Ch. 13 Difference Eqiations

(March 5, 2018)

The original use of time series analysis is the methodology developed to decompose a series into a trend, seasonal and an irregular components. As you can see,[1] the trend change the mean of the series and the seasonal component imparts a regular cyclical patterns. In practice, the trend and seasonal components will not be simplistic deterministic functions. However, for the time being, it is wise to sidestep these complications so the trend and seasonal components are regarded as deterministic at the moment and is even ignored now.

Notice that the irregular component, while not having a well-defined pattern, is somewhat predictable. One of parametric stochastic process model that express the value of a variable as a function of its own lagged value and other variable is the *difference equation*.

1 First-Order Difference Equations

Suppose we are given a dynamic equation relating the value Y takes on at date t to another variables W_t and to the value Y took in the previous period:

$$Y_t = \phi Y_{t-1} + W_t, \tag{13-1}$$

where ϕ is a constant. Equation (13-1) is a *linear first-order difference equation*. A difference equation is an expression relating a variable Y_t to its previous values. This is a *first-order* difference equation because only the first lag of the variable (Y_{t-1}) appears in the equation. Note that it expresses Y_t as a linear function of Y_{t-1} and W_t.

[1] See Non-Stationary and Linear Trend Example on p.3 in Ch.12.

In Chapter 14 the input variable W_t will be regarded as a random variable, and the implication of (13-1) for the statistical properties of the output variables Y_t will be explored. In preparation for this discussion, it is necessary first to understand the mechanics of the difference equations. For the present discussion in Chapter 13, the values for the input variables $\{W_1, W_2, ...\}$ will simply be regarded as a sequence of deterministic numbers. Our goal is to answer the following question: *If a dynamic system is described by (13-1), what are the effects on Y of changes in the value of W ?*

1.1 Solving a Difference equation by Recursive Substitution

The presumption is that the dynamic equation (13-1) governs the behavior of Y for all dates t, that is

$$\{Y_t = \phi Y_{t-1} + W_t, \ \ t \in \mathcal{T}\}.$$

We now consider the index set $\mathcal{T} = \{..., -2, -1, 0, 1, 2, 3, ...\}$. By direct substitution

$$\begin{aligned} Y_t &= \phi Y_{t-1} + W_t = \phi(\phi Y_{t-2} + W_{t-1}) + W_t \\ &= \phi^2 Y_{t-2} + \phi W_{t-1} + W_t = \phi^2(\phi Y_{t-3} + W_{t-2}) + \phi W_{t-1} + W_t \\ &= \phi^3 Y_{t-3} + \phi^2 W_{t-2} + \phi W_{t-1} + W_t \\ &= \phi^{t+1} Y_{-1} + \phi^t W_0 + \phi^{t-1} W_1 + \phi^{t-2} W_2 + ... + \phi W_{t-1} + W_t \\ &= ...... \end{aligned}$$

If we assume the value of Y for date $t = -1$ is known (Y_{-1} here is an "initial value"), we can express (13-1) by repeated substitution in the form

$$Y_t = \phi^{t+1} Y_{-1} + \phi^t W_0 + \phi^{t-1} W_1 + \phi^{t-2} W_2 + ... + \phi W_{t-1} + W_t. \tag{13-2}$$

This procedure is known as solving the difference equation (13-1) by *recursive substitution.*

1.2 Dynamic Multipliers

Note that (13-2) expresses Y as a linear function of the initial value Y_{-1} and the historical value of W. This makes it very easy to calculate the effect of W_0 (say) on Y_t. If W_0 were to change with Y_{-1} and $W_1,\ W_2, ...,\ W_t$ taken as unaffected (this is the reason that we need the error term to be a white noise sequence in the ARMA model in the subsequent chapters), the effect on Y_t would be given by

$$\frac{\partial Y_t}{\partial W_0} = \phi^t \ \ (backwards).$$

Note that the calculation would be exactly the same if the dynamic simulation were started at date t (taking Y_{t-1} as given); then Y_{t+j} can be described as a function of Y_{t-1} and $W_t,\ W_{t+1}, ...,\ W_{t+j}$:

$$Y_{t+j} = \phi^{j+1}Y_{t-1} + \phi^j W_t + \phi^{j-1}W_{t+1} + \phi^{j-2}W_{t+2} + ... + \phi W_{t+j-1} + W_{t+j}. \quad (13\text{-}3)$$

The effect of W_t on Y_{t+j} is given by

$$\frac{\partial Y_{t+j}}{\partial W_t} = \phi^j \ \ (forewords). \quad (13\text{-}4)$$

Thus the *dynamic multiplier* (or also refereed as the *impulse-response function*) (13-4) depends only on j, the length of time separating the disturbance to the input variable W_t and the observed value of output Y_{t+j}. The multiplier does not depend on t; that is, it does not depend on the dates of the observations themselves. This is true for any difference equation.

Different value of ϕ in (13-1) can produce a variety of dynamic responses of Y to W. If $0 < \phi < 1$, the multiplier $\partial Y_{t+j}/\partial W_t$ in (13-4) decays geometrically toward zero. If $-1 < \phi < 0$, the absolute value of the multiplier $\partial Y_{t+j}/\partial W_t$ in (13-4) also decays geometrically toward zero. If $\phi > 1$, the dynamic multiplier increase exponentially over time and if $\phi < -1$, the multiplier exhibit explosive oscillations.

