A New Linear Programming Approach to the Cutting Stock Problem

HARALD DYCKHOFF
Fernuniversität Hagen, Federal Republic of Germany
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A new approach to the one-dimensional cutting stock problem is described and compared to the classical model for which Gilmore and Gomory have developed a special column-generation technique. The new model is characterized by a dynamic use of simply structured cutting patterns. Nevertheless, it enables the representation of complex combinations of cuts. It can be advantageous in practical applications where many different stock lengths or a relatively large number of order lengths have to be dealt with. The new approach is applied to a real problem where the "trim loss" is not valueless, since it can be used for further demands arising in later planning periods.

CUTTING (STOCK) PROBLEMS rank among the first applications of operations research methods (see Kantorovich [1960], Eisemann [1957], and Förstner [1959]) and continue to arouse considerable interest. (For example, six of the contributions at the 1980 EURO IV meetings in Cambridge, England were cutting problems.) The following problem may be designated as the *Standard Problem* of the one-dimensional cutting stock optimization (compare Gilmore and Gomory [1961]):

An unlimited number of pieces of a stock with different *standard lengths* (*stock lengths*) s_k, $k = 1, \cdots, K$, is to be cut lengthwise so that at least n_i pieces of the *order lengths* (*demand lengths*) d_i, $i = 1, \cdots, I$, are produced and a given objective function is optimized.

(For example, the problem may be to cut tubes or rolls of paper, plastic film, cellophane, or steelplate.) The Standard Problem is solvable if for one k and all i: $s_k \geq d_i$.

The feature common to all papers is that each possible cutting pattern is identified by one variable. Dropping integrality conditions leads to linear programming (LP) problems characterized by an enormous number of potential variables and a small number of constraints. In order to solve these LP-problems, Gilmore and Gomory [1961, 1963, 1965] have developed a special technique similar to the "column-generation procedures" known from the Dantzig-Wolfe decomposition principle. Considering

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knapsack problems at each stage of the revised simplex method one obtains improved cutting patterns; furthermore, only a few columns need to be generated.

The algorithms of Gilmore and Gomory have successfully been applied to a broad class of cutting problems (compare Lasdon [1970], p. 207). There are, however, many real problems for which these algorithms are not appropriate due to their structure or size. In many cases, this is caused by special restrictions, be it a constraint on the maximum number of each product type (Christofides and Whitlock [1977]), or the permission of only a few distinct cutting patterns (Haessler [1971], Coverdale and Warton [1976]). For such problems, special methods—often heuristic ones—are used. (For a survey, see Hinxman [1980].)

One such heuristic procedure, proposed in Gehring et al. [1979] and similarly in Heicken and Koenig [1980] is the a priori generation of a small number of "good" cutting patterns. Thus, the corresponding linear program can be solved with a general LP-code. In this case only suboptimal solutions are obtained. The problem in Heicken and Koenig is characterized by a number of different demand sizes which is large in comparison with the largest stock size. Furthermore in Gehring et al., some of the "trim loss" has value for further demands arising in later planning periods. Hence, the problem in Gehring et al. features many stock sizes and an objective function containing the value of the "trim loss." (In both Gehring et al. and Heicken and Koenig, the methods of Gilmore and Gomory have not been applied. Nevertheless, it is possible to use them for these problems as pointed out by one of the referees.)

In this paper we propose a new LP-approach to cutting stock problems. This approach has been developed to obtain an optimal solution for the problem in Gehring et al., but it is also useful for other problems, especially those of the type described in Heicken and Koenig.

To facilitate a comparison of our new approach to the classical approach, both are illustrated for the Standard Problem. In the classical case we obtain the well known LP-model referred to here as "Model *I*" (Section 1). The new approach leads to a different LP-model which we call "Model *II*" (Section 2). Both are illustrated by an example (Section 3). A comparison of the two models by theoretical considerations yields an initial idea of their essential properties (Section 4). Special advantage can be taken of the structural differences of both approaches when applied to real cutting problems deviating from the Standard Problem, e.g. the problem described in Gehring et al. (Section 5).

