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# Chap. 1 de Rham Theory

## § 1. The de Rham Complex on $\mathbb{R}^n$

$$\underline{\Omega}^*$$

$\mathbb{R}^n$  の座標  $x_1, \dots, x_n$  に対して定義される.

$dx_1, \dots, dx_n$  によって  $\mathbb{R}$  上生成された代数

$$(dx_i)^2 = 0$$

$$dx_i dx_j = -dx_j dx_i \quad i \neq j$$

これは

$$1, dx_i, dx_i dx_j, \dots, dx_1 \cdots dx_n$$

$i < j$

を基底とする  $\mathbb{R}$  上の線形空間  $\Omega^*$  である.

$$\Omega^*(\mathbb{R}^n) = \{ \mathbb{R} \text{ 上の } C^\infty \text{ 関数 } f \} \otimes_{\mathbb{R}} \Omega^*$$

の元  $\omega \in \mathbb{R} \text{ 上の } C^\infty \text{ 微分形式}$  とし.

$$\omega = \sum f_{i_1 \dots i_q} dx_{i_1} \cdots dx_{i_q} = \sum f_I dx_I$$

⊗

省略かと.

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{k=0}^n \Omega^k(\mathbb{R}^n)$$

↑  $\omega \in \Omega^k$  is  $k$ -form  $\omega$

$$\sum_{i_1, \dots, i_k} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$\sum f_i dx_i \text{ etc.}$$

微分演算子

$$d: \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n)$$

$$f \in \Omega^0(\mathbb{R}^n) \rightsquigarrow df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\omega = \sum f_i dx_i \rightsquigarrow d\omega = \sum df_i \wedge dx_i$$

$$\text{Is } \mathbb{R}^3 \text{ } x_1, x_2, x_3 \rightsquigarrow x, y, z$$

$$f \in \Omega^0(\mathbb{R}^3)$$

$$\rightsquigarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$f_1 dx + f_2 dy + f_3 dz \in \Omega^1(\mathbb{R}^3)$$

$$\rightsquigarrow d(f_1 dx + f_2 dy + f_3 dz)$$

$$= \left( \frac{\partial f_1}{\partial x} \cancel{dx} + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz \right) dx \\ + \left( \frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} \cancel{dy} + \frac{\partial f_2}{\partial z} dz \right) dy \\ + \left( \frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} \cancel{dz} \right) dz$$

$$= \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) dy dz - \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) dx dz \\ + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy$$

$$f_1 dy dz - f_2 dx dz + f_3 dx dy \in \Omega^2(\mathbb{R}^3)$$

$$\rightsquigarrow d(f_1 dy dz - f_2 dx dz + f_3 dx dy)$$

$$= \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx dy dz$$

$$\tau = \sum f_i dx_i, \quad \omega = \sum g_j dx_j$$

$$\tau \wedge \omega = \sum f_i g_j dx_i dx_j$$

$$\tau \cdot \omega = (-1)^{\deg \tau \deg \omega} \omega \cdot \tau$$

$$d(\tau \cdot \omega) = d\tau \cdot \omega + (-1)^{\deg \tau} \tau \cdot d\omega$$

$$d^2 = 0$$

$$d^2 f = d\left(\sum \frac{\partial f}{\partial x_i} dx_i\right) = \sum \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0$$

↖ ↗  
対称 反対称

$$\exists i, j \text{ して } d(dx_i) = 0$$

complex

$d \in \Omega^*(\mathbb{R}^n)$  をあわせて  $\mathbb{R}^n$  上の de Rham 複体という

$\ker d$  の元を closed

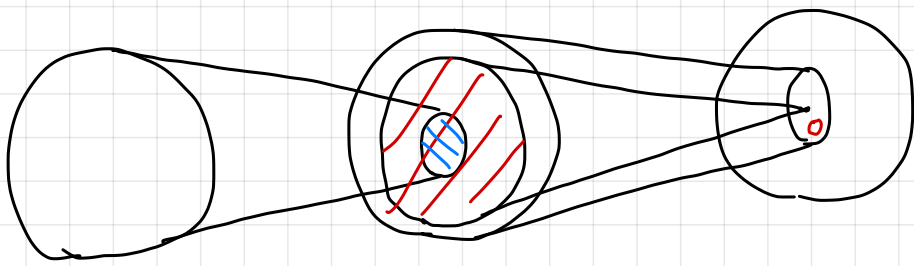
$\operatorname{im} d$  の元を exact という.

