How to uniquely specify a pitched baseball's seam orientation

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Being able to describe how the seams of a baseball are oriented relative to the spin axis is now possible thanks to optical baseball measurement technologies. This paper presents an easy-to-understand convention for describing the seam texture orientation of the ball relative to the spin axis (Figure 1) in the form of two angles *SeamLat* and *SeamLong* (herein referred to as θ_{lat} and θ_{long}, respectively), provides the mathematical derivation and equations necessary to implement and verify this method, and provides a link to the source code [1] of a reference implementation of this method.

I. MOTIVATION

With the development of optical baseball tracking systems, it is now possible for the orientation of the seams on the baseball to be measured.

This has important implications for training as, for instance, there is a difference between a two-seam fastball and a four-seam baseball, in addition to other kinds of pitches where the orientation of the seam texture relative to the spin axis is significant.

The spin axis relative to the texture on the ball also has implications for what the ball looks like as it comes towards the batter, with the appearance of the ball often being used to judge the kind of pitch that has been thrown.

To describe how a baseball rotates as it flies through the air, the spin rate and 3D spin axis can be used. This is important for determining the effects of aerodynamic forces on the ball, causing the ball's trajectory to deviate from that of an non-spinning ball. But the spin rate and axis say nothing about how the seams of the ball are oriented relative to the spin axis, which recent work [2] is indicating can have an effect on the aerodynamics of the ball.

The Four Seam Rotation Index (FSRI) [3] is one way of describing the seam orientation, specifically whether it was a four-seam ball (FSRI=100) or a two-seam ball (FSRI=0). However this single number is not sufficient for uniquely specifying how the seams are oriented relative to the spin axis, hence the need for this specification.

To uniquely specify the orientation of an object in 3D, three pieces of information are required. Whether it's Euler angles, Euler parameters, unit quaternions, or rotation matrices, they all boil down to needing three pieces of information to specify the orientation.

For a spinning ball, the orientation of the ball is constantly changing with time. Due to gyroscopic stiffening of the spinning ball, the spin axis is relatively constant throughout the ball flight relative to the world frame. This constant spin axis forms the foundation on which the orientation of the seams can be specified.

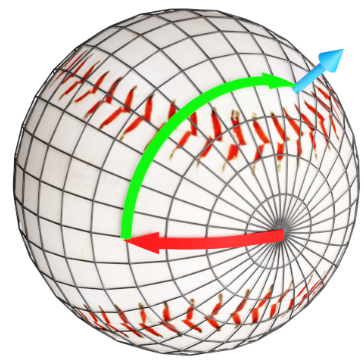

FIG. 1. Seam Orientation Angles. Spin axis is blue. $\theta_{lat} = 45°$ is red. $\theta_{long} = 112.5°$ is green.

A frame transformation can be defined whereby the observer travels along with the ball, with the spin axis along one of the principal directions of the observer frame. The observer frame translates but does not rotate in time.

At any moment in time the orientation of the ball can be specified with three pieces of information, but one piece of information, the rotation of the seam texture about the spin axis, can be discarded and it's constantly changing and is not considered to be important.

Thus two pieces of information are required to uniquely specify the orientation of the seams relative to the spin axis.

II. SPECIFICATION

Choosing these pieces of information to be physically meaningful as well as easy to understand is important.

The FSRI of the ball is a physically meaningful parameter, so it would be good if the seam orientation specification could describe whether the ball is a two-seam or four-seam or something in between.

Being able easily to understand where on the ball's surface the spin axis pierces it also important, as the texture close to the piercing will be blurred least in time by the rotation of the ball.

To begin with, a standard ball texture and coordinate frame need to be introduced. Figure 2 shows the

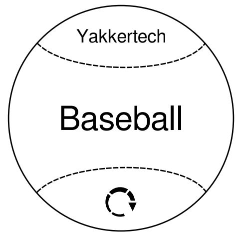

FIG. 2. Standard Baseball Orientation

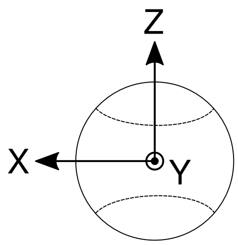

FIG. 3. Baseball Coordinate Frame

standard orientation of the ball in the observer frame. This orientation is the most natural orientation when one holds the ball, with the logos facing towards the observer and the text upright. All baseballs have the seams running the same way relative to this standard orientation.

Next a coordinate frame is needed. This choice is arbitrary, but one should be chosen that is easy to remember and with which people are familiar.

Although the observer frame travels along with the ball, and the baseball frame is attached to the surface of the ball, to remember how the axes are oriented, a simple convention has been chosen.

