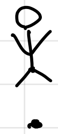
2023

教養数学

§1. 座標

次元と座標

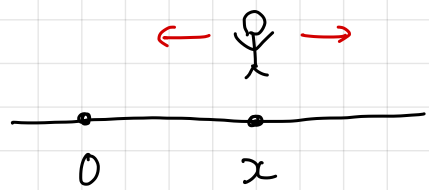
点



動けない

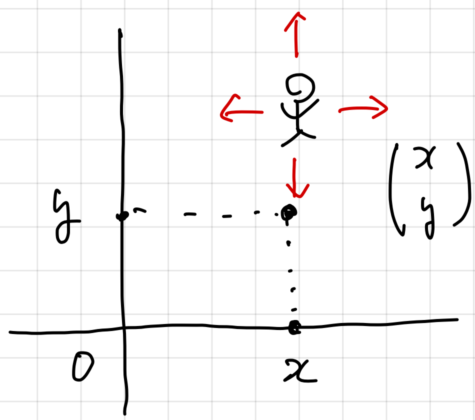
0次元

直線



1次元

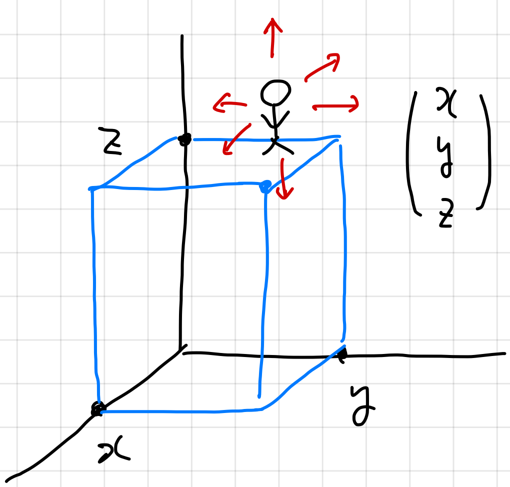
平面



$\begin{pmatrix} x \\ y \end{pmatrix}$

(x, y) とも

2次元

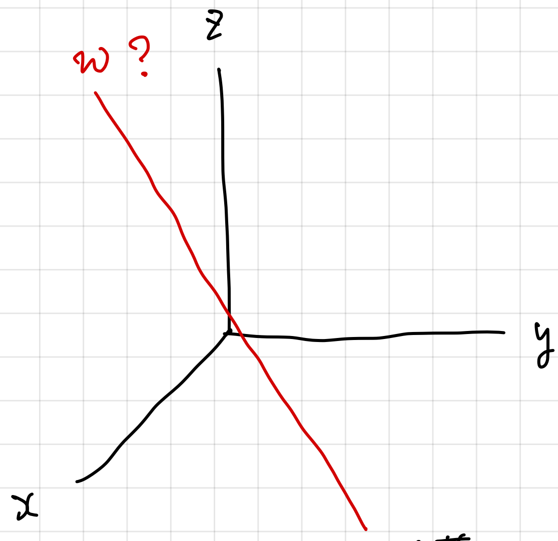


$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(x, y, z)

3次元

4次元



描けない…

$\wr$

(x, y, z, w)

$\wr$

(x^1, x^2, x^3, x^4)

5次元，6次元…

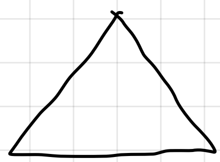
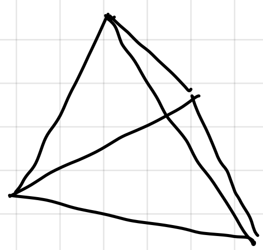
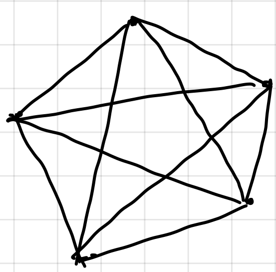
n次元空間

実数

$$\mathbb{R}^n = \{(x^1, \ldots, x^n) \mid x^i \in \mathbb{R}\}$$

$x = (x^1, \ldots, x^n)$ などと書く

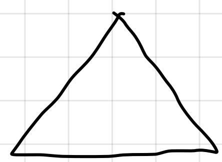
三角形の仲間たち

			頂点	辺	面	"四面体"
0次元	点					
1次元	線分		2	1	0	0
2次元	三角形		3	3	1	0
3次元	四面体		4	6	4	1
4次元	五胞体		5	10	10	5

