Bott-Tu ノート

Chap. 1 de Rham Theory

§1. The de Rham Complex on $\mathbb{R}^n$

$\underline{\Omega^*}$

$\mathbb{R}^n$の座標 $x_1, \dots, x_n$ に対して定義される.

$dx_1, \dots, dx_n$ によって $\mathbb{R}$上生成される代数

$$(dx_i)^2 = 0$$

$$dx_i dx_j = -dx_j dx_i \quad i \neq j$$

これは

$$1,\ dx_i,\ \underset{i<j}{dx_i dx_j},\ \dots,\ dx_1 \cdots dx_n$$

を基底とする $\mathbb{R}$上の線形空間である.

$$\Omega^*(\mathbb{R}^n) = \{\mathbb{R}\text{上の}C^\infty\text{関数たち}\} \otimes_{\mathbb{R}} \Omega^*$$

の元を$\mathbb{R}$上のC^∞微分形式という.

$$\omega = \sum f_{i_1 \dots i_q}\, dx_{i_1} \cdots dx_{i_q} = \sum f_I\, dx_I$$

$\otimes$

省略かと.

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n)$$

この元を q-form とよぶ

$$\sum_{i_1, \dots i_q} f_{i_1 \dots i_q} \, dx_{i_1} \cdots dx_{i_q}$$

$\sum f_I \, dx_I$ とも書く.

微分演算子

$$d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n)$$

$$f \in \Omega^0(\mathbb{R}^n) \rightsquigarrow df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$\omega = \sum f_I \, dx^I \rightsquigarrow d\omega = \sum df_I \, dx_I$$

例 $\mathbb{R}^3$ 上 $\quad x_1, x_2, x_3 \rightsquigarrow x, y, z$

$$f \in \Omega^0(\mathbb{R}^3)$$

$$\rightsquigarrow \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$f_1 dx + f_2 dy + f_3 dz \in \Omega^1(\mathbb{R}^3)$$

$$\rightsquigarrow \quad d(f_1 dx + f_2 dy + f_3 dz)$$

$$\begin{aligned}
&= \left(\frac{\partial f_1}{\partial x} \cancel{dx} + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz\right) dx \\
&\quad + \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} \cancel{dy} + \frac{\partial f_2}{\partial z} dz\right) dy \\
&\quad + \left(\frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} \cancel{dz}\right) dz \\
&= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy\, dz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx\, dz \\
&\qquad + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\, dy
\end{aligned}$$

$$f_1 dy\, dz - f_2 dx\, dz + f_3 dx\, dy \in \Omega^2(\mathbb{R}^3)$$

$$\rightsquigarrow \quad d(f_1 dy\, dz - f_2 dx\, dz + f_3 dx\, dy)$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz$$

$$\tau = \sum f_I dx_I \quad , \quad \omega = \sum g_J dx_J$$

$\swarrow$ ∧ とも

$$\tau \cdot \omega = \sum f_I g_J \, dx_I \, dx_J$$

$$\tau \cdot \omega = (-1)^{\deg \tau \deg \omega} \, \omega \cdot \tau$$

$$d(\tau \cdot \omega) = d\tau \cdot \omega + (-1)^{\deg \tau} \tau \cdot d\omega$$

$$d^2 = 0$$

$$d^2 f = d\left(\sum \frac{\partial f}{\partial x_i} dx_i\right) = \sum \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \, dx_i = 0$$

対称.　　反対称.