$\mathbb{R}^n$  の de Rham cohomology ベクトル空間

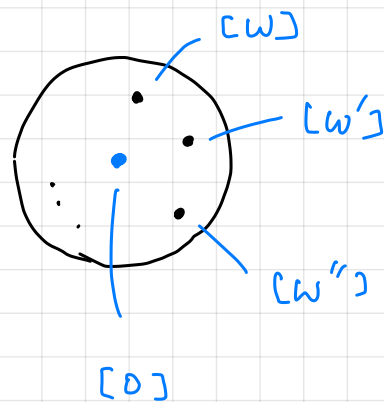
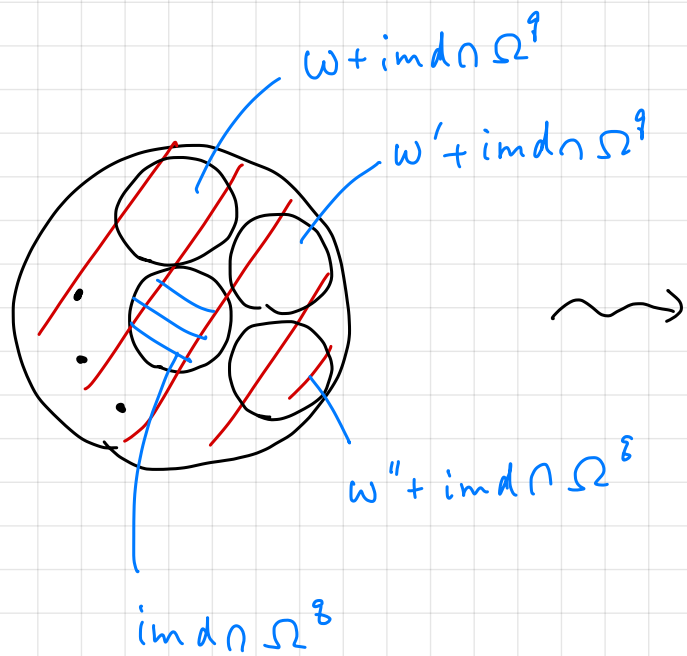
$$H_{\text{DR}}^k(\mathbb{R}^n) = \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-forms}\}$$

$$\Omega^{q-1} \xrightarrow{d} \Omega^q \xrightarrow{d} \Omega^{q+1}$$

$$\ker d \cap \Omega^q$$



$$\operatorname{im} d \cap \Omega^q$$



$$\ker d \cap \Omega^q / \operatorname{im} d \cap \Omega^q$$

$$\begin{aligned}
0 &\rightarrow H^0(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n) \rightarrow \\
0 &\xrightarrow{d} \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \cdots \\
&\quad \rightarrow H^{n-1}(\mathbb{R}^n) \rightarrow H^n(\mathbb{R}^n) \rightarrow 0 \\
\cdots &\xrightarrow{d} \Omega^{n-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} 0
\end{aligned}$$

$$H^0(\mathbb{R}^n) = \ker d \cap \Omega^0(\mathbb{R}^n)$$

$$H^q(\mathbb{R}^n) = \ker d \cap \Omega^q(\mathbb{R}^n) / \operatorname{im} d \cap \Omega^q(\mathbb{R}^n)$$

$$0 < q < n$$

$$H^n(\mathbb{R}^n) = \Omega^n(\mathbb{R}^n) / \operatorname{im} d \cap \Omega^n(\mathbb{R}^n)$$

$\mathbb{R}^n$  の開部分集合  $U$  に対して同様に

$\Omega^q(U)$ ,  $H^q(U)$  を定義される.

$$n = 0$$

$$0 \rightarrow \Omega^0(\mathbb{R}^0) \rightarrow 0$$

$$\mathbb{R}^0 = \{0\}$$

$$\text{"}\mathbb{R}^0 \text{ 上の関数"} = \mathbb{R}$$

$$H^0(\mathbb{R}^0) = \Omega^0(\mathbb{R}^0) = \mathbb{R}$$

他に 0.