Imagine the observer being a pitcher standing on the pitching mound facing home plate with the ball held out in front of them in the standard orientation.

The coordinate axes are the same as those used by pitchF/X and Trackman, with the origin at the tip of home plate, Z pointing up anti-parallel to the gravity vector, Y pointing from the tip of home plate towards the middle of the pitching rubber, while being perpendicular to the gravity vector, and X being orthogonal to both of those pointing towards the first base side of the field. The origin of the frames used in this paper are at the centre of the ball instead of the tip of home plate.

By remembering how these axes appear relative to the pitcher and the ball in front of them, the axes of the observer frame and baseball frame are the same (when the ball appears in the standard orientation to the pitcher).

Figure 3 depicts these two frames that happen to coincide when the ball is held in the standard orientation.

In the observer frame the ball is always considered to be rotating about the positive Y axis as described by the right-hand rule. This means that the ball will be rotating counter-clockwise in the observer frame.

Finally a specification of the seam orientation is needed. Because the baseball coordinate frame will not, in general, be the same as the observer frame, a coordinate frame transformation from one to the other can be specified by rotating a ball from the standard orientation in the observer frame, to the actual orientation as seen in the observer frame.

As was seen in Section I, this frame transformation can be specified with two pieces of information.

One requirement of this frame transformation is that one of the frame transformation angles directly relates to the FSRI of the ball. The FSRI is related to the angle that the four-seam plane of the ball makes with the spin axis. In the baseball coordinate frame, the four-seam plane is the X-Z plane. The angle that the normal of this plane (the baseball frame's Y axis) makes with the spin axis of the ball (the observer frame's Y axis) is the angle that is directly related to the FSRI. Thus one of the transformations made from the observer frame to the baseball frame should produce this angle.

With the ball in the standard orientation in the observer frame, rotating about either the observer frame's X or Z axes will cause the ball's Y axis to be rotated by the same amount relative to the spin axis of the ball (the observer frame's Y axis). Which one to choose comes down to what people are familiar with, and will be described shortly.

The second angle must be chosen such that the Y axis in the baseball frame does not change relative to the Y axis of the observer frame, as any change would render the physical interpretation of the first angle meaningless. The only possible axis about which the second rotation can be done and preserve the meaning of the first is the baseball frame's Y axis.

Consider Figure 1. This specifies the location of where the spin axis (cyan arrow in figure) pierces the surface of the ball at point P relative to the middle front of the ball in the standard orientation using two angles θ_{lat} and θ_{long} in a polar coordinate frame.

The first angle θ_{lat} (red arrow in the figure) defines the latitude at which the piercing occurs, going from 0 degrees at front centre of the ball, to 90 degrees along the equator of the ball, to 180 degrees at the back middle of the ball.

The second angle θ_{long} (green arrow in the figure) defines the longitude at which the piercing occurs, measured relative to baseball frame's positive X axis, starting at 0 degrees along the X axis and increasing with clockwise rotation.

To use a world globe analogy, the north pole would be the front middle of the ball, the prime meridian would be along the baseball frame's X axis, and the latitude angle would go from 0 (90 degrees north on the globe) to 180 (90 degrees south on the globe).

The Trackman convention of the SpinAxis metric is defined as the angle the spin axis makes relative to the world frame's positive X axis, with a positive angle being clockwise from the pitcher's point of view. This Trackman SpinAxis convention also matches up with the θ_{long} angle used in this specification.

The θ_{lat} angle first moves the piercing along the surface of the ball in the positive X direction, and the θ_{long} angle then moves it along the surface of the ball on a line of constant latitude (in the ball's X-Z plane) to the spot where the spin axis pierces the ball.

Thus the question of whether to rotate about the X or Z axis in the observer frame has been answered: rotating by θ_{lat} about the Z axis fits in with a convention people are already familiar with.

Note that the θ_{lat} and θ_{long} angles are the angles the ball needs to rotate about the ball's local coordinate frame as it is transformed from the standard orientation in the observer frame, to the actual orientation in the observer frame. Recall that a third rotation about the spin axis (positive Y in the observer frame) is necessary to get to the final seam orientation observed at an instant of time in the observer frame.

It is by design that these angles are simple to reason about when transforming the ball, agree with conventions that people are already familiar with, preserve the meaning of the θ_{lat} angle relative to the FSRI, and also describe the location of the spin axis piercing in polar coordinates on the ball.