ちなみに、$d(dx_i) = 0$

d と $\Omega^*(\mathbb{R}^n)$ をあわせて $\mathbb{R}^n$ 上の de Rham 複体(complex)という

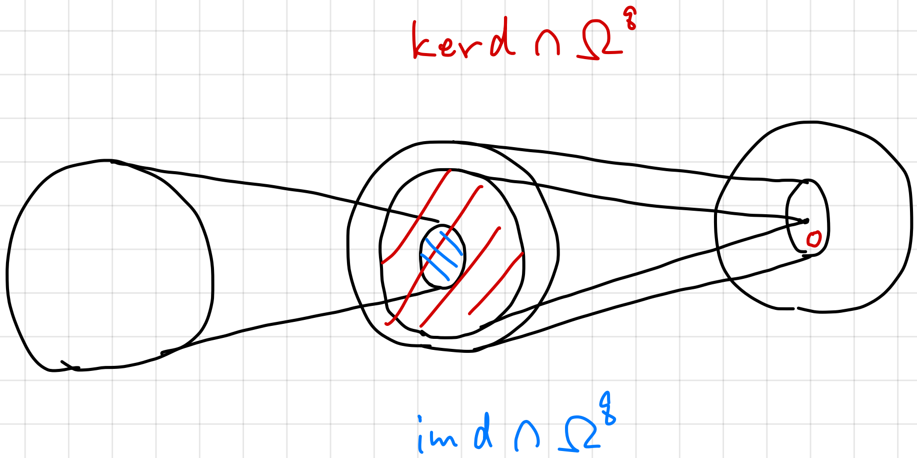
$\ker d$ の元を closed

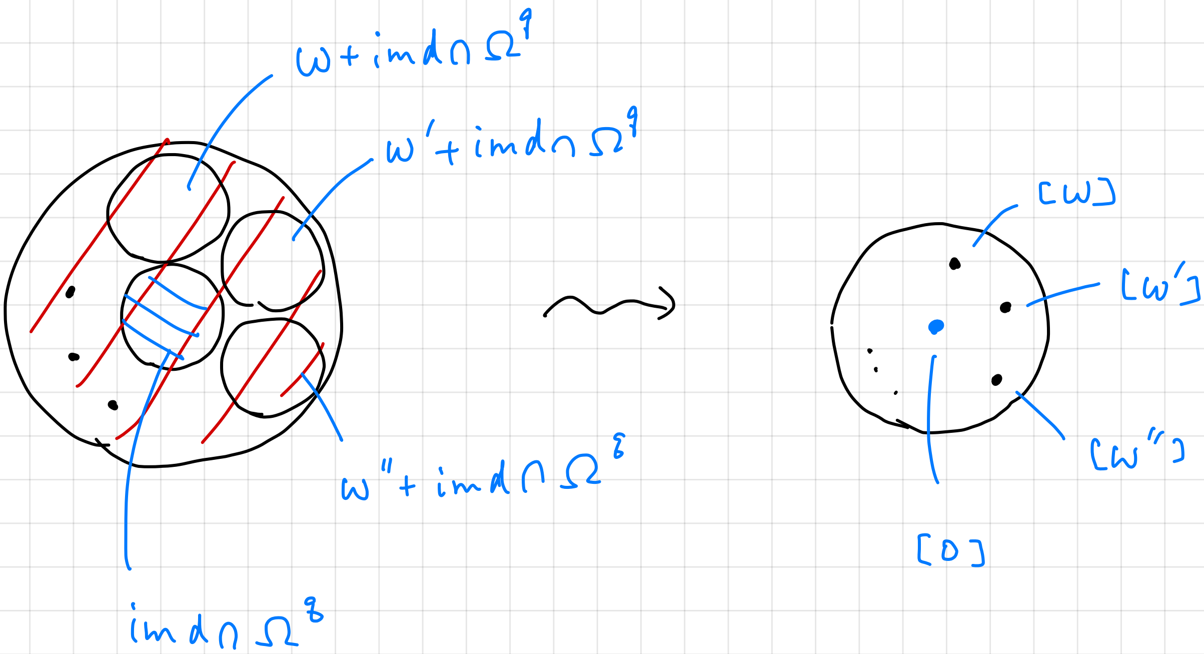
$\operatorname{im} d$ の元を exact　という.

$\mathbb{R}^n$ の de Rham cohomology　ベクトル空間

$$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\} / \{\text{exact } q\text{-forms}\}$$

$$\Omega^{q-1} \xrightarrow{d} \Omega^{q} \xrightarrow{d} \Omega^{q+1}$$





$$\ker d \cap \Omega^q \,/\, \operatorname{im} d \cap \Omega^q$$

$$0 \to H^0(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \to$$

$$0 \xrightarrow{d} \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \cdots$$

$$\to H^{n-1}(\mathbb{R}^n) \to H^n(\mathbb{R}^n) \to 0$$

$$\cdots \xrightarrow{d} \Omega^{n-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} 0$$

$$H^0(\mathbb{R}^n) = \ker d \cap \Omega^0(\mathbb{R}^n)$$

$$H^q(\mathbb{R}^n) = \ker d \cap \Omega^q(\mathbb{R}^n) / \mathrm{im}\, d \cap \Omega^q(\mathbb{R}^n)$$

$$0 < q < n$$

$$H^n(\mathbb{R}^n) = \Omega^n(\mathbb{R}^n) / \mathrm{im}\, d \cap \Omega^n(\mathbb{R}^n)$$

$\mathbb{R}^n$の開部分集合Uに対しても同様に

$\Omega^q(U)$, $H^q(U)$が定義される.

