Université Nangui-Abrogoua

Année Universitaire 2019-2020

UFR - SFA

Licence 2 - MI

TD d'ARITHMETIQUE

Exercice 1

Soit n un entier naturel non nul et $q \in \mathbb{R}$ ou $\mathbb{C}$.

On pose : $(n)_q = 1 + q + \cdots + q^{n-1}$, $(n!)_q = (1)_q (2)_q \cdots (n)_q$.

$$\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q\,((n-k)!)_q} \quad \text{et} \quad (0!)_q = 1.$$

a) Montrons que : $(n!)_q = \dfrac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$, avec $q \neq 1$.

Comme $(n)_q = 1 + q + \cdots + q^{n-1} = \sum_{k=0}^{n-1} q^k$ et $q \neq 1$,

alors $(n)_q = \dfrac{1-q^n}{1-q} = \dfrac{q^n-1}{q-1}$.

Donc $(n!)_q = \left(\dfrac{q-1}{q-1}\right)\left(\dfrac{q^2-1}{q-1}\right)\cdots\left(\dfrac{q^n-1}{q-1}\right) = \dfrac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$

b) Montrons que : $\binom{n}{k}_q = \binom{n}{n-k}_q$.

On sait que : $\begin{pmatrix} n \\ k \end{pmatrix}_q = \dfrac{(n!)_q}{(k!)_q \, ((n-k)!)_q}$

or $k = n - (n-k)$;

alors $\begin{pmatrix} n \\ k \end{pmatrix}_q = \dfrac{(n!)_q}{([n-(n-k)]!)_q \, ((n-k)!)_q}$

d'où $\begin{pmatrix} n \\ k \end{pmatrix}_q = \dfrac{(n!)_q}{((n-k)!)_q \, ([n-(n-k)]!)_q}$

ainsi $\begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n \\ n-k \end{pmatrix}_q$

c) <u>Montrons que</u> : $\begin{cases} \begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix}_q \\ \begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n-1 \\ k \end{pmatrix}_q + q^{n-k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q \end{cases}$

(*) $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \dfrac{((n-1)!)_q}{((k-1)!)_q \, ((n-k)!)_q} + q^k \dfrac{((n-1)!)_q}{(k!)_q \, ((n-k-1)!)_q}$

or $(n!)_q = \dfrac{(q-1)(q^2-1)\dots(q^{n-1}-1)}{(q-1)^{n-1}} \times \dfrac{q^n-1}{q-1} = ((n-1)!)_q \dfrac{q^{n-1}-1}{q-1}$

donc $((n-1)!)_q = \dfrac{q-1}{q^n-1}(n!)_q$;

de même, on a: $((k-1)!)_q = \dfrac{q-1}{q^k-1}(k!)_q$

ainsi $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \dfrac{\left(\frac{q-1}{q^n-1}\right)(n!)_q}{\left(\frac{q-1}{q^k-1}\right)(k!)_q((n-k)!)_q} + q^k \dfrac{\left(\frac{q-1}{q^n-1}\right)(n!)_q}{(k!)_q\left(\frac{q-1}{q^{n-k}-1}\right)((n-k)!)_q}$

$\Rightarrow \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \left(\dfrac{q^k-1}{q^n-1}\right) \dfrac{(n!)_q}{(k!)_q((n-k)!)_q} + q^k \left(\dfrac{q^{n-k}-1}{q^n-1}\right) \dfrac{(n!)_q}{(k!)_q((n-k)!)_q}$

$\Rightarrow \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \left(\dfrac{q^k-1}{q^n-1}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q + \left(\dfrac{q^n-q^k}{q^n-1}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q$

$\Rightarrow \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \left(\dfrac{q^k-1}{q^n-1} + \dfrac{q^n-q^k}{q^n-1}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q$

$\Rightarrow \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \left(\dfrac{q^n-1}{q^n-1}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n \\ k \end{pmatrix}_q .$

3

$$(**)\quad \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-2}_q = \frac{((n-1)!)_q}{(k!)_q\,((n-k-1)!)_q} + q^{n-k}\frac{((n-1)!)_q}{((k-1)!)_q\,((n-k)!)_q}$$

$$\Rightarrow \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \frac{\left(\frac{q-1}{q^n-1}\right)(n!)_q}{(k!)_q\left(\frac{q-1}{q^{n-k}-1}\right)((n-k)!)_q} + q^{n-k}\frac{\left(\frac{q-1}{q^n-1}\right)(n!)_q}{\left(\frac{q-1}{q^k-1}\right)(k!)_q((n-k)!)_q}$$

$$\Rightarrow \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \left(\frac{q^{n-k}-1}{q^n-1}\right)\frac{(n!)_q}{(k!)_q\,((n-k)!)_q} + q^{n-k}\left(\frac{q^k-1}{q^n-1}\right)\frac{(n!)_q}{(k!)_q((n-k)!)_q}$$

$$\Rightarrow \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \left(\frac{q^{n-k}-1}{q^n-1}\right)\binom{n}{k}_q + \left(\frac{q^n-q^{n-k}}{q^n-1}\right)\binom{n}{k}_q$$

$$\Rightarrow \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \left(\frac{q^{n-k}-1}{q^n-1} + \frac{q^n-q^{n-k}}{q^n-1}\right)\binom{n}{k}_q$$

$$\Rightarrow \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \left(\frac{q^n-1}{q^n-1}\right)\binom{n}{k}_q$$

Donc $\binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = \binom{n}{k}_q$