

# Binary Choice

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# Roadmap of Talk

# Binary Choice: Overview

Many problems we are interested in look at discrete rather than continuous outcomes.

- To move or not.
- Entering a market.
- Getting married.
- Going to college.

In this lecture we will

- Review the well known limitations of the linear probability model (LPM)
- Discuss interpretation of estimates of popular alternatives.
- Discussion options for dealing with endogeneity

# Linear Probability Model

OLS is called the **linear probability model**

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

because:

$$\begin{aligned} E[Y|X] &= 1 \times \Pr(Y = 1|X) + 0 \times \Pr(Y = 0|X) \\ \Pr(Y = 1|X) &= \beta_0 + \beta_1 X_i + \varepsilon_i \end{aligned}$$

The predicted value is a **probability** and

$$\beta_1 = \frac{\Pr(Y = 1|X = x + \Delta x) - \Pr(Y = 1|X = x)}{\Delta x}$$

So  $\beta_1$  represents the average change in probability that  $Y = 1$  for a unit change in  $X$ .

## Well known problems

- Constant marginals
- Not bounded between 0 and 1
- Relatedly, literally cannot satisfy OLS condition that  $\epsilon$  uncorrelated with  $X$  for covariates with wide support.
- Nevertheless ....

## Some well known textbooks

(Baby) Wooldrige:

*“Even with these problems, the linear probability model is useful and often applied in economics. It usually works well for values of the independent variables that are near the averages in the sample.” (2009, p. 249)*

- Mentions heteroskedasticity of error (which is binomial given  $X$ ) but does not address the violation of the first LSA.

## Some well known textbooks

Angrist and Pischke (MHE)

- several examples where marginal effects of probit and LPM are “indistinguishable”.

*...while a nonlinear model may fit the CEF (conditional expectation function) for LDVs (limited dependent variable models) more closely than a linear model, when it comes to marginal effects, this probably matters little. This optimistic conclusion is not a theorem, but as in the empirical example here, it seems to be fairly robustly true. (2009, p. 107)*

and continue...

*...extra complexity comes into the inference step as well, since we need standard errors for marginal effects. (ibid.)*



## Counter example: Lewbel Dong and Yang (2012)

- LPM is not just about taste and convenience.
- Three treated observations, three untreated
- Assume that  $f(\varepsilon) \sim N(0, \sigma^2)$

$$D = I(1 + Treated + R + \varepsilon \geq 0)$$

- Each individual treatment effect given by:

$$I(2 + R + \varepsilon \geq 0) - I(1 + R + \varepsilon \geq 0) = I(0 \leq 1 + R + \varepsilon \leq 1)$$

- All treatment effects are positive for all  $(R, \varepsilon)$ .
- Construct a sample where true effect = 1 for 5th individual, 0 otherwise.  $ATE = \frac{1}{6}$ .

# Lewbel Dong and Yang (2012)

```
. list
|      R      Treated   D |
1. |   -1.8           0   0 |
2. |    -.9           0   1 |
3. |   -.92           0   1 |
4. |  -2.1           1   0 |
5. | -1.92           1   1 |
6. |    10           1   1 |
```

```
. reg D Treated R, robust
```

```
Linear regression               Number of obs   =           6
                                F(2, 3)        =           1.02
                                Prob > F         =           0.4604
                                R-squared        =           0.1704
                                Root MSE     =           .60723
```

	D	Coef.	Robust Std. Err.	t	P> t	[95% Conf. Interval]
Treated		-.1550841	.5844637	-0.27	0.808	-2.015108 1.70494
R		.0484638	.0419179	1.16	0.331	-.0849376 .1818651
_cons		.7251463	.3676811	1.97	0.143	-.4449791 1.895272

```
. nlcom _b[Treated]/_b[R]
      _nl_1:  _b[Treated]/_b[R]
```

	D	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
_nl_1		-3.2	10.23042	-0.31	0.754	-23.25125 16.85125