1. CLASSICAL APPROACH: MODEL *I*

The *classical approach* to cutting problems is characterized by its definition of a cutting pattern. For the Standard Problem, the jth *cutting pattern* (k, j) for length s_k is determined by a vector $(a_{kj1}, \cdots, a_{kjI})$ in the following way: divide a piece of length s_k into a_{kj1} pieces of length d_1, a_{kj2} pieces of length d_2, etc., with a remainder of length r_{kj}. Hence, the set of all possible patterns for length s_k equals the set of solutions $(a_{k1}, \cdots, a_{kI})$ to the restrictions:

$$\textstyle\sum_{i=1}^{I} d_i a_{ki} \leq s_k;\ a_{ki} \in \{0, 1, 2, \cdots\}, \qquad i = 1, \cdots, I. \tag{1}$$

The number $M_k = M(s_k; d_1, \cdots, d_I)$ of possible cutting patterns (k, j) for the standard length s_k and given order lengths d_i is finite but may be very large: "For example, with a standard roll of 200 in. and demands for 40 different lengths ranging from 20 in. to 80 in., the number of cutting patterns can easily exceed 10 or even 100 million. Problems of these sizes are often encountered in practice." (See Gilmore and Gomory [1963], p. 865; compare Comtet [1974], p. 94.)

Let x_{kj} denote how often pattern (k, j) is used and the coefficient c_{kj} indicate the costs incurred by cutting a length according to pattern (k, j). If we choose $c_{kj} = s_k$, the goal is *input minimization*; if $c_{kj} = r_{kj}$, it is *trim loss minimization*. With these definitions *Model I* becomes:

$$\text{Minimize } z = \textstyle\sum_{k=1}^{K} \sum_{j=1}^{M_k} c_{kj} x_{kj} \tag{2}$$

$$\text{subject to} \quad \textstyle\sum_{k=1}^{K} \sum_{j=1}^{M_k} a_{kji} x_{kj} \geq n_i, \qquad i = 1, \cdots, I, \tag{3}$$

$$\text{and } x_{kj} \geq 0,\ k = 1, \cdots, K, \qquad j = 1, \cdots, M_k. \tag{4}$$

(Compare above to Lasdon, p. 207.) In view of the difficulties otherwise involved, we only consider the LP-relaxation obtained by neglecting the restrictions that the variables x_{kj} have to be integer. In many cases, especially in those with high demands, the optimal fractional values are large enough so that usually little is lost in rounding to integers.

Model *I* consists of *I* rows and $\sum_{k=1}^{K} M_k$ columns. In order to solve this LP-problem it is not necessary to formulate all potential cutting patterns explicitly. Starting with a (simple) feasible solution one can generate improved cutting patterns within the frame work of the revised simplex method. At each Pivot step *K* knapsack problems:

$$\text{Maximize} \quad \textstyle\sum_{i=1}^{I} \pi_i a_{ki} \qquad \text{subject to (1).} \tag{5}$$

are considered where π_i is the actual simplex multiplier for the ith constraint (3). Any solution with a negative reduced cost coefficient $\bar{c}_{kj} < 0$, $\bar{c}_{kj} = c_{kj} - \sum_{i=1}^{I} \pi_i a_{kji}$, determines a possible, new basic variable. For the knapsack problems (5), Gilmore and Gomory [1961, 1963, 1965] have developed special algorithms which have successfully been applied to many cutting problems (compare to Lasdon, p. 209).

2. A NEW APPROACH: MODEL *II*

In looking for a real cutting process which would best represent Model *I*, one could imagine the *simultaneous cutting of all pieces with an* *unlimited number of knives.* In the following, however, we have in mind a different cutting process: *cutting with only one knife but in an unlimited sequence of cutting operations.* In detail, the assumptions on the cutting process are:

(A) The number of cutting operations is not limited.
(B) In each cutting operation only two new, shorter pieces are produced, at least one of which corresponds to an order length.
(C) Residual pieces can be further divided in subsequent cutting operations.
(D) Pieces of a given length are not distinguished from each other; to be more precise, it does not matter whether they are cut from a piece of larger length or whether they are already in stock as a standard length.
(E) Standard lengths are available in any quantity.
(F) The demands for order lengths must be satisfied.
(G) The production costs are linearly dependent on the net consumption of standard lengths.
(H) All order lengths are shorter than the largest standard length ($d_i < s_{\max}$ for all i).
(I) The smallest standard length is larger than the smallest order length ($s_{\min} > d_{\min}$), and no order length is identical with a standard length ($d_i \neq s_k$ for all i and k).