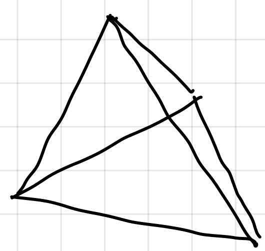
四面体を5個もってきて
はりあわせてください。

1次元 線分

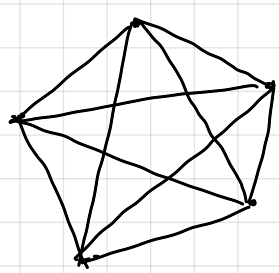
2次元 三角形



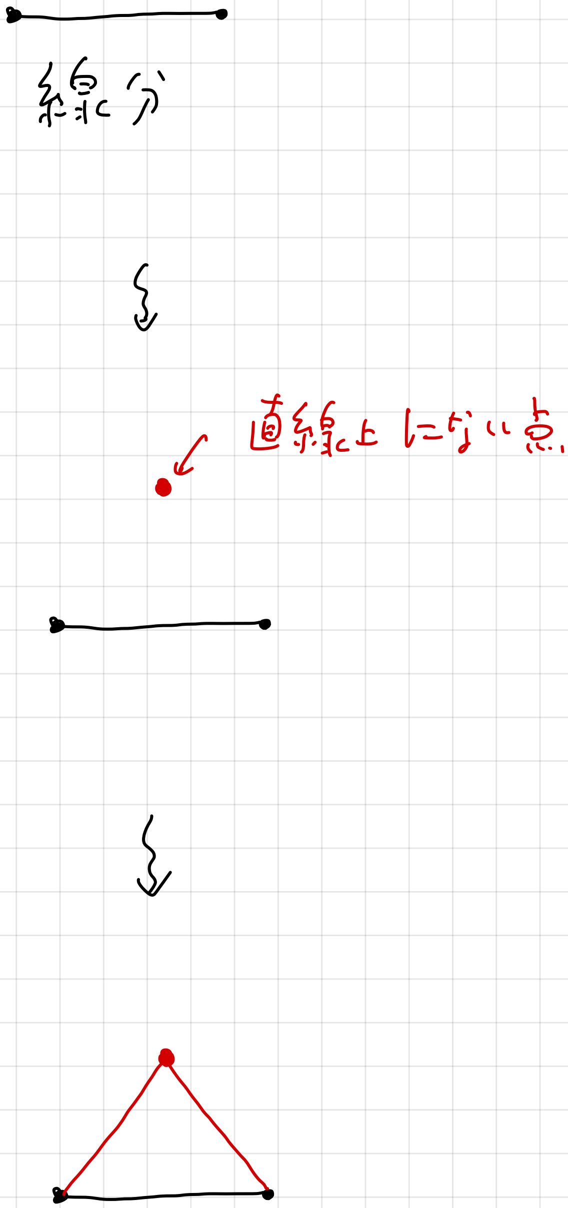
3次元 四面体



4次元 五胞体

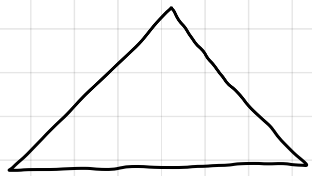


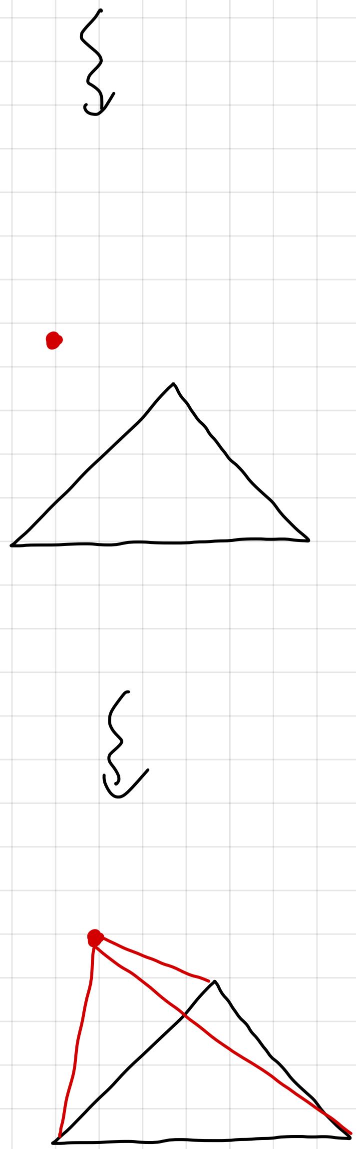
1次元



2次元

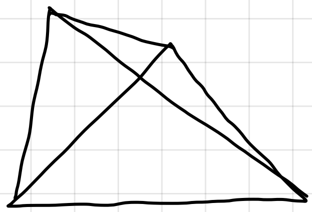
2次元

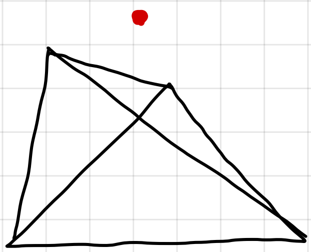




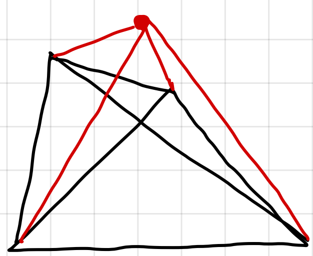
3次元

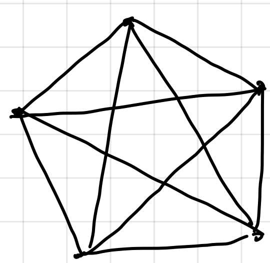
3次元



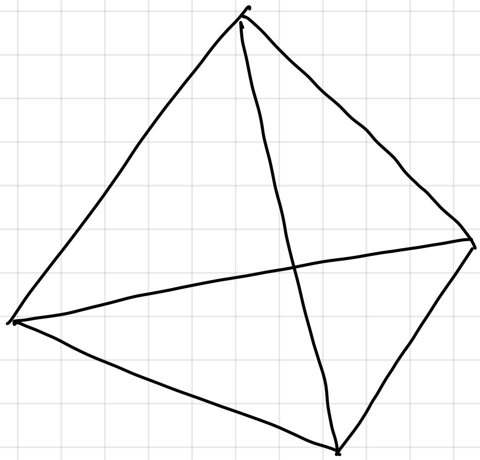


4次元



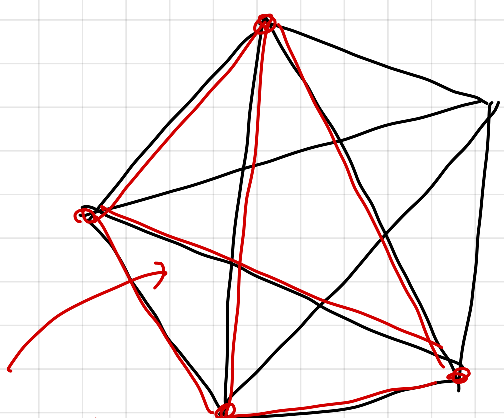


見やすい？



四つの面

四面体

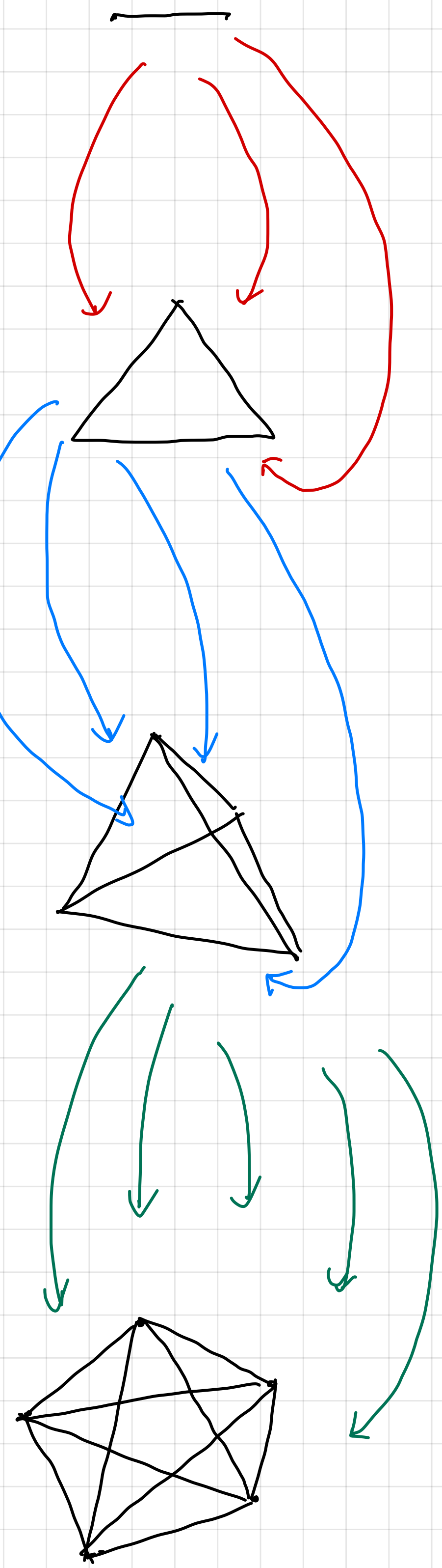


五つの四面体

四面体

五胞体

1次元 線分



2次元 三角形

3つ集めて

3次元 四面体

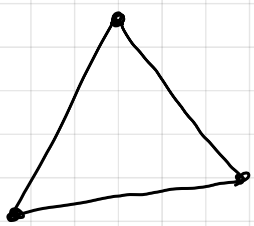
4つ集めて

4次元 五胞体

5つ集めて

	点	線分	三角形	四面体	五胞体
線分	2	1	0	0	0
三角形	3	3	1	0	0
四面体	4	6	4	1	0
五胞体	5	10	10	5	1

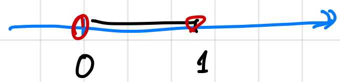
例



点　3
線分　3
三角形　1

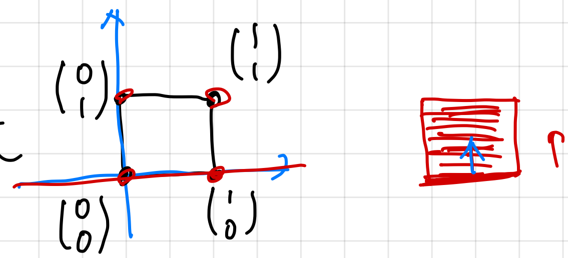
四角形の仲間たち

1次元



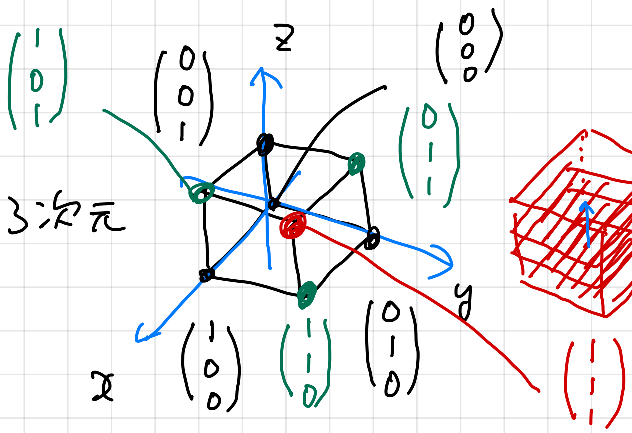
2

2次元



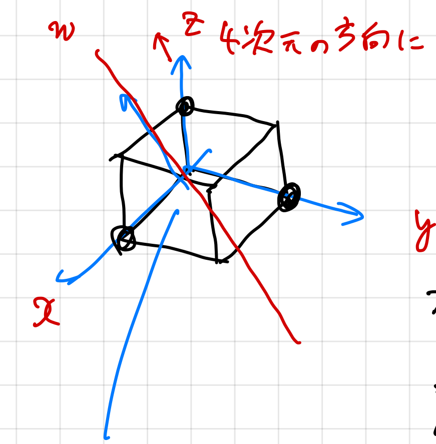
2^2

3次元



2^3

4次元



4次元の方向にずらす.

4次元の直方体.

$2^4 = 16$ 個

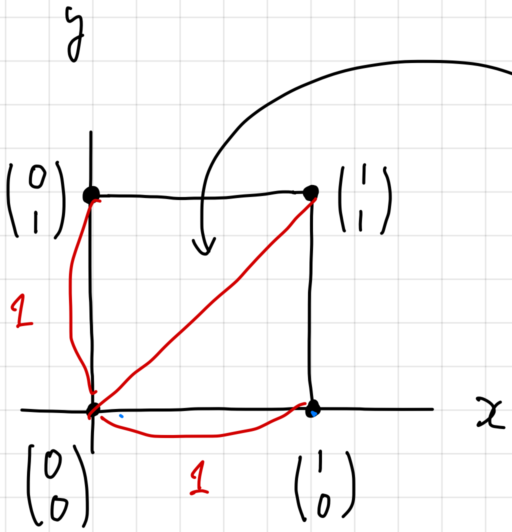
$$\begin{array}{c} x \\ y \\ z \\ w \end{array} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

頂点 $\begin{pmatrix}○\\○\\○\\○\end{pmatrix}$ ○ = 0か1 $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

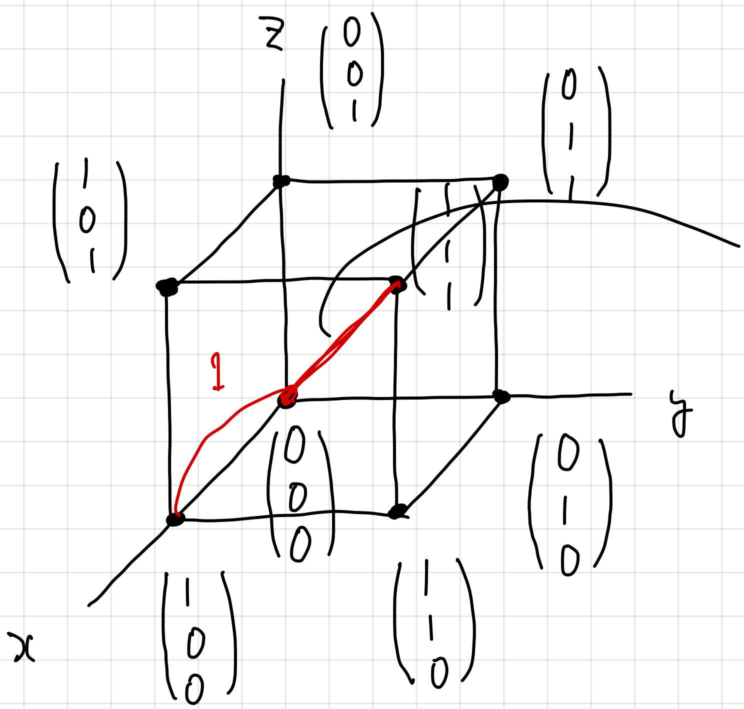
頂点と $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ のきょり $= \sqrt{1^2+1^2+1^2+1^2}$

$= 2$



$$\sqrt{(1-0)^2+(1-0)^2}$$

$$=\sqrt{2}$$

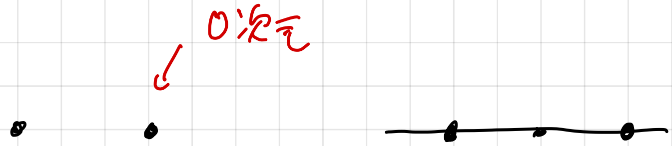


$$\sqrt{(1-0)^2+(1-0)^2+(1-0)^2}$$

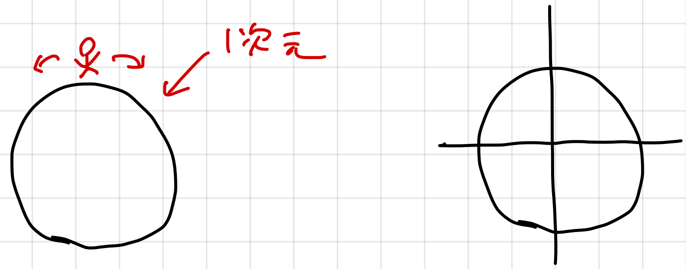
$$=\sqrt{3}$$

円・球面の仲間たち

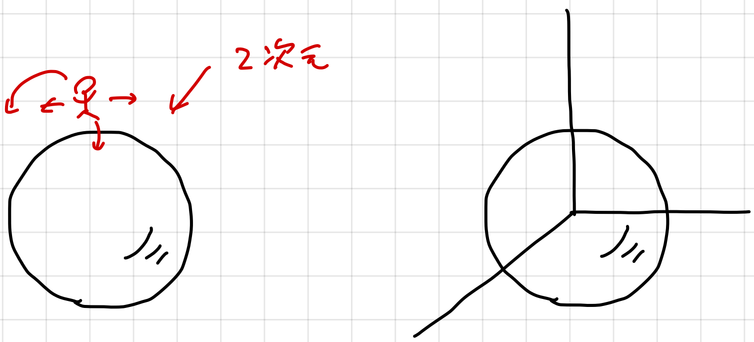
1次元内



2次元内



3次元内

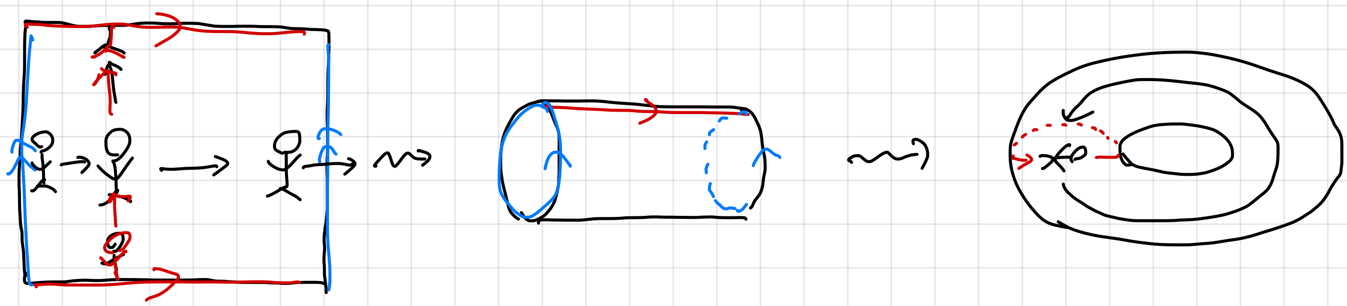


4次元内

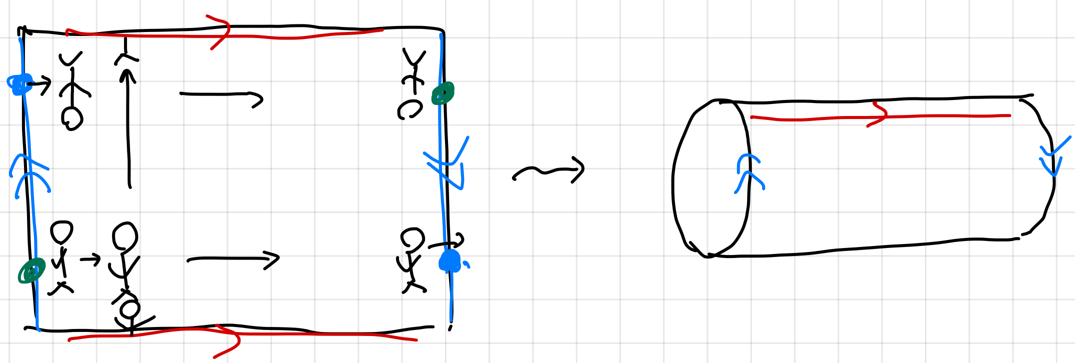
3次元

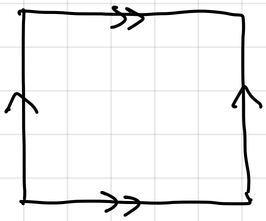
?