$n=0$

$$0 \longrightarrow \Omega^0(\mathbb{R}^0) \longrightarrow 0$$

$\mathbb{R}^0 = \{0\}$

"$\mathbb{R}^0$ 上の関数" $= \mathbb{R}$

$H^0(\mathbb{R}^0) = \Omega^0(\mathbb{R}^0) = \mathbb{R}$

他は 0.

$n=1$

$$0 \longrightarrow \Omega^0(\mathbb{R}^1) \xrightarrow{d} \Omega^1(\mathbb{R}^1) \longrightarrow 0$$

$\ker d \cap \Omega^0(\mathbb{R}^1) = \mathbb{R}$

$H^0(\mathbb{R}^1) = \mathbb{R}$

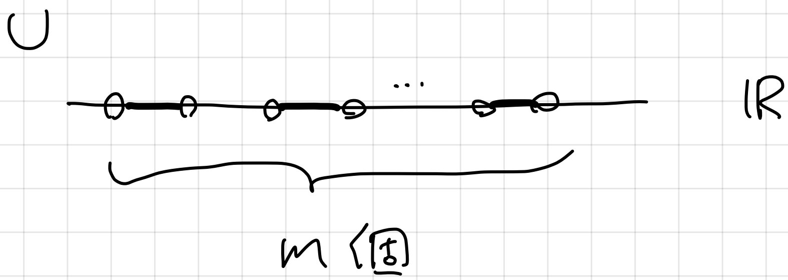
$\omega = g(x)\,dx$ のとき　$f(x) = \int_0^x g(u)\,du$ とおくと

$df = g(x)\,dx$

よって $\Omega^1(\mathbb{R}) = \mathrm{im}\, d$　全部 exact

$H^1(\mathbb{R}^1) = \Omega^1(\mathbb{R}) / \mathrm{im}\, d \cap \Omega^1(\mathbb{R}) = 0$

$\{0\}$



$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow 0$$

$H^0(U) = \mathbb{R}^m$ 　m個の場所で定数.

$H^1(U) = 0$

一般に

$H^0(\mathbb{R}^n) = \mathbb{R}$ 　Poincaré lemma

$H^q(\mathbb{R}^n) = 0 \quad q \neq 0$

differential complex

$$\longrightarrow C^{q-1} \xrightarrow{d} C^{q} \xrightarrow{d} C^{q+1} \longrightarrow$$

$$H^{q}(C) = (\ker d \cap C^{q}) / (\mathrm{im}\, d \cap C^{q})$$

$$\begin{tikzcd}
A = \arrow[r] & A^{q-1} \arrow[r, "d_A"] \arrow[d, "f_{q-1}"] \arrow[dr, phantom, "\circlearrowleft"] & A^{q} \arrow[r, "d_A"] \arrow[d, "f_q"] \arrow[dr, phantom, "\circlearrowleft"] & A^{q+1} \arrow[r] \arrow[d, "f_{q+1}"] & {} \\
B = \arrow[r] & B^{q-1} \arrow[r, "d_B"] & B^{q} \arrow[r, "d_B"] & B^{q+1} \arrow[r] & {}
\end{tikzcd}$$

のとき

$f = \{f_q\}$ は A B 間の chain map という

$$\longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_{i} \xrightarrow{f_i} V_{i+1} \longrightarrow$$