The mapping from the θ_{lat} angle to the FSRI is very simple. When θ_{lat} is 0 degrees, it is a four-seam ball with FSRI=100. When θ_{lat} is 45 degrees, it is a partially four-seam and partially two-seam ball, with FSRI=50. When θ_{lat} is 90 degrees, it is a two-seam ball with FSRI=0. As a single equation it is $\text{FSRI} = 100\frac{|90-\theta_{lat}|}{90}$ when θ_{lat} is in degrees.

It is reasonable to ask if this choice of polar coordinates is the best. It would certainly be more natural to define coordinates like that of a globe, with the pole at the top of the ball instead of the centre of the front face, and the zero angles origin of the frame would still be on the centre of the front face of the ball. The problem with this is that one of the angles is no longer directly related to the FSRI. A two-seam ball would no longer be identified by a single unique θ_{lat} number, but would be identified by two different θ_{long} numbers. A ball somewhere between a two and four-seam could be identified by a whole range of θ_{lat} and θ_{long} combinations, and would not be possible to identify easily.

It is also reasonable to ask if the Y axis is the correct axis to serve as the coordinate origin of the θ_{lat} angle. Perhaps another axis could be used, and instead of talking about four-seam-rotation-index a two-seam-rotation-index could be used. The problem with this is that there are two planes that could be defined as the two-seam rotation plane, making such a formulation ambiguous. Using any other axis as the pole origin would still create the problem of a range of parameters being able to specify a two-seam ball.

Since the four-seam rotation plane is unique, and using its normal (the baseball frame's Y axis) as the pole origin results in a parameter that is directly related to the FSRI and can uniquely specify a two-seam (or otherwise) ball, the convention proposed in this paper is the best one to use.

TABLE I. Coordinate Frames

Number	**Description**	**Transformation to get there**
0	World	None
1	Observer	Spin axis to Y axis
2	Intermediate 1	Rotate about ball/observer Z axis
3	Intermediate 2	Rotate about ball Y axis
4	Baseball	Rotate about spin axis

III. SEAM ANGLE CALCULATIONS

In the following angle derivations, it will be helpful to have a reference of the coordinate frames being discussed. Refer to Table I for a listing of the different coordinate frames used.

To transform the baseball from the standard orientation in the world frame (as if the pitcher were holding it) to the baseball frame, one must undergo several different transformations about specific axes.

The unit vector $\hat{\alpha}$ is the spin axis of the ball. The unit vectors $\hat{x}_i$, $\hat{y}_i$, and $\hat{z}_i$ where $i \in \{0, 1, 2, 3, 4\}$ represent the coordinate axes that define the i^{th} frame.

A. Frame 0 to 1

The first transformation is from Frame 0 (World) to Frame 1 (Observer). This is done by lining up the $\hat{y}_0$ axis with the spin axis $\hat{\alpha}$. This can easily be accomplished by rotating Frame 0 about a unit vector $\hat{\kappa}$ by an amount θ_κ, where $\hat{\kappa} = \frac{\vec{\kappa}}{|\vec{\kappa}|}$ and $\theta_\kappa = \arcsin(|\vec{\kappa}|)$ and $\vec{\kappa} = \hat{y}_0 \times \hat{\alpha}$. The cross product here provides a convenient orthogonal vector about which to rotate, and the arc sine of its magnitude the angle. Note that this transformation does not enforce any specific orientation of the observer frame about the spin axis, as the orientation of the ball about the spin axis doesn't matter.

There is actually some additional consideration that needs to be paid to this transformation, as the cross product alone does not provide enough information to uniquely transform the coordinate frame, and the unit vector is singular when the coordinate axes are already lined up. A more thorough treatment is presented in Equation 1.

$$\theta_\kappa = \begin{cases} \arcsin(|\hat{y}_0 \times \hat{\alpha}|) & \text{if } \hat{y}_0 \cdot \hat{\alpha} \geq 0 \\ \pi - \arcsin(|\hat{y}_0 \times \hat{\alpha}|) & \text{if } \hat{y}_0 \cdot \hat{\alpha} < 0 \\ 0 & \text{if } \hat{y}_0 \cdot \hat{\alpha} = 1 \\ \pi & \text{if } \hat{y}_0 \cdot \hat{\alpha} = -1 \end{cases} \tag{1}$$

Note that $\theta_\kappa = 0$ when $\hat{y}_0 \cdot \hat{\alpha} = 1$ and the rotation