トーラス

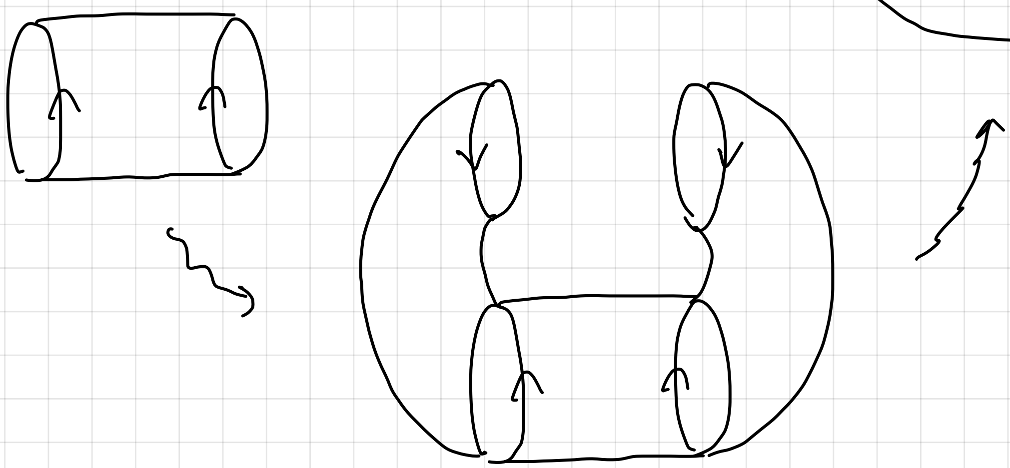


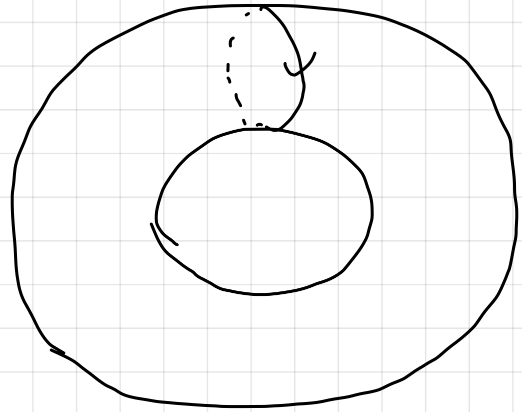
クラインの壺

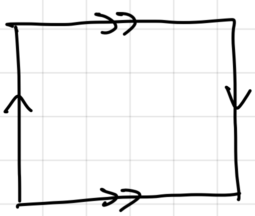




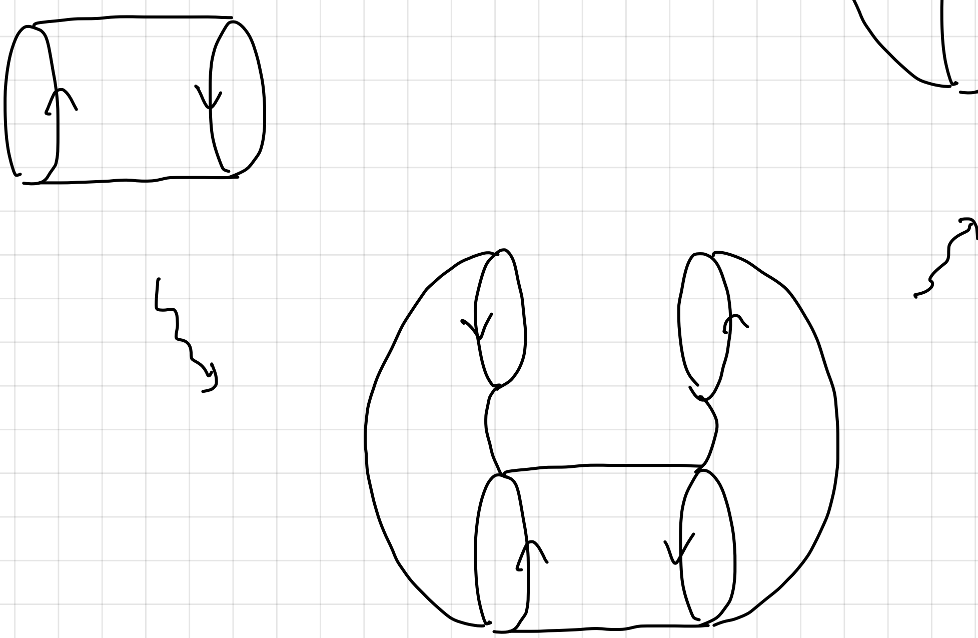
トーラス

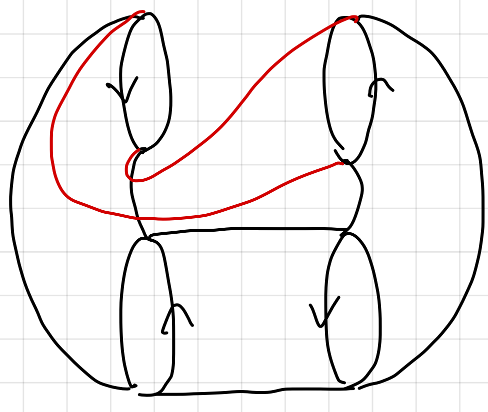






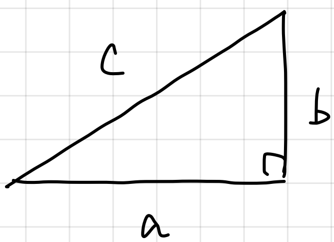
クラインの壺





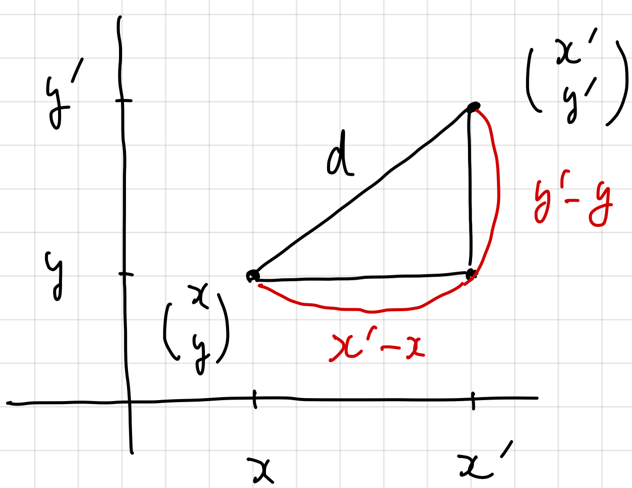
§2 距離とユークリッド空間

三平方の定理（ピタゴラスの定理）



$$a^2 + b^2 = c^2$$

$$c = \sqrt{a^2 + b^2}$$



$$d^2 = (x'-x)^2 + (y'-y)^2$$

$$d = \sqrt{(x'-x)^2 + (y'-y)^2}$$

n次元ユークリッド空間

実数

$$\mathbb{R}^n = \{(x^1, \dots, x^n) \mid x^i \in \mathbb{R}\}$$

$$\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

$x = (x^1, \dots, x^n)$ と $y = (y^1, \dots, y^n)$ の距離

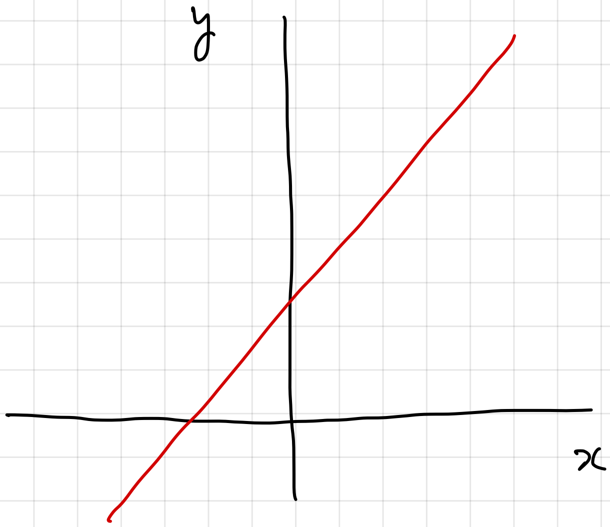
$$d(x, y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$$

$\mathbb{R}^n$ を n次元数空間などとも言う.