Thus, if $|\phi| < 1$, the system is *stable*; the consequence of a given change in W_t will eventually die out. If $|\phi| > 1$, the system is explosive. An interesting possibility is the borderline case, $|\phi| = 1$. In this case, the solution (13-3) becomes

$$Y_{t+j} = Y_{t-1} + W_t + W_{t+1} + W_{t+2} + ... + W_{t+j-1} + W_{t+j}.$$

Here the output variables Y is the sum of the historical input W. A one-unit increase in W will cause a *permanent* one-unit increase in Y:

$$\frac{\partial Y_{t+j}}{\partial W_t} = 1 \quad for\ j = 0, 1, ... \ \ (unit\ root).$$

Example.

See Figure 1.1 on p.4 of Hamilton (1994). ■

2 pth-Order Difference Equations

Let us now generalize the dynamic system (13-1) by allowing the value of Y at date t to depend on p of its own lags along with the current value of the input variable W_t:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + W_t, \;\; t \in \mathcal{T}. \tag{13-5}$$

Equation (13-5) is a linear pth-order difference equation.

It would be a cumbersome task to solve the p-th order difference equation by recursive substitution. However, it is often convenient to rewrite the pth-order difference equation (13-5) in the scalar Y_t as a first-order difference equation in a vector $\boldsymbol{\xi}_t$. Define the $(p \times 1)$ vector $\boldsymbol{\xi}_t$ by

$$\boldsymbol{\xi}_t \equiv \begin{bmatrix} Y_t \\ Y_{t-1} \\ Y_{t-2} \\ . \\ . \\ . \\ Y_{t-p+1} \end{bmatrix},$$

the $(p \times p)$ matrix $\mathbf{F}$ by

$$\mathbf{F} \equiv \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & . & . & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \end{bmatrix},$$

and the $(p \times 1)$ vector $\mathbf{v}_t$ by

$$\mathbf{v}_t \equiv \begin{bmatrix} W_t \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}.$$

Consider the following first-order vector difference equation:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t, \tag{13-6}$$

or

$$\begin{bmatrix} Y_t \\ Y_{t-1} \\ Y_{t-2} \\ . \\ . \\ . \\ Y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & . & . & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \\ Y_{t-3} \\ . \\ . \\ . \\ Y_{t-p} \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}.$$

This is a system of p equations. The first equation in this system is identical to equation (13-5). The remaining $p-1$ equations is simply the identity

$$Y_{t-j} = Y_{t-j}, \quad j = 1, 2, ..., p-1.$$

Thus, the first-order vector system (13-6) is simply an alternative representation of the pth-order scalar system (13-5). The advantage of rewriting the pth-order system in (13-5) in the form of a first-order (vector) system (13-6) is that first-order systems are often *easier* to work with than pth-order systems.

A dynamic multiplier for (13-5) can be found in exactly the same way as was done for the first-order scalar system of section 1. If we knew the value of $\boldsymbol{\xi}_{-1}$, then proceeding recursively in this fashion as in the scalar first order difference equation produce a generalization of (13-2):

$$\boldsymbol{\xi}_t = \mathbf{F}^{t+1}\boldsymbol{\xi}_{-1} + \mathbf{F}^t\mathbf{v}_0 + \mathbf{F}^{t-1}\mathbf{v}_1 + \mathbf{F}^{t-2}\mathbf{v}_2 + ... + \mathbf{F}\mathbf{v}_{t-1} + \mathbf{v}_t. \tag{13-7}$$

Writing this out in terms of the definition of $\boldsymbol{\xi}_t$ and $\mathbf{v}_t$,

$$\begin{aligned} \begin{bmatrix} Y_t \\ Y_{t-1} \\ Y_{t-2} \\ . \\ . \\ . \\ Y_{t-p+1} \end{bmatrix} &= \mathbf{F}^{t+1} \begin{bmatrix} Y_{-1} \\ Y_{-2} \\ Y_{-3} \\ . \\ . \\ . \\ Y_{-p} \end{bmatrix} + \mathbf{F}^t \begin{bmatrix} W_0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} + \mathbf{F}^{t-1} \begin{bmatrix} W_1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} + ... \\ & + \mathbf{F}^1 \begin{bmatrix} W_{t-1} \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} + \begin{bmatrix} W_t \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}. \end{aligned} \tag{13-8}$$

Consider the first scalar equation of this system, which characterize the value of Y_t. Let f_{11}^t denote the $(1,1)$ elements of $\mathbf{F}_t$, f_{12}^t denote the $(1,2)$ elements of $\mathbf{F}_t$, and so on. Then the first equation of (12-8) states that

$$Y_t = f_{11}^{t+1}Y_{-1} + f_{12}^{t+1}Y_{-2} + ... + f_{1p}^{t+1}Y_{-p} + f_{11}^{t}W_0 + f_{11}^{t-1}W_1 + ... + f_{11}^{1}W_{t-1} + W_t. \tag{13-9}$$

This describe the value of Y at date t as a linear function of p initial value of Y $(Y_{-1}, Y_{-2}, ..., Y_{-p})$ and the history of the input variables W since date 0. $(W_0, W_1, ..., W_t)$

Result. (Initial values)
Note that whereas only one initial value for Y was needed in the case of a first-order difference equation, p initial values for Y are needed in the case of a pth-order difference equation. ■