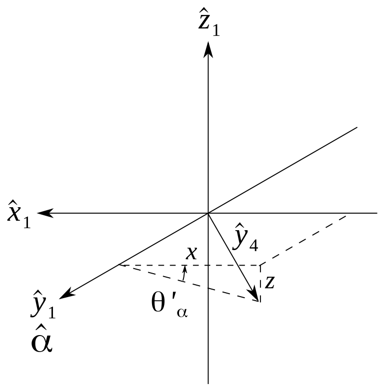

FIG. 4. Spin axis rotation

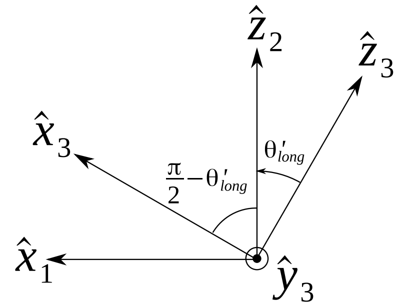

FIG. 5. Y axis rotation

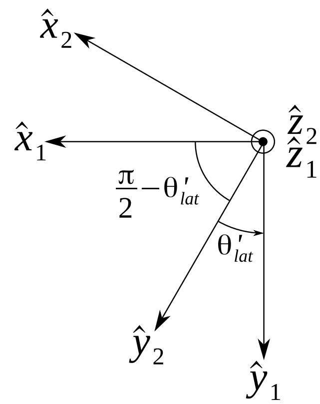

FIG. 6. Z axis rotation

unit vector $\hat{\kappa}$ is undefined. In this case, $\hat{y}_0$ is already aligned with $\hat{\alpha}$ and no frame transformation needs to be performed.

When $\hat{y}_0 \cdot \hat{\alpha} = -1$ then $\theta_\kappa = \pi$ and the rotation unit vector $\hat{\kappa}$ is also undefined. In this case, $\hat{y}_0$ is anti-parallel with $\hat{\alpha}$ so simply let $\hat{y}_1 = -\hat{y}_0$, $\hat{x}_1 = -\hat{x}_0$, and $\hat{z}_1 = \hat{z}_0$.

For the remaining frame transformations it is necessary to derive them in reverse order, going from Frame 4 (Baseball) to Frame 1 (Observer), as the objective and mathematics of each step is clearer. All of the angles derived in the following subsections will be the negative of the angles that are ultimately used, as the subsections are done in reverse order. In the equations and figures below, primed angles will be used, which are the negative of the desired angles. Thus $\theta' = -\theta$ for all of the various angles.

At each stage the new coordinate axes are calculated from the old by rotating them the appropriate amount about the appropriate axis. Appendix B contains equations for the rotation matrices.

B. Frame 4 to 3

The first reverse step is to go from Frame 4 (Baseball) to Frame 3 (Intermediate 2) by placing the Baseball $\hat{y}_4$ axis into the $\hat{x}_1 - \hat{y}_1$ plane on the negative $\hat{x}_1$ side by rotating about the spin axis $\hat{\alpha}$ by an angle θ'_α, as depicted in Figure 4. This angle is actually discarded, as rotations about the spin axis are not ultimately used, but this transformation is required to get the basis unit vectors of Frame 3 (Intermediate 2).

The reason the $\hat{y}_4$ vector is axis placed on the negative $\hat{x}_1$ side is because this is necessary for the θ_{lat} angle to be positive, which is needed for the latitude and longitude analogy to work.

The quantities in the triangle are $x = \hat{y}_4 \cdot (-\hat{x}_1)$ and $z = \hat{y}_4 \cdot (-\hat{z}_1)$. The rotation angle is then $\theta'_\alpha = \arctan\left(\frac{z}{x}\right) = \arctan\left(\frac{\hat{y}_4 \cdot (-\hat{z}_1)}{\hat{y}_4 \cdot (-\hat{x}_1)}\right)$. Note that when using these equations in computer software, the two argument arc tangent function, often called **atan2**, should be used to determine correctly the quadrant of the angle.

C. Frame 3 to 2

To transform from Frame 3 (Intermediate 2) to Frame 2 (Intermediate 1) the coordinate frame must now be rotated about the $\hat{y}_3$ axis so that the $\hat{z}_3$ axis lines up with the $\hat{z}_2$ axis, as depicted in Figure 5. This brings $\hat{x}_3$ into the $\hat{x}_1 - \hat{y}_1$ plane. Using two separate dot products allows for the determination of the correct quadrant for the angle. Note that $\hat{z}_2 = \hat{z}_1$ and since $\hat{z}_1$ is in the observer frame, it will be used instead of $\hat{z}_2$. By using the fact that $\hat{z}_3 \cdot \hat{z}_1 = \cos\theta'_{long}$ and $\hat{x}_3 \cdot \hat{z}_1 = \cos\left(\frac{\pi}{2} - \theta'_{long}\right) = \sin\left(\theta'_{long}\right)$ one can write $\theta'_{long} = \arctan\left(\frac{\hat{x}_3 \cdot \hat{z}_1}{\hat{z}_3 \cdot \hat{z}_1}\right)$ and make use of the two argument arc tangent function.