Assumptions (H) and (I) are purely technical and have been established to exclude trivial cases. Assumptions (E) and (F) are always correct for Model *I*, too; as far as the assumptions (D) and (G) are concerned, the objective function coefficients in (2) would have to be adapted accordingly. The essential differences from the classical approach result from assumptions (A)–(C).

Assumptions (A)–(C) allow consideration of cutting patterns of a very simple structure $[k;\, l]$ only, where l and k-l are the lengths of the two sections into which the length k is divided and where, without loss of generality, l is an order length. We call them "one-cut-patterns" or "*one-cuts*" in short.

Despite this simple structure, any complicated cutting combination (like the patterns of Model *I*) can be constructed successively. For example, dividing a standard length 9 into one piece of order length 4, two pieces of order length 2 and a residual length 1 can be represented by using the one-cuts [9; 4], [5; 2] and [3; 2]. An equivalent solution can be obtained by [9; 6], [6; 4] and [3; 2]. Owing to assumption (B), however, the one-cut [9; 6] is considered only if 6 (or 3) is another order length. One-cut [6; 4] is an example of the fact that both sections have an order length. Hence, the one-cuts [6; 4] and [6; 2] are identical; in such cases only one of the two one-cuts needs to be considered.

In practice, standard and order lengths are measured by rational numbers. Therefore, there always exists a standardized measure such that all appearing lengths are subsets of the set of positive integers $\mathbb{N}$. For the Standard Problem let S be the set of all standard lengths $l \in \{s_1, \cdots, s_K\} \subset \mathbb{N}$, D be the set of all order lengths $l \in \{d_1, \cdots, d_I\} \subset \mathbb{N}$, and R be the set of all residual lengths $l \in \mathbb{N}$ which are not shorter than the shortest demand length $d_{\min} \in D$ and which can be produced by one-cuts with the standard and order lengths S and D being given.

Because of assumption (I): $S \cap D = \phi$. The set R of *relevant residual lengths* can be determined by a simple algorithm. For example, for standard lengths 9, 6, and 5 and order lengths 2, 3 and 4 one obtains 7, 6, 5, 4, 3, and 2 as relevant residual lengths.

Let us further assume that the following data of the cutting problem are given:

N_l = level of demand (number of pieces) for length l (for $l \notin D$: $N_l = 0$; for $l \in D$ with $l = d_i$: $N_l = n_i$),

c_l = the cost of consumption of one piece of the standard length $l \in S$.

Then our approach to the Standard Problem results in the following (integer) linear program that we call *Model II.* The variables are

$y_{k,l}$ = number of pieces of length k each of which is divided into a section of the order length $l \in D$, $l < k$, and a section of the residual length $k - l$,

i.e., $y_{k,l}$ indicates how often pieces of the standard or residual length k are divided according to the one-cut $[k; l]$. Again neglecting the restrictions to integers we have

$$y_{k,l} \geqq 0, \qquad k \in S \cup R; \qquad l \in D, \qquad l < k. \tag{6}$$

The other constraints of Model *II* result from the conditions that the quantitative output per length l, where l is an order and/or a residual length, cannot be larger than the input:

$$\begin{aligned} &\textstyle\sum_{k \in A_l} y_{k,l} + \sum_{k \in B_l} y_{k+l,k} \\ &\qquad \geqq \textstyle\sum_{k \in C_l} y_{l,k} + \mathrm{N}_l, \quad \text{for all} \quad l \in (D \cup R) \setminus S. \end{aligned} \tag{7}$$

(Note that $\sum_\phi := 0$.) These "balance constraints" need not be formulated for the standard lengths since they are available in any number of pieces due to assumption (E). The left side of each of the inequalities (7) indicates the input as the number of newly generated pieces of length l, either as order length $l \in D$ from the one-cut $[k; l]$ when cutting a larger standard or residual length k, i.e.,

$$A_l := \{k \in S \cup R \mid k > l\} \quad \text{with} \quad A_l := \phi \quad \text{for} \quad l \notin D,$$

or as residual length l of the one-cut $[k + l; k]$ which is produced by dividing the standard or residual length $k + l$ into the order length k, i.e.,

$$B_l := \{k \in D \mid k + l \in S \cup R\}.$$

The right side of each of the inequalities (7) indicates those parts of the output of length l (in number of pieces) either being further cut in order to satisfy the demands for shorter lengths k, i.e.,

$$C_l := \{k \in D | k < l\},$$

or directly meeting the demand for length l at the level N_l.