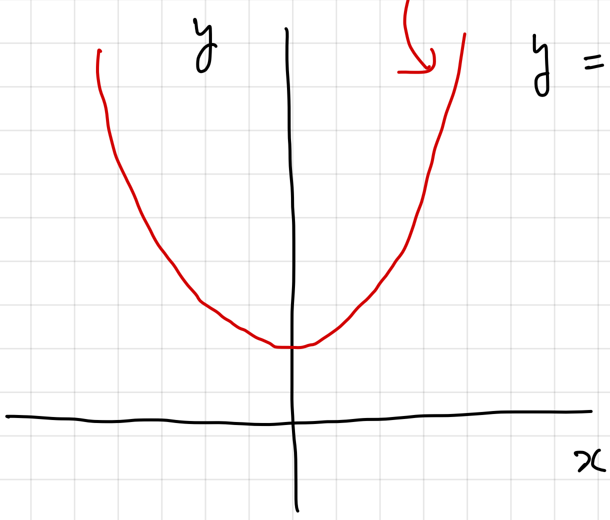
§3 図形と式

直線

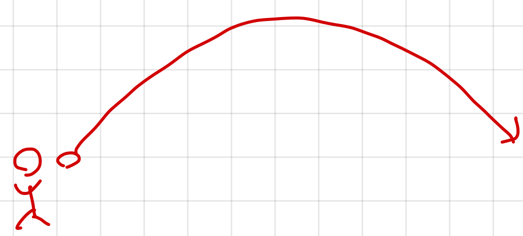


$y = 2x + 1$

放物線



$y = x^2 + 1$

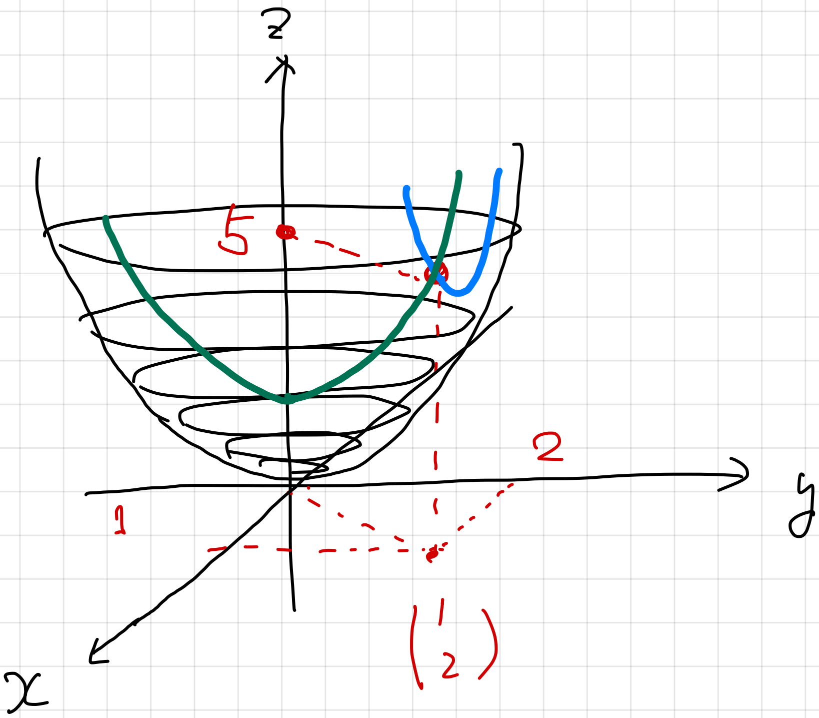


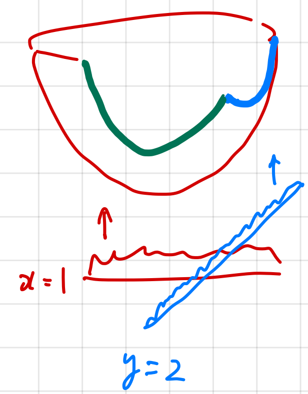
$\mathbb{R}^2 \to \mathbb{R}$ の例

定義域 $\mathbb{R}^2$

値域 $\{z \in \mathbb{R} \mid z \geq 0\}$

$z = x^2 + y^2$

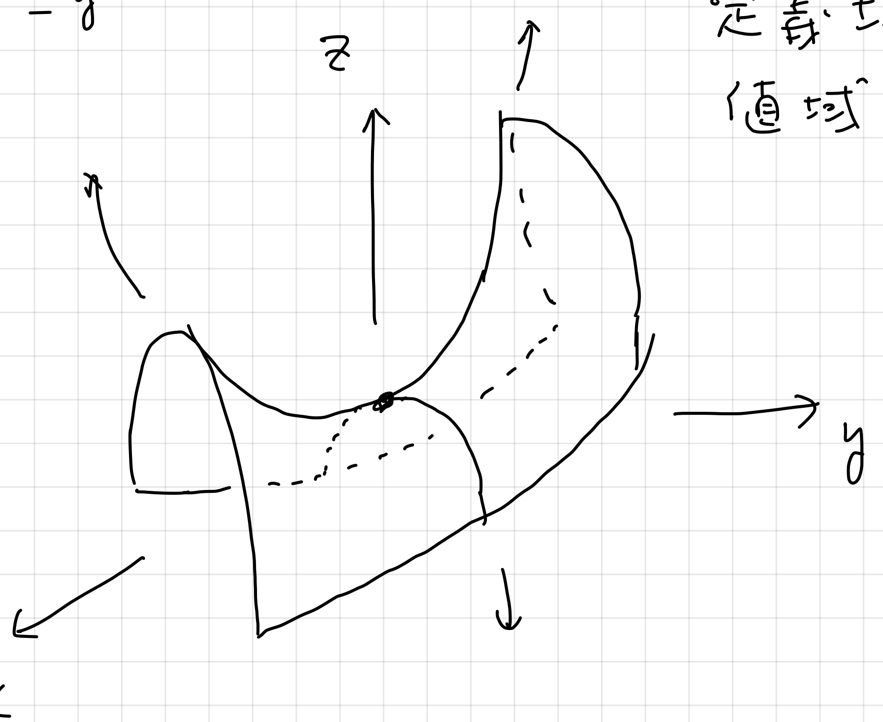


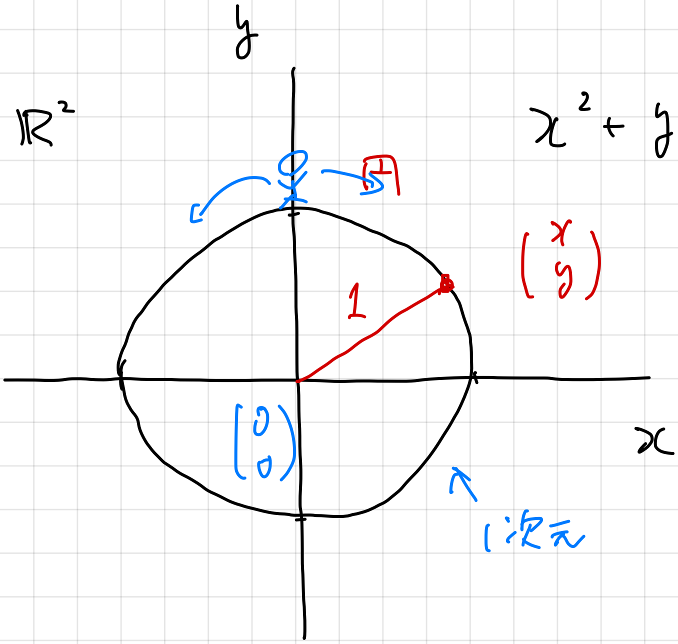


$z = x^2 - y^2$

定義域 $\mathbb{R}^2$

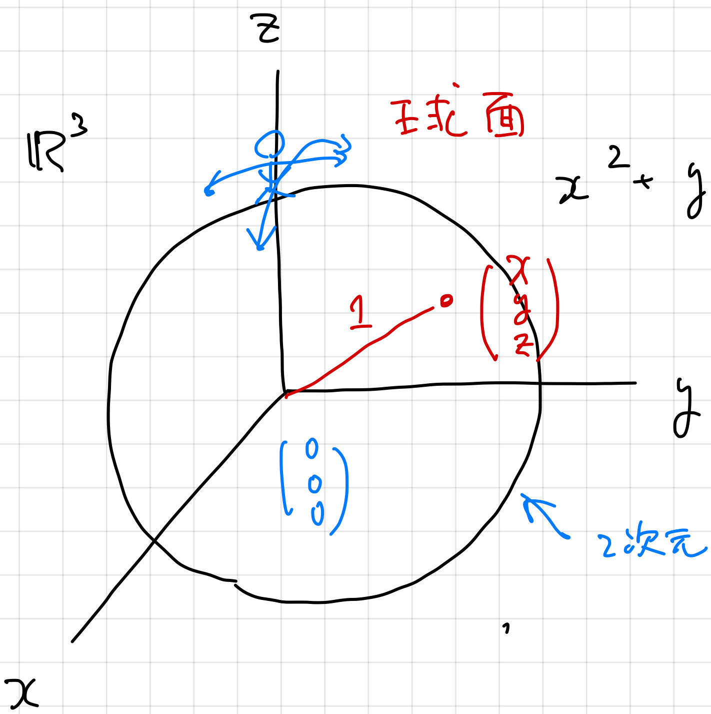
値域 $\mathbb{R}$





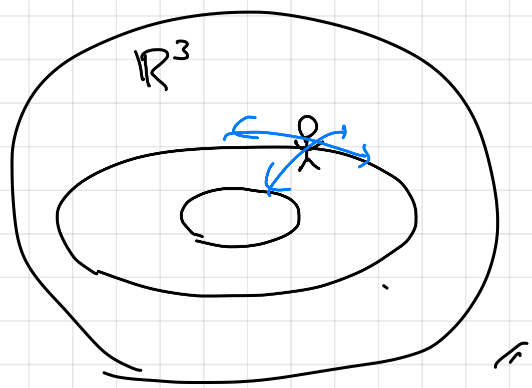
$x^2 + y^2 = 1$ S^1

$1 = \sqrt{x^2 + y^2}$



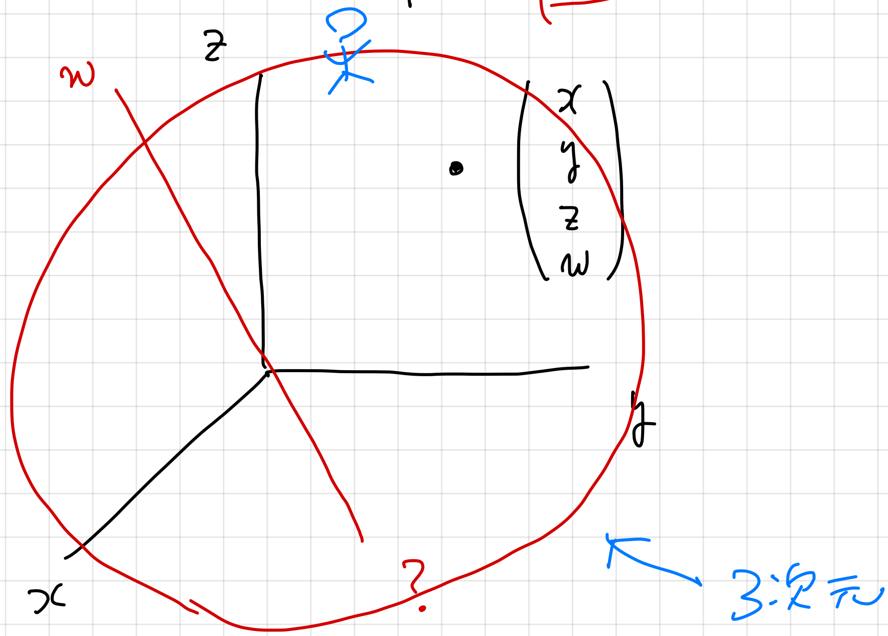
$x^2 + y^2 + z^2 = 1$ S^2

$1 = \sqrt{x^2 + y^2 + z^2}$



$\mathbb{R}^4$

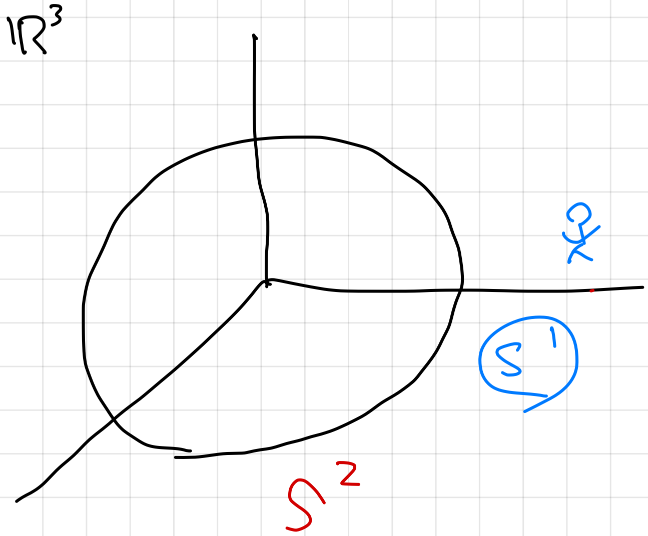
4次元内の3次元 球面 $\boxed{S^3}$

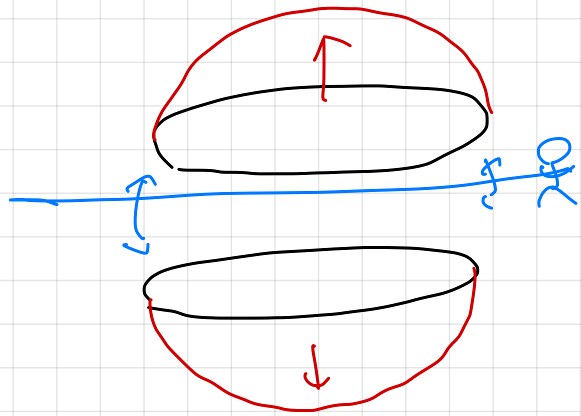


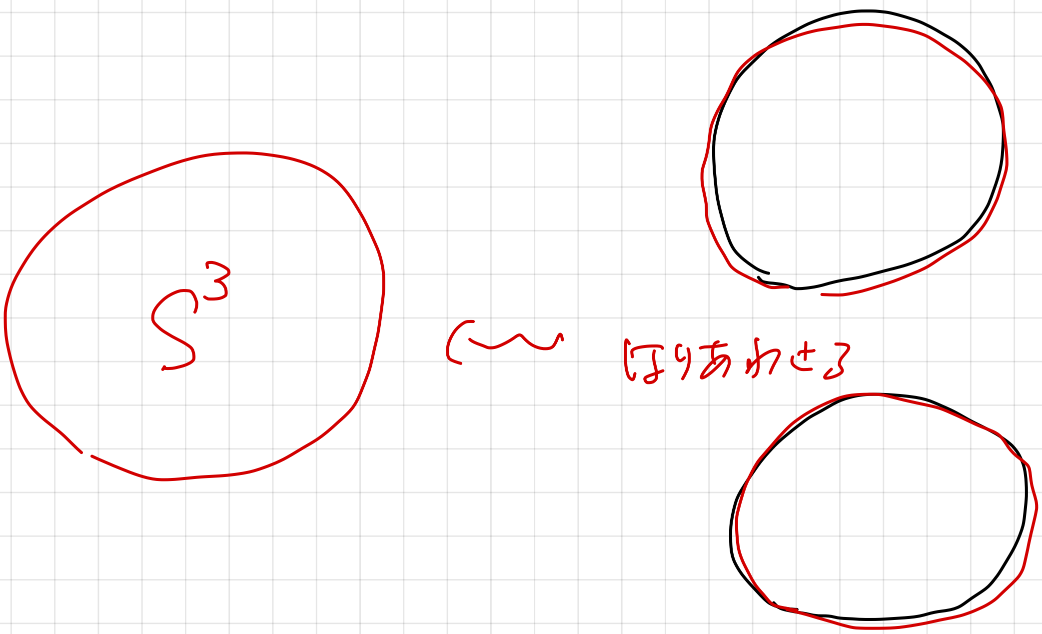
$$x^2 + y^2 + z^2 + w^2 = 1$$

座標の番号

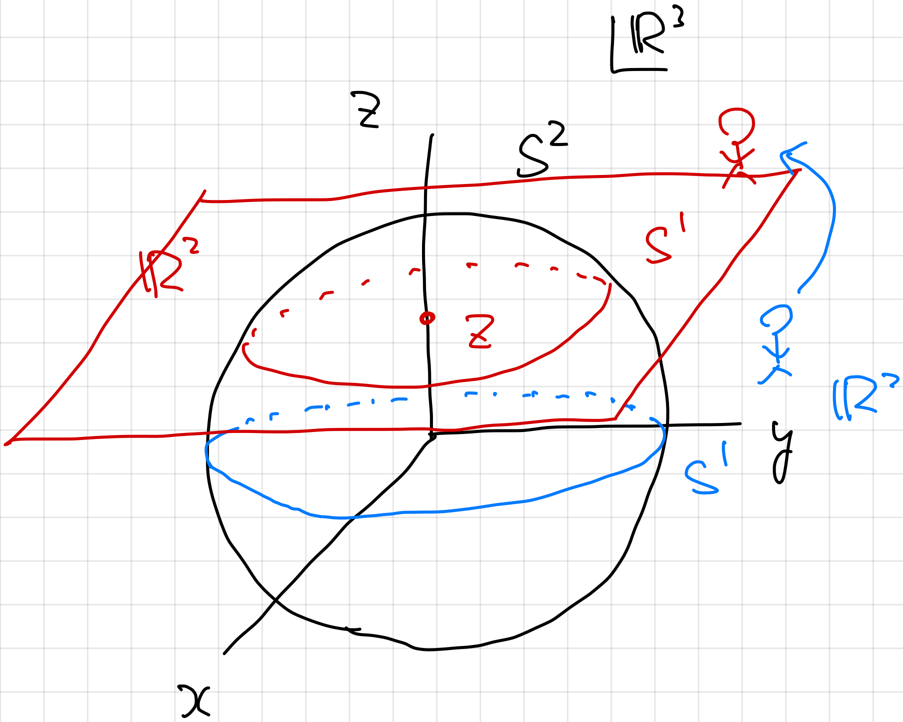
$$\left((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \right)$$







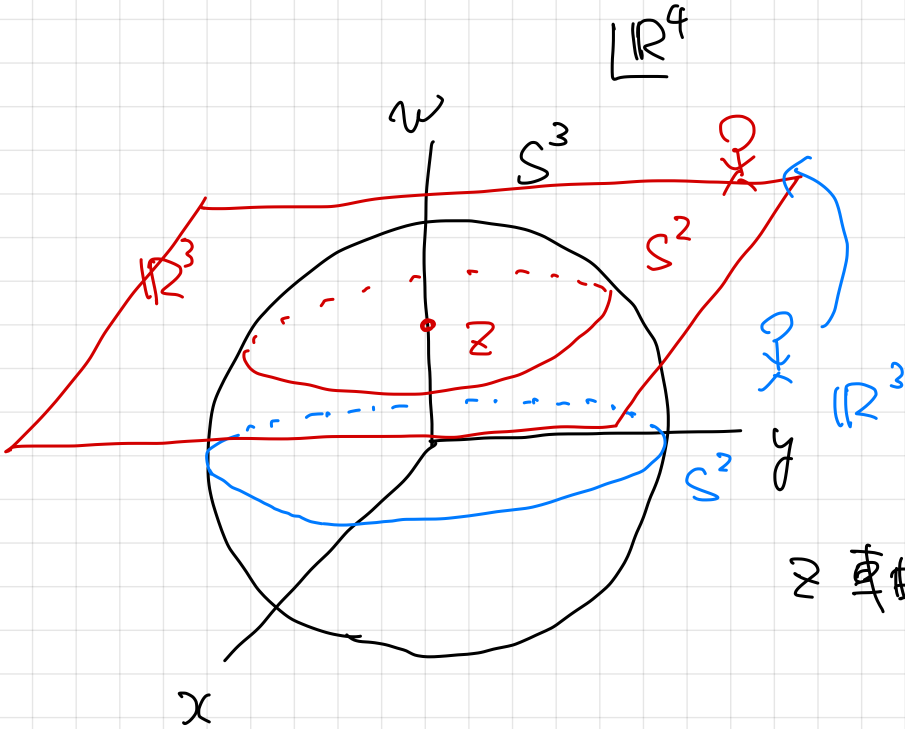
中身のつまった
3次元の球
表面はS^2



$$x^2+y^2+z^2=1$$

$$\underbrace{x^2+y^2=1-z^2}_{\text{半径}\sqrt{1-z^2}\text{の円}} \qquad -1 \leq z \leq 1$$