D. Frame 2 to 1

The final step from Frame 2 (Intermediate 1) to Frame 1 (Observer) is to rotate about the $\hat{z}_2$ axis such that $\hat{x}_2$ and $\hat{y}_2$ line up with $\hat{x}_1$ and $\hat{y}_1$, respectively, as depicted in Figure 6.

This will be expressed as an arc tangent to facilitate using the two argument arc tangent function. Since $\hat{y}_2 \cdot \hat{y}_1 = \cos\left(\theta'_{lat}\right)$ and $\hat{y}_2 \cdot \hat{x}_1 = \cos\left(\frac{\pi}{2} - \theta'_{lat}\right) = \sin\left(\theta'_{lat}\right)$

TABLE II. Coordinate Frames

Frame Transformation	Axis	Angle
$0 \rightarrow 1$	$\vec{\kappa} = \frac{\hat{y}_0 \times \hat{\alpha}}{\lvert \hat{y}_0 \times \hat{\alpha} \rvert}$	$\theta_\kappa = \arcsin(\lvert\vec{\kappa}\rvert)$
$1 \rightarrow 2$	$\hat{z}_1$	$\theta_{lat} = -\arctan\left(\frac{\hat{y}_2 \cdot \hat{x}_1}{\hat{y}_2 \cdot \hat{y}_1}\right)$
$2 \rightarrow 3$	$\hat{y}_2$	$\theta_{long} = -\arctan\left(\frac{\hat{x}_3 \cdot \hat{z}_1}{\hat{z}_3 \cdot \hat{z}_1}\right)$
$3 \rightarrow 4$	$\hat{\alpha}$	$\theta_\alpha = -\arctan\left(\frac{\hat{y}_4 \cdot (-\hat{z}_1)}{\hat{y}_4 \cdot (-\hat{x}_1)}\right)$

one can write $\theta'_{lat} = \arctan\left(\frac{\hat{y}_2 \cdot \hat{x}_1}{\hat{y}_2 \cdot \hat{y}_1}\right)$.

Note that this figure depicts $\hat{y}_2$ on the positive $\hat{x}_1$ side of the $\hat{x}_1 - \hat{y}_1$ which should never actually happen in the steps described here. But for the determination of the appropriate rotation angle this does not matter as long as the two argument arc tangent function is used.

E. Summary

Since the axis and angle convention laid out in Section II operates on the forward transformation, all of the primed angles derived above must be negated. The desired unprimed angles, the rotation axes, and the coordinate transformations effected by these rotations are summarized in Table II.

F. Corner Case

There is one corner cases that need to be considered when implementing these algorithms.

If $\theta_{lat} = 0$ then there is an ambiguity between whether the ball should be rotated about $\hat{\alpha}$ by θ_α or about $\hat{y}_3$ by θ_{long} since in this case $\hat{\alpha} = \hat{y}_3$. Here the equation in subsection III B will choose $\theta_\alpha = 0$ because both dot products are zero and the atan2 function will return 0. Thus any rotation between the Frame 3 and Frame 2 will be expressed as a rotation about $\hat{y}_3$ by θ_{long}. But in this case the rotation should be done by the θ_α rotation about $\hat{\alpha}$ so it gets discarded. Thus set $\theta_\alpha = \theta_{long}$ so it gets discarded, then set $\theta_{long} = 0$.

G. Texture Degeneracies

An important consideration in specifying the orientation of the seams of the ball is that the seam texture is degenerate. If the ball had no markings on it, there are four different orientations of the ball that would produce an identical seam pattern, despite having very different ball rotations.

The logos on the ball don't affect the flight of the ball, so it may be desirable to know the seam orientation of the ball that produces the smallest angles relative to the nearest texture degeneracy. In such a case the angles $(\theta_{lat}, \theta_{long}) = (0,0), (\pi,0), (0,\pi), (\pi,\pi)$ would all produce the same seam appearance relative to the rotation axis of the ball.

In such a case the smallest θ_{lat} and θ_{long} angles can be found simply be recalculating the angles for the four different texture degeneracies. Four new frames called Frame 5_j (Degeneracy j) can be defined where Frame 4 (Baseball) is transformed by one of the degeneracy rotations and the θ_{z_j} and θ_{y_j} angles calculated. Then simply choosing the angles where their sum is minimized will give the smallest angles to the closest texture degeneracy.