The slack variables of the inequalities (7) represent the number of pieces of the length l which are neither consumed in further cuttings nor used to satisfy the demand. Hence they represent the "trim loss." Their explicit formulation in (7) leads to *balance equations*. Since there are no requirements that have to be met by the trim loss, it is not necessary to formulate any constraints for lengths l which are shorter than the shortest order length $d_{\min}$.

If the total costs of the net consumption of standard lengths are to be minimized, Model *II* as a whole becomes:

$$\text{Minimize} \quad Z = \textstyle\sum_{l \in S} c_l \left(\sum_{k \in C_l} y_{l,k} - \sum_{k \in B_l} y_{k+l,k}\right) \qquad (8)$$

subject to the constraints (6) and (7).

The *net consumption* per standard length $l \in S$ results from the number of pieces l which are further cut in order to produce shorter order lengths $k \in C_l$, less the number of pieces which are obtained as residual pieces l in this standard length by cutting larger lengths $k + l$ with $k \in B_l$.

3. AN EXAMPLE

Example. Let 5, 6, and 9 be standard lengths with the respective unit costs 6, 7, and 10. These lengths are to be cut in such a way that the demand for 20 pieces of length 2, 10 pieces of length 3, and 20 pieces of length 4 is satisfied, total costs being kept as low as possible.

This small example was given by Gilmore and Gomory ([1961], p. 856) to illustrate their algorithm. In this case, Model *I* has 3 constraints and 32 variables. Gilmore and Gomory generate 6 columns only. The optimal solution is: "... to cut each of 10 pieces of stock length 6 into 1 piece of length 4 and 1 piece of length 2 and each of 10 pieces of the stock length 9 into 1 piece of length 2, 1 piece of length 3, and 1 piece of length 4. The cost is 170. That integers should result as the solution of the example is, of course, fortuitous" (Gilmore and Gomory [1961], p. 858).

The coefficients of the objective function (2) are chosen as: $c_{kj} = c_l$ for all patterns (k, j) of a standard length s_k (with $l = s_k$). This choice deviates from our assumptions (D) and (G). Gilmore and Gomory regard residual pieces of standard length as useless trim. If assumptions (D) and (G) are changed to comply with their intentions, then the appropriate objective function of Model *II* to be minimized is

$$\tilde{Z} = \textstyle\sum_{l \in S} c_l \cdot \max\{0, \sum_{k \in C_l} y_{l,k} - \sum_{k \in B_l} y_{k+l,k}\}$$

instead of (8). This change leads to a modified Model *II* with additional $K-1$ equality constraints for the standard lengths s_k (smaller than $s_{\max}$, the largest standard length) and additional $2(K-1)$ variables. For the above example, we have to solve a LP-problem of 6 constraints (for the lengths 2 to 7) and 16 non-negative variables (as relaxation of the integer problem). An optimal solution to this linear program yields the same cost (170) and the same cutting patterns as described above.

If we do not change the assumptions (D) and (G), the small example can be stated in terms of Model *II* as follows:

$$S = \{5, 6, 9\} \quad \text{with} \quad c_5 = 6, \qquad c_6 = 7, \qquad c_9 = 10 \quad \text{and}$$

$$D = \{2, 3, 4\} \quad \text{with} \quad N_2 = 20, \qquad N_3 = 10, \qquad N_4 = 20.$$

