lceil n/2 \rceil \end{cases} && \text{substituting in } |E| \text{ and } |S|
\end{aligned}
$$

$\square$

Problem 2

Proof. Let X be the RV representing the number of successes we get. Then,

$$X = \sum_{i=1}^{n} X_i \qquad\qquad \text{definition of } X$$

$$X \sim Binomial(n, p) \qquad\qquad \text{since each coin flip is i.i.d. Bernoulli trial}$$

Examining $\bar{X}$, the average number of successes, we have:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad\qquad \text{definition of } \bar{X}$$

$$= \frac{1}{n} X \qquad\qquad \text{definition of } X$$

Using the linearity of expectation, we can compute the expected value of $\bar{X}$:

$$E(\bar{X}) = E(\frac{1}{n} X) \qquad\qquad \text{definition of } \bar{X}$$

$$= \frac{1}{n} E(X) \qquad\qquad \text{linearity of expectation}$$

$$= \frac{1}{n}(np) \qquad\qquad X \sim Binomial(n, p)$$

$$= p \qquad\qquad \text{simplifying}$$

Next, we can compute the variance of $\bar{X}$:

$$Var(\bar{X}) = Var(\frac{1}{n} X) \qquad\qquad \text{definition of } \bar{X}$$

$$= \left(\frac{1}{n}\right)^2 Var(X) \qquad\qquad \text{property of variance}$$

$$= \frac{1}{n^2}(np(1-p)) \qquad\qquad X \sim Binomial(n, p)$$

$$= \frac{p(1-p)}{n} \qquad\qquad \text{simplifying}$$

Using Chebyshev's inequality, we can give an upper bound on how likely the value of $\bar{X}$ differs from the bias of the coin p by at least $\frac{p}{10}$:

$$Pr(|\bar{X} - E(\bar{X})| \geq \frac{p}{10}) \leq \frac{Var(\bar{X})}{(p/10)^2} \qquad \text{Chebyshev's inequality}$$

$$\Rightarrow Pr(|\bar{X} - p| \geq \frac{p}{10}) \leq \frac{Var(\bar{X})}{(p/10)^2} \qquad \text{substituting in } E(\bar{X})$$

$$= \frac{\frac{p(1-p)}{n}}{(p/10)^2} \qquad \text{substituting in } Var(\bar{X})$$

$$= \frac{100(1-p)}{np} \qquad \text{simplifying}$$

Therefore:

$$Pr(|\bar{X} - p| \geq \frac{p}{10}) \leq \frac{100(1-p)}{np}$$

$\square$

Problem 3

Claim. *Let f and g be valid probability distributions defined over the same domain, S. Then, the convex combination of f and g*

$$h := \lambda f + (1 - \lambda)g$$

for some lambda $\in [0, 1]$ is also a valid probability distribution.

Proof. To show that h is a valid probability distribution, we need to show that:

  (i) $h(x) \geq 0 \forall x \in S$

  (ii) $\sum_{x \in S} h(x) = 1$

Since the domains of f and g are the same, h is also defined over the same domain, S.
For (i), we have:

$$
\begin{aligned}
h(x) &= \lambda f(x) + (1 - \lambda)g(x) && \text{definition of } h \\
\sum_{x \in S} h(x) &= \sum_{x \in S} \lambda f(x) + (1 - \lambda)g(x) && \text{applying summation} \\
&= \sum_{x \in S} \lambda f(x) + \sum_{x \in S} (1 - \lambda)g(x) && \text{linearity of summation} \\
&= \lambda \sum_{x \in S} f(x) + (1 - \lambda) \sum_{x \in S} g(x) && \text{linearity of summation} \\
&= \lambda(1) + (1 - \lambda)(1) && \text{since } f \text{ and } g \text{ are valid probability distributions} \\
&= \lambda + 1 - \lambda && \text{simplifying} \\
&= 1 && \text{simplifying}
\end{aligned}
$$

Thus, h satisfies (i).
For (ii), we have:

$$
\begin{aligned}
\lambda f(x) &\geq 0 && \text{since } \lambda \in [0, 1] \text{ and } f(x) \geq 0 \\
(1 - \lambda)g(x) &\geq 0 && \text{since } (1 - \lambda) \in [0, 1] \text{ and } g(x) \geq 0 \\
\Rightarrow h(x) = \lambda f(x) + (1 - \lambda)g(x) &\geq 0 + 0 && \text{adding the two inequalities} \\
&= 0 && \text{simplifying}
\end{aligned}
$$

Thus, h satisfies (ii).
Since h satisfies both (i) and (ii), h is a valid probability distribution. $\square$

