



University of Wrocław

UWr 1

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- 4 Number theory
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# Contest (1)

template.cpp38 lines

```
#pragma GCC optimize("O3")
#include "bits/stdc++.h"
using namespace std;

#define rep(i, b, e) for(int i = (b); i <= (e); i++)
#define per(i, b, e) for(int i = (e); i >= (b); i--)
#define FOR(i, b, e) rep(i, b, (e) - 1)
#define SZ(x) int(x.size())
#define all(x) x.begin(), x.end()
#define pb push_back
#define mp make_pair
#define st first
#define nd second
using ll = long long;
using vi = vector<int>;
using pii = pair<int, int>;

auto &operator<<(auto &o, pair<auto, auto> p) {
    return o << "(" << p.st << ", " << p.nd << ")"; }
auto operator<<(auto &o, auto x)->decltype(end(x), o) {
    o << "{"; int i=0; for(auto e: x) o << ", " + 2*!i++ << e;
    return o << "}"; }
#ifndef LOCAL
#define deb(x...) cerr << "[" #x "]: ", [](auto...$) { \
    ((cerr << $ << "; ",...) << endl; }(x)
#else
#define deb(...)
#endif

void solve() { }
```

int main() {
 cin.tie(0)->sync\_with\_stdio(0);
 int tt = 1;
 // cin >> tt;
 FOR(te, 0, tt) solve();
 return 0;
}

1 Makefile2 lines

```
CXXFLAGS=-std=c++20 -O3 -g -DLOCAL -Wall -Wextra -Wshadow
-fsanitize=address,undefined
```

3 hash.sh3 lines

```
# Hashes a file, ignoring all whitespace and comments.
# Use for verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]' | md5sum | cut -c-6
```

8 test.sh9 lines

```
# Example usage: "./test.sh prog"
for i in {1..1000}
do
    ./gen > $1.in
    ./$1 < $1.in > $1.out
    ./brute < $1.in > $1.brute
    diff $1.out $1.brute || break
    echo $i
done
```

13

19

20

Data structures (2)

HashMap.h

Description: Hash map with mostly the same API as unordered\_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

3e474b, 12 lines

```
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
// for places where hacking might be a problem:
const int R = chrono::high_resolution_clock::
    now().time_since_epoch().count();
struct chash { // to use most bits rather than just the lowest
    const uint64_t C = ll(4e18 * acos(0)) | 71;
    ll operator()(ll x) const {
        return __builtin_bswap64((x^R)*C); }
};
// 1 << 16 is initial size, don't put it if you want small map
gp_hash_table<ll, int, chash> h({},{},{},{},{}, {1 << 16});
```

OrderStatisticTree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null\_type.

Time:  $\mathcal{O}(\log N)$ , join probably  $\mathcal{O}(N)$

023ea1, 17 lines

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
    tree_order_statistics_node_update>;

void example() {
    Tree<int> t, t2; t.insert(8);
    auto it = t.insert(10).st;
    assert(it == t.lower_bound(9));
    assert(t.order_of_key(10) == 1);
    assert(t.order_of_key(11) == 2);
    assert(*t.find_by_order(0) == 8);
    t.join(t2); //  $\mathcal{O}(N)$ , assuming  $T \prec T2$  or  $T \succ T2$ , merge t2 into t
}
```

UnionFindRollback.h

Description: Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().

Usage: int t = uf.time(); ...; uf.rollback(t);

Time:  $\mathcal{O}(\log(N))$

2cca9c, 19 lines

```
struct RollbackUF {
    vi e; vector<pii> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return SZ(st); }
    void rollback(int t) {
        for(int i = time(); i --> t;) e[st[i].st] = st[i].nd;
        st.resize(t);
    }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if(a == b) return 0;
        if(e[a] > e[b]) swap(a, b);
        st.pb({a, e[a]}); st.pb({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return 1;
    }
};

LineContainer.h
```

Description: Container where you can add lines of the form  $kx+m$ , and query maximum values at points  $x$ . Useful for dynamic programming (“convex hull trick”).Time:  $\mathcal{O}(\log N)$ 

8ec1c7, 29 lines