The same kind of vector representation can be applied to the foreward solutions. The obvious generalization of (13-3) is

$$\boldsymbol{\xi}_{t+j} = \mathbf{F}^{j+1}\boldsymbol{\xi}_{t-1} + \mathbf{F}^{j}\mathbf{v}_t + \mathbf{F}^{j-1}\mathbf{v}_{t+1} + \mathbf{F}^{j-2}\mathbf{v}_{t+2} + ... + \mathbf{F}\mathbf{v}_{t+j-1} + \mathbf{v}_{t+j} \tag{13-10}$$

from which

$$\begin{aligned} Y_{t+j} &= f_{11}^{j+1}Y_{t-1} + f_{12}^{j+1}Y_{t-2} + ... + f_{1p}^{j+1}Y_{t-p} + \\ &\quad f_{11}^{j}W_t + f_{11}^{j-1}W_{t+1} + ... + f_{11}^{1}W_{t+j-1} + W_{t+j}. \end{aligned} \tag{13-11}$$

Thus, for a pth-order difference equation, the dynamic multiplier is given by

$$\frac{\partial Y_{t+j}}{\partial W_t} = f_{11}^j,$$

where f_{11}^j denotes the $(1,1)$ element of $\mathbf{F}^j(=\underbrace{\mathbf{F}\times\mathbf{F}\times\cdots\times\mathbf{F}}_{j})$.

Example.
The $(1,1)$ elements of $\mathbf{F}^1$ is ϕ_1 and the $(1,1)$ elements of $\mathbf{F}^2(=[\phi_1,\phi_2,...,\phi_p][\phi_1,1,0,...,0]')$ is $\phi_1^2+\phi_2$. Thus,

$$\begin{aligned} \frac{\partial Y_{t+1}}{\partial W_t} &= \phi_1; \quad \textit{and} \\ \frac{\partial Y_{t+2}}{\partial W_t} &= \phi_1^2 + \phi_2 \end{aligned}$$

in a pth-order system. ■

For larger values of j, an easy way to obtain a numerical value for the dynamic multiplier $\partial Y_{t+j}/\partial W_t$ is in terms the eigenvalues of the matrix $\mathbf{F}$. Recall that the eigenvalues of a matrix $\mathbf{F}$ are those numbers λ for which

$$|\mathbf{F} - \lambda \mathbf{I}_p| = 0. \tag{13-12}$$

For example, for $p = 2$ the eigenvalues are the solutions to

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

or

$$\left| \begin{bmatrix} (\phi_1 - \lambda) & \phi_2 \\ 1 & -\lambda \end{bmatrix} \right| = \lambda^2 - \phi_1 \lambda - \phi_2 = 0. \tag{13-13}$$

For a general pth-order system, the determinant in (13-12) is a pth-order ploynominal in λ whose p solutions characterize the p eigenvalues of $\mathbf{F}$. This polynomial turns out to take a very similar form to (13-13).

Result. (Eigenvalues of $\mathbf{F}$):
The eigenvalues of the matrix $\mathbf{F}$ defines in equation (13-12) are the values of λ that satisfy the following equation:

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - ... - \phi_{p-1} \lambda - \phi_p = 0.$$ ■

2.1 General Solution of a pth-order Difference Equation with Distinct Eigenvalues

Recall that if the eigenvalues of a $(p \times p)$ matrix $\mathbf{F}$ are distinct,[2] there exists a nonsingular $(p \times p)$ matrix $\mathbf{X}$ such that[3]

$$\mathbf{F} = \mathbf{X} \mathbf{\Lambda} \mathbf{X}^{-1}$$

[2] Since the eigenvalues are distinct by assumption, the associated eigenvectors are linear independent so $\mathbf{X}^{-1}$ below exist.

[3] See p.36, section 4 of Chapter 1.

where $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_p]$, $\mathbf{x}_i, i = 1, 2, ..., p$ are the eigenvectors of $\mathbf{F}$ corresponding to its eigenvalues λ_i; and $\mathbf{\Lambda}$ is a $(p \times p)$ matrix such that

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \lambda_p \end{bmatrix}.$$

This enables us to characterize the dynamic multiplier (the (1,1) elements of $\mathbf{F}^j$) very easily. In general, we have

$$\begin{aligned} \mathbf{F}^j &= \underbrace{\mathbf{X\Lambda X}^{-1} \times \mathbf{X\Lambda X}^{-1} \times ... \times \mathbf{X\Lambda X}^{-1}}_{\text{j terms}} \\ &= \mathbf{X\Lambda}^j\mathbf{X}^{-1}, \end{aligned} \tag{13-14}$$

where

$$\mathbf{\Lambda}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2^j & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \lambda_p^j \end{bmatrix}.$$

Let t_{ij} denote the row i column j element of $\mathbf{X}$ and let t^{ij} denote the row i column j element of $\mathbf{X}^{-1}$. Equation (13-14) written out explicitly become

$$\begin{aligned} \mathbf{F}^j &= \begin{bmatrix} t_{11} & t_{12} & . & . & . & . & t_{1p} \\ t_{21} & t_{22} & . & . & . & . & t_{2p} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ t_{p1} & t_{p2} & . & . & . & . & t_{pp} \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2^j & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & . & . & . & . & t^{1p} \\ t^{21} & t^{22} & . & . & . & . & t^{2p} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ t^{p1} & t^{p2} & . & . & . & . & t^{pp} \end{bmatrix} \\ &= \begin{bmatrix} t_{11}\lambda_1^j & t_{12}\lambda_2^j & . & . & . & . & t_{1p}\lambda_p^j \\ t_{21}\lambda_1^j & t_{22}\lambda_2^j & . & . & . & . & t_{2p}\lambda_p^j \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ t_{p1}\lambda_1^j & t_{p2}\lambda_2^j & . & . & . & . & t_{pp}\lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & . & . & . & . & t^{1p} \\ t^{21} & t^{22} & . & . & . & . & t^{2p} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ t^{p1} & t^{p2} & . & . & . & . & t^{pp} \end{bmatrix} \end{aligned}$$

from which the (1,1) element of $\mathbf{F}^j$ is given by

$$f_{11}^j = c_1\lambda_1^j + c_2\lambda_2^j + ... + c_p\lambda_p^j$$

where

$$c_i = t_{1i}t^{i1}$$

and

$$c_1 + c_2 + ... + c_p = t_{11}t^{11} + t_{12}t^{21} + ... + t_{1p}t^{p1} = 1.$$