## Lewbel Dong and Yang (2012)

- That went well, except that:
  - we got the wrong sign of  $\beta_T$
  - $\beta_1/\beta_2$  was the wrong sign and three times too big.
- this is not because of small sample size or  $\beta_1 \approx 0$ .
- As  $n \rightarrow \infty$  we can get an arbitrarily precise wrong answer.
- We don't even get the sign right!
- This is still in OLS (not much hope for 2SLS).

```
. expand 30
(...)
. reg D Treated R, robust
```

```
Linear regression               Number of obs   =          180
                                F(2, 177)      =          59.93
                                Prob > F        =          0.0000
                                R-squared        =          0.1704
                                Root MSE     =          .433
```

			Robust				
	D	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
Treated		-.1550841	.0760907	-2.04	0.043	-.3052458	-.0049224
R		.0484638	.0054572	8.88	0.000	.0376941	.0592334
_cons		.7251463	.047868	15.15	0.000	.6306808	.8196117

# Roadmap of Talk

## Moving Away from LPM

Problem with the LPM/OLS is that it requires that **marginal effects are constant** or that probability can be written as linear function of parameters.

$$Pr(Y = 1|X) = \beta_0 + \beta_1 X + \epsilon$$

Want a function  $F(z) : (-\infty, \infty) \rightarrow [0, 1]$ .

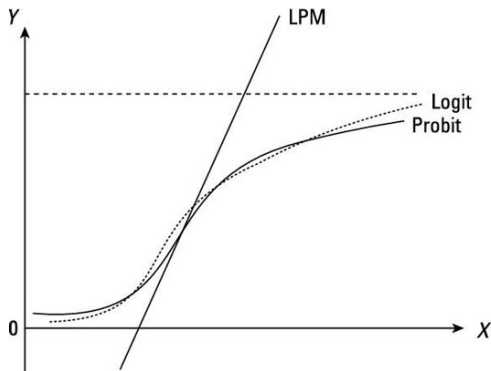
What function will work?

## Choosing a transformation

- One  $F(\cdot)$  that works is  $\Phi(z)$  the normal CDF. This is the **probit** model.
  - Actually any CDF would work but the normal is convenient.
- Another  $F(\cdot)$  that works is  $\frac{e^z}{1+e^z} = \frac{1}{1+e^{-z}}$  the logistic function . This is the **logit** model.
- Both of these give 'S'-shaped curves.
- This  $F(\cdot)$  is often called a **link function**.

## A quick comparison

- LPM prediction departs greatly from CDF long before  $[0, 1]$  limits.
- We get probabilities that are too extreme even for  $X\hat{\beta}$  “in bounds”.
- Logit and Probit are highly similar



## We really care about marginal effects

**Table 14.3.** *Binary Outcome Data: Commonly Used Models*

Model	Probability ( $p = \Pr[y = 1 \mathbf{x}]$ )	Marginal Effect ( $\partial p / \partial x_j$ )
Logit	$\Lambda(\mathbf{x}'\boldsymbol{\beta}) = \frac{e^{\mathbf{x}'\boldsymbol{\beta}}}{1 + e^{\mathbf{x}'\boldsymbol{\beta}}}$	$\Lambda(\mathbf{x}'\boldsymbol{\beta})[1 - \Lambda(\mathbf{x}'\boldsymbol{\beta})]\beta_j$
Probit	$\Phi(\mathbf{x}'\boldsymbol{\beta}) = \int_{-\infty}^{\mathbf{x}'\boldsymbol{\beta}} \phi(z)dz$	$\phi(\mathbf{x}'\boldsymbol{\beta})\beta_j$
Complementary log-log	$C(\mathbf{x}'\boldsymbol{\beta}) = 1 - \exp(-\exp(\mathbf{x}'\boldsymbol{\beta}))$	$\exp(-\exp(\mathbf{x}'\boldsymbol{\beta})) \exp(\mathbf{x}'\boldsymbol{\beta})\beta_j$
Linear probability	$\mathbf{x}'\boldsymbol{\beta}$	$\beta_j$