```
int rank = rand(), size = 0;
int val = 0, mn = 1e9, sum = 0; // Subtree aggregates
bool flip = 0; int add = 0; // Lazy tags
Node() {}
Node(int v) : size(1), val(v), mn(v), sum(v) {}
};
vector<Node> t;
Treap() : t(1) {}

void pull(int v) {
    auto [l, r] = t[v].ch;
    t[v].size = t[l].size + 1 + t[r].size;
    t[v].mn = min({t[l].mn, t[v].val, t[r].mn});
    t[v].sum = t[l].sum + t[v].val + t[r].sum;
}

int apply(int v, bool flip, int add) {
    if(!v) return 0;
    // t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    if(flip) t[v].flip ^= 1, swap(t[v].ch[0], t[v].ch[1]);
    t[v].val += add; t[v].mn += add;
    t[v].sum += add * t[v].size;
    t[v].add += add;
    return v;
}

void push(int v) {
    FOR(i, 0, 2) {
        t[v].ch[i] = apply(t[v].ch[i], t[v].flip, t[v].add);
    }
    t[v].flip = 0; t[v].add = 0;
}

pii split(int v, int k) {
    if(!v) return {0, 0};
    push(v);
    auto [l, r] = t[v].ch;
    if(k <= t[l].size) {
        // if(k <= t[v].val) { // <- by values
        auto [p, q] = split(l, k);
        t[v].ch[0] = q, pull(v);
        return {p, v};
    }
    else {
        auto [p, q] = split(r, k - t[l].size - 1);
        // auto [p, q] = split(r, k); // <- by values
        t[v].ch[1] = p, pull(v);
        return {v, q};
    }
}

int merge(int v, int u) {
    if(!v || !u) return v ^ u;
    push(v), push(u);
    if(t[v].rank > t[u].rank) {
        t[v].ch[1] = merge(t[v].ch[1], u);
        return pull(v), v;
    }
    else {
        t[u].ch[0] = merge(v, t[u].ch[0]);
        return pull(u), u;
    }
}

void insert(int &v, int pos, int val) {
    // if(v) t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    auto [p, q] = split(v, pos);
    t.pb(Node(val)); int u = SZ(t) - 1;
    // t.pb(Node(pos)); int u = SZ(t) - 1; // <- by values
```

```
        v = merge(merge(p, u), q);
    }

void erase(int &v, int l, int r) {
    // if(v) t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    auto [p, q] = split(v, l);
    auto [u, s] = split(q, r - l + 1);
    // auto [u, s] = split(q, r + 1); // <- by values
    v = merge(p, s);
}

void modify(int &v, int l, int r, bool flip, int add) {
    // if(v) t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    auto [p, q] = split(v, l);
    auto [u, s] = split(q, r - l + 1);
    // auto [u, s] = split(q, r + 1); // <- by values
    u = apply(u, flip, add);
    v = merge(merge(p, u), s);
}

pii get(int &v, int l, int r) {
    // if(v) t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    auto [p, q] = split(v, l);
    auto [u, s] = split(q, r - l + 1);
    // auto [u, s] = split(q, r + 1); // <- by values
    int mn = t[u].mn, sum = t[u].sum;
    v = merge(merge(p, u), s);
    return {mn, sum};
}

// only when by values
int join(int v, int u) {
    if(!v || !u) return v ^ u;
    if(t[v].rank < t[u].rank) swap(v, u);
    auto [p, q] = split(u, t[v].val);
    push(v);
    t[v].ch[0] = join(t[v].ch[0], p);
    t[v].ch[1] = join(t[v].ch[1], q);
    return pull(v), v;
}

// only when by values, persistency destroys complexity
void increase(int &v, int l, int r, int increase) {
    // if(v) t.pb(t[v]), v = SZ(t) - 1; // <- persistency
    auto [p, q] = split(v, l);
    auto [u, s] = split(q, r + 1);
    u = apply(u, 0, increase);
    v = join(merge(p, s), u);
}