(In order to make the results comparable, we do not alter the coefficients c_l although a larger standard length should have a larger unit value than a shorter one because of the greater flexibility of use.) Relevant residual lengths are $R = \{7, 6, 5, 4, 3, 2\}$. Owing to their identity with [7; 4], [6; 4] and [5; 3] the one-cuts [7; 3], [6; 2], [5; 2] can be deleted in the formulation of Model *II*; hence the following LP-relaxation consisting of 4 constraints and 12 variables results:

$$\begin{aligned} \text{Minimize } Z = {} & 4\,y_{9,4} + 3\,y_{9,3} + 10\,y_{9,2} + 7\,y_{6,4} + 7\,y_{6,3} \\ & + 6\,y_{5,4} + 6\,y_{5,3} - 6\,y_{7,2} \\ \text{subject to } & y_{9,2} - y_{7,4} - y_{7,2} \geqq 0 \\ & y_{9,4} + y_{7,4} + y_{6,4} + y_{5,4} - y_{4,3} - y_{4,2} \geqq 20 \\ & y_{9,3} + y_{7,4} + 2\,y_{6,3} + y_{5,3} + y_{4,3} - y_{3,2} \geqq 10 \\ & y_{9,2} + y_{7,2} + y_{6,4} + y_{5,3} + 2\,y_{4,2} + y_{3,2} \geqq 20 \\ & y_{k,l} \geq 0 \quad \text{for all one-cuts} \quad [k; l]. \end{aligned}$$

An optimal solution to this linear program yields the following, fortuitously integer values:

$$y_{9,2} = 10, \qquad y_{7,2} = 10, \qquad y_{9,4} = 20, \qquad y_{9,3} = 10, \qquad z = 150.$$

Consequently, only pieces of the standard length 9 are cut; in detail: each of 10 pieces into 2 pieces of length 2 and a residual piece of length 5, each of 20 pieces into 1 piece of length 4 and a residual piece of length 5, as well as each of 10 pieces into 1 piece of length 3 and a residual piece of length 6. The stock of standard length 9 is reduced by 40 pieces in total while costs of 400 arise; at the same time the stock of standard lengths 5 and 6 is increased by 30 and 10 pieces, respectively, so that net costs amount to $400 - 180 - 70 = 150$ only.

If one wished to solve the above example according to assumptions (D) and (G) but using Model *I*, the coefficients c_{kj} of the objective function (2) would have to be differentiated to take account of the fact that residual pieces of standard lengths will reduce costs. For this purpose, however, the algorithms of Gilmore and Gomory [1961, 1963] would have to be modified (compare with Stainton [1977], p. 140).

4. COMPARISON OF BOTH MODELS

Model *I* and Model *II* are now compared with regard to various criteria such as: set of feasible solutions, optimal solution, model size, computational effort, special properties, and possible extensions.

Set of Feasible Solutions

As far as the Standard Problem is concerned both approaches comprise all relevant cutting combinations, although in a different way: Model *I* implies the idea of onestage cutting with any number of knives, Model *II* the idea of cutting with only one knife as often as required. Evidently, *both models lead to (integer) linear programs with equivalent sets of feasible solutions.*

Optimal Solution

Both approaches can be combined with the usual goal criteria such as minimization of cost, input, or trim loss: Model *I* by choosing the coefficients of the objective function correspondingly, Model *II* by a modified objective function. *If the objective functions of both models are formulated accordingly, both models yield to the same optimal solutions for the Standard Problem.*

Model Size

Model *I* and Model *II* represent the Standard Problem by equivalent (integer) linear programs. But they differ in their "dimensions." *Generally, Model II contains more constraints and clearly fewer variables than Model I. For sufficiently complex problems,*

- —The *number of constraints* increases from the number of order lengths to the number of relevant residual and order lengths which are not identical with a standard length, and
- —The *number of variables* decreases from the number of possible cutting patterns for each standard length to the number of possible one-cuts for each standard or residual length.

It is impossible to give precise, universally valid estimates of the dimensions of the two models since they are not only dependent on the number of standard and order lengths but also, to a great extent, on the actual relations of the sizes of the individual lengths. A rough estimate of the *dimensions of Model II* can be made by proceeding as follows: Let all standard lengths $s \in S$ and all order lengths $d \in D$ be given as positive integers having no common divisor, and let $s_{\max}{:} = \max\{s|s \in S\}$ be the largest standard length and $d_{\min}{:} = \min\{d|d \in D\}$ the smallest order length. Then, the number of constraints of Model *II* is definitely smaller than $s_{\max} - 2\ d_{\min}$, and the number of variables is smaller than $I \cdot (K + s_{\max} - 2\ d_{\min})$, $K = |S|$ being the number of standard lengths and $I = |D|$ the number of order lengths.