## Latent Variables/ Limited Dependent Variables

An alternative way to think about this problem is that there is a continuously distributed  $Y^*$  that we as the econometrician don't observe.

$$Y_i = \begin{cases} 1 & \text{if } Y^* > 0 \\ 0 & \text{if } Y^* \leq 0 \end{cases}$$

- Instead we only see whether  $Y^*$  exceeds some threshold (in this case 0).
- We can think about  $Y^*$  as a **latent variable**.
- Now, the distribution of  $\epsilon$  implies probit/logit.

# Index Models

We sometimes call these single index models or threshold crossing models

$$Z_i = X_i\beta$$

- We start with a potentially large number of regressors in  $X_i$  but  $X_i\beta = Z_i$  is a **scalar**
- We can just calculate  $F(Z_i)$  for Logit or Probit (or some other CDF).
- $Z_i$  is the **index**. if  $Z_i = X_i\beta$  we say it is a **linear index** model.

## Challenge: Where to evaluate?

$$\frac{\partial E[Y_i|X_i]}{\partial X_{ik}} = f(X_i'\beta)\beta_k$$

- The whole point was that we wanted marginal effects not to be constant
- So where do we evaluate?
  - Software often plugs in mean or median values for each component
  - Alternatively we can integrate over  $X$  and compute:

$$E_{X_i}[f(X_i'\beta)\beta_k]$$

- The right thing to do is probably to plot the response surface (either probability) or change in probability over all  $X$ .

# How to compare across models?

Natural to compare log-Likelihood

$$L_N(\hat{\beta}) = \sum_i \{y_i \ln \hat{p}_i + (1 - y_i) \ln(1 - \hat{p}_i)\}$$

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What about the different marginals?

Coefficients will be different because of the different formulas and normalizations.

According to CT, this works well for  $0.1 < p < 0.9$

$$\hat{\beta}_{Logit} \approx 4\hat{\beta}_{OLS}$$

$$\hat{\beta}_{Probit} \approx 2.5\hat{\beta}_{OLS}$$

$$\hat{\beta}_{Logit} \approx 1.6\hat{\beta}_{Probit}$$

## A word of caution re interaction effects

- Often we want to include interaction effects in our models.

$$Y = \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

- A leading example is difference in differences
- With LPM, interaction terms directly interpretable as the change in probabilities.
- Conversely, Ai and Norton (2003) show that the marginal of the interaction term has NO INTERPRETATION!

## Consider the probit

$$E[y|X] = \Phi(\beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2)$$

where  $\Phi$  is the standard normal cdf.

Consider continuous  $x_1, x_2$ . The true interaction effect is

$$\frac{\partial E[y|X]}{\partial x_1 \partial x_2} = \beta_{12} \Phi'(\cdot) + (\beta_1 + \beta_{12} x_1) \Phi''(\cdot)$$

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However, most applied economists instead report the marginal effect of the interaction term,

$$\partial \Phi(\cdot) / \partial (x_1 x_2) = \beta_{12} \Phi'(\cdot)$$

Note that this can easily give the wrong sign, and that the true partial can be nonzero even if  $\beta_{12} = 0$ .