};
```

FenwickTree.h  
**Description:** Computes partial sums  $a[0] + a[1] + \dots + a[pos - 1]$ , and updates single elements  $a[i]$ , taking the difference between the old and new value.  
**Time:**  $\mathcal{O}(\log N)$ .

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < SZ(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos)
        ll res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
```

```
    // Returns n if no sum is >= sum, or -1 if empty sum is.
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1) {
        if (pos + pw <= SZ(s) && s[pos + pw-1] < sum)
            pos += pw, sum -= s[pos-1];
    }
    return pos;
}
};
```

FenwickTree2D.h  
**Description:** Computes sums  $a[i,j]$  for all  $i < I, j < J$ , and increases single elements  $a[i,j]$ . Requires that the elements to be updated are known in advance (call `fakeUpdate()` before `init()`).  
**Time:**  $\mathcal{O}(\log^2 N)$  (use persistent segment trees for  $\mathcal{O}(\log N)$ )

```
"FenwickTree.h" cc92f9, 21 lines

struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for(; x < SZ(ys); x |= x + 1) ys[x].pb(y);
    }
    void init() {
        for(vi &v: ys) sort(all(v)), ft.emplace_back(SZ(v));
    }
    int ind(int x, int y) {
        return int(lower_bound(all(ys[x]), y) - ys[x].begin());
    }
    void update(int x, int y, ll dif) {
        for(; x < SZ(ys); x |= x + 1) ft[x].update(ind(x,y), dif);
    }
    ll query(int x, int y) {
        ll sum = 0;
        for(; x; x &= x - 1) sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

RMQ.h  
**Description:** Range Minimum Queries on an array. Returns  $\min(V[a], V[a + 1], \dots V[b - 1])$  in constant time.  
**Time:**  $\mathcal{O}(|V| \log |V| + Q)$

```
1e70c8, 16 lines

template<class T>
struct RMQ {
    vector<vector<T>> jmp;
    RMQ(const vector<T> &V) : jmp(1, V) {
        for(int pw = 1, k = 1; pw * 2 <= SZ(V); pw *= 2, ++k) {
            jmp.emplace_back(sz(V) - pw * 2 + 1);
            FOR(j, 0, SZ(jmp[k]))
                jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
        }
    }
    T query(int a, int b) {
        assert(a < b); // or return inf if a == b
        int dep = 31 - __builtin_clz(b - a);
        return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
    }
};
```

MoQueries.h  
**Description:** Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge  $(a, c)$  and remove the initial add call (but keep in).  
**Time:**  $\mathcal{O}(N\sqrt{Q})$

```
c9cf1e, 48 lines

void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
```

```
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
    int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
    vi s(SZ(Q)), res = s;
#define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) {
        pii q = Q[qi];
        while(L > q.st) add(--L, 0);
        while(R < q.nd) add(R++, 1);
        while(L < q.st) del(L++, 0);
        while(R > q.nd) del(--R, 1);
        res[qi] = calc();
    }
    return res;
}

vi moTree(vector<array<int,2>> Q, vector<vi> &ed, int root=0){
    int N = SZ(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
    vi s(SZ(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
    add(0, 0), in[0] = 1;
    auto dfs = [&](int x, int p, int dep, auto& f) -> void {
        par[x] = p; L[x] = N;
        if(dep) I[x] = N++;
        for(int y: ed[x]) if (y != p) f(y, x, !dep, f);
        if(!dep) I[x] = N++;
        R[x] = N;
    };
    dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for(int qi: s) FOR(end, 0, 2) {
        int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
        else { add(c, end); in[c] = 1; } a = c; }
        while(!(L[b] <= L[a] && R[a] <= R[b])) {
            I[i++] = b, b = par[b]; }
        while(a != b) step(par[a]);
        while(i--) step(I[i]);
        if(end) res[qi] = calc();
    }
    return res;
}
```

ExtendedLiChao.h

**Description:** Supports Range Line Insertion, Range Line Addition, Point Query. Can be modified to support Range Maximum Query, if Line Addition is replaced by Sum Addition.

**Time:**  $\mathcal{O}(\log^2 N)$

31b9af, 90 lines