球(S^2)の輪切りが円(S^1)



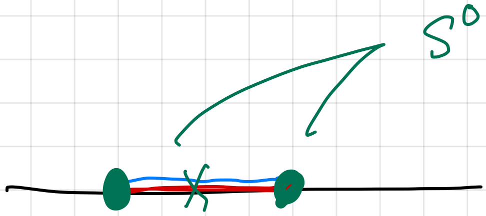
z軸は描いていない

$$x^2+y^2+z^2+w^2=1$$

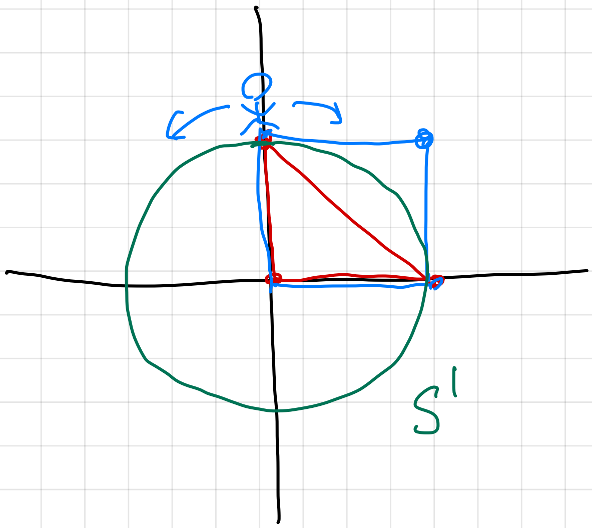
$$\underbrace{x^2+y^2+z^2=1-w^2}_{\text{半径 }\sqrt{1-w^2}\text{ の球}} \qquad -1\leq w\leq 1$$

$\mathbb{R}^0$

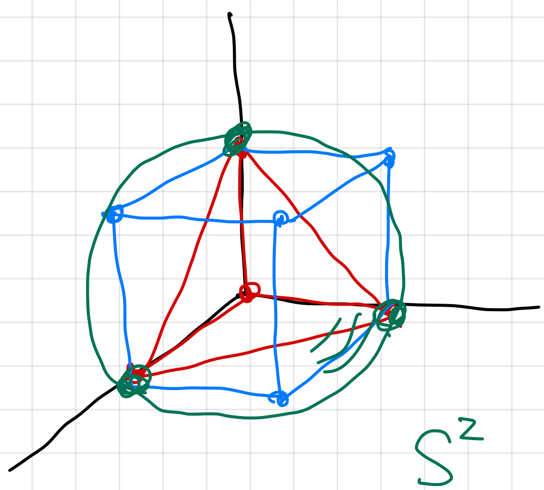
$\mathbb{R}^1$



$\mathbb{R}^2$



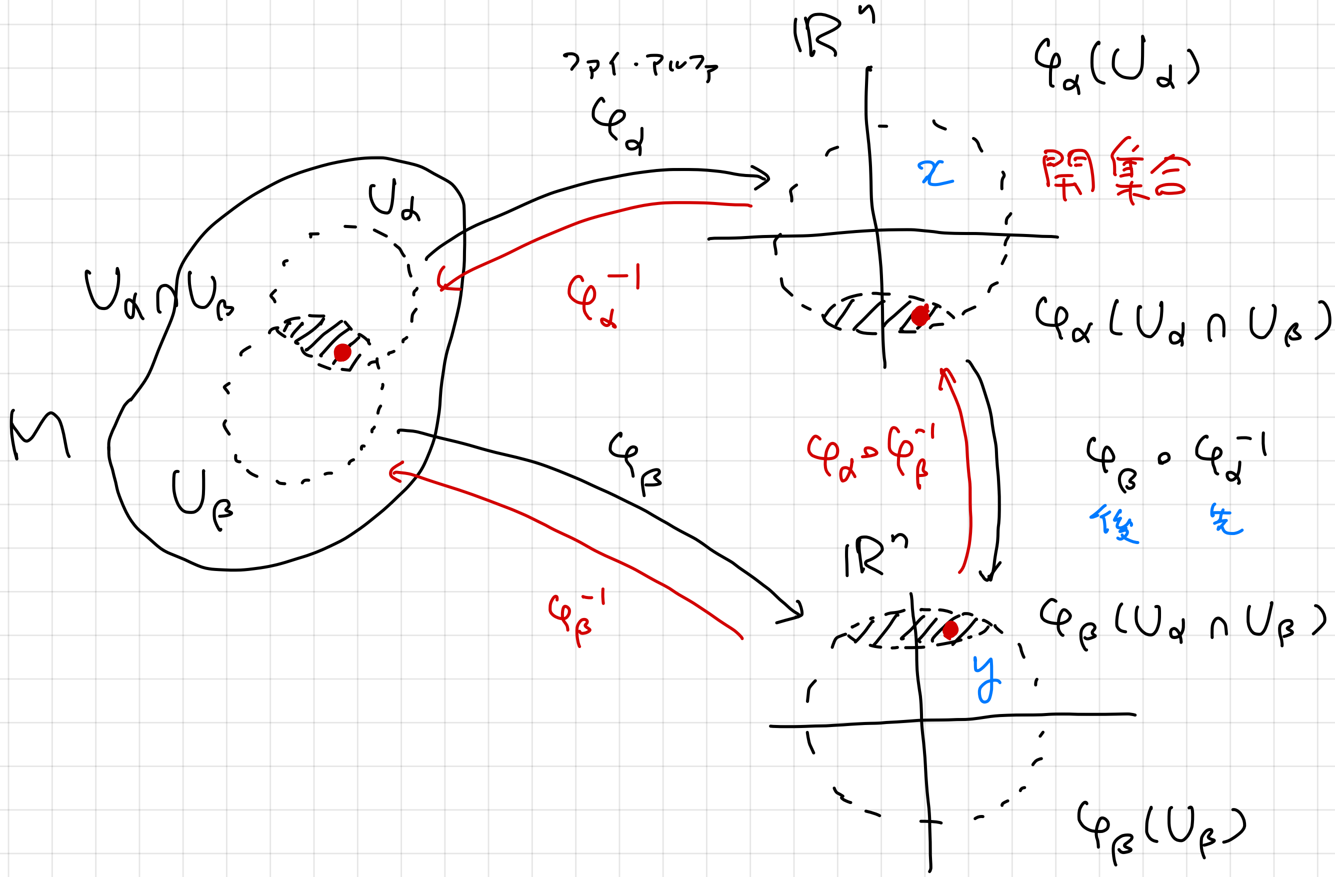
$\mathbb{R}^3$



§4 多様体

$(U_\alpha, \varphi_\alpha)$ 座標近傍

ユークリッド空間



$$\varphi_\beta \circ \varphi_\alpha^{-1} : \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}} \to \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}}$$

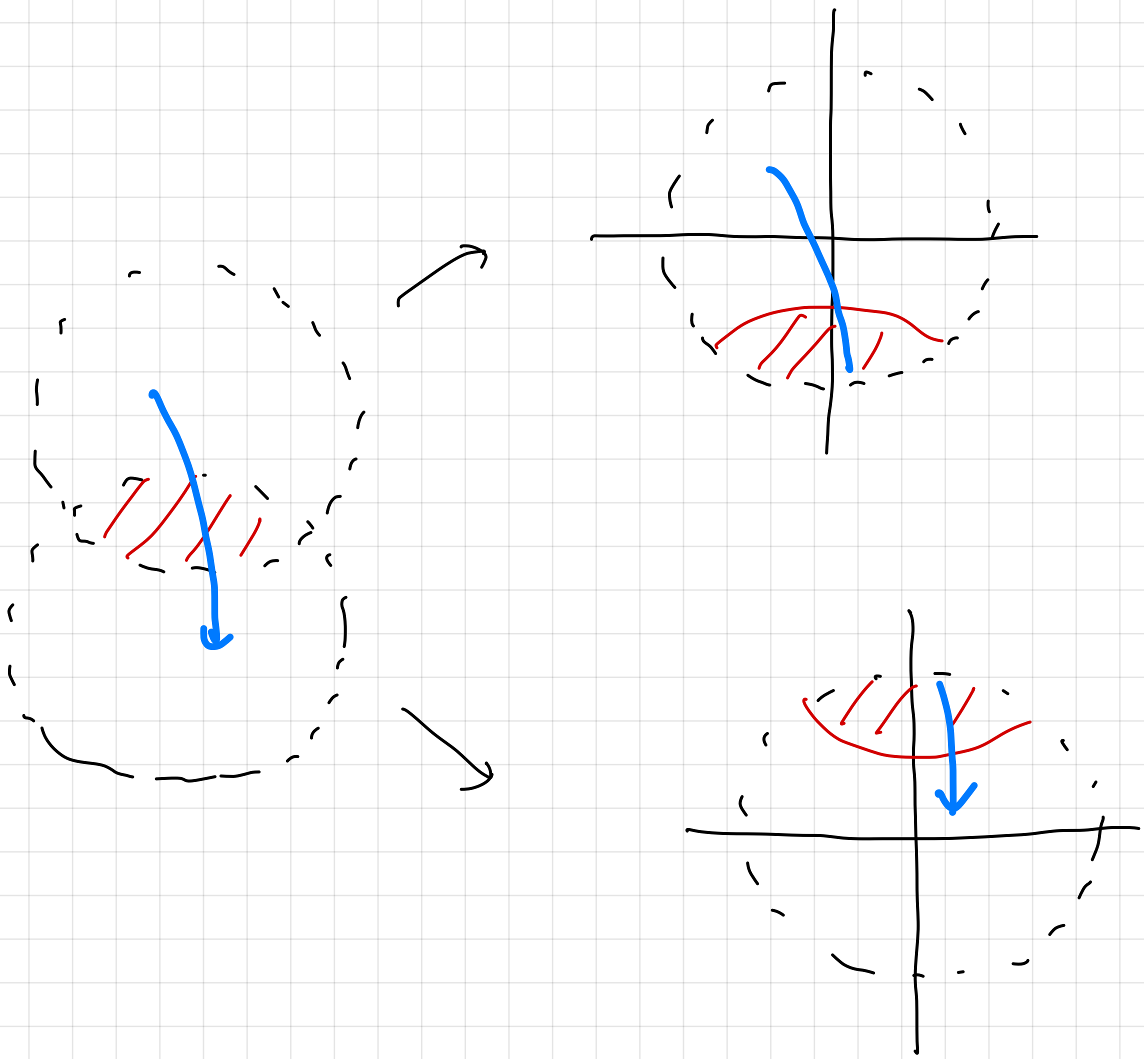
微分可能

$$(x^1, x^2, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$$

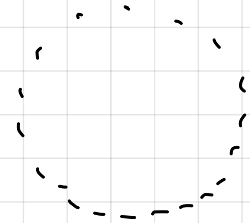
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}} \to \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}}$$