Problem 4

To find the MLE for p^* and μ^*, we will optimize the log-likelihood function:

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_{ij} \left(\log \pi_j + \log f(x_i; \vec{\theta_j}) \right)$$

$$s.t. \sum_{j=1}^{k} \pi_j = 1$$

Using the Langrangian multiplier method, we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_{ij} \left(\log \pi_j + \log f(x_i; \vec{\theta_j}) \right) - \lambda \left(\sum_{j=1}^{k} \pi_j - 1 \right)$$

$$\frac{\partial}{\partial \vec{\theta_w}} \left[\sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_{ij} \left(\log \pi_j + \log f(x_i; \vec{\theta_j}) \right) - \lambda \left(\sum_{j=1}^{k} \pi_j - 1 \right) \right] = 0 \qquad \text{differentiating w.r.t. } \vec{\theta_w}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial \vec{\theta_w}} \log \left(Pr[x_i | \theta_w] \right) \qquad\qquad \text{derived in lecture}$$

For the Geometric distribution, we have:

$$Pr[x_i | p] = (1 - p)^{x_i - 1} p$$

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$$\sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial \vec{\theta_w}} \log\left(Pr[x_i|\theta_w]\right) \hspace{6cm} \text{above}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial p} \log\left((1-p)^{x_i-1}p\right) \hspace{4cm} \text{substituting in } Pr[x_i|p]$$

$$= \sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial p} \left((x_i - 1)\log(1-p) + \log(p)\right) \hspace{3cm} \text{log property}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \left(-(x_i - 1)\frac{1}{1-p} + \frac{1}{p}\right) \hspace{4cm} \text{differentiating}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \left(\frac{(1-p) - (x_i - 1)p}{p(1-p)}\right) \hspace{3cm} \text{combining the two fractions}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \left(\frac{1 - px_i}{p(1-p)}\right) \hspace{5cm} \text{simplifying}$$

$$= \frac{1}{p(1-p)} \sum_{i=1}^{n} \gamma_{iw}(1 - px_i) \hspace{3.5cm} \text{factoring out } \frac{1}{p(1-p)}$$

$$\frac{1}{p(1-p)} \sum_{i=1}^{n} \gamma_{iw}(1 - px_i) = 0 \hspace{4cm} \text{setting equal to } 0$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_{iw}(1 - px_i) = 0 \hspace{3.5cm} \text{multiplying both sides by } p(1-p)$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_{iw} - \sum_{i=1}^{n} \gamma_{iw}px_i = 0 \hspace{4cm} \text{distributing the summation}$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_{iw} = p \sum_{i=1}^{n} \gamma_{iw}x_i \hspace{3cm} \text{moving the second term to the right side}$$

$$\Rightarrow p^* = \frac{\sum_{i=1}^{n} \gamma_{iw}}{\sum_{i=1}^{n} \gamma_{iw}x_i} \hspace{3.5cm} \text{dividing both sides by } \sum_{i=1}^{n} \gamma_{iw}x_i$$

For the Borel distribution, we have:

$$Pr[x_i|\mu] = \frac{e^{-\mu x_i}(\mu x_i)^{x_i-1}}{x_i!}$$

$$\sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial \vec{\theta_w}} \log\left(Pr[x_i|\theta_w]\right) \qquad\qquad \text{above}$$

$$= \sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial \mu} \log\left(\frac{e^{-\mu x_i}(\mu x_i)^{x_i - 1}}{x_i!}\right) \qquad\qquad \text{substituting in } Pr[x_i|\mu]$$

$$= \sum_{i=1}^{n} \gamma_{iw} \frac{\partial}{\partial \mu}\left(-\mu x_i + (x_i - 1)\log(\mu x_i) - \log(x_i!)\right) \qquad\qquad \text{log property}$$

$$= \sum_{i=1}^{n} \gamma_{iw}\left(-x_i + (x_i - 1)\frac{1}{\mu}\right) \qquad\qquad \text{differentiating}$$

$$\sum_{i=1}^{n} \gamma_{iw}\left(-x_i + (x_i - 1)\frac{1}{\mu}\right) \qquad\qquad \text{setting equal to 0}$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_{iw}(x_i - 1)\frac{1}{\mu} - \sum_{i=1}^{n} \gamma_{iw} x_i = 0 \qquad\qquad \text{splitting the summation and rearranging}$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_{iw}(x_i - 1)\frac{1}{\mu} = \sum_{i=1}^{n} \gamma_{iw} x_i \qquad\qquad \text{moving the second term to the right side}$$

$$\Rightarrow \frac{1}{\mu} = \frac{\sum_{i=1}^{n} \gamma_{iw} x_i}{\sum_{i=1}^{n} \gamma_{iw}(x_i - 1)} \qquad\qquad \text{dividing both sides by } \sum_{i=1}^{n} \gamma_{iw}(x_i - 1)$$

$$\Rightarrow \mu^* = \frac{\sum_{i=1}^{n} \gamma_{iw}(x_i - 1)}{\sum_{i=1}^{n} \gamma_{iw} x_i} \qquad\qquad \text{taking the reciprocal of both sides}$$