```
template<typename T>
struct Line {
    T a, b;
    Line() : a(0), b(-inf) {}
    Line(T a, T b) : a(a), b(b) {}
    T get(T x) { return a * x + b; }
    void add(Line x) { a += x.a; b += x.b; }
};

struct Node {
    Line<T> line = Line<T>();
    Line<T> lazy = Line<T>(0, 0);
    Node *lc = nullptr, *rc = nullptr;
    void apply(T l, T r, Line<T> v) { line.add(v); lazy.add(v); }
};
```

```
void PushLazy(Node* &n, T tl, T tr) {
    if(n == nullptr) return;
    if(n->lc == nullptr) n->lc = new Node();
    if(n->rc == nullptr) n->rc = new Node();
    T mid = (tl + tr) / 2;
    n->lc->apply(tl, mid, n->lazy);
    n->rc->apply(mid + 1, tr, n->lazy);
    n->lazy = Line<T>(0, 0);
}

void PushLine(Node* &n, T tl, T tr) {
    if(n == nullptr) return;
    T mid = (tl + tr) / 2;
    InsertLineKnowingly(n->lc, tl, mid, n->line);
    InsertLineKnowingly(n->rc, mid + 1, tr, n->line);
    n->line = Line<T>();
}

void InsertLineKnowingly(Node* &n, T tl, T tr, Line<T> x) {
    if(n == nullptr) n = new Node();
    if(n->line.get(tl) < x.get(tl)) swap(n->line, x);
    if(n->line.get(tr) >= x.get(tr)) return;
    if(tl == tr) return;
    T mid = (tl + tr) / 2;
    PushLazy(n, tl, tr);
    if (n->line.get(mid) > x.get(mid)) {
        InsertLineKnowingly(n->rc, mid + 1, tr, x); }
    else {
        swap(n->line, x);
        InsertLineKnowingly(n->lc, tl, mid, x);
    }
}

void InsertLine(Node* &n, T tl, T tr, T l, T r, Line<T> x) {
    if(tr < l || r < tl || tl > tr || l > r) return;
    if(n == nullptr) n = new Node();
    if(l <= tl && tr <= r) return InsertLineKnowingly(n, tl, tr, x);
    T mid = (tl + tr) / 2;
    PushLazy(n, tl, tr);
    InsertLine(n->lc, tl, mid, l, r, x);
    InsertLine(n->rc, mid + 1, tr, l, r, x);
}

void AddLine(Node* &n, T tl, T tr, T l, T r, Line<T> x) {
    if(tr < l || r < tl || tl > tr || l > r) return;
    if(n == nullptr) n = new Node();
    if(l <= tl && tr <= r) return n->apply(tl, tr, x);
    T mid = (tl + tr) / 2;
    PushLazy(n, tl, tr);
    PushLine(n, tl, tr);
    AddLine(n->lc, tl, mid, l, r, x);
    AddLine(n->rc, mid + 1, tr, l, r, x);
}

T Query(Node* &n, T tl, T tr, T x) {
    if(n == nullptr) return -inf;
    if(tl == tr) return n->line.get(x);
    T res = n->line.get(x);
    T mid = (tl + tr) / 2;
    PushLazy(n, tl, tr);
    if(x <= mid) res = max(res, Query(n->lc, tl, mid, x));
    else res = max(res, Query(n->rc, mid + 1, tr, x));
    return res;
}

void InsertLine(T l, T r, Line<T> x) {
    return InsertLine(root, 0, sz - 1, l, r, x);
}
```

```
}

void AddLine(T l, T r, Line<T> x) {
    return AddLine(root, 0, sz - 1, l, r, x);
}

T Query(T x) { return Query(root, 0, sz - 1, x); }
```

Numerical (3)

3.1 Polynomials and recurrences

Polynomial.h

81a226, 17 lines

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for (int i = SZ(a); i--;) (val *= x) += a[i];
        return val;
    }
    void diff() {
        FOR(i,1,SZ(a)) a[i-1] = i*a[i];
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=SZ(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};
```