# Roadmap of Talk

## Probit Model: Log Likelihood Derivation

- Binary choice model:  $y_i \in \{0, 1\}$

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- Binary choice model:  $y_i \in \{0, 1\}$
- Latent variable representation:

$$y_i^* = x_i' \beta + \epsilon_i, \quad \epsilon_i \sim N(0, 1)$$

$$y_i = \begin{cases} 1 & \text{if } y_i^* > 0 \\ 0 & \text{if } y_i^* \leq 0 \end{cases}$$

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$$y_i = \begin{cases} 1 & \text{if } y_i^* > 0 \\ 0 & \text{if } y_i^* \leq 0 \end{cases}$$

- For observation  $i$ :

$$\begin{aligned} P(y_i = 1 | x_i) &= P(y_i^* > 0) \\ &= P(x_i' \beta + \epsilon_i > 0) \\ &= P(\epsilon_i > -x_i' \beta) \\ &= 1 - \Phi(-x_i' \beta) \\ &= \Phi(x_i' \beta) \end{aligned}$$

# Probit Model: Estimation

Log likelihood:

$$\min_{\beta} \ln L(\beta) = \sum_{i=1}^n \left[ y_i \ln \Phi(x'_i \beta) + (1 - y_i) \ln(1 - \Phi(x'_i \beta)) \right]$$

- Score function (gradient):

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^n \left[ y_i \frac{\phi(x'_i \beta)}{\Phi(x'_i \beta)} - (1 - y_i) \frac{\phi(x'_i \beta)}{1 - \Phi(x'_i \beta)} \right] x_i$$

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- Information matrix (negative expected Hessian):

$$\begin{aligned} \mathcal{I}(\beta) &= E \left[ -\frac{\partial^2 \ln L}{\partial \beta \partial \beta'} \right] \\ &= \sum_{i=1}^n \frac{\phi(x'_i \beta)^2}{\Phi(x'_i \beta)(1 - \Phi(x'_i \beta))} x_i x'_i \end{aligned}$$



# Computation

- Computation using Newton-Raphson:

$$\beta^{(k+1)} = \beta^{(k)} + \mathcal{I}(\beta^{(k)})^{-1} s(\beta^{(k)})$$

where  $s(\beta)$  is the score vector

## Key Implementation Notes

- Need to handle numerical issues when  $\Phi(x_i'\beta)$  is close to 0 or 1
- Can use BHHH approximation:  $\mathcal{I}(\beta) \approx \sum_{i=1}^n s_i(\beta) s_i(\beta)'$

# Roadmap of Talk

# What About Endogeneity?

Lewbel, Dong, Yang (2012) discuss four “convenient” options for binary choice models with endogenous regressors.

1. Close eyes, run the LPM with instruments (Suggested by MHE).
2. Specify the distribution of errors in first and second stage and do MLE (`biprobit` in STATA).
3. Control Function Estimation
4. ‘Special Regressor’ Methods

# Setup: Endogeneity

## Setup:

- Binary variable  $D$ : the outcome of interest
- $X$  is a vector of observed regressors with coefficient  $\beta$ 
  - (Can think about  $X^e$ : endogenous and  $X^0$ : exogenous).
  - In a treatment model we might have that  $T$  is a binary treatment indicator within  $X$
- $\epsilon$  is unobserved error. Specifying  $f(e)$  can give logit/probit.
- Threshold Crossing / Latent Variable Model:

$$D = \mathbf{1}(X\beta + \epsilon \geq 0)$$

- Goal is not usually  $\hat{\beta}$  or it's CI, but rather  $P(D = 1|X)$  or  $\frac{\partial P[D=1|x]}{\partial X}$  (marginal effects).

## Solution #0 : LPM

### Advantages

- Just like 2SLS.
- Computationally easy (no numerical searches)
- Missing  $Z$  is about efficiency not consistency.
- $X^e$  can be discrete or continuous (same estimator)
- allows general heteroskedasticity (random coefficients)

### Disadvantages

- $\hat{F}_D$  is linear not S-shaped; can be outside  $[0, 1]$ .
- no element of  $X$  can have  $\infty$  support (e.g. no normally distributed regressors).
- $\varepsilon$  not independent of any regressors (even the exogenous ones). How do we also get  $E[X^0 \varepsilon] = 0$  ?

