Chap. 1 de Rham Theory

§1. The de Rham Complex on $\mathbb{R}^n$

$\underline{\Omega^*}$

$dx_1, \dots, dx_n$ によって $\mathbb{R}$ 上生成される代数

$$(dx_i)^2 = 0$$

$$dx_i dx_j = -dx_j dx_i \quad i \neq j$$

これは

$$1,\ dx_i,\ \underset{i<j}{dx_i dx_j},\ \dots,\ dx_1 \cdots dx_n$$

を基底とする $\mathbb{R}$ 上の線形空間である.

$$\Omega^*(\mathbb{R}^n) = \{\mathbb{R} \text{ 上の } C^\infty \text{ 関数たち}\} \otimes_{\mathbb{R}} \Omega^*$$

の元を $\mathbb{R}$ 上の C^∞ 微分形式という.

$$\omega = \sum f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_q} = \sum f_I dx_I$$

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n)$$

微分演算子

$$d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n)$$

$$f \in \Omega^0(\mathbb{R}^n) \rightsquigarrow df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$\omega = \sum f_I dx^I \rightsquigarrow d\omega = \sum df_I dx_I$$

例) $\mathbb{R}^3$ 上 $\quad x_1, x_2, x_3 \rightsquigarrow x, y, z$

$$f \in \Omega^0 \rightsquigarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$f_1 dx + f_2 dy + f_3 dz \in \Omega^1$$

$$\rightsquigarrow \quad d(f_1 dx + f_2 dy + f_3 dz)$$

$$= \left(\frac{\partial f_1}{\partial x} \cancel{dx} + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz\right) dx$$

$$+ \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} \cancel{dy} + \frac{\partial f_2}{\partial z} dz\right) dy$$

$$+ \left(\frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} \cancel{dz}\right) dz$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy\, dz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx\, dz$$

$$+ \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\, dy$$

$$f_1 dy\, dz - f_2 dx\, dz + f_3 dx\, dy \in \Omega^2(\mathbb{R})$$

$$\rightsquigarrow \quad d(f_1 dy\, dz - f_2 dx\, dz + f_3 dx\, dy)$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz$$

$\tau = \sum f_I dx_I \quad , \quad \omega = \sum g_J dx_J$

$\wedge$ とも

$\tau \cdot \omega = \sum f_I g_J \, dx_I \, dx_J$

$\tau \cdot \omega = (-1)^{\deg \tau \deg \omega} \, \omega \cdot \tau$

$d(\tau \cdot \omega) = d\tau \cdot \omega + (-1)^{\deg \tau} \, \tau \cdot d\omega$

$d^2 = 0$

$d^2 f = d\left(\sum \frac{\partial f}{\partial x_i} dx_i\right) = \sum \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \, dx_i = 0$

対称　　交対称

ちなみに、$d(dx_i) = 0$

d と $\Omega^*(\mathbb{R}^n)$ をあわせて $\mathbb{R}^n$ 上の de Rham 複体 (complex) という

$\ker d$ の元を closed

$\operatorname{im} d$ の元を exact　という。

$\mathbb{R}^n$ の de Rham cohomology

$$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\} / \{\text{exact } q\text{-forms}\}$$

$\mathbb{R}^n$ の開部分集合 U に対しても同様

$n = 0$

$$0 \longrightarrow \Omega^0(\mathbb{R}^0) \longrightarrow 0$$

$\mathbb{R}^0 = \{0\}$ だよね…

"$\mathbb{R}^0$ 上の関数" $= \mathbb{R}$

$H^0(\mathbb{R}^0) = \mathbb{R}$　他は 0.

$n = 1$

$$0 \longrightarrow \Omega^0(\mathbb{R}^1) \xrightarrow{d} \Omega^1(\mathbb{R}^1) \longrightarrow 0$$

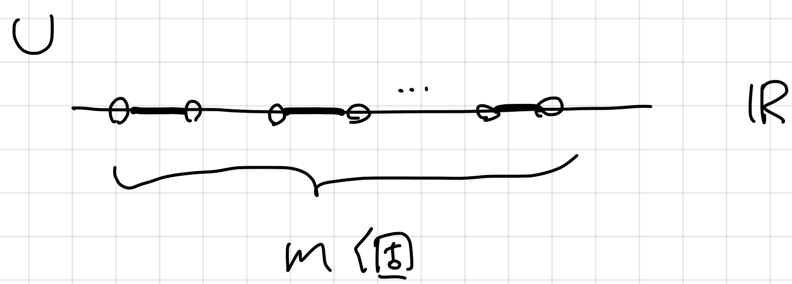
$\ker d = \mathbb{R}$　　$H^0(\mathbb{R}^1) = \mathbb{R}$

$\omega = g(x)dx$ のとき　$f(x) = \int_0^x g(u)du$ とおくと

$df = g(x)dx$

よって $\Omega^1(\mathbb{R}) = \mathrm{im}\, d$　全部 exact

$H^1(\mathbb{R}^1) = 0$



$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow 0$$

$H^0(U) = \mathbb{R}^m$　　m個の場所で定数.