$$(y^1, \ldots, y^n) \mapsto (x^1, \ldots, x^n)$$

地図は"ゆがんで"いてもよい.
つながりが切れておらず
折れまがったりしなければよい



位相空間 ⇝ 開集合

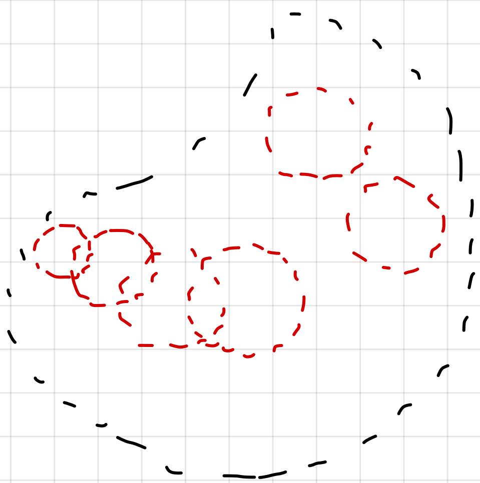


第2可算公理をみたす

ハウスドルフ空間

"自然数の個数"程度の開集合を決めておくと

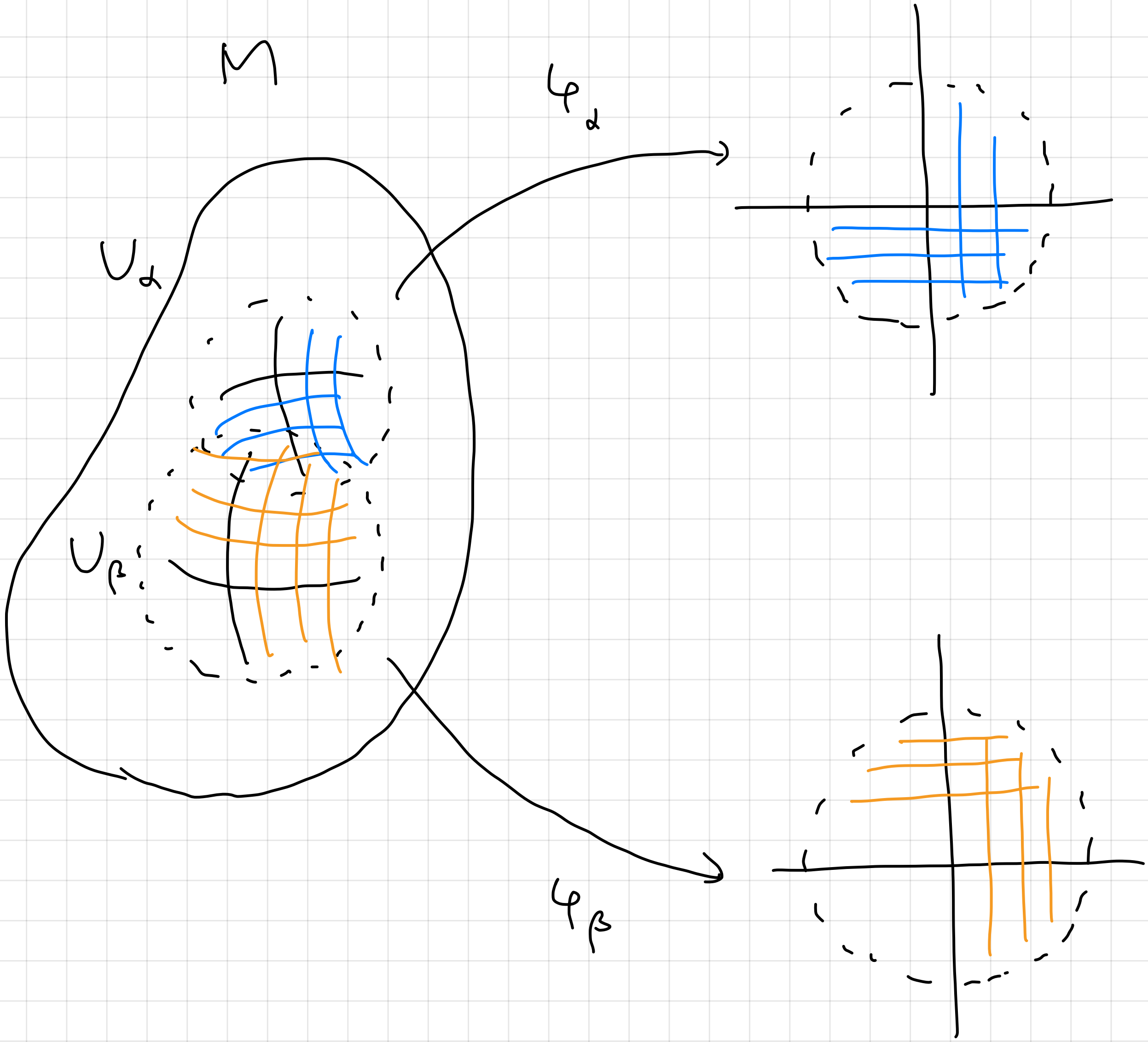
他の開集合はそれらの和で表すことができる。



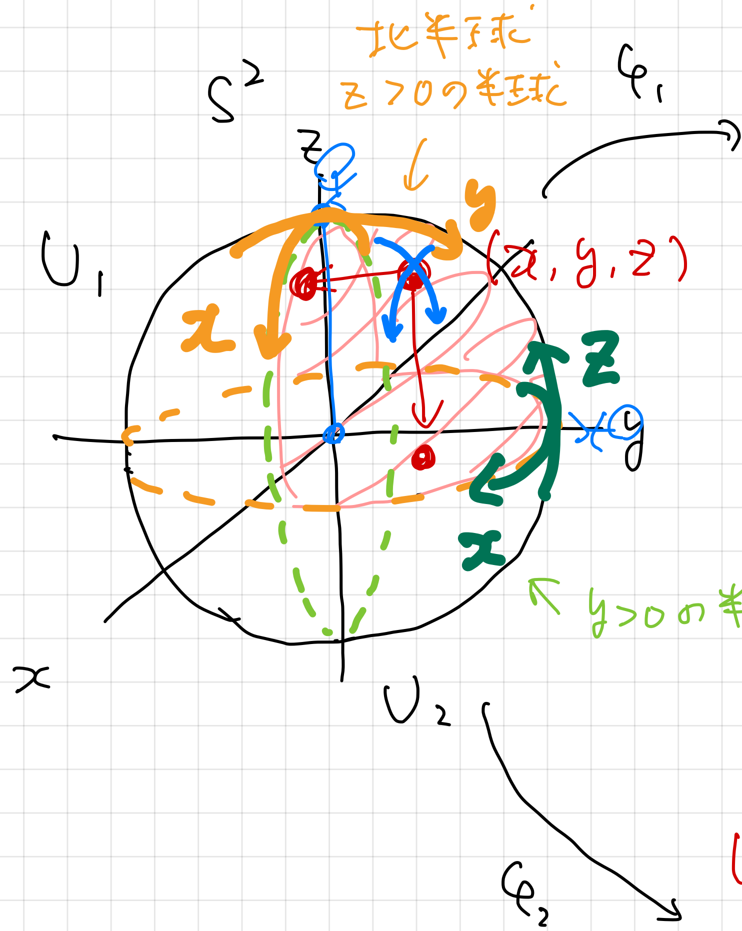
任意の2点に対し、それらを含む交わらない

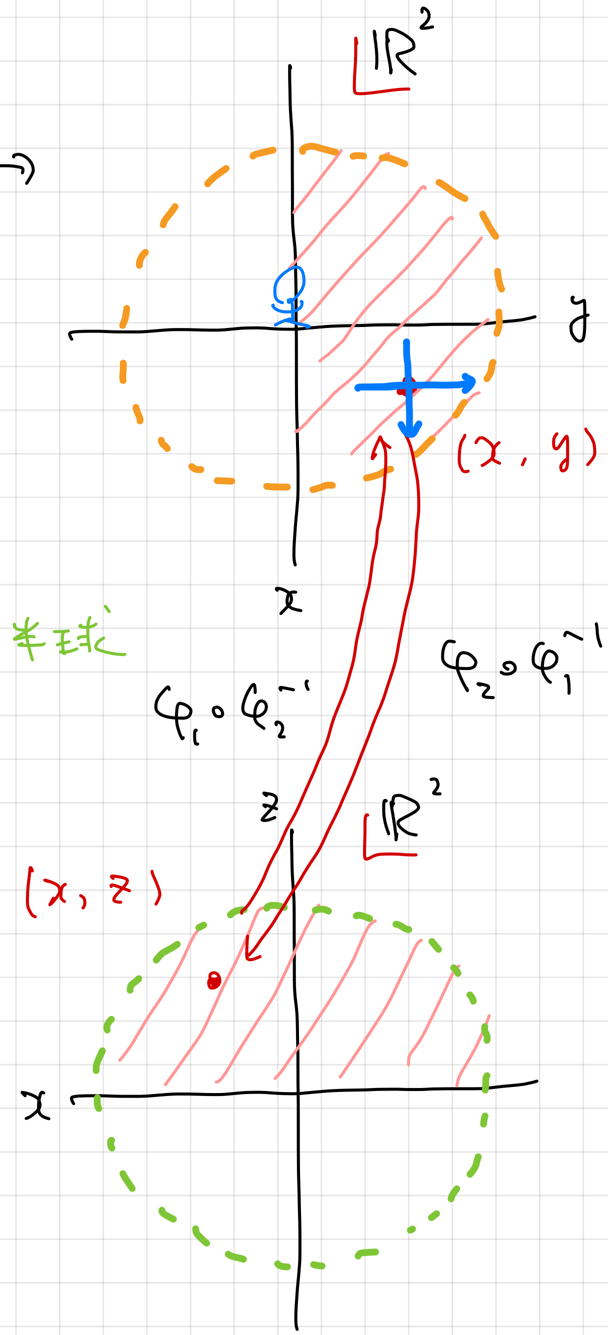
開集合がある

⇝ $\mathbb{R}^2$のような"普通の空間"



M
U_α
U_β
φ_α
φ_β





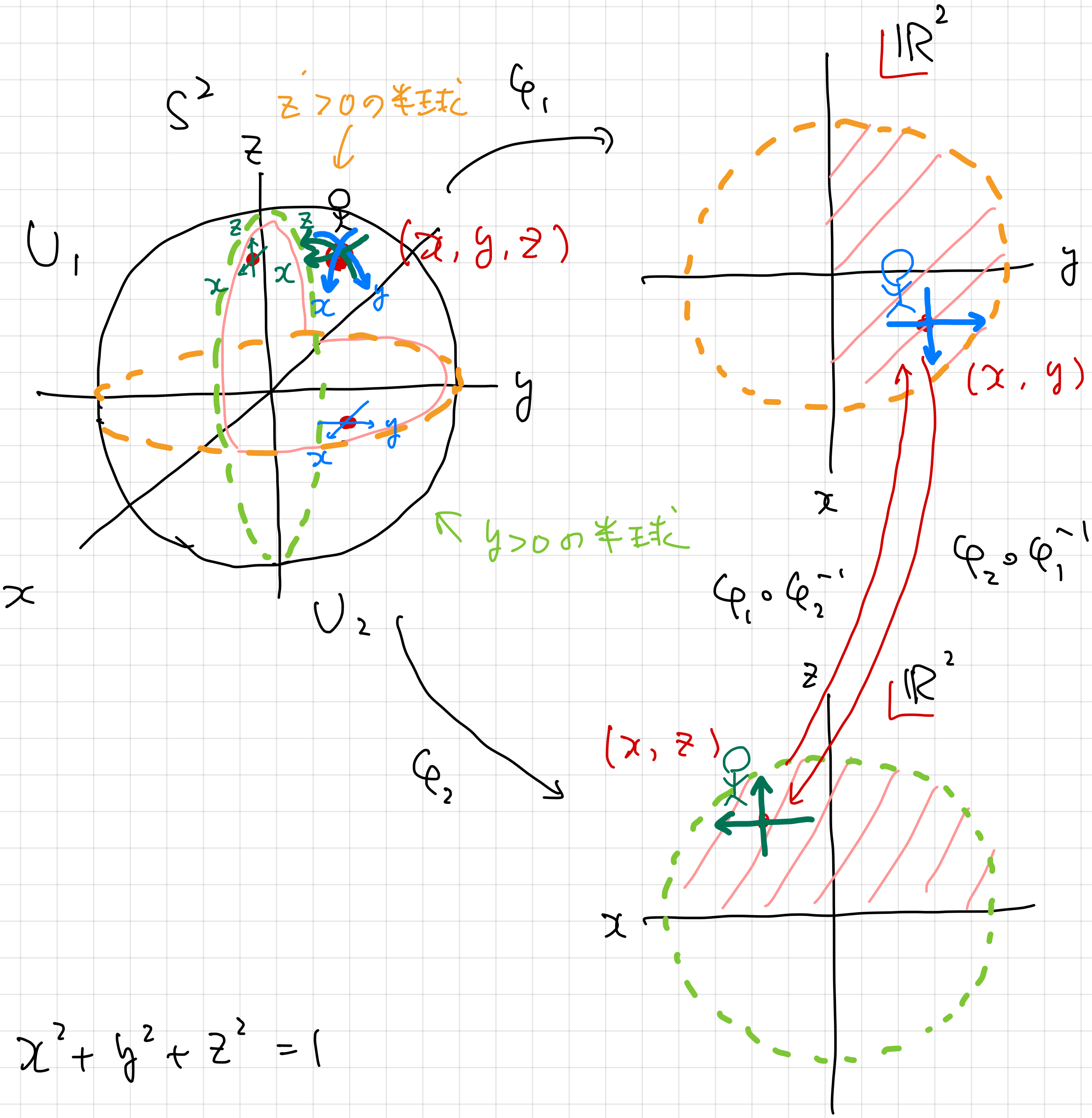
$$x^2 + y^2 + z^2 = 1$$

$$z = \sqrt{1 - x^2 - y^2}$$

$$y = \sqrt{1 - x^2 - z^2}$$

$$\varphi_2 \circ \varphi_1^{-1}(x, y) = (\overset{x'}{x}, \overset{z'}{\sqrt{1 - x^2 - y^2}})$$