## Solution #1 : MLE

$$D = I(X'\beta + \varepsilon \geq 0) \quad \text{and} \quad X^e = G(Z, \theta, e)$$

- Fully specified  $G$  (could be vector). Could be linear if  $X^e$  continuous or probit if  $X^e$  binary.
- Need to fully specify distribution of  $(\varepsilon, e, |Z)$ , parametrized.
- Implementation (see CT book), `biprobit` for joint normal in Stata.

# Solution #1 : MLE

## Advantages

- Nests logit, probit, etc. as special cases.
- Can have any kind of  $X^e$
- Allows heteroskedasticity, random coefficients
- Asymptotically efficient (if correctly specified)

## Disadvantages

- Need to parametrize everything  $G, F_{\varepsilon, e|Z}$ .
- Numerical optimization issues
- Many nuisance parameters, sometimes poorly identified, especially with discrete  $X^e$ , correlation between latent  $(\varepsilon, e)$ .
- Need to know all required instruments  $Z$ . Omitting just one  $Z$  causes inconsistency in  $G$ .

## Solution #2 : Control Functions

Setup:

$$D = I(X'\beta + \varepsilon \geq 0)$$

Assume first stage relationship identified and invertible in  $e$

$$X^e = G(Z) + e \quad \text{or} \quad X^e = G(Z, e)$$

With conditions  $U \perp Z, e$

$$\varepsilon = \lambda'e + U \quad \text{or} \quad \varepsilon = H(U, e)$$

Simple Case:

- Estimate a vector of functions  $G$  in the  $X^e$  models, get estimated errors  $\hat{e}$ .
- Estimate the  $D$  model including  $\hat{e}$  as additional regressors in addition to  $X$ .
- This “cleans” the errors in  $U$ .



## CF Example: Linear I

From Wooldridge. Second stage:

$$y_1 = z_1' \delta + \alpha_1 y_2 + u_1$$

$y_2$  is endogenous, but  $E(z' u_1) = 0$

$z$  is  $L$  by 1 vector of instruments, with  $z_1$  a strict subvector

Write reduced form first stage as

$$y_2 = z' \pi_2 + v_2$$

with  $E(z' v_2) = 0$

Engogeneity of  $y_2$  arises iff  $u_1$  and  $v_2$  are correlated.

## CF Example: Linear II

Write the projection  $u_1 = \rho_1 v_2 + e_1$

By definition,  $E(v_2 e_1) = 0$  and by construction  $E(z' e_1) = 0$  (because  $z$  uncorrelated with  $u$  and  $v$ ).

We now have

$$y_1 = z_1' \delta + \alpha_1 y_2 + \rho_1 v_2 + e_1$$

Procedure

1. Regress  $y_2$  on  $z$  to recover  $\hat{v}_2$
2. Regress  $y_1$  on  $z_1, y_2$ , and  $\hat{v}_2$

This regression is a **control function** estimate.

## Binary CF (Stata Version)

- ivprobit assumes that  $G(Z, e)$  is linear, and  $(e, \varepsilon)$  jointly normal, independent of  $Z$ .
- It is actually Control Function not IV

$$\begin{aligned} D &= I(X^e \beta_e + X^0 \beta_0 + \varepsilon \geq 0) \\ X^e &= \gamma Z + e \end{aligned}$$

Run first-stage OLS and get residuals  $\hat{e}$ . Then plug into

$$D = I(X^e \beta_e + X^0 \beta_0 + \lambda \hat{e} + U \geq 0)$$

and do a conventional probit estimator.

- God help you if  $X^e$  isn't continuous.

## Stronger requirements than 2SLS

$$\begin{aligned} D &= I(X'\beta + \varepsilon \geq 0); & X^e &= G(Z, e), \\ \varepsilon &= H(U, e) & ; & \quad U \perp X, e \end{aligned}$$

Much stronger requirements than 2SLS

- Must be able to solve for errors  $e$  in  $X^e$  equations (not just orthogonality)
- Endogeneity must be caused only by  $\varepsilon$  relation to  $e$  so after conditioning on  $e$  must be that  $f(\varepsilon|e, X^e) = f(\varepsilon|e)$ .
- I need a consistent estimator for  $e$  which means nothing is omitted.