のとき

$\ker f_i = \mathrm{im}\, f_{i-1}$

完全という。

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

完全

短完全列 SES

$$\rightsquigarrow$$

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^{q+1} \arrow[u] \arrow[r, "f_{q+1}"] & B^{q+1} \arrow[u] \arrow[r, "g_{q+1}"] & C^{q+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^{q} \arrow[u, "d_A"] \arrow[r, "f_q"] & B^{q} \arrow[u, "d_B"] \arrow[r, "g_q"] & C^{q} \arrow[u, "d_C"] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^{q+1} \arrow[u] \arrow[r] & B^{q+1} \arrow[u] \arrow[r] & C^{q+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^{q} \arrow[u, "d_A"] \arrow[r] & B^{q} \arrow[u, "d_B"] \arrow[r] & C^{q} \arrow[u, "d_C"] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^{q}(A) \arrow[r, "f_q"] & H^{q}(B) \arrow[r, "g_q"] & H^{q}(C) \arrow[dll, out=0, in=180] \\
 & H^{q+1}(A) \arrow[r, "f_{q+1}"] & H^{q+1}(B) \arrow[r, "g_{q+1}"] & H^{q+1}(C) \arrow[r] & {}
\end{tikzcd}$$

$$
\begin{tikzcd}
& H^{q+1}(A) \arrow[r] & {} & \\
{} \arrow[r] & H^{q}(A) \arrow[r, "f^*"] & H^{q}(B) \arrow[r, "g^*"] & H^{q}(C) \arrow[ull, "d^*"']
\end{tikzcd}
$$

長完全列 ES

$$\operatorname{Supp} f = \overline{\{ p \in X \mid f(p) \neq 0 \}}$$

$$\Omega_c^*(\mathbb{R}^n) = \{ C^\infty \text{ funcs. on } \mathbb{R}^n \text{ with compact supp.} \} \otimes_{\mathbb{R}} \Omega^*$$

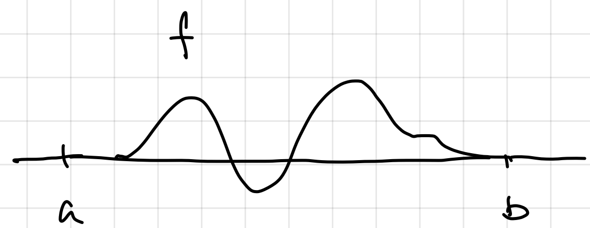
Ex. 1.6

(a) $H_c^*(\mathbb{R}^0) = \begin{cases} \mathbb{R} & \text{in 0 dim.} \\ 0 & \text{elsewhere} \end{cases}$

(b) $H_c^0(\mathbb{R}^1) = 0$ const. func. の supp. はコンパクトでない

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \longrightarrow \mathbb{R}^1 \quad \text{surj.}$$

$$\int_{\mathbb{R}^1} df = f(b) - f(a) = 0$$



$\int_{\mathbb{R}^1} g(u)\,du = 0$ のとき

$f(x) = \int_{-\infty}^{x} g(u)\,du$ と与えて

これは comp. supp. で $df = g\,dx$.

$\therefore \{ \text{exact forms.} \} = \ker \int_{\mathbb{R}^1}$

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1$$

$$\int_{\mathbb{R}^1} g(u)\,du = s \neq 0$$

$$\int_{\mathbb{R}^1} g'(u)\,du = s$$

$$\int_{\mathbb{R}^1} (g - g')(u)\,du = 0$$

つまり $g\,dx = g'\,dx \mod \ker \int_{\mathbb{R}^1}$

$\lrcorner$

一般に

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dim. } n \\ 0 & \text{eleswhere} \end{cases}$$ Poincaré lemma.