Therefore the dynamic multiplier of a pth-order difference equation is:

$$\frac{\partial Y_{t+j}}{\partial W_t} = f_{11}^j = c_1\lambda_1^j + c_2\lambda_2^j + ... + c_p\lambda_p^j, \tag{13-15}$$

that is the dynamic multiplier is a *weighted average* of each of the p eigenvalues raised to the jth power.

The following result provides a closed-form expression for the constant $c_1, c_2, ..., c_p$.

Result.

If the eigenvalues $(\lambda_1, \lambda_2, ..., \lambda_p)$ of the matrix $\mathbf{F}$ are distinct, then the magnitude c_i can be written as

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{k=1,\ k\neq i}^{p}(\lambda_i - \lambda_k)}. \qquad \blacksquare$$

Example.

In then case $p = 2$, we have

$$c_1 = \frac{\lambda_1^{(2-1)}}{\lambda_1 - \lambda_2} = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \text{ and } c_2 = \frac{\lambda_2^{(2-1)}}{\lambda_2 - \lambda_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1}. \qquad \blacksquare$$

2.1.1 Real Roots

Suppose first that all the eigenvalues of $\mathbf{F}$ are **real** and *all these real eigenvalues are less than one in absolute value*, then the system is stable, and its dynamics are represented as a weighted average of decaying exponentially or decaying exponentially oscillating in sign. The dynamic multiplier converges to zero as $j \to \infty$, i.e.

$$\lim_{j\to\infty} \frac{\partial Y_{t+j}}{\partial W_t} = f_{11}^j = c_1\lambda_1^j + c_2\lambda_2^j + ... + c_p\lambda_p^j \longrightarrow 0.$$

Example. (Distinct real roots)
Consider the following second-order difference equation:

$$Y_t = 0.6Y_{t-1} + 0.2Y_{t-2} + W_t.$$

The eigenvalues are the solutions of the polynomial

$$\lambda^2 - 0.6\lambda - 0.2 = 0,$$

which are

$$\begin{aligned}\lambda_1 &= \frac{0.6 + \sqrt{(0.6)^2 + 4(0.2)}}{2} = 0.84\\ \lambda_2 &= \frac{0.6 - \sqrt{(0.6)^2 + 4(0.2)}}{2} = -0.24.\end{aligned}$$

The dynamic multiplier for this system,

$$\frac{\partial Y_{t+j}}{\partial W_t} = c_1\lambda_1^j + c_2\lambda_2^j = c_1(0.84)^j + c_2(-0.24)^j$$

is geometrically decaying and is plotted as a function of j in panel (a) of Hamilton, p.15. Note that as j becomes larger, the pattern is dominated by the larger eigenvalues (λ_1), approximating a simple geometric decay at rate (λ_1). ■

If the eigenvalue are *all real but at least one is greater than one in absolute value*, the system is explosive. If λ_1 denotes the eigenvalue that is largest in absolute value, the dynamic multiplier is eventually dominated by an exponential function of that eigenvalues:

$$\lim_{j\to\infty} \frac{\partial Y_{t+j}}{\partial W_t} \cdot \frac{1}{\lambda_1^j} = c_1.$$

2.1.2 Complex Roots

It is possible that the eigenvalue of $\mathbf{F}$ are complex (Since $\mathbf{F}$ is not symmetric. For a symmetric matrix, its eigenvalues are all real.[4]). Whenever this is the case, they appear

[4] see the proof on p.22 of Chapter 1.

as complex conjugates. For example if $p = 2$ and $\phi_1^2 + 4\phi_2 < 0$, then the solutions λ_1 and λ_2 are complex conjugates. Suppose that λ_1 and λ_2 are complex conjugates, written as

$$\lambda_1 = a + bi, \text{ and } \lambda_2 = a - bi.$$

By rewritten the definition of the sine and the cosine function we have

$$\begin{aligned} a &= r\cos(\theta) \ \ and \\ b &= r\sin(\theta), \end{aligned}$$

where for a given angle θ and r are defined in terms of a and b by

$$r = \sqrt{a^2+b^2}, \ \ \cos(\theta) = \frac{a}{r}, \ \ \sin(\theta) = \frac{b}{r}.$$

Therefore, we have (Polar form)

$$\begin{aligned} \lambda_1 &= r[\cos\theta + i\sin\theta], \\ \lambda_2 &= r[\cos\theta - i\sin\theta]. \end{aligned}$$

By *Eular relations*[5] (see for example, Chiang, A.C. (1984), p. 520) we further have

$$\begin{aligned} \lambda_1 &= r[\cos\theta + i\sin\theta] = Re^{i\theta} \\ \lambda_2 &= r[\cos\theta - i\sin\theta] = Re^{-i\theta}, \end{aligned}$$

and when they are raised to the jth power,

$$\begin{aligned} \lambda_1^j &= r^j[\cos(\theta j) + i\sin(\theta j)] = r^j e^{i\theta j} \\ \lambda_2^j &= r^j[\cos(\theta j) - i\sin(\theta j)] = r^j e^{-i\theta j}. \end{aligned}$$