$$n = 1$$

$$0 \rightarrow \Omega^0(\mathbb{R}^1) \xrightarrow{d} \Omega^1(\mathbb{R}^1) \rightarrow 0$$

$$\ker d \cap \Omega^0(\mathbb{R}^1) = \mathbb{R}$$

$$H^0(\mathbb{R}^1) = \mathbb{R}$$

$$\omega = g(x) dx \text{ の } \tau \approx \quad f(x) = \int_0^x g(u) du \quad \tau \text{ 上 } < \tau$$

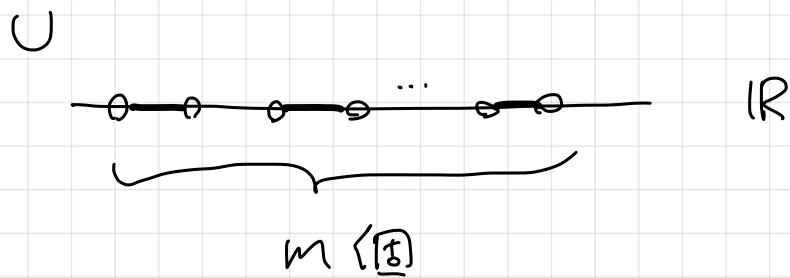
$$df = g(x) dx$$

$$\text{よって } \Omega^1(\mathbb{R}) = \text{im } d \quad \text{全部 exact}$$

$$H^1(\mathbb{R}^1) = \Omega^1(\mathbb{R}) / \text{im } d \cap \Omega^1(\mathbb{R}) = 0$$

$$\{0\}$$





$$0 \rightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \rightarrow 0$$

$$H^0(U) = \mathbb{R}^m \quad m \text{ 個の場所での定数.}$$

$$H^1(U) = 0$$

一般に

$$H^0(\mathbb{R}^n) = \mathbb{R}$$

Poincaré lemma

$$H^k(\mathbb{R}^n) = 0 \quad k \neq 0$$

differential complex

$$\longrightarrow C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \longrightarrow$$

$$H^i(C) = (\ker d \cap C^i) / (\operatorname{im} d \cap C^i)$$

$$\begin{array}{ccccccc} A = & \longrightarrow & A^{i-1} & \xrightarrow{d_A} & A^i & \xrightarrow{d_A} & A^{i+1} \longrightarrow \\ & & f_{i-1} \downarrow & \circlearrowleft & f_i \downarrow & \circlearrowleft & f_{i+1} \downarrow \\ B = & \longrightarrow & B^{i-1} & \xrightarrow{d_B} & B^i & \xrightarrow{d_B} & B^{i+1} \longrightarrow \end{array} \quad \text{の } \subset \supseteq$$

$f = \{f_i\}$  は  $A \rightarrow B$  の chain map といふ

$$\longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \quad \text{の } \subset \supseteq$$

$$\ker f_i = \operatorname{im} f_{i-1}$$

完全 といふ.

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

完全

短完全列 SES

$\{$

$$0 \rightarrow A^{i+1} \xrightarrow{f_{i+1}} B^{i+1} \xrightarrow{g_{i+1}} C^{i+1} \rightarrow 0$$

$$0 \rightarrow A^i \xrightarrow{f_i} B^i \xrightarrow{g_i} C^i \rightarrow 0$$

$\}$

$$0 \rightarrow A^{i+1} \xrightarrow{f_{i+1}} B^{i+1} \xrightarrow{g_{i+1}} C^{i+1} \rightarrow 0$$

$$0 \rightarrow A^i \xrightarrow{f_i} B^i \xrightarrow{g_i} C^i \rightarrow 0$$

$d^*$

是完金列 ES

$$\text{Supp } f = \overline{\{p \in X \mid f(p) \neq 0\}}$$

$$\Omega_c^*(\mathbb{R}^n) = \{C^\infty \text{ funcs. on } \mathbb{R}^n \text{ with compact supp.}\} \otimes_{\mathbb{R}} \Omega^*$$

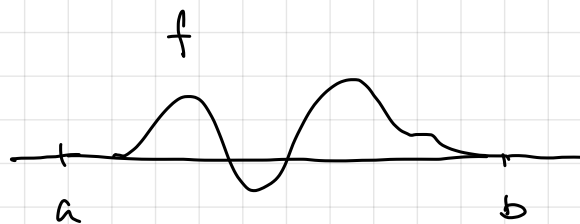
Ex. 1.6

$$(a) \quad H_c^*(\mathbb{R}^0) = \begin{cases} \mathbb{R} & \text{in 0 dim.} \\ 0 & \text{elsewhere} \end{cases}$$

$$(b) \quad H_c^0(\mathbb{R}^1) = 0 \quad \text{const. func. of supp. } \{x \in \mathbb{R}^1 \mid x \leq 2\}$$