マイヤー　ヴィートリス　列

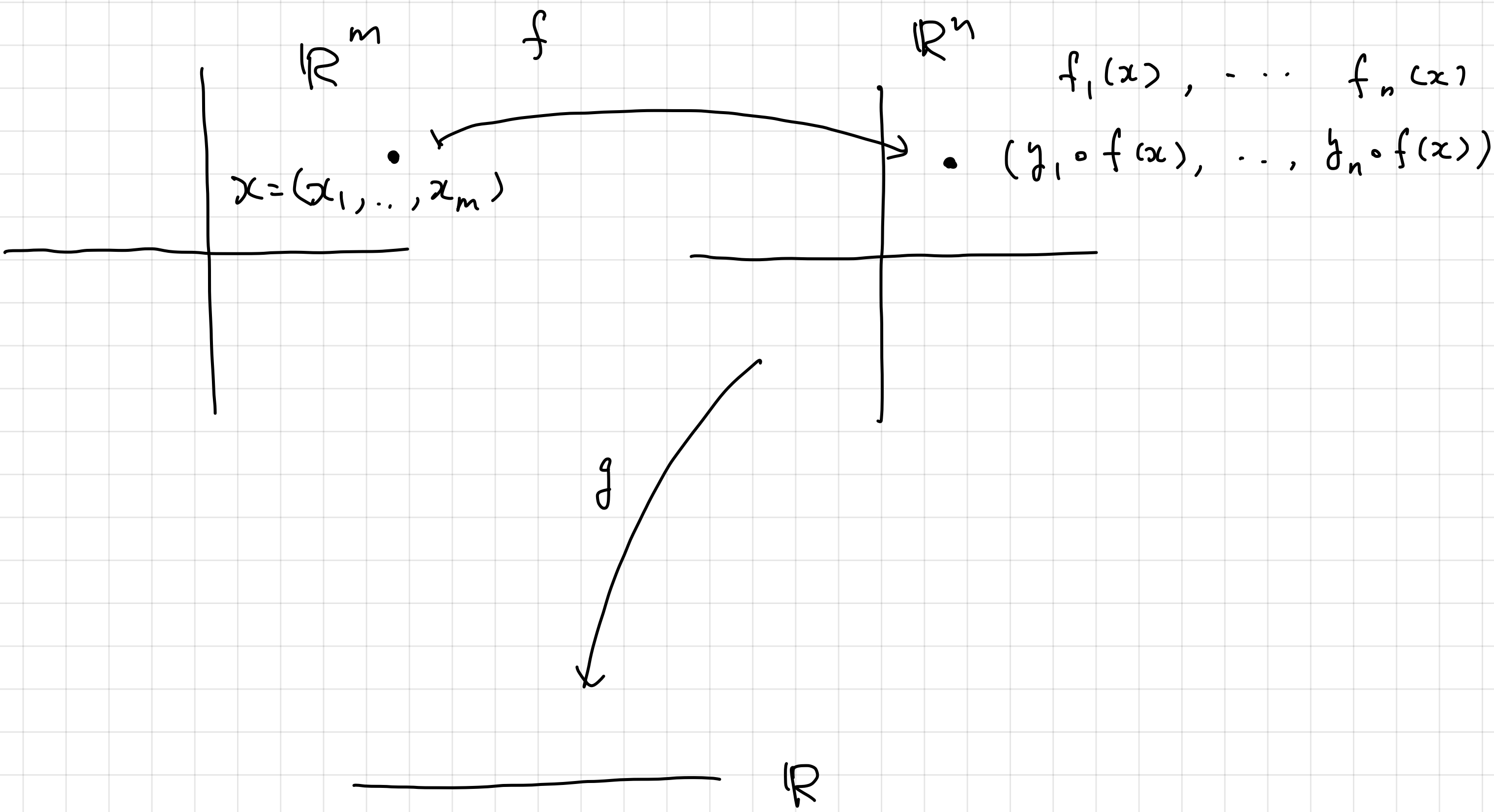
§2 The Mayer-Vietoris Sequence

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad C^\infty$$

$$x_1, \dots, x_m \qquad y_1, \dots, y_n$$

$$f^* : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^0(\mathbb{R}^m)$$

$$g \longmapsto g \circ f$$



$$f^* : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^q(\mathbb{R}^m)$$

$$\sum g_I \, dy_{i_1} \cdots dy_{i_q} \longmapsto \sum (g_I \circ f) \, df_{i_1} \cdots df_{i_q}$$

ただし $f_i = g_i \circ f$

このとき $d f^*(\omega) = f^*(d\omega)$ 次ページcf.

座標変換

$$(x_1, \ldots, x_n) \rightsquigarrow (u_1, \ldots, u_n)$$

$$\sum_i \frac{dg}{\partial u_i} du_i = \sum_i \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial x_j} dx_j = \sum \frac{\partial g}{\partial x_j} dx_j$$

よって dg は座標のとり方によらない.