PolyRoots.h

**Description:** Finds the real roots to a polynomial.  
**Usage:** polyRoots({{2,-3,1}},-1e9,1e9) // solve x<sup>2</sup>-3x+2 = 0  
**Time:**  $\mathcal{O}(n^2 \log(1/\epsilon))$

"Polynomial.h"

aa01df, 21 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax) {
    if (SZ(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
    vector<double> ret;
    Poly der = p; der.diff();
    auto dr = polyRoots(der, xmin, xmax);
    dr.pb(xmin-1); dr.pb(xmax+1);
    sort(all(dr));
    FOR(i,0,SZ(dr)-1) {
        double l = dr[i], h = dr[i+1];
        bool sign = p(l) > 0;
        if (sign ^ (p(h) > 0)) {
            FOR(it,0,60) { // while (h - l > 1e-8)
                double m = (l + h) / 2, f = p(m);
                if ((f <= 0) ^ sign) l = m;
                else h = m;
            }
            ret.pb((l + h) / 2);
        }
    }
    return ret;
}
```

PolyInterpolate.h

**Description:** Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them:  $p(x) = a[0] * x^0 + ... + a[n - 1] * x^{n-1}$ . For numerical precision, pick  $x[k] = c * \cos(k/(n - 1) * \pi), k = 0 ... n - 1$ .  
**Time:**  $\mathcal{O}(n^2)$

c57814, 13 lines

```
using vd = vector<double>;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
```

```
FOR(k, 0, n-1) FOR(i, k+1, n) {
    y[i] = (y[i] - y[k]) / (x[i] - x[k]); }
double last = 0; temp[0] = 1;
FOR(k, 0, n) FOR(i, 0, n) {
    res[i] += y[k] * temp[i];
    swap(last, temp[i]);
    temp[i] -= last * x[k];
}
return res;
}
```

BerlekampMassey.h

**Description:** Recovers any  $n$ -order linear recurrence relation from the first  $2n$  terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size  $\leq n$ .  
**Usage:** berlekampMassey{0, 1, 1, 3, 5, 11} // {1, 2}  
**Time:**  $\mathcal{O}(N^2)$

```
vector<ll> berlekampMassey(vector<ll> s) {
    int n = SZ(s), L = 0, m = 0;
    vector<ll> C(n), B(n), T;
    ll b = C[0] = B[0] = 1;
    FOR(i, 0, n) {
        ++m; ll d = s[i] % mod;
        FOR(j, 1, L+1) d = (d + C[j] * s[i - j]) % mod;
        if(!d) continue;
        T = C; ll coef = d * modpow(b, mod-2) % mod;
        FOR(j, m, n) C[j] = (C[j] - coef * B[j - m]) % mod;
        if(2 * L > i) continue;
        L = i + 1 - L; B = T; b = d; m = 0;
    }
    C.resize(L + 1); C.erase(C.begin());
    for(ll &x: C) x = (mod - x) % mod;
    return C;
}
```

LinearRecurrence.h

**Description:** Generates the  $k$ 'th term of an  $n$ -order linear recurrence  $S[i] = \sum_j S[i - j - 1]tr[j]$ , given  $S[0 \dots \geq n - 1]$  and  $tr[0 \dots n - 1]$ . Faster than matrix multiplication. Useful together with Berlekamp–Massey.  
**Usage:** linearRec{0, 1}, {1, 1}, k) // k'th Fibonacci number  
**Time:**  $\mathcal{O}(n^2 \log k)$

```
using Poly = vector<ll>;
ll linearRec(Poly S, Poly tr, ll k) {
    int n = SZ(tr);
    auto combine = [&](Poly a, Poly b) {
        Poly res(n * 2 + 1);
        FOR(i, 0, n+1) FOR(j, 0, n+1) {
            res[i + j] = (res[i + j] + a[i] * b[j]) % mod; }
        for(int i = 2 * n; i > n; --i) FOR(j, 0, n) {
            res[i-1 - j] = (res[i-1 - j] + res[i] * tr[j]) % mod; }
        res.resize(n + 1);
        return res;
    };
    Poly pol(n + 1), e(pol); pol[0] = e[1] = 1;
    for(++k; k; k /= 2) {
        if(k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }
    ll res = 0;
    FOR(i, 0, n) res = (res + pol[i + 1] * S[i]) % mod;
    return res;
}
```

3.2 Optimization

Integrate.h

**Description:** Simple integration of a function over an interval using Simpson's rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