The contribution of the complex conjugates to the dynamic multiplier $\partial Y_{t+j}/\partial W_t$:

$$\begin{aligned} c_1\lambda_1^j + c_2\lambda_2^j &= c_1 r^j[\cos(\theta j) + i\sin(\theta j)] + c_2 r^j[\cos(\theta j) - i\sin(\theta j)] \\ &= (c_1+c_2)r^j \cdot \cos(\theta j) + i(c_1 - c_2)r^j \cdot \sin(\theta j). \end{aligned}$$

From last Result we know that if λ_1 and λ_2 are complex conjugates, then c_1 and c_2 are also complex conjugates; that is they can be written as

$$\begin{aligned} c_1 &= \alpha + \beta i, \\ c_2 &= \alpha - \beta i, \end{aligned}$$

[5]We may also apply De Moivre's Formula directly.

for some real number α and β. Therefore, the dynamic multiplier $\partial Y_{t+j}/\partial W_t$ can further be expressed as

$$\begin{aligned} c_1\lambda_1^j + c_2\lambda_2^j &= [(\alpha+\beta i)+(\alpha-\beta i)]\cdot r^j \cdot \cos(\theta j) + i[(\alpha+\beta i)-(\alpha-\beta i)]\cdot r^j \cdot \sin(\theta j) \\ &= (2\alpha)r^j \cdot \cos(\theta j) + i\cdot(2\beta i)r^j \cdot \sin(\theta j) \\ &= 2\alpha r^j \cos(\theta j) - 2\beta r^j \sin(\theta j). \end{aligned}$$

Result. (Distinct Complex Roots)
Thus, when some of the (distinct) eigenvalues are complex, then if
(a). $r = 1$, that is the complex eigenvalues have unit modulus, the multipliers are *periodic sine and cosine functions of* j;
(b). $r < 1$, that is the complex eigenvalues are less then one in modulus, the impulse again follows a sinusoidal pattern though its amplitude *decays at the rate* r^j;
(c). $r > 1$, that is the complex eigenvalues are greater then one in modulus, its amplitude of the sinusoids *explodes at the rate* r^j. ■

Example.:
Consider the following second-order difference equation:

$$Y_t = 0.5Y_{t-1} - 0.8Y_{t-2} + W_t.$$

The eigenvalues are the solutions the polynomial

$$\lambda^2 - 0.5\lambda + 0.8 = 0$$

which are

$$\begin{aligned} \lambda_1 &= \frac{0.5+\sqrt{(0.5)^2-4(0.8)}}{2} = 0.25 + 0.86i \\ \lambda_2 &= \frac{0.5-\sqrt{(0.5)^2-4(0.8)}}{2} = 0.25 - 0.86i, \end{aligned}$$

with modulus

$$r = \sqrt{(0.25)^2 + (0.86)^2} = 0.9.$$

Since $r < 1$, the dynamic multiplier follows a pattern of damped oscillation as plotted in panel (b) of Figure 1.4 of Hamilton, p. 15. ■

2.2 General Solution of a pth-order Difference Equation with Repeated Eigenvalues

By Jordan decomposition, the conclusion is the same that for the pth-order difference equation to be stable, all the eigenvalues of $\mathbf{F}$ must be smaller than 1 in modulus.

Example.

The dynamic multiplier in the second order difference equation with repeated roots is

$$\frac{\partial Y_{t+j}}{\partial W_t} = k_1\lambda^j + k_2 j\lambda^{j-1},$$

where k_1 and k_2 are constants.[6] ∎

[6] $\lim_{j\to\infty} j \cdot \lambda^{(j-1)} = \lim_{j\to\infty} \frac{j}{\lambda^{-(j-1)}} \left(\frac{\infty}{\infty}\right) = \lim_{j\to\infty} \frac{\frac{dj}{dj}}{\frac{d\lambda^{-(j-1)}}{dj}} = \lim_{j\to\infty} \frac{1}{\lambda^{-(j-1)}\ln\lambda} = 0.$

3 Lag Operators

3.1 Introduction

As we have defined that a stochastic process (or time series) is a sequence of random variables denoted by $\{X_t, t \in \mathcal{T}\}$. A time series *operator* transforms one time series into a new time series. It accepts as input a sequence such as $\{X_t, t \in \mathcal{T}\}$ and has an output a new sequence $\{Y_t, t \in \mathcal{T}\}$.

An example of a time series operator is the multiplication operator, represented as

$$Y_t = \beta X_t; \tag{13-16}$$

Although it is written exactly the same way as simply scalar multiplication, equation (13-16) is actually shorthand for an infinite sequence of multiplication, one for each date t. The operator multiplies whatever value x the random variable X takes on at any date t by some constant β to generate the value y for that date. Therefore, it is important to keep in mind that equation (13-16) has better be read as

$$\{Y_t = \beta X_t; \quad t \in \mathcal{T}\}. \tag{13-17}$$

A highly useful operator is the *lag operator*. Suppose that we start a time series $\{X_t, t \in \mathcal{T}\}$ and generate a new sequence $\{Y_t, t \in \mathcal{T}\}$ where the value of y for date t is equal to the value x took on at date $t-1$:

$$y_t = x_{t-1}.$$

This is described as applying the *lag operator* to $\{X_t\}$. The operator is represented by the symbol L:

$$Y_t = LX_t = X_{t-1}.$$

Consider the result of applying the lag operator twice to a series:

$$L(LX_t) = L(X_{t-1}) = X_{t-2}.$$

Such a double application of the lag operator is indicated by "L^2":