These four degeneracy rotation matrices R_{d_j} are listed in Equation B5 and modify ${}^0P_4$ into ${}^0P_{5_j}$ thus:

$$ {}^0P_{5_j} = {}^0P_4 R_{d_j} \tag{2} $$

IV. VERIFICATION

To verify that the conversion from coordinate frame axes to angles described in Section III is correct, it is helpful to do the forward transformation from Frame 0 (World) to Frame 4 (Baseball).

To do this, rotation matrices and linear algebra will be used to represent the frame transformations. The appendix contains the derivations of the matrix algebra used in this section.

First the transformation from Frame 0 to Frame 1 must be done by rotating $\hat{y}_0$ to line up with $\hat{y}_1 = \hat{\alpha}$ by rotating about $\hat{\kappa}$ by θ_κ using Equation B4.

The projection of both $\hat{y}_0$ and $\hat{\alpha}$ onto Frame 0, ${}^0y_0$ and ${}^0\alpha$, respectively, will be used.

Here ${}^0\alpha = {}^0y_1 = R_{\hat{\kappa}}(\theta_\kappa)\ {}^0y_0$, which can then be expanded by the $\bar{e}_0$ basis to yield $\hat{y}_1 = \bar{e}_0\ {}^0y_1 = \bar{e}_0 R_{\hat{\kappa}}(\theta_\kappa)\ {}^0y_0$.

By extending this so that we collect the projections of the row vector $[\hat{x}_0\ \hat{y}_0\ \hat{z}_0] = \bar{e}_0$ onto frame 0 as the matrix $\left[{}^0x_0\ {}^0y_0\ {}^0z_0\right] = {}^0P_0$ the $\bar{e}_1$ frame can be defined in terms of the $\bar{e}_0$ frame as $\bar{e}_1 = \bar{e}_0\ {}^0P_1 = \bar{e}_0 R_{\hat{\kappa}}(\theta_\kappa)\ {}^0P_0$. Thus ${}^0P_1 = R_{\hat{\kappa}}(\theta_\kappa)\ {}^0P_0$. Since ${}^0P_0 = I$ this can be written as:

$$ {}^0P_1 = R_{\hat{\kappa}}(\theta_\kappa) \tag{3} $$

For subsequent transformations it will be helpful to work in reverse again, as a useful identity falls out.

To transform from Frame 4 to Frame 3 a rotation around $\hat{\alpha}$ by θ'_α is performed. Since $\hat{\alpha} = \hat{y}_1$ by construction, one can project $\bar{e}_4$ onto $\bar{e}_1$, then rotate by θ'_α around the $\hat{y}_1$ axis, then expand using $\bar{e}_1$. This can be summarized as $\bar{e}_3 = \bar{e}_1 R_y(\theta'_\alpha)\ {}^1P_4 = \bar{e}_1\ {}^1P_3$.

To transform from Frame 3 to 2, rotate around the $\hat{y}_3$ vector by θ'_{long}. This can be written as $\bar{e}_2 = \bar{e}_3 R_y(\theta'_{long})$.

To transform from Frame 2 to 1, rotate around the $\hat{z}_2$ vector by θ'_{lat}. This can be written as $\bar{e}_1 = \bar{e}_2 R_z(\theta'_{lat})$.

Chaining all of these definitions together yields $\bar{e}_1 = \bar{e}_1 R_y(\theta'_{long})\ {}^1P_4 R_y(\theta'_{long}) R_z(\theta'_{lat})$. Since this is writing

$\bar{e}_1$ in terms of itself, the matrix components of the right hand side must be equal to the identity matrix, viz. $R_y(\theta'_{long})\ {}^1P_4 R_y(\theta'_{long}) R_z(\theta'_{lat}) = I$. Then pre and post multiply the transposes of these matrices to recover ${}^1P_4$. Since the transpose of a rotation matrix is just a rotation in the opposite direction, $R(\phi)^T = R(-\phi)$, and since $\theta' = -\theta$, this then yields:

$$ {}^1P_4 = R_y(\theta_\alpha) R_z(\theta_{lat}) R_y(\theta_{long}) \tag{4}$$

Since the columns of the projection matrix are just the components of the $\bar{e}_4$ basis vectors written in terms of the $\bar{e}_1$ basis vectors, all the information necessary to derive the seam orientation angles using the procedures of Section III is now present.