In order to test their algorithm, Gilmore and Gomory ([1963], p. 869) used "typical test problems" with order lengths not smaller than 21.50 in., standard lengths of 218 in. at the most, and ¼ in. as unit of measure. For such problems, Model *II* contains less than 700 constraints and less than $(700 + |S|) \cdot |D|$ variables, whereas Model *I* has $|D|$ constraints and usually millions of variables.

Gilmore and Gomory ([1963], p. 873) explicitly quote an example with $|D| = 30$ order lengths and *one* standard length of 218 in. In this case, Model *II* has 528 constraints and 12,300 variables. Introducing additional standard lengths of e.g. 200, 180, 150, 120, and 100 in. we obtained 535 constraints and 12,660 variables. Further 20 order lengths, randomly chosen between 21.50 in. and 218 in. with ¼ in. as unit of measure, yielded 574 constraints and 22,838 variables. When, in the last example including 6 standard lengths and 50 order lengths, the sizes of the order lengths were (randomly) changed so that ½ in. became the unit of measure then the size of Model *II* was reduced to 319 constraints and 13,144 variables.

Computational Effort

As far as Model *I* and the Gilmore and Gomory method are concerned, a larger number of standard and order lengths correspondingly increases the number of knapsack problems (5) and constraints (3), respectively. Changing the unit of measure alters the complexity of the knapsack problems. Hence, the examples quoted show that for a sufficiently complex cutting problem the effort necessary to find a solution increases with each further standard length $s < s_{\max}$ and each further order length $d > d_{\min} (d < s_{\max})$ to a greater extent with respect to Model *I* than to Model *II*. In the hypothetical, extreme case that all lengths, ranging from the smallest order length $d_{\min}$ to one single standard length $s_{\max}$, are all order lengths both models have a nearly equal number of constraints; with each additional standard length, smaller than $s_{\max}$, the problem size still increases as to Model *I* whereas it continues to decrease as far as Model *II* is concerned.

On the basis of these theoretical considerations it is reasonable to state the following *hypothesis*:

Model II will prove to be more favorable than Model I

(i) *If many standard lengths* (*stock sizes*) *exist and/or*

(ii) *If the set of relevant residual lengths is not of much larger cardinality than the set of order lengths (demand sizes), e.g. if* $|D|/s_{max}$ *tends to one.*

This hypothesis has to be tested by comparative runs on computers. Especially, the rather fuzzy assertions (i) and (ii) should be sharpened by more precise criteria indicating in which cases the one model needs less computational effort than the other. Such experience is not yet available. For their "typical test problems" with 20 to 50 order lengths and *one* standard length, Gilmore and Gomory [1963] obtained very good computer times with their algorithm so that Model *II* would probably come off worse in these instances. Nevertheless, there are real problems for which the classical approach seems to be not as efficient as the new one. (Compare Section 5 and the following.)

As a real example for case (ii) of our hypothesis consider the cutting problem (Heicken and Koenig) of a firm producing central heating radiator units. Tubes of the order lengths $D = \{5, 6, 7, \cdots, 46\}$ are to be cut from tubes of one standard length $s = 98$ so that weekly demands are satisfied and input is minimized. (Heicken and Koenig did not apply the Gilmore-Gomory method; they use Model *I* formulated for only 200 "meaningful," a priori chosen cutting patterns.) In order to obtain an optimal solution Model *II* is an useful approach for the following reasons: it consists of 89 constraints—instead of 42 for Model *I*—and less than 3,000 variables. *Once having been generated, the LP-matrix would not have to be revised;* only the "right-hand side" of the linear program would have to be modified every week according to the actual demands. With the demands showing a typical behavior (higher demands for odd sizes and for sizes smaller than 30) an optimal solution could be determined in a short computing time.