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \rightarrow \mathbb{R}^1 \quad \text{surj.}$$

$$\int_{\mathbb{R}^1} df = f(b) - f(a) = 0$$



$$\int_{\mathbb{R}^1} f(u) du = 0 \quad \text{if } f \in \mathcal{D}'$$

$$f(x) = \int_{-\infty}^x g(u) du \quad \text{if } f \in \mathcal{D}'$$

$$\text{if } f \in \mathcal{D}' \text{ comp. supp. then } df = g dx.$$

$$\therefore \{ \text{exact forms.} \} = \ker \int_{\mathbb{R}^1}$$

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1$$

$$\int_{\mathbb{R}^1} g(u) du = s \neq 0$$

$$\int_{\mathbb{R}^1} g'(u) du = s$$

$$\int_{\mathbb{R}^1} (g - g')(u) du = 0$$

$$\Rightarrow \exists \gamma \quad g dx = g' dx \quad \text{mod } \ker \int_{\mathbb{R}^1}$$

└

-  $\mathbb{A}_2^2 \hookrightarrow$

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dim. } n \\ 0 & \text{elsewhere} \end{cases}$$

Poincaré lemma.

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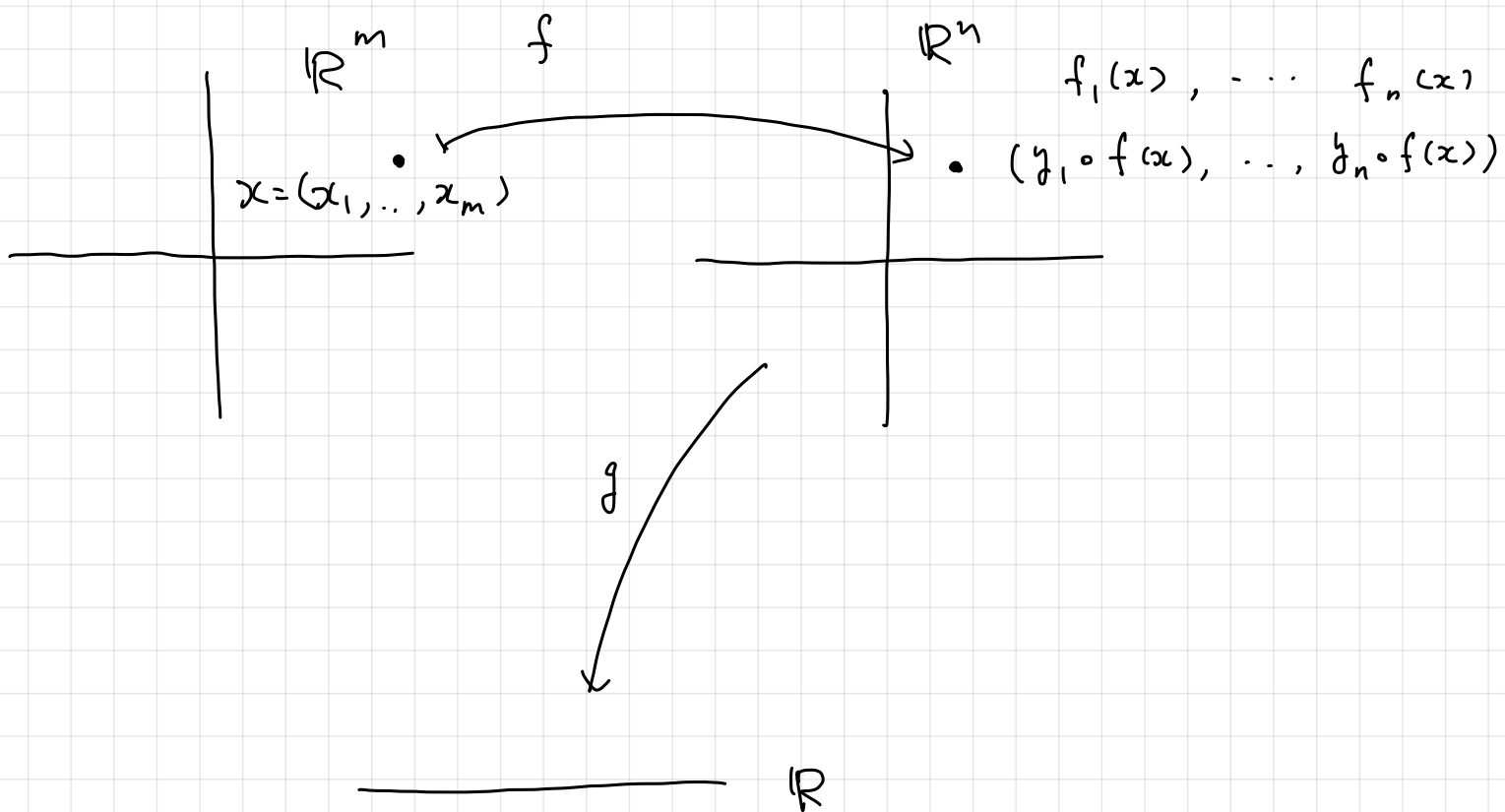
## § 2 The Mayer-Vietoris Sequence