$$\varphi_1 \circ \varphi_2^{-1}(x, z) = (x, \sqrt{1 - x^2 - z^2})$$



$$x^2 + y^2 + z^2 = 1$$

$$z = \sqrt{1 - x^2 - y^2}$$

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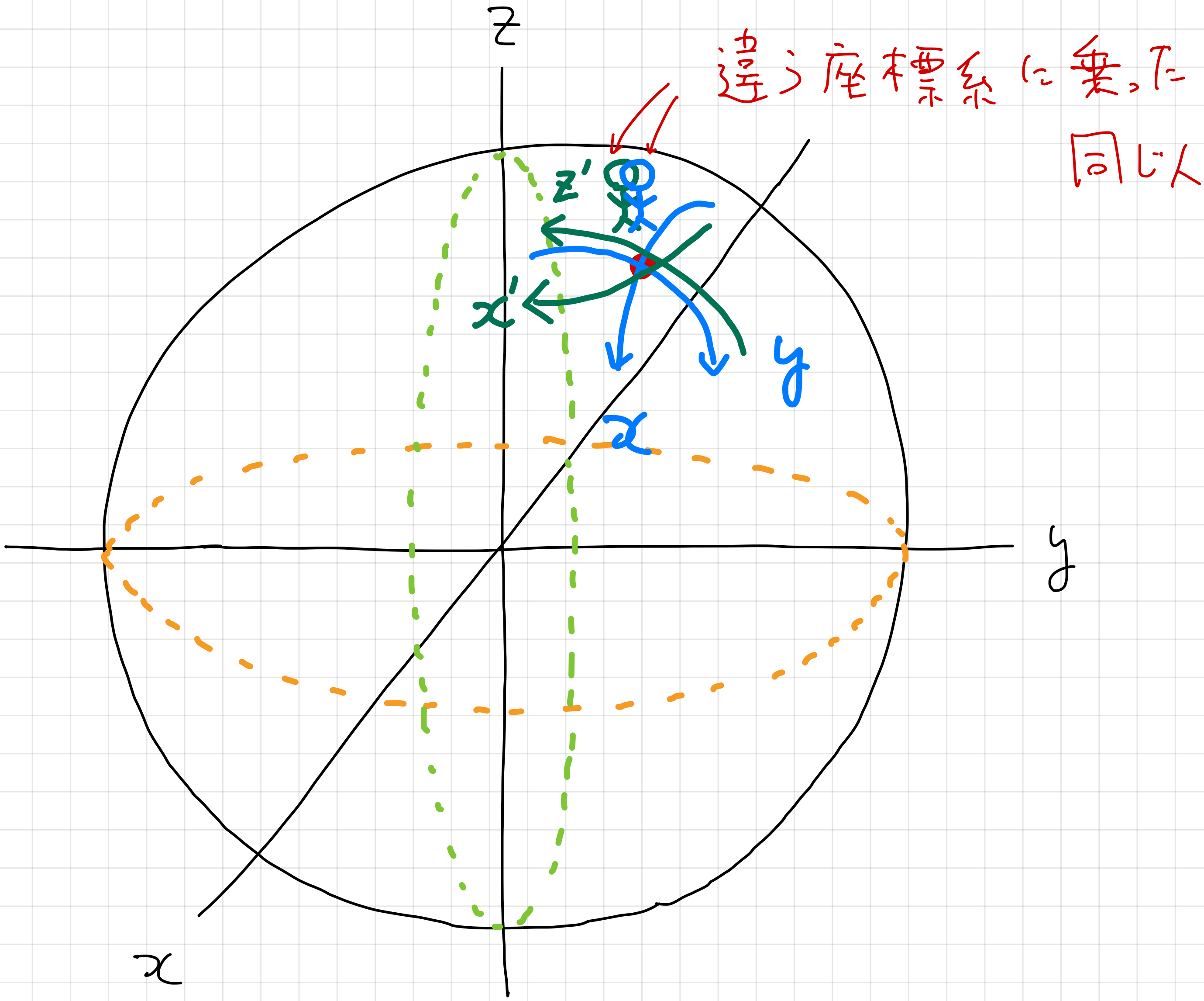
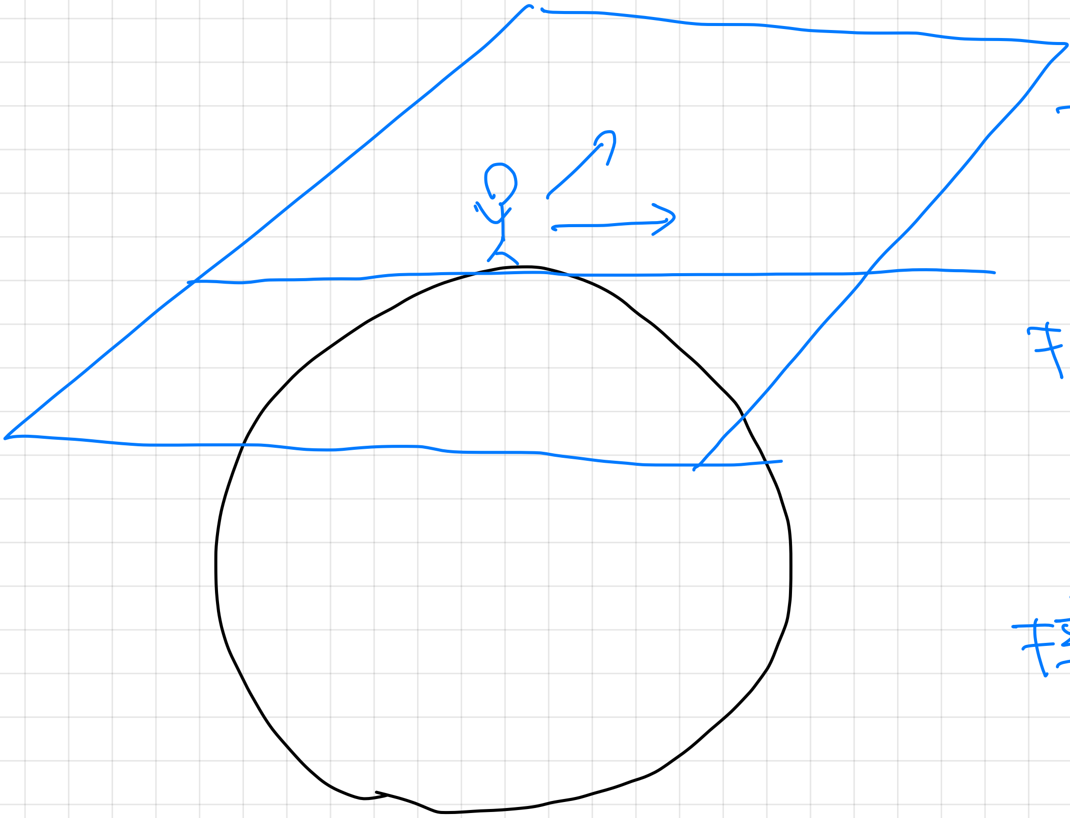
$$\varphi_2 \circ \varphi_1^{-1}(x, y) = (\overset{x'}{x}, \overset{z'}{\sqrt{1 - x^2 - y^2}})$$