Special Properties of Model *II*

It is an advantage of Model *II* that in treating cutting problems of sizes encountered in practice *general purpose commercial LP-software suffices.* Standard routines such as parametric procedures can be used without any extra effort in implementation. Since dual degeneration often occurs, it might sometimes be easy to find an integer optimal solution. A few experiences show that it may be a useful heuristic procedure to choose the value 1 as pivot element as long as it is possible. (Note that at the beginning of the iterations, nearly each column consists of only 3 nonzero elements, each having the value 1 or -1.)

Possible Extensions

Both approaches can be applied to *multidimensional cutting problems* provided that the usual restrictive assumptions are made, such as the assumption that only "guillotine cuts" are allowed (compare with Gilmore and Gomory [1965] and Dyckhoff [1979], respectively).

As for Model *I limitations with respect to the number of cutting knives* can very easily be allowed for within the framework of solving the knapsack problems (5). As far as Model *II* is concerned it is impossible to recognize how often a residual length has already been cut before. Hence, Model *II* will not be practical if only one cutting operation is possible (or multiple cutting is very expensive) *and* the number of knives available is clearly smaller than $s_{\max}/d_{\min}$ (compare Gilmore and Gomory [1963]).

Since the slack variables of the constraints (7) of Model *II* indicate the remainder of the respective length, it is easy to impose additional *restrictions on the "trim loss"* and to consider an objective function containing the "trim loss," whereas for the classical approach the Gilmore-Gomory algorithm in its present known form does not work. It may be possible to extend the Gilmore-Gomory method as to take such additional constraints into account; but it would probably not be as efficient as the new approach. In the following we outline a real problem of this type which is also an example for case (i) of our hypothesis (also see Stainton).

5. PRACTICAL APPLICATION

A German textile firm buys rolls of a synthetic cloth of the constant width 366 cm from a producer in the United States. The rolls must be cut because of the customers' demand for smaller widths (and lengths). With "guillotine cuts" only being allowed and customers not caring much whether their demand for a certain width is fulfilled with one or several rolls of this width, the two-dimensional problem can be simplified to a one-dimensional cutting problem with respect to the widths of rolls.

The cutting process yields residual rolls (widths) which can be used to cover further demands arising in later planning periods. Hence most of the residual rolls are not valueless. Their value is determined by the discounted net returns of the future. This amount must be estimated on the basis of experience and rules of thumb; also, rolls of larger widths have higher unit values than smaller ones. Given the rolls in stock and the demands at the beginning of a planning period the problem is to maximize the sum of net returns and the change in value of the stock until the end of the period. The demands need not be satisfied. There is a constraint on the maximum level of all cloth in inventory.

This cutting problem is described in Gehring et al. (and in more detail in Dyckhoff). It deviates from the Standard Problem in that there are many stock sizes with only a finite amount of each available and also there are restrictions on the residual rolls. Both approaches—the classical and the new one—lead to LP-models of nearly the same number of constraints, with many variables for the classical approach. By eliminating the equality constraints which define the remainder of a width, the number of constraints of the classical model would clearly decrease (to the number of stock and demand sizes, plus several other constraints), but the formulation would become much more complicated. In order to obtain an optimal solution the Gilmore-Gomory algorithm would have to be generalized. Gehring et al. content themselves with suboptimal solutions by heuristically choosing a small number of "good" cutting patterns.

The new approach (Dyckhoff) to this cutting problem yields an optimal solution. It has been tested with real data (Dyckhoff, and Dyckhoff and Gehring [1981]). For a problem of 6 stock and 12 order widths, a linear program of 335 constraints and about 2,000 variables (with a matrix density of 0.9%) has been solved in 12 seconds of CPU-time (using APEX III, version 1.201, on CYBER 175).

Although it is not a trivial matter to generate a LP-matrix of this size, the management of the textile firm prefers the new approach to the classical (heuristic) one as it guarantees an optimal solution and yields *a more realistic representation of the actual cutting process*: only one knife exists such that, indeed, only one-cuts are possible. Nevertheless, the management decided to wait with the implementation until, in a few years, the cutting problems will become more complex because of higher expected sales so that the decision problem cannot be solved efficiently enough without applying a simultaneous planning approach.

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