Not Quite MLE

- First stage can be semi/non-parametric .
- Don't need to fully specify joint distribution of  $(\varepsilon, e)$  (Stata does though!).

## Control Functions: Advantages

- Nests logit, probit, etc. as special cases
- Requires less parametric information than MLE
- Some versions are computationally easy without numerical optimization (Bootstrap!)
- Less efficient than MLE due to less restrictions, but can be semiparametrically efficient given information.

## Control Functions (Disadvantages: Not well known)

- Need to correctly specify vector  $G(Z, e)$  including all  $Z$ . Omitting a  $Z$  or misspecified  $G$  causes inconsistency because we need to have joint conditions on  $(\varepsilon, e)$ .
- Generally inconsistent for  $X^e$  that is discrete, censored, limited, or not continuous.
- If you cannot solve for a latent  $e$  in  $G(Z, e)$  then you can't get  $\hat{e}$  for the censored observations (e.g.:  $X^e = \max(0, Z'\gamma + e)$ ).
- An observable  $e$  is  $e = X^e - E[X^e|Z]$  but for discontinuous  $X^e$  that  $e$  violates assumptions (except in very strange cases)
  - Ex:  $\varepsilon = [X^e - E[X^e|Z]]\lambda + U$  satisfies CF, but if  $X^e$  is discrete then  $e$  has some strange distribution that depends on regressors.
  - Hard to generate a model of behavior that justifies this!

## Solution #2 : Control Functions Generalized Residuals

What if  $X^e$  isn't continuous? Technically possible...

- Given the probit estimate in first stage we could construct a generalized residual (see Imbens and Wooldridge notes)
- $e^g \propto E[\varepsilon|Z, e]$ . An estimate  $\hat{e}^g$  of  $e^g$  can be included as a regressor in the model to fix the endogeneity problem, just as  $\hat{e}$  would have been used if the endogenous regressor were continuous.

Why would you ever want to do this..

- In the linear model we should just do IV with far fewer restrictions
- In the nonlinear model,  $\hat{e}^g$  requires almost as many assumptions as MLE which is efficient!

## Solution #3 : Special Regressor

This section draws on Dong and Lewbel (2015),



## Solution #3 : Special Regressor

This section draws on Dong and Lewbel (2015), but there are many other papers on this....

Binary, ordered, and multinomial choice, censored regression, selection, and treatment models (Lewbel 1998, 2000, 2007a), truncated regression models (Khan and Lewbel 2007), binary panel models with FE (Honore and Lewbel 2002), dynamic choice models (Heckman and Navarro 2007, Abbring and Heckman 2007), contingent valuation models (Lewbel, Linton, and McFadden 2008), market equilibrium models of multinomial choice (Berry and Haile 2009a, 2009b), models with (partly) nonseparable errors (Lewbel 2007b, Matzkin 2007, Briesch, Chintagunta, and Matzkin 2009).

Other empirical applications: Anton, Fernandez Sainz, and Rodriguez-Poo (2002), Cogneau and Maurin (2002), Goux and Maurin (2005), Stewart (2005), Lewbel and Schennach (2007), and Tiwari, Mohnen, Palm, and van der Loeff (2007).

Precursors: Matzkin (1992, 1994) and Lewbel (1997).

Recent theory: Magnac and Maurin (2007, 2008) Khan and Tamer (2010), and Khan and Nekipelov (2010a, 2010b).