$$L^2 X_t = X_{t-2}.$$

In general, for any integer k,

$$L^k X_t = X_{t-k}.$$

Notice that if we first apply the multiplication operator and then the lag operator, as in:

$$X_t \to \beta X_t \to \beta X_{t-1},$$

the result will be exactly the same if we had applied the lag operator first and then the multiplication operator:

$$X_t \to X_{t-1} \to \beta X_{t-1}.$$

Thus the lag operator and multiplication operator are commutative:

$$L(\beta X_t) = \beta \cdot LX_t.$$

Similarly, if we first add two series and then apply the lag operator to the result,

$$(X_t, W_t) \to X_t + W_t \to X_{t-1} + W_{t-1},$$

the result is the same as if we had applied the lag operator before adding:

$$(X_t, W_t) \to (X_{t-1}, W_{t-1}) \to X_{t-1} + W_{t-1}.$$

Thus, *the lag operator is distributive over the addition operator*:

$$L(X_t + W_t) = LX_t + LW_t.$$

We therefore see that the lag operator **follows exactly the same algebraic rule as the multiplication operator**. For this reason, it is tempting to use the expression "multiply Y_t by L" rather than "operate on $\{Y_t; t \in \mathcal{T}\}$ by L".

Faced with a time series defined in terms of compound operators, we are free to use the standard *commutative*, *associative*, and *distributive* algebraic laws for multiplication and addition to express the compound operator in an alternative form. For example, the process defined by

$$Y_t = (a + bL)LX_t$$

is exactly the same as

$$Y_t = (aL + bL^2)X_t = aX_{t-1} + bX_{t-2}.$$

To take another example,

$$\begin{aligned}
(1 - \lambda_1 L)(1 - \lambda_2 L)X_t &= (1 - \lambda_1 L - \lambda_2 L + \lambda_1\lambda_2 L^2)X_t \\
&= (1 - [\lambda_1 + \lambda_2]L + \lambda_1\lambda_2 L^2)X_t \\
&= X_t - (\lambda_1 + \lambda_2)X_{t-1} + (\lambda_1\lambda_2)X_{t-2}.
\end{aligned}$$

An expression such as $(aL + bL^2)$ is referred to as a *polynomial in the lag operator*. It is algebraically to a simple polynomial $(az + bz^2)$ where z is a scalar. The difference is that the simple polynomial $(az + bz^2)$ refers to a particular number, whereas a polynomial in the lag operator $(aL + bL^2)$ refers to an operator that would applied to one time series $\{X_t; t \in \mathcal{T}\}$ to produce a new time series $\{Y_t; t \in \mathcal{T}\}$.

3.2 Solving First-Order Difference Equation by Lag Operator

Let now return to the first-order difference equation in section 1:

$$Y_t = \phi Y_{t-1} + W_t,$$

which can now be rewritten using lag operator as

$$Y_t = \phi L Y_t + W_t.$$

This equation, in turn, can be rearranged using standard algebra,

$$Y_t - \phi L Y_t = W_t$$

or

$$(1 - \phi L)Y_t = W_t. \tag{13-18}$$

3.2.1 The Case $\mathcal{T} = \{-1, 0, 1, 2, ...\}$, and an Initial Value Y_{-1} Is Given

We first consider "multiplying" both side of (13-18) by the following operator:

$$(1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t),$$

the result would be

$$(1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t) \times (1 - \phi L)Y_t = (1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t) \times W_t$$

or

$$(1 - \phi^{t+1} L^{t+1})Y_t = (1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t)W_t. \tag{13-19}$$

Writing (13-19) out explicitly produces

$$Y_t - \phi^{t+1}Y_{t-(t+1)} = W_t + \phi W_{t-1} + \phi^2 W_{t-2} + ... + \phi^t W_{t-t}$$

or

$$Y_t = \phi^{t+1}Y_{-1} + W_t + \phi W_{t-1} + \phi^2 W_{t-2} + ... + \phi^t W_0. \tag{13-20}$$

Notice that equation (13-20) is identical to equation (13-2). Applying the lag operator is performing the same set of recursive substitution that were employed in the previous section.

3.2.2 The Case $\mathcal{T} = \{... - 3, -2, -1, 0, 1, 2, 3, ...\}$ and t is large

It is interesting to reflect on the nature of the operator as t become large. We saw that

$$(1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t)(1 - \phi L)Y_t = Y_t - \phi^{t+1}Y_{-1}.$$

That is, $(1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t)(1 - \phi L)Y_t$ differs from Y_t by the term $\phi^{t+1}Y_{-1}$. If $|\phi| < 1$ *and if* Y_{-1} *is a finite number*, this residual $\phi^{t+1}Y_{-1}$ will be become negligible as t became large:

$$(1 + \phi L + \phi^2 L^2 + ... + \phi^t L^t)(1 - \phi L)Y_t \cong Y_t \quad for\ t\ large.$$

Definition.