$$f: \mathbb{R}^m \rightarrow \mathbb{R}^n \quad C^\infty$$

$$x_1, \dots, x_m \quad y_1, \dots, y_n$$

$$f^*: \Omega^0(\mathbb{R}^n) \rightarrow \Omega^0(\mathbb{R}^m)$$

$$g \longmapsto g \circ f$$



$$f^* : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^m)$$

$$\sum g_I dy_{i_1} \cdots dy_{i_k} \longmapsto \sum (g_I \circ f) df_{i_1} \cdots df_{i_k}$$

$$I = (i_1, \dots, i_k) \quad f_{i_j} = y_{i_j} \circ f$$

このこと  $df^*(\omega) = f^*(d\omega)$  次ページ参照.

座標変換

$$(x_1, \dots, x_n) \rightsquigarrow (u_1, \dots, u_n)$$

$$\sum_i \frac{dg}{du_i} du_i = \sum_i \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial x_j} dx_j = \sum_j \frac{\partial g}{\partial x_j} dx_j$$

よって  $dg$  は座標のとり方によらない.

$$\omega = \sum g_I du_I \quad \text{のとき}$$

$$d\omega = \sum dg_I du_I$$



cf.

$$df^*(g_1 dy_{i_1} \cdots dy_{i_g}) = d(g_1 \circ f) df_{i_1} \cdots df_{i_g}$$

$$f^*(dg_1 dy_{i_1} \cdots dy_{i_g})$$

$$= f^*\left(\sum_i \frac{\partial g_1}{\partial y_i} dy_i dy_{i_1} \cdots dy_{i_g}\right)$$

$$= \left(\sum_i \frac{\partial g_1 \circ f}{\partial y_i}\right) df_i df_{i_1} \cdots df_{i_g}$$

$$\stackrel{(*)}{=} \underbrace{d(g_1 \circ f) df_{i_1} \cdots df_{i_g}}$$

$$\stackrel{(*)}{=} dg_1(f(x)) = \sum_i \frac{\partial}{\partial x_i} g_1(f(x)) dx_i$$

$$= \sum_{i,j} \underbrace{\frac{\partial (y_i \circ f)}{\partial x_i}} \frac{\partial}{\partial y_j} g_1(f(x)) \underbrace{dx_i}_{(*)}$$

$$= \sum \frac{\partial}{\partial y_j} g_1(f(x)) \underbrace{d(y_j \circ f)}$$

$$\stackrel{(**)}{=} d(y_i \circ f) = \sum_j \frac{\partial (y_i \circ f)}{\partial x_j} dx_j$$