In order to get the representation of Frame 4 (Baseball) in terms of Frame 0 (World), chain ${}^0P_1$ and ${}^1P_4$ together using Equation A6 to yield:

$$ {}^0P_4 = {}^0P_1\ {}^1P_4 = R_{\hat{\kappa}}(\theta_\kappa) R_y(\theta_\alpha) R_z(\theta_{lat}) R_y(\theta_{long}) \tag{5}$$

Where the columns of the projection matrix are the components of $\bar{e}_4$ written in terms of the $\bar{e}_0$ basis vectors.

V. CONCLUSIONS

Given the ability of new imaging technologies to provide heretofore unavailable levels of detail on the flight of a baseball, this paper has presented a physically meaningful and easy to remember convention for describing the seam orientation of a baseball relative to its spin axis.

This information is useful for both baseball aerodynamic studies, and training purposes, and is presented in a way that is understandable by analysts and coaches alike.

This paper also provides the technical details necessary to calculate the new metrics given measurement of the baseball, as well as the mathematics needed to simulate these measurements to verify the calculation of the metrics.

Finally this paper provides a link to a reference implementation of the methods used in this paper.

Appendix A: Frame Transformation Algebra

A vector $\vec{V}$ has a length and direction. A unit vector $\hat{V} = \frac{\vec{V}}{|\vec{V}|}$ has a length of 1 and a direction.

A coordinate frame is usually defined by three orthogonal basis unit vectors. For example, the A frame would have $\hat{x}_A$, $\hat{y}_A$, and $\hat{z}_A$ as the basis vectors. These basis vectors can be collected into a row vector to form a basis frame $\bar{e}_A = [\hat{x}_A\ \hat{y}_A\ \hat{z}_A]$. Note that this is not a unit vector, nor is it a row vector of numbers. It is a row vector of unit vectors. Thus $\bar{e}_A$ is written with a bar as a reminder of its special character and its ability to project from a component representation to a vector representation and vice-versa.

Typically, vectors are expressed as components in a particular coordinate frame. 3D vectors are usually expressed as a column vector of three numbers. This is a projection of a vector onto a basis frame. Only when a vector is projected onto a basis does it take on a numerical representation.

The transpose of a basis row vector can be defined as ${}^B\underline{e} = \bar{e}_B^T$. The vector $\vec{V}$ can be projected onto the basis frame $\bar{e}_A$ thus:

$$ {}^AV = \begin{bmatrix} \hat{x}_A \cdot \vec{V} \\ \hat{y}_A \cdot \vec{V} \\ \hat{z}_A \cdot \vec{V} \end{bmatrix} = {}^A\underline{e}\vec{V} \tag{A1}$$

where dot products are used instead of scalar multiplication when combining the transposed basis vector ${}^A\underline{e}$ with the vector $\vec{V}$. This can also be expressed using index notation as ${}^AV_i = \hat{e}_{A_i} \cdot \vec{V}$ where $i \in \{0, 1, 2\}$ and the basis unit vectors summarized as:

$$ \begin{aligned} \hat{e}_{A_0} &= \hat{x}_A \\ \hat{e}_{A_1} &= \hat{y}_A \\ \hat{e}_{A_2} &= \hat{z}_A \end{aligned} \tag{A2}$$

Note that this projection can be reversed by *expanding* the projection using the basis vectors of a particular frame. This technique converts the numerical representation of a vector back into the non-numerical vector itself. To do this, simply multiply the components of the vector in a particular frame by the basis unit vectors of that frame. Then make use of the fact that $\bar{e}_A$ is a row vector of unit basis vectors and ${}^AV$ is column vector of the components of $\vec{V}$ projected onto the $\bar{e}_A$ unit basis vectors to write:

$$ \vec{V} = \bar{e}_A\ {}^AV \tag{A3}$$

Not only can a vector be projected onto a basis, but one basis can be projected onto another. Instead of producing a vector of numbers, this produces a matrix of numbers, with each column of the matrix representing the projection of one frame onto the other.

Write ${}^BP_A$ as the projection of the basis unit vectors of frame A onto the basis unit vectors of frame B. The index notation used in the vector projection can easily be extended for this purpose: ${}^BP_{A_{ij}} = \hat{e}_{B_i} \cdot \hat{e}_{A_j}$. This can be thought of as a matrix whose columns are as follows:

$$ {}^BP_A = \left[\begin{array}{l|l|l} \hat{x}_A & \hat{y}_A & \hat{z}_A \\ \text{in B} & \text{in B} & \text{in B} \\ \text{frame} & \text{frame} & \text{frame} \end{array}\right] \tag{A4}$$

Using the transpose of the basis row vector ${}^B\underline{e}$ the projection equation can then be written using a dot product version of the outer product of the basis vectors as:

$$ {}^BP_A = \bar{e}_B^T \bar{e}_A = {}^B\underline{e}\bar{e}_A \tag{A5}$$

where the outer product of the column vector ${}^B\underline{e}$ with row vector $\bar{e}_A$ yields a matrix. Since vectors of unit basis vectors are being multiplied instead of vectors of numbers, instead of using scalar multiplication like the regular outer product, this one uses the vector dot product.