## Special Regressor: Requirements

$$D = I(X'\beta + V + \varepsilon \geq 0)$$

- Exogenous  $E[\varepsilon|V] = 0$  (Strict Exogeneity) **This is the key!**
- Additively separable in the model
  - NOT interacted with other regressors.
  - enters LINEARLY, e.g.  $V$  must be continuously distributed after conditioning on other regressors
- Continuous distribution and large support (such as Normal)
- Helpful to have thick-tails (kurtosis). Why? We want to trace  $Pr(D|V)$  from  $[0, 1]$ .
- Can normalize its coefficient to 1
- **In contrast:** 2SLS Assumptions:  $E[\varepsilon|Z] = 0$  and  $E[Z'X]$  is full rank.

# Simple SR Estimation Procedure

1. Model:  $Y^* = V + X\beta + \epsilon$ ,  $Y = 1[Y^* > 0]$
2. Transformation Steps:
  - a. `v_centered = v .- mean(v)`
  - b. `y_transformed = y ./ pdf.(Normal(), v_centered)`
  - c. Trim extreme values to get `x_trim`, `y_trim`
  - d. Run weighted linear regression:
    - `weights = pdf.(Normal(), v_centered[keep])`
    - `β_hat = (x_trim'Diagonal(weights)x_trim)\(x_trim'Diagonal(weights)y_trim)`

# A simple SR estimator with instruments (Lewbel and Dong, 2015)

1. Demean or center  $V$  at zero.
  - Project  $V$  onto all the instruments  $Z$  and regressors  $X$  to recover  $\hat{U}_i = V_i - S_i' \hat{b}$
2. Sort  $n$  observations by  $\hat{U}_i$ . Then define  $\hat{f}_i = 2 / [(\hat{U}_i^+ - \hat{U}_i^-)n]$ 
  - this estimates the density using only the observations on either side of  $i$
  - alternatively, can use a kernel density estimator
3. Construct  $\hat{T}_i = [D_i - I(V_i \geq 0)] / \hat{f}_i$
4. Linear 2SLS regression of  $\hat{T}$  on  $X$  using instruments  $Z$  to get the estimated coefficients  $\hat{\beta}$ .

By properly adjusting  $T_i$  we guarantee to stay in  $[0, 1]$ .

Lewbel/Dong shows that by using instruments,  $V$  does not need full support, so this is sort of a hybrid approach.

## Why does this work?

- $V = S'b + U$
- Define  $T = [D - I(V \geq 0)]/f(U)$
- Under these assumptions, can be shown that  $T = X'\beta + \tilde{\epsilon}$  with  $E(Z\tilde{\epsilon}) = 0$
- Therefore  $E(ZT) = E(ZX')\beta$

## Special Regressor Advantages

- Unlike LPM it stays “in bounds” and is consistent with threshold crossing models.
- Unlike MLE and CF, does not require correctly specified first stage model: any valid set of instruments may be used, with only efficiency at stake.
- Unlike MLE, the SR method has a linear form, not requiring iterative search
- Unlike CF, the SR method can be used when endogenous regressors  $X^e$  are discrete or limited; unlike ML there is a single estimation method, regardless of the characteristics of  $X^e$
- Unlike MLE, the SR method permits unknown heteroskedasticity in the model errors.

## Special Regressor Disadvantages

- Because assumptions are weaker we give up a lot of potential efficiency (larger SEs).
- Of course this presumes the assumptions were valid and alternatives were consistent.
- Finding a suitable  $V$  (or  $V$  and  $Z$ ) can be difficult.
- might be sensitive to trimming or density estimation in first step.

## Binary Choice with Endogenous Regressors

- Linear probability models, Maximum Likelihood, and Control functions (including `ivprobit` have more drawbacks and limitations than are usually recognized.
- Special Regressor estimators are a viable alternative (or at least they have completely different drawbacks and may be more generally applicable than has been recognized).
- In practice, best might be to try all estimators and check robustness of results. Can use marginal effects to normalize them the same when comparing.
- Average Index Functions can be used to construct estimated probabilities and comparable marginal effects across estimators, often simpler to calculate than Average Structural Functions.
- Implementation of special regressor in Stata is done in `sspecialreg`.



