# Exercice 9

1. Montrons que pour tout $n \in \mathbb{N}$ et tout $p \in \mathbb{Z}$

$$p\binom{n}{p} = n\binom{n-1}{p-1}$$

$$p\binom{n}{p} = p\left(\frac{n!}{p!\,(n-p)!}\right) = \frac{n!}{(p-1)!\,(n-p)!}$$

$$p\binom{n}{p} = n\,\frac{(n-1)!}{(p-1)!\,[(n-1)-(p-1)]!}$$

or $$\frac{(n-1)!}{(p-1)!\,[(n-1)-(p-1)]!} = \binom{n-1}{p-1}$$

donc $$p\binom{n}{p} = n\binom{n-1}{p-1}.$$

2. Calculons pour tout $n$

$$S_0 = \sum_{p=0}^{n}\binom{n}{p} \; ; \; S_1 = \sum_{p=1}^{n} p\binom{n}{p} \; ; \; S_2 = \sum_{p=0}^{n} p^2\binom{n}{p}$$

$$S_0 = \sum_{p=0}^{n} \binom{n}{p}$$

$$S_0 = \sum_{p=0}^{n} \binom{n}{p} 1^p \times 1^{n-p}$$

$$S_0 = (1+1)^n = 2^n$$

$$\underline{\underline{S_0 = 2^n}}$$

$$S_1 = \sum_{p=0}^{n} p \binom{n}{p} = \sum_{p=1}^{n} p \binom{n}{p}$$

or $p \binom{n}{p} = n \binom{n-1}{p-1}$

donc $S_1 = \displaystyle\sum_{p=1}^{n} n \binom{n-1}{p-1}$

d'où $S_1 = n \left( \displaystyle\sum_{p=1}^{n} \binom{n-1}{p-1} \right)$

$$S_1 = n \left[ \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-2} + \binom{n-1}{n-1} \right]$$

23 $\quad S_1 = n \left( \displaystyle\sum_{p=0}^{n-1} \binom{n-1}{p} \right)$

$$S_1 = n \times 2^{n-1} \quad \left(\text{car } \sum_{p=0}^{n-1} \binom{n-1}{p} = 2^{n-1}\right)$$

$$\underline{\underline{S_1 = n\, 2^{n-1}}}$$

$$S_2 = \sum_{p=1}^{n} p^2 \binom{n}{p} = \sum_{p=1}^{n} p\left(p\binom{n}{p}\right)$$

or $p\binom{n}{p} = n\binom{n-1}{p-1}$

donc $S_2 = \displaystyle\sum_{p=1}^{n} p\, n \binom{n-1}{p-1}$

d'où $S_2 = n\left(\displaystyle\sum_{p=1}^{n} p\binom{n-1}{p-1}\right)$

or $\displaystyle\sum_{p=1}^{n} p\binom{n-1}{p-1} = 1\binom{n-1}{0} + 2\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1}$

$$= \sum_{p=0}^{n-1} (p+1)\binom{n-1}{p}$$

24 donc $S_2 = n\left[\displaystyle\sum_{p=0}^{n-1} (p+1)\binom{n-1}{p}\right]$

$$S_2 = n\left[\sum_{p=0}^{n-1} p\binom{n-1}{p} + \sum_{p=0}^{n-1}\binom{n-1}{p}\right]$$

or $\displaystyle\sum_{p=0}^{n-1} p\binom{n-1}{p} = (n-1)\,2^{n-2}$ et $\displaystyle\sum_{p=0}^{n-1}\binom{n-1}{p} = 2^{n-1}$

alors $S_2 = n\left((n-1)\,2^{n-2} + 2^{n-1}\right)$

d'où $\underline{\underline{S_2 = n(n-1)\,2^{n-2} + n\,2^{n-1}}}$