Therefore, the MLEs are:

$$p^* = \frac{\sum_{i=1}^{n} \gamma_{iw}}{\sum_{i=1}^{n} \gamma_{iw} x_i}$$

$$\mu^* = \frac{\sum_{i=1}^{n} \gamma_{iw}(x_i - 1)}{\sum_{i=1}^{n} \gamma_{iw} x_i}$$

Problem 5

Proof. Let X be the RV representing the total number of successes we get after repeating the experiement n times. Since each coin flip is i.i.d., we have: $X \sim Binomial(N, p)$.

Here, $N = nm$, since we have n experiements, each with m coin flips. Let x_{ij} be the outcome of the jth coin flip in the ith experiement, then:

$$x_{ij} = \begin{cases} x_{ij} = 1 & \text{if the } j^{th} \text{ coin flip in the } i^{th} \text{ experiement is success} \\ x_{ij} = 0 & \text{otherwise} \end{cases}$$

$$x_{ij} \qquad\qquad\qquad\qquad\qquad\qquad \sim Bernoulli(p) \text{each coin flip is i.i.d. Bernoulli trial}$$

$$\Rightarrow X_i = \sum_{j=1}^{m} x_i j \qquad\qquad\qquad\qquad \text{definition of } X_i$$

$$\Rightarrow X_i \sim Binomial(m, p) \quad \text{since each } x_{ij} \text{ is i.i.d. Bernoulli trial}$$

$$\Rightarrow X = \sum_{i=1}^{n} X_i \qquad\qquad\qquad\qquad \text{definition of } X$$

$$\Rightarrow X = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \qquad\qquad\qquad \text{substituting in } X_i$$

$$\Rightarrow X \sim Binomial(nm, p) \quad \text{since each } X_i \text{ is i.i.d. Binomial trial}$$

Let k be the total number of successes we observed:

$$k = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}$$

Then, we can use the likelihood function to derive the MLE for p:

$$\mathcal{L}(p) = Pr[X = k|p] \qquad\qquad \text{definition of likelihood function}$$

$$= \binom{nm}{k} p^k (1-p)^{nm-k} \qquad\qquad X \sim Binomial(nm, p)$$

$$\mathcal{LL}(p) = \log\left(\binom{nm}{k} p^k (1-p)^{nm-k}\right) \qquad\qquad \text{taking the log of the likelihood function}$$

$$= \log\left(\binom{nm}{k}\right) + k\log(p) + (nm-k)\log(1-p) \qquad\qquad \text{log property}$$

$$\frac{\partial}{\partial p}\mathcal{LL}(p) = \frac{\partial}{\partial p}\left[\log\left(\binom{nm}{k}\right) + k\log(p) + (nm-k)\log(1-p)\right] \qquad\qquad \text{differentiating}$$

$$= 0 + k\frac{1}{p} - (nm-k)\frac{1}{1-p} \qquad\qquad \text{differentiating}$$

$$= k\frac{1}{p} - (nm-k)\frac{1}{1-p} \qquad\qquad \text{simplifying}$$

$$k\frac{1}{p} - (nm-k)\frac{1}{1-p} = 0 \qquad\qquad \text{setting equal to 0}$$

$$\Rightarrow \frac{k}{p} = \frac{nm-k}{1-p} \qquad\qquad \text{moving the second term to the right side}$$

$$\Rightarrow \frac{1-p}{p} = \frac{nm-k}{k} \qquad\qquad \text{cross-multiplying}$$

$$\Rightarrow \frac{1}{p} - 1 = \frac{nm-k}{k} \qquad\qquad \text{rewriting the left side}$$

$$\Rightarrow \frac{1}{p} = \frac{nm-k}{k} + 1 \qquad\qquad \text{adding 1 to both sides}$$

$$\Rightarrow \frac{1}{p} = \frac{nm}{k} \qquad\qquad \text{combining the right side}$$

$$\Rightarrow p^* = \frac{k}{nm} \qquad\qquad \text{taking the reciprocal of both sides}$$

Therefore, the MLE for p is:

$$p^* = \frac{k}{nm}$$

where k is the total number of successes we observed after repeating the experiement n times, each with m coin flips. $\qquad\square$