$\omega = \sum g_I \, du_I$ のとき

$$d\omega = \sum dg_I \, du_I$$

cf.

$$d f^*(g_I \, dy_{i_1} \cdots dy_{i_q}) = d(g_I \circ f)\, df_{i_1} \cdots df_{i_q}$$

$$f^*(dg_I \, dy_{i_1} \cdots dy_{i_q})$$

$$= f^*\Big(\sum_i \frac{\partial g_I}{\partial y_i} dy_i \, dy_{i_1} \cdots dy_{i_q}\Big)$$

$$= \underbrace{\Big(\sum_i \frac{\partial g_I}{\partial y_i} \circ f\Big) df_i}\, df_{i_1} \cdots df_{i_q}$$

$$= \underbrace{d(g_I \circ f)}\, df_{i_1} \cdots df_{i_q}$$

(*)

(*) $$d\, g_I(f(x)) = \sum_i \frac{\partial}{\partial x_i} g_I(f(x))\, dx_i$$

$$= \sum_{i,j} \underbrace{\frac{\partial (y_j \circ f)}{\partial x_i}} \frac{\partial}{\partial y_j} g_I(f(x))\, \underbrace{dx_i}$$

(**)

$$= \sum \frac{\partial}{\partial y_j} g_I(f(x))\, \underbrace{d(y_j \circ f)}$$

(**) $$d(y_i \circ f) = \sum_j \frac{\partial (y_i \circ f)}{\partial x_j} dx_j$$

cf 終わり

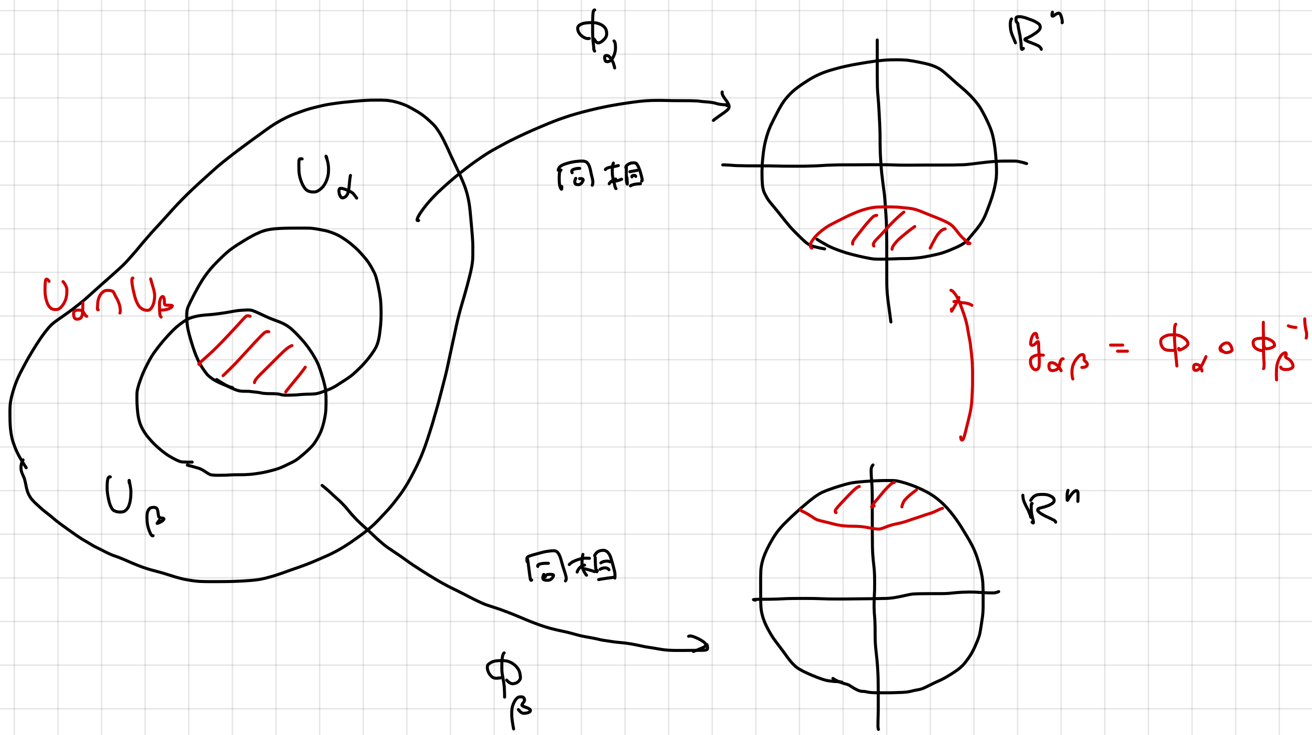
Ω^* は関手.