$$\varphi_1 \circ \varphi_2^{-1}(x, z) = (x, \sqrt{1 - x^2 - z^2})$$



$x' = x$

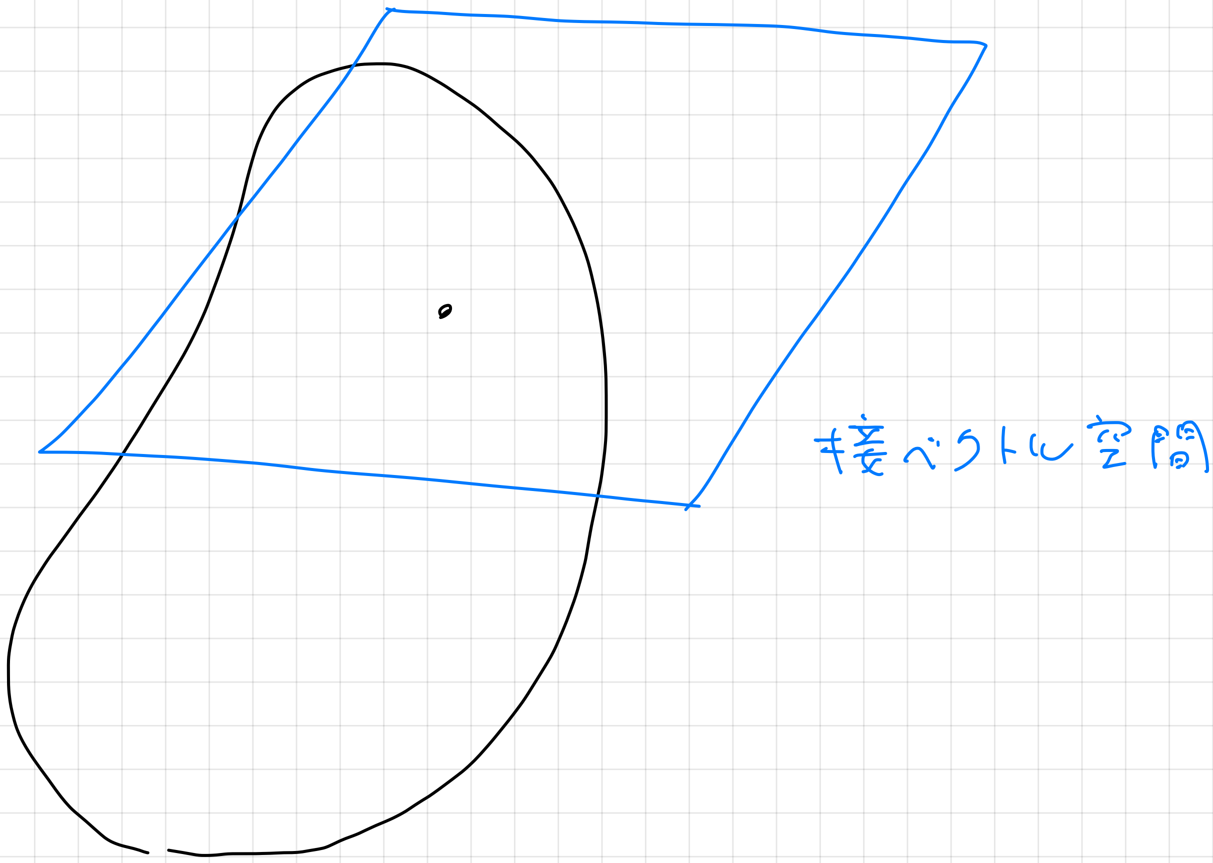
$z' = \sqrt{1 - x^2 - y^2}$



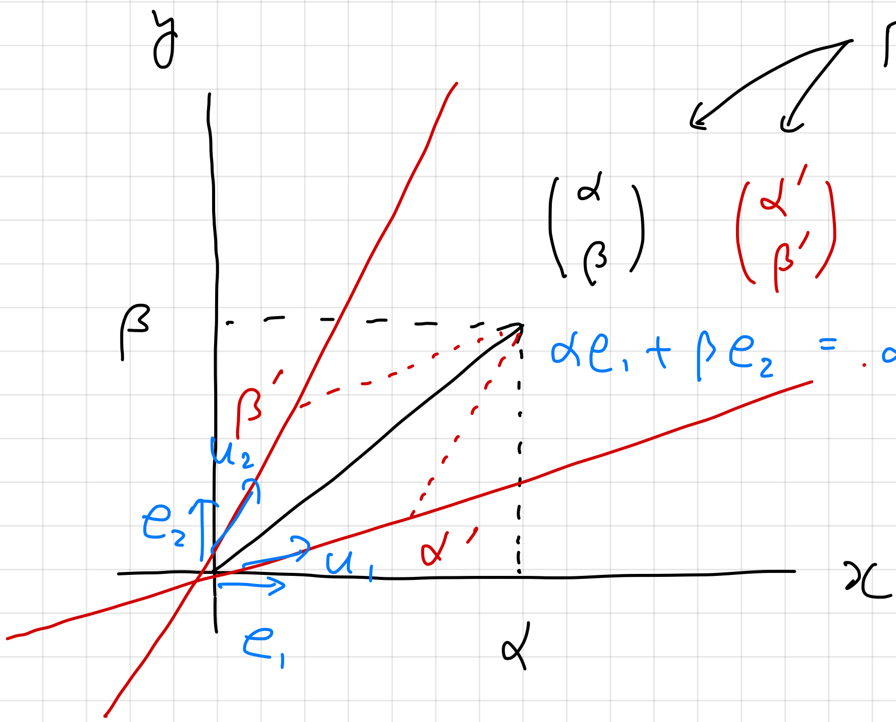
接平面

接線

接ベクトル空間


接ベクトル空間

ベクトル空間



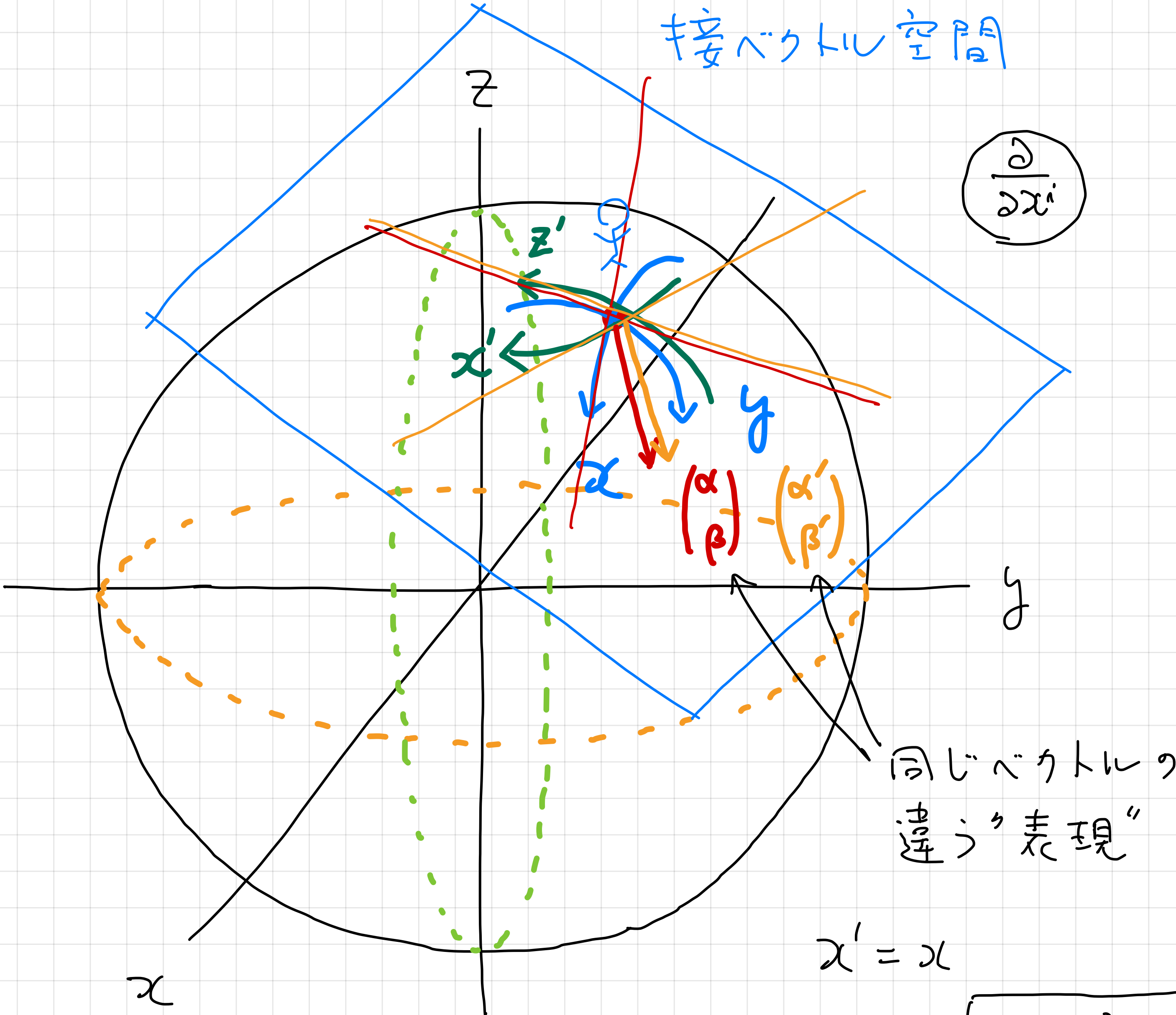
同じベクトルを表す.

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$　$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$

$\alpha e_1 + \beta e_2 = \alpha' u_1 + \beta' u_2$

e_1, e_2 は 一つの基底
u_1, u_2 も 一つの基底

座標軸は 斜めでよい.



$$x' = x$$

$$z' = \sqrt{1 - x^2 - y^2}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightsquigarrow \begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \qquad \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} \rightsquigarrow \begin{pmatrix}\Delta x'\\ \Delta z'\end{pmatrix}$$

Δxに対して

$$\Delta x' = \frac{\Delta x'}{\Delta x}\Delta x$$

$$\Delta z' = \frac{\Delta z'}{\Delta x}\Delta x$$

Δyに対して

$$\Delta x' = \frac{\Delta x'}{\Delta y}\Delta y$$

$$\Delta z' = \frac{\Delta z'}{\Delta y}\Delta y$$

$$\frac{\Delta x'}{\Delta x} \rightsquigarrow \frac{\partial x'}{\partial x} \text{ など}$$

偏微分

$$\overset{\alpha'}{\Delta x'} = \overset{\frac{\partial x'}{\partial x}}{\frac{\Delta x'}{\Delta x}} \overset{\alpha}{\Delta x} + \overset{\frac{\partial y'}{\partial y}}{\frac{\Delta y'}{\Delta y}} \overset{\beta}{\Delta y}$$

$$\overset{\beta'}{\Delta z'} = \overset{\frac{\partial z'}{\partial x}}{\frac{\Delta z'}{\Delta x}} \overset{\alpha}{\Delta x} + \overset{\frac{\partial z'}{\partial y}}{\frac{\Delta z'}{\Delta y}} \overset{\beta}{\Delta y}$$

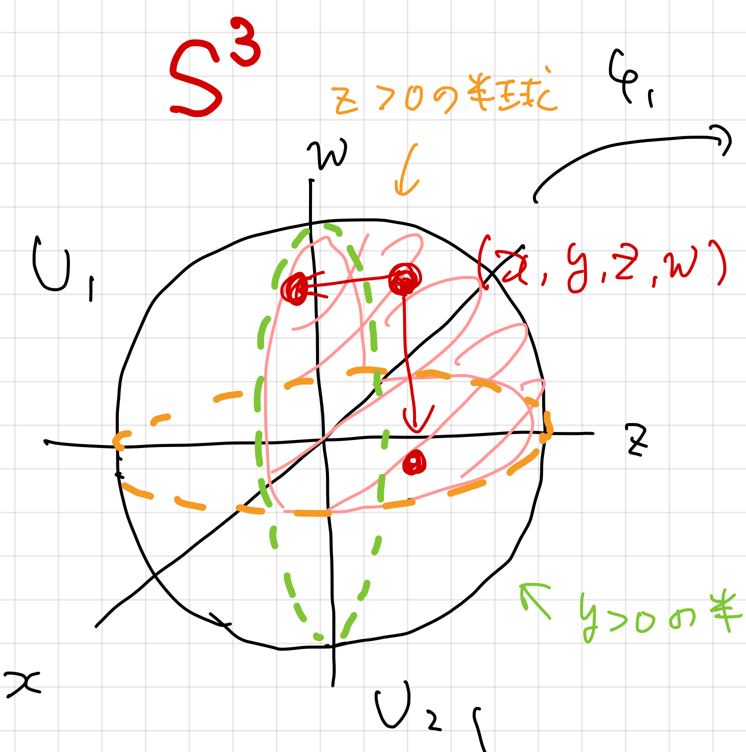
$$\rightsquigarrow \quad \alpha' = \alpha$$

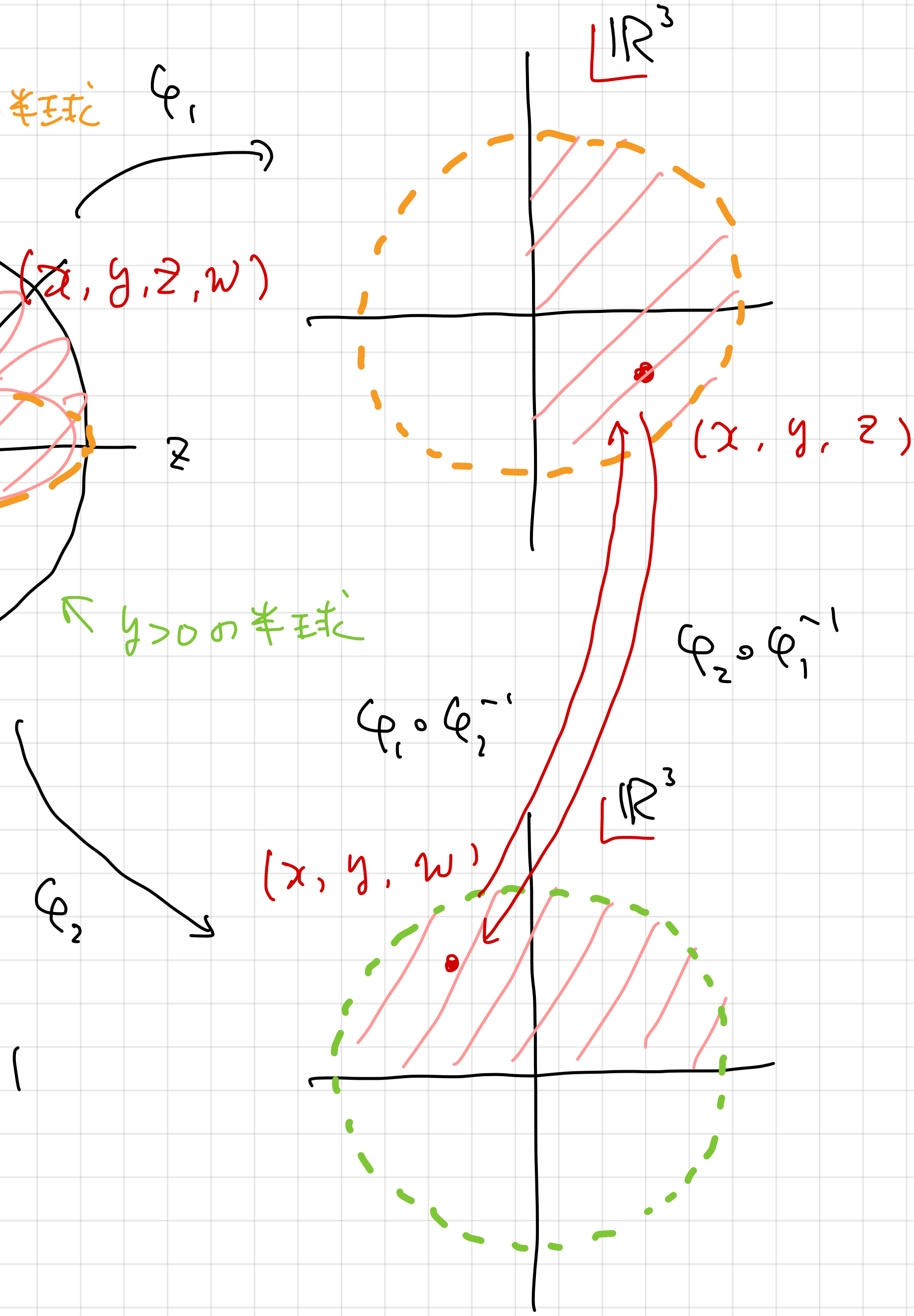
$$\beta' = -\frac{x}{\sqrt{1-x^2-y^2}}\alpha - \frac{y}{\sqrt{1-x^2-y^2}}\beta$$

具体的に

$$(1, 0) \longmapsto \left(1, -\frac{x}{\sqrt{1-x^2-y^2}}\right)$$

$$(0, 1) \longmapsto \left(0, -\frac{y}{\sqrt{1-x^2-y^2}}\right)$$





y軸を省略した。

$$x^2 + y^2 + z^2 + w^2 = 1$$

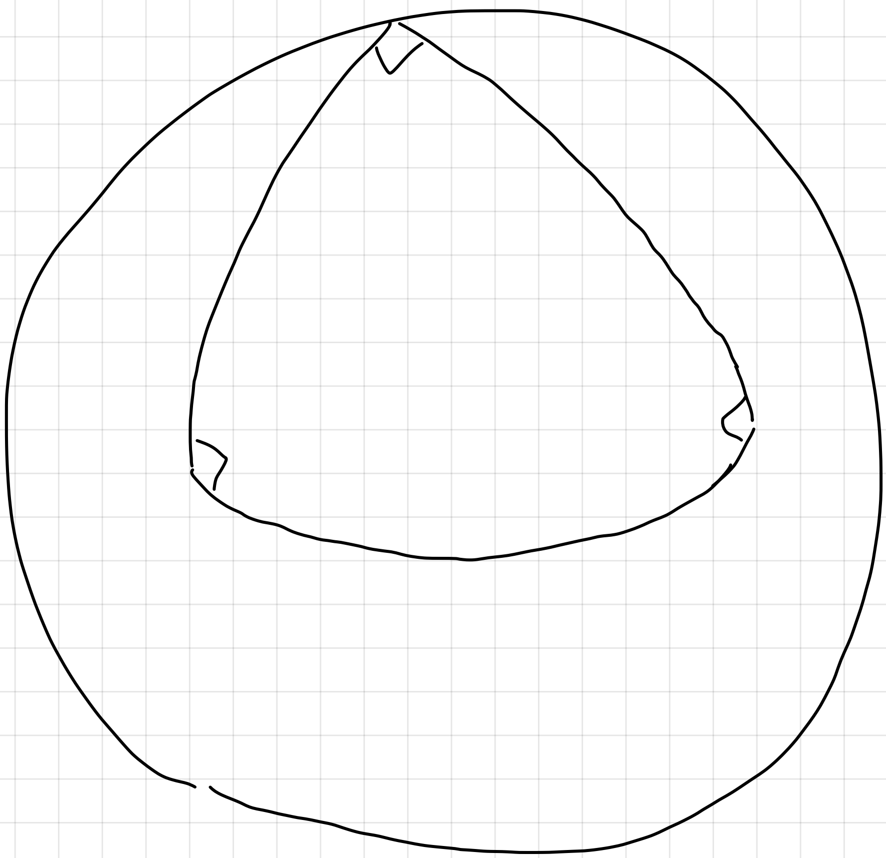
$$w = \sqrt{1 - x^2 - y^2 - z^2}$$

$$z = \sqrt{1 - x^2 - y^2 - w^2}$$

以下 S^2 と同様

三角形の内角の和は π (180°) か?

三平方の定理は正しいか?



曲った空間では正しいとは言えない.