A sequence $\{X_t; t \in \mathcal{T},\ \mathcal{T} = \{..., -2, -1, 0, 1, 2, ...\}\}$ is said to be *bounded* if there exists number $\bar{X}$ such that

$$|X_t| < \bar{X} \quad for\ all\ t.$$

■

Thus when $|\phi| < 1$ and when we are considering applying an operator to a bounded sequence,[7] we can think of

$$(1 + \phi L + \phi^2 L^2 + ... + \phi^j L^j)$$

[7] Remember that when $|\phi| < 1$, the stochastic process Y_t is stable.

as approximating the inverse of the operator $(1-\phi L)$, with this approximation made arbitrarily accurate by choosing j sufficiently large:

$$(1-\phi L)^{-1} = \lim_{j\to\infty}(1+\phi L+\phi^2L^2+....+\phi^jL^j).$$

This operator $(1-\phi L)^{-1}$ has the property

$$(1-\phi L)^{-1}(1-\phi L) = 1,$$

where "1" denotes the identity operator:

$$1Y_t = Y_t.$$

Provided that $|\phi|<1$ and we restrict ourselves to bounded sequence, the solution to the first-order difference equation would be

$$(1-\phi L)^{-1}(1-\phi L)Y_t = (1-\phi L)^{-1}W_t$$

or

$$\begin{aligned} Y_t &= (1-\phi L)^{-1}W_t \\ &= (1+\phi L+\phi^2L^2+....+\phi^iL^i+...)W_t \\ &= W_t+\phi W_{t-1}+\phi^2W_{t-2}+\phi^3W_{t-3}+..... \end{aligned}$$

3.3 Solving Second-Order Difference Equation by Lag Operator

Consider next a second-order difference equation:

$$Y_t = \phi_1Y_{t-1}+\phi_2Y_{t-2}+W_t.$$

Rewriting this in lag operator form produces

$$(1-\phi_1L-\phi_2L^2)Y_t = W_t. \tag{13-21}$$

The left side of (13-21) contains a second-order polynomial in the lag operator L. Suppose we factor this polynomial and find its roots. That is, find numbers γ_1 and γ_2 such that

$$(1-\phi_1L-\phi_2L^2) = (1-\gamma_1L)(1-\gamma_2L) = 0,$$

we obtains $L_1 = (\gamma_1)^{-1}$ and $L_2 = (\gamma_2)^{-1}$. Substituting both roots back the equation we should get the identity that

$$1 - \phi_1\gamma_1^{-1} - \phi_2\gamma_1^{-2} = 0$$

and

$$1 - \phi_1\gamma_2^{-1} - \phi_2\gamma_2^{-2} = 0.$$

Equivalently,

$$\gamma_1^2 - \phi_1\gamma_1^1 - \phi_2 = 0$$

and

$$\gamma_2^2 - \phi_1\gamma_2^1 - \phi_2 = 0,$$

theses two equation is the same calculation as in find the eigenvalues of $\mathbf{F}$ in (13-12) (That is, $\gamma_i = \lambda_i, i = 1, 2.$). This finding is summarized in the following result.

Result.

Factoring the polynomial

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \gamma_1 L)(1 - \gamma_2 L) \tag{13-22}$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$ in (13-12) when $p = 2$. (i.e. λ_1 and λ_2) λ_1 and λ_2 are the same as the parameters γ_1 and γ_2 in (13-22). ■

There are one source of possible semantic confusion about which we have to be careful. Recall from section 1 that the system is stable if both λ_1 and λ_2 are less than 1 in modulus and explosive if either λ_1 and λ_2 is greater than 1 in modulus. Some times this is described as the requirement that the roots of

$$\lambda^2 - \phi_1\lambda - \phi_2 = 0 \tag{13-23}$$

lie **inside the unit circle**.

The possible confusion is that it is often convenient to work directly with the polynomial in the lag operator in which it appears as

$$1 - \phi_1 L - \phi_2 L^2 = 0, \tag{13-24}$$

where roots, $L = (\lambda)^{-1}$.

Thus, we could say with equal accuracy that the difference equation is stable whenever the roots of (13-23) **lie inside the unit circle** or that the difference equation is stable whenever the roots of (13-24) **lie outside the unit circle**. The two statement mean exactly the same thing.

This note will follow the convention of using the term "eigenvalues" to refer to the roots of (13-23). Whenever the term "roots" is used, we will indicate explicitly the equation (13-24) whose roots are being described.

From now on this section, it is assumed that the second-order difference equation is stable, with the eigenvalue λ_1 and λ_2 distinct and both **inside the unit circle**. Where this is the case, the inverse

$$(1-\lambda_1 L)^{-1} = 1 + \lambda_1^1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + ...$$
$$(1-\lambda_2 L)^{-1} = 1 + \lambda_2^1 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + ...$$

are well defined for bounded sequence. Written the second-order difference in factored form:

$$(1-\lambda_1 L)(1-\lambda_2 L)Y_t = W_t$$

and operate on both side by $(1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}$:

$$Y_t = (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1} W_t. \tag{13-25}$$

Notice that an alternative way of writing the operator is:

$$\begin{aligned}
(\lambda_1-\lambda_2)^{-1}\left\{\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}\right\} &= (\lambda_1-\lambda_2)^{-1}\left\{\frac{\lambda_1(1-\lambda_2 L) - \lambda_2(1-\lambda_1 L)}{(1-\lambda_1 L)\cdot(1-\lambda_2 L)}\right\} \\
&= \frac{1}{(1-\lambda_1 L)\cdot(1-\lambda_2 L)}.
\end{aligned}$$

Thus, equation (13-25) can be written as

$$\begin{aligned}
Y_t &= (\lambda_1-\lambda_2)^{-1}\left\{\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}\right\} W_t \\
&= \left\{\frac{\lambda_1}{\lambda_1-\lambda_2}[1+\lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + ...] - \frac{\lambda_2}{\lambda_1-\lambda_2}[1+\lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + ...]\right\} W_t
\end{aligned}$$

or

$$Y_t = (c_1+c_2)W_t + (c_1\lambda_1 + c_2\lambda_2)W_{t-1} + (c_1\lambda_1^2 + c_2\lambda_2^2)W_{t-2} + ..., \tag{13-26}$$

where

$$\begin{aligned} c_1 &= \lambda_1/(\lambda_1 - \lambda_2) \\ c_2 &= -\lambda_2/(\lambda_1 - \lambda_2). \end{aligned}$$

From (13-26) the dynamic multiplier can be read off directly as

$$\frac{\partial Y_{t+j}}{\partial W_t} = c_1\lambda_1^j + c_2\lambda_2^j,$$

the same result arrived at in previous sections.