## Empirical Example: Dong and Lewbel (2015)

- Binary dependent variable: does  $i$  migrate from one state to another.
- Special Regressor  $V_i$ : age. Human capital theory suggests it should appear linearly (or at least monotonic) in a threshold crossing model
- Migration is drive by maximizing expected lifetime income and potential gain from a permanent change in income declines linearly in age.
- $V_i$  is defined as negative of age, demeaned so that coefficient is positive with mean zero.
- Other endogenous regressors: family income pre migration, home ownership.

## Empirical Example: Dong and Lewbel (2015)

As a reminder, normally we would be in trouble here:

- MLE would be very complicated with multiple endogenous variables
- Control functions `ivprobit` won't work with 0/1 homeowner variable.

## Empirical Example: Dong and Lewbel (2015)

### 1990 PSID

- male head of household (23-59 years), completed education and not-retired (key!)
- $D = 1$  indicates migration during 1991-1993.
- 4689 Individuals, 807 migrants.
- Exogenous regressors: years of education, # of children, white, disabled, married.
- Instruments: level of govt benefits in 1989-1990, state median residential tax rate.

# Empirical Example: Dong and Lewbel (2015)

## Specifications

- Special Regressors: kernel density vs. sorted data density.
- Special Regressor: homoskedastic vs. heteroskedastic errors.
- LPM vs 2SLS
- Probit (assuming exogeneity)
- Control Function (ivprobit) misspecified for homeowner endogenous binary variable.

# Empirical Example

Table: Marginal effects: binary outcome, binary endogenous regressor

	kdens	sortdens	kdens_hetero	sortdens_hetero	IV-LPM	probit	ivprobit
age	0.0146 (0.003)***	0.0112 (0.003)***	0.0071 (0.003)*	0.0104 (0.003)***	-0.0010 (0.002)	0.0019 (0.001)**	-0.0005 (0.007)
log income	-0.0079 (0.028)	0.0024 (0.027)	0.0382 (0.024)	0.0176 (0.026)	0.0550 (0.080)	-0.0089 (0.007)	0.1406 (0.286)
homeowner	0.0485 (0.072)	-0.0104 (0.065)	-0.0627 (0.059)	-0.0111 (0.061)	-0.3506 (0.204)	-0.0855 (0.013)***	-1.0647 (0.708)
white	0.0095 (0.008)	0.0021 (0.010)	0.0021 (0.007)	0.0011 (0.008)	0.0086 (0.018)	-0.0099 (0.012)	0.0134 (0.065)
disabled	0.1106 (0.036)**	0.0730 (0.042)	0.0908 (0.026)***	0.0916 (0.037)*	0.0114 (0.055)	-0.0122 (0.033)	0.0104 (0.203)
education	-0.0043 (0.002)*	-0.0023 (0.003)	-0.0038 (0.002)*	-0.0036 (0.002)	0.0015 (0.004)	0.0004 (0.002)	0.0047 (0.015)
married	0.0628 (0.020)**	0.0437 (0.028)	0.0258 (0.013)	0.0303 (0.020)	0.0322 (0.031)	-0.0064 (0.017)	0.0749 (0.114)
nr. children	-0.0169 (0.005)***	-0.0117 (0.005)*	0.0006 (0.002)	-0.0021 (0.003)	0.0137 (0.006)*	0.0097 (0.005)*	0.0502 (0.023)*

Note: bootstrapped standard errors in parentheses (100 replications)

## Empirical Example: Dong and Lewbel (2015)

- SEs of MFX are computed from 100 bootstrap replications
- MFX of special regressor (age) is estimated as positive and significant but LPM and ivprobit estimate negative effects!
- household income and home ownership status do not seem to play significant roles in migration decision.
- Kernel density estimator seems to give most significant results.