$(\{\mathbb{R}^n\}_{n\in\mathbb{N}}$, C^∞ 写像$)$

$\longrightarrow$ (comutative differential graded algebra たち, homomorphism たち)

$$\begin{array}{ccc} \mathbb{R}^n & & \Omega^*(\mathbb{R}^m) \\ f \downarrow & \rightsquigarrow & \downarrow f^* \\ \mathbb{R}^m & & \Omega^*(\mathbb{R}^n) \end{array}$$

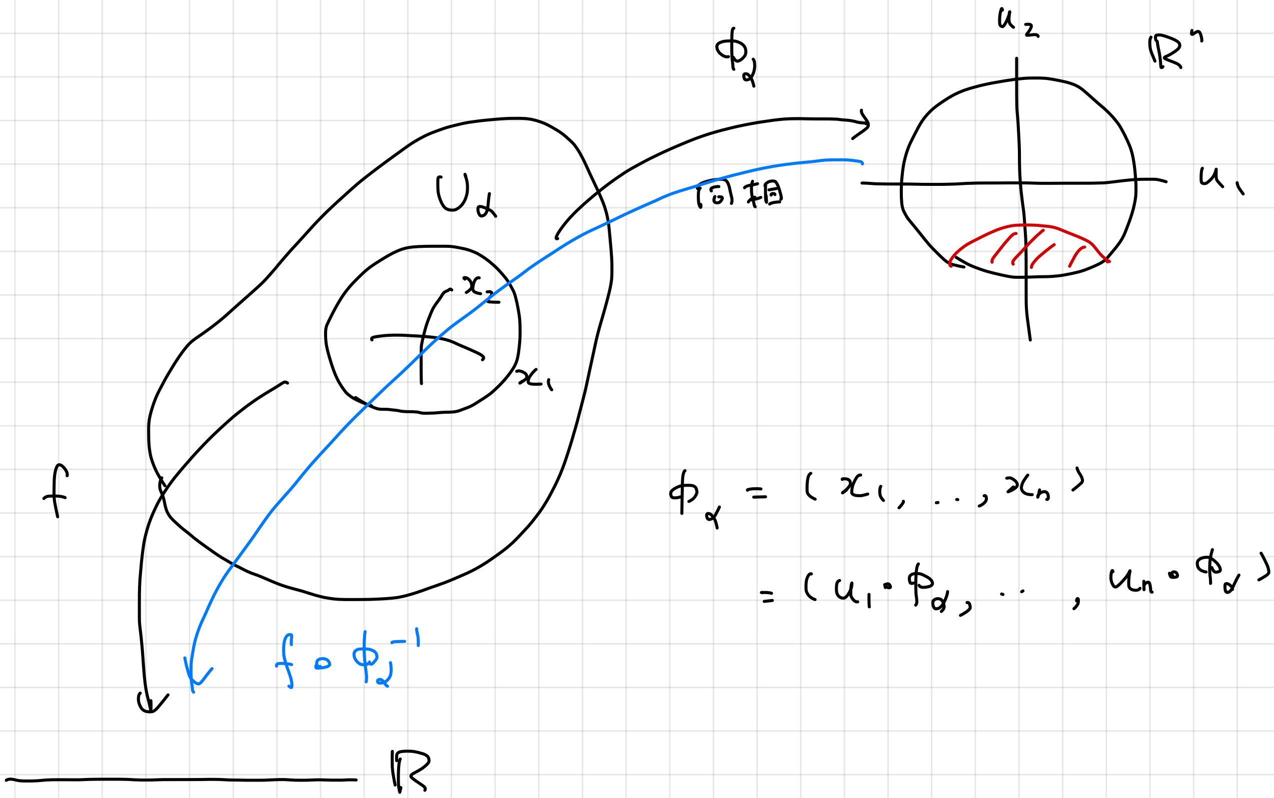
多様体 manifold



$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

Hausdorff

可算な開基をもつ.



$$\phi_\alpha = (x_1, \ldots, x_n)$$

$$= (u_1 \circ \phi_\alpha, \ldots, u_n \circ \phi_\alpha)$$

$f \circ \phi_\alpha^{-1}$ が微分可能のとき

f は微分可能という。

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p))$$