3.4 Solving pth-Order Difference Equations by Lag Operator

The techniques generalize in a straightforward way to a pth-order difference equation of the form:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + W_t,$$

and it can be written in terms of lag operator as

$$(1 - \phi_1 L - \phi_2 L^2 - .... - \phi_p L^p) Y_t = W_t.$$

Factorizing the polynomial in the lag operator as

$$(1 - \phi_1 L - \phi_2 L^2 - .... - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)....(1 - \lambda_p L)$$

we obtain the roots of this polynomial as $L_i = \lambda_i^{-1}, i = 1, 2, ..., p$. From the identity that for any roots of a polynomial

$$\begin{aligned} 1 - \phi_1\lambda_i^{-1} - \phi_2\lambda_i^{-2} - .... - \phi\lambda_i^{-p} &= 0 \\ \Rightarrow \lambda_i^p - \phi_1\lambda_i^{p-1} - .... - \phi_{p-1}\lambda_i - \phi_p &= 0, \end{aligned}$$

where λ_i are the eigenvalues of $\mathbf{F}$ as we defined before. Thus, the last **results** readily generalizes.

Result.

Factoring the polynomial

$$(1 - \phi_1 L - \phi_2 L^2 - .... - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)....(1 - \lambda_p L)$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$ in (13-12). The eigenvalue of $\mathbf{F}$, $(\lambda_1, \lambda_2, ..., \lambda_p)$ are the same as the parameters $(\lambda_1, \lambda_2, ..., \lambda_p)$ in (13-12). ■

Assuming that the eigenvalues are **inside** the unit circle and we are restricting ourselves to considering bounded sequence, the inverse $(1-\lambda_1 L)^{-1}$, $(1-\lambda_2 L)^{-1}$, ..., $(1-\lambda_p L)^{-1}$ all exist, permitting the difference equation

$$(1-\lambda_1 L)(1-\lambda_2 L)....(1-\lambda_p L)Y_t = W_t$$

to be written as

$$Y_t = (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}....(1-\lambda_p L)^{-1}W_t.$$

The dynamic multiplier can be read directly as (13-15) to be

$$\frac{\partial Y_{t+j}}{\partial W_t} = c_1\lambda_1^j + c_2\lambda_2^j + ... + c_p\lambda_p^j.$$

3.5 Unbounded Sequences

As we have shown that given a first-order difference equation in the lag operator form:

$$(1-\phi L)Y_t = W_t. \tag{13-27}$$

When $|\phi| < 1$, it is advised to solve the equation "backward" by

$$(1-\phi L)^{-1} = (1 + \phi L + \phi^2 L^2 + ...).$$

However, when $|\phi| > 1$, Sargent (1987) advice to solve the equation "forward" by

$$\begin{aligned}(1-\phi L)^{-1} &= \frac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}}, \ \ (since\ for\ |\phi| > 1,\ |\phi^{-1}| < 1)\\ &= -\phi^{-1}L^{-1}(1 + \phi^{-1}L^{-1} + \phi^{-2}L^{-2} + ...),\end{aligned}$$

where as defined, $L^{-k}Y_t = Y_{t+k}$.[8] We expression (13-27) as

$$\begin{aligned} Y_t &= (-\phi^{-1}L^{-1} - \phi^{-2}L^{-2} - \phi^{-3}L^{-3} + ...)W_t \\ &= -\phi^{-1}W_{t+1} - \phi^{-2}W_{t+2} - \phi^{-3}W_{t+3} + ... \end{aligned}$$

In a economic agent with (rational) expectation, the price at date t depends possibly on future price (expected) at date $t+k, k > 0$.

Notation:

Y_t: random variable,

y_t: the value of the random variable Y_t take,

$\mathbf{y}_t$: a random vector,

$\mathbf{Y}_t$: a random matrix.

[8] Here,

$$\begin{aligned} (1-\phi L)^{-1} \times (1-\phi L) &= \{\lim_{j\to\infty}[-\phi^{-1}L^{-1}(1+\phi^{-1}L^{-1}+\phi^{-2}L^{-2}+...+\phi^{-j}L^{-j})] \cdot [1-\phi L]\} \\ &= \lim_{j\to\infty}[-\phi^{-1}L^{-1}][(1+\phi^{-1}L^{-1}+\phi^{-2}L^{-2}+...+\phi^{-j}L^{-j}) \\ &\quad -(\phi L + 1 + \phi^{-1}L^{-1} + \phi^{-2}L^{-2} + ... + \phi^{-j+1}L^{-j+1})] \\ &= \lim_{j\to\infty}(-\phi^{-1}L^{-1})(-\phi L + \phi^{-j}L^{-j}) \\ &= \lim_{j\to\infty}[1-\phi^{-(j+1)}L^{-(j+1)}] \\ &= \lim_{j\to\infty}[1-(\phi^{-1})^{(j+1)}L^{-(j+1)}] \\ &= 1 \quad \textit{since} \ \ |\phi^{-1}| < 1. \end{aligned}$$