cf. Ex 4.9

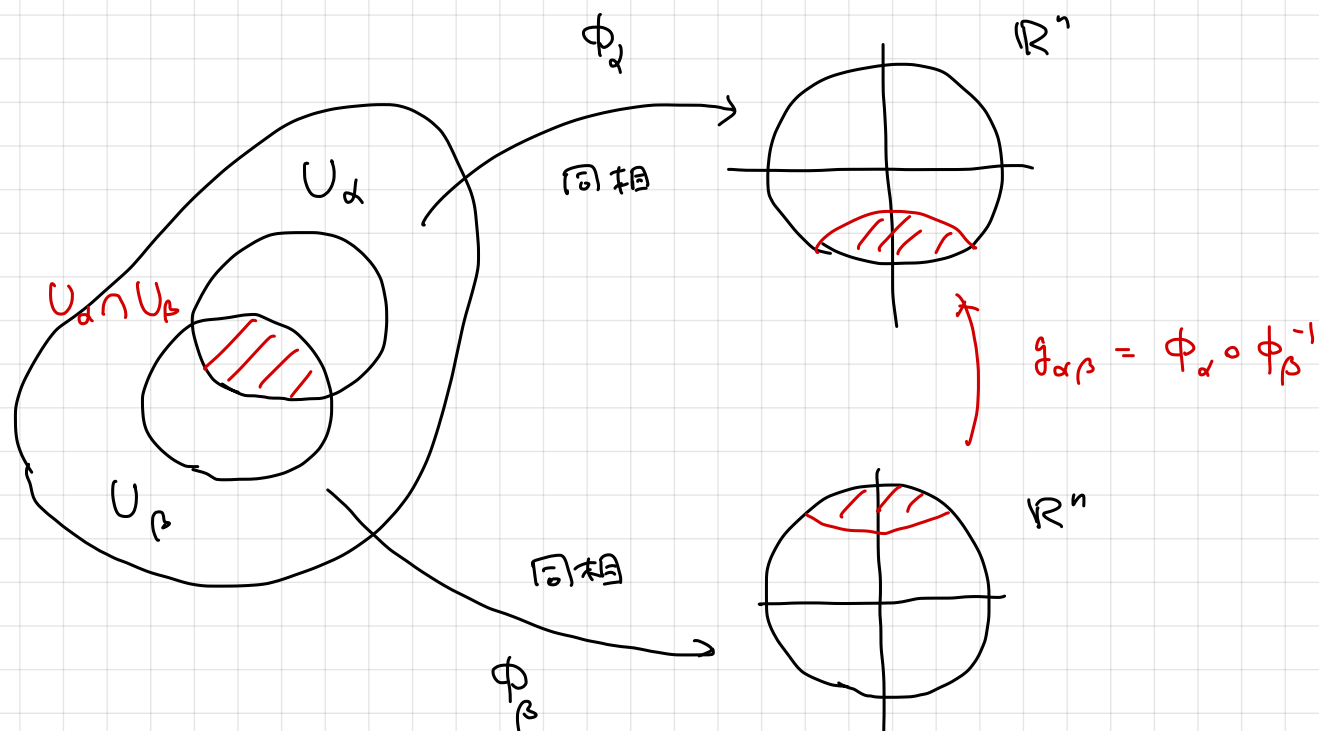
$\Omega^*$  は閉手.

$(\{\mathbb{R}^n\}_{n \in \mathbb{N}}, C^\infty \text{写像})$

$\rightarrow$  (commutative differential  
graded algebra  $T = \mathcal{I}$ , homomorphism  $T \rightarrow \mathcal{I}$ )

$$\begin{array}{ccc} \mathbb{R}^n & & \Omega^*(\mathbb{R}^n) \\ f \downarrow & \rightsquigarrow & \downarrow f^* \\ \mathbb{R}^m & & \Omega^*(\mathbb{R}^m) \end{array}$$