$H^1(U) = 0$

一般に

$H^0(\mathbb{R}^n) = \mathbb{R}$　　　Poincaré lemma

$H^q(\mathbb{R}^n) = 0 \quad q \neq 0$

differential complex

$$\longrightarrow C^{q-1} \xrightarrow{d} C^{q} \xrightarrow{d} C^{q+1} \longrightarrow$$

$$H^{q}(C) = (\ker d \cap C^{q}) / (\mathrm{im}\, d \cap C^{q})$$

$$\begin{array}{ccccccccc} \longrightarrow & A^{q-1} & \xrightarrow{d_A} & A^{q} & \xrightarrow{d_A} & A^{q+1} & \longrightarrow \\ & f_{q-1}\downarrow & \circlearrowleft & f_{q}\downarrow & \circlearrowleft & f_{q+1}\downarrow & \\ \longrightarrow & B^{q-1} & \xrightarrow{d_B} & B^{q} & \xrightarrow{d_B} & B^{q+1} & \longrightarrow \end{array}$$

のとき

$f = \{f_q\}$ は A B 間の chain map という

$$\longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow$$

のとき

$\ker f_i = \mathrm{im}\, f_{i-1}$

完全という.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

完全

短完全列 SES

$$\begin{tikzcd}
 & \uparrow & \uparrow & \uparrow & \\
0 \arrow[r] & A^{q+1} \arrow[r, "f"] \arrow[u] & B^{q+1} \arrow[r, "g"] \arrow[u] & C^{q+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^{q} \arrow[r, "f"] \arrow[u, "d"] & B^{q} \arrow[r, "g"] \arrow[u, "d"] & C^{q} \arrow[r] \arrow[u, "d"] & 0 \\
 & \uparrow \arrow[u] & \uparrow \arrow[u] & \uparrow \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^{q+1}(A) \arrow[r] & {} \\
{} \arrow[r] & H^{q}(A) \arrow[r, "f^*"] & H^{q}(B) \arrow[r, "g^*"] & H^{q}(C) \arrow[llu, "d^*"']
\end{tikzcd}$$

長完全列 ES

$$\operatorname{Supp} f = \overline{\{p \in X \mid f(p) \neq 0\}}$$

$$\Omega_c^*(\mathbb{R}^n) = \{C^\infty \text{ funcs. on } \mathbb{R}^n \text{ with compact supp.}\} \otimes_{\mathbb{R}} \Omega^*$$

Ex. 1.6

$$H_c^*(\mathbb{R}^1) = \begin{cases} \mathbb{R} & \text{in 1 dim.} \\ 0 & \text{elsewhere} \end{cases}$$

$H_c^0(\mathbb{R}^1) = 0$ const. func. の supp. はコンパクトでない

$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \to \mathbb{R}^1$ surj.

$\int_{\mathbb{R}^1} df = 0$

$\int_{\mathbb{R}^1} g(u)\,du = 0$ のとき

$f(x) = \int_{-\infty}^{x} g(u)\,du$ とおくと

これは comp. supp. で $df = g\,dx$.

$\therefore$ $\{\text{exact forms.}\} = \ker \int_{\mathbb{R}^1}$

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1$$

$$\int_{\mathbb{R}^1} g(u)\,du = s \neq 0$$

$$\int_{\mathbb{R}^1} g'(u)\,du = s$$

$$\int_{\mathbb{R}^1} (g - g')(u)\,du = 0$$

つまり $g\,dx = g'\,dx \mod \ker \int_{\mathbb{R}^1}$

$\lrcorner$

一般に

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dim. } n \\ 0 & \text{eleswhere} \end{cases}$$ Poincaré lemma.

マイヤー　ヴィートリス　列

§2 The Mayer-Vietoris Sequence

$$\frac{\partial g_I}{\partial y_i}(f(x))\, d f_i$$

$$d\, g_I(f(x)) = \sum_i \frac{\partial}{\partial x_i} g_I(f(x))\, d x_i$$

$$= \sum_{i,j} \frac{\partial (y_j \circ f)}{\partial x_i} \frac{\partial}{\partial y_j} g_I(f(x))\, d x_i$$

$$= \sum \frac{\partial}{\partial y_j} g_I(f(x))\, d(y_j \circ f)$$

$$d(y_i \circ f) = \sum_j \frac{\partial (y_i \circ f)}{\partial x_j}\, d x_j$$