Write the transpose of a projection matrix as ${}^BP_A^T = {}^AP_B$.

Multiple basis projections can be chained together. Given $\bar{e}_B = \bar{e}_C \ {}^CP_B$ this can be transposed as ${}^B\underline{e} = {}^CP_B^T \ {}^C\underline{e} = {}^BP_C \ {}^C\underline{e}$. Then given ${}^BP_A = {}^B\underline{e}\bar{e}_A$ it can be written as ${}^BP_A = {}^BP_C \ {}^C\underline{e}\bar{e}_A$. But since ${}^CP_A = {}^C\underline{e} \ \bar{e}_A$ it can be written that:

$$ {}^BP_A = {}^BP_C \ {}^CP_A \tag{A6}$$

Notice how the upper and lower frame letters of adjacent projections match up and form a chain of projection frames, and how transposing simply changes the side and location of the frame indicator. This matching of upper and lower frames of adjacent matrices and vectors works for most things in this algebra, and helps to add a layer of abstraction onto the underlying matrix algebra that makes the frame transformations easier to reason about.

Notice how the special bars of a basis frame when above in row vector form (like $\bar{e}_A$) give vector symbols to scalars when expanding, and how the when transposed and below in column vector form (like ${}^A\underline{e}$) take them away and convert to scalars when projecting.

Next, to rotate the components of a vector from one frame's projection to another, simply define the following transformation:

$$ {}^BV = {}^BR_A \ {}^AV \tag{A7}$$

where ${}^BR_A$ is a rotation matrix that transforms vectors components expressed in the A frame to vector components expressed in the B frame.

Since $\vec{V} = \bar{e}_A {}^AV = \bar{e}_B \ {}^BV = \bar{e}_B \ {}^BR_A \ {}^AV$, then:

$$ \bar{e}_A = \bar{e}_B \ {}^BR_A \tag{A8}$$

which rotates the unit basis vectors of frame B into frame A.

Multiply both sides by ${}^B\underline{e}$, use the fact that ${}^B\underline{e}\bar{e}_B = I$, to get:

$$ {}^BR_A = {}^B\underline{e}\bar{e}_A \tag{A9}$$

Notice how Equations A5 and A9 are exactly the same. Thus the rotation matrix from frame A to frame B is the same as the projection of frame A onto frame B. They can also be used interchangeably and converted from one to the other.

There is another way of transforming a frame's basis unit vectors using a rotation in an intermediate frame. By projecting the unit vectors of frame A onto frame C, then rotating the components of those vectors about an axis defined in frame C to go from frame A to frame B, then expanding back using the basis vectors of frame C, an expression for frame B can be derived:

$$ \bar{e}_B = \bar{e}_C \ {}^BR_A \ {}^CP_A \tag{A10}$$

Appendix B: Rotation Matrices

For reference, several different rotation matrices will be reproduced here.

To rotate about the X axis by an angle θ:

$$ R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{B1}$$

To rotate about the Y axis by an angle θ:

$$ R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{B2}$$

To rotate about the Z axis by an angle θ:

$$ R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B3}$$

To rotate about a unit vector $\hat{u} = [u_x u_y u_z]^T$ by an angle θ [4, 5]:

$$ \begin{gathered} R_{\hat{u}}(\theta) = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \\ R_{11} = \cos\theta + u_x^2(1-\cos\theta) \\ R_{12} = u_x u_y(1-\cos\theta) - u_z\sin\theta \\ R_{13} = u_x u_z(1-\cos\theta) + u_y\sin\theta \\ R_{21} = u_y u_x(1-\cos\theta) + u_z\sin\theta \\ R_{22} = \cos\theta + u_y^2(1-\cos\theta) \\ R_{23} = u_y u_z(1-\cos\theta) - u_x\sin\theta \\ R_{31} = u_z u_x(1-\cos\theta) - u_y\sin\theta \\ R_{32} = u_z u_y(1-\cos\theta) + u_x\sin\theta \\ R_{33} = \cos\theta + u_z^2(1-\cos\theta) \end{gathered} \tag{B4}$$

The seam texture degeneracy matrices are:

$$
\begin{aligned}
R_{d_0} &= I \\
R_{d_1} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
R_{d_2} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
R_{d_3} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{B5}
$$

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