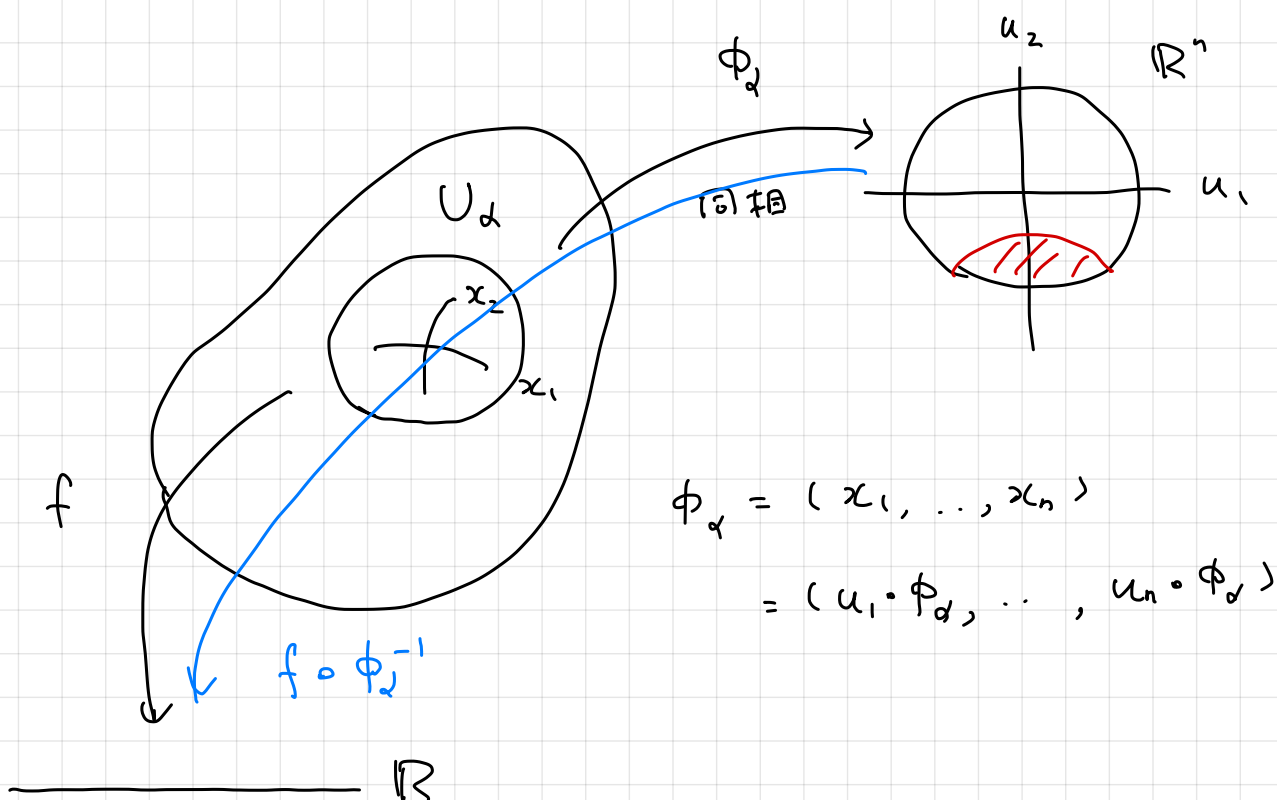
多様体 manifold



$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

Hausdorff

可算な開基をもつ。



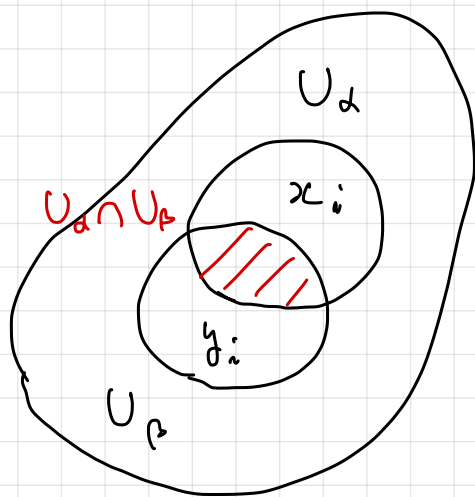
$f \circ \phi_\alpha^{-1}$  が微分可能でないとき

$f$  は微分可能という。

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p))$$

$(C^\infty)$  vector field on  $U_\alpha$

$$X_\alpha = \sum_i f_i \frac{\partial}{\partial x_i} \quad f_i : C^\infty$$



$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$$

$$X_\alpha = \sum_i f_i \frac{\partial}{\partial x_i}$$

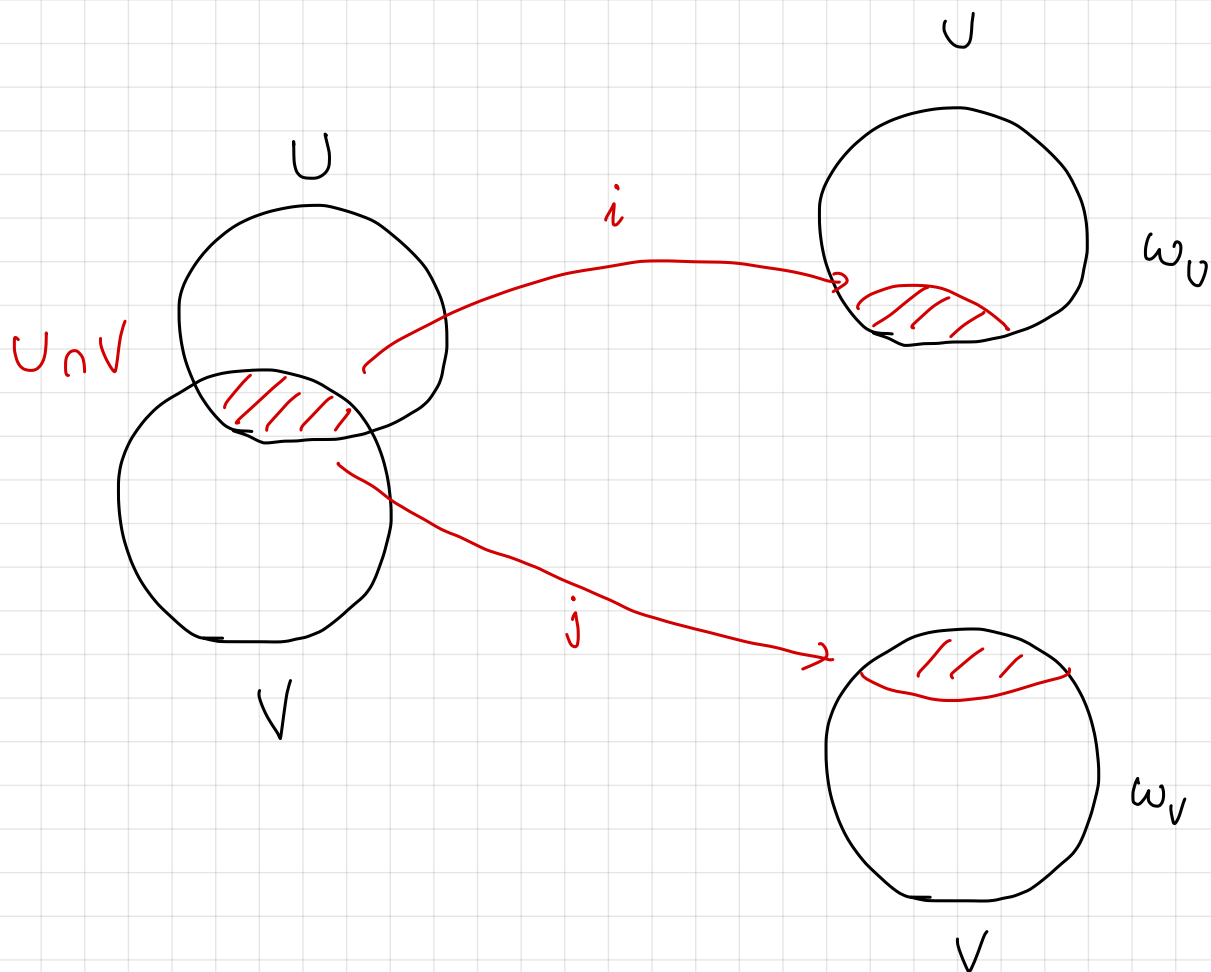
$$= \sum_j \left( \sum_i f_i(x_i(y)) \frac{\partial y_j}{\partial x_i} \right) \frac{\partial}{\partial y_j}$$

$X_\alpha = X_\beta$  on  $U_\alpha \cap U_\beta$  のとき

あわせて  $U_\alpha \cup U_\beta$  上の vector field とした。

$\leadsto M$  上の vector field として

あなただけの  $U_\alpha \cap U_\beta$  上でも一致するもの



$$i^* \omega_U = j^* \omega_V \quad \text{in } \Omega^*(U \cap V) \text{ かつ}$$

$\omega$  は  $M$  上に定義され  $\omega|_{U \cap V} = i^* \omega_U = j^* \omega_V \in \Omega^*(U \cap V)$

$d$  は wedge 積で、各  $U_i$  上で成り立つ。

$$f: M \rightarrow N \text{ に対し } f^*: \Omega^*(N) \rightarrow \Omega^*(M)$$