```
template<class F>
double quad(double a, double b, F f, const int n = 1000) {
    double h = (b - a) / 2 / n, v = f(a) + f(b);
    FOR(i, 1, n*2) v += f(a + i*h) * (i&1 ? 4 : 2);
    return v * h / 3;
}

template<class F>
D rec(F &f, D a, D b, D eps, D S) {
    D c = (a + b) / 2, S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
    if(abs(T - S) <= 15 * eps || b - a < 1e-10) {
        return T + (T - S) / 15; }
    return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}

template<class F>
D quad(D a, D b, F f, D eps = 1e-8) {
    return rec(f, a, b, eps, S(a, b));
}
```

```
using D = double;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template<class F>
D rec(F &f, D a, D b, D eps, D S) {
    D c = (a + b) / 2, S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
    if(abs(T - S) <= 15 * eps || b - a < 1e-10) {
        return T + (T - S) / 15; }
    return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}

template<class F>
D quad(D a, D b, F f, D eps = 1e-8) {
    return rec(f, a, b, eps, S(a, b));
}
```

Simplex.h

**Description:** Solves a general linear maximization problem: maximize  $c^T x$  subject to  $Ax \leq b, x \geq 0$ . Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of  $c^T x$  otherwise. The input vector is set to an optimal  $x$  (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that  $x = 0$  is viable.  
**Usage:** vvd A = {{1,-1}, {-1,1}, {-1,-2}};  
vd b = {1,1,-4}, c = {-1,-1}, x;  
T val = LPSolver(A, b, c).solve(x);  
**Time:**  $\mathcal{O}(NM * \#pivots)$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case.

```
using T = double; // long double, Rational, double + modP>...
using vd = vector<T>;
using vvd = vector<vvd>;

const T eps = 1e-8, inf = 1/.0;
#define ltj(X) if(s == -1 || mp(X[j],N[j]) < mp(X[s],N[s])) s=j

struct LPSolver {
    int m, n; vi N, B; vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(SZ(b)), n(SZ(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        FOR(i,0,m) FOR(j,0,n) D[i][j] = A[i][j];
        FOR(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
        FOR(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }

    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
```

```
FOR(i, 0, m+2) if(i != r && abs(D[i][s]) > eps) {
    T *b = D[i].data(), inv2 = b[s] * inv;
    FOR(j, 0, n+2) b[j] -= a[j] * inv2;
    b[s] = a[s] * inv2;
}
FOR(j, 0, n+2) if(j != s) D[r][j] *= inv;
FOR(i, 0, m+2) if(i != r) D[i][s] *= -inv;
D[r][s] = inv; swap(B[r], N[s]);
}

bool simplex(int phase) {
    int x = m + phase - 1;
    for(;;) {
        int s = -1;
        FOR(j, 0, n+1) if(N[j] != -phase) ltj(D[x]);
        if(D[x][s] >= -eps) return 1;
        int r = -1;
        FOR(i,0,m) {
            if(D[i][s] <= eps) continue;
            if(r == -1 || mp(D[i][n+1] / D[i][s], B[i])
                < mp(D[r][n+1] / D[r][s], B[r])) r = i;
        }
        if(r == -1) return 0;
        pivot(r, s);
    }
}
```

```
T solve(vd &x) {
    int r = 0;
    FOR(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
        pivot(r, n);
        if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        FOR(i,0,m) if (B[i] == -1) {
            int s = 0;
            FOR(j,1,n+1) ltj(D[i]);
            pivot(i, s);
        }
    }
    bool ok = simplex(1); x = vd(n);
    FOR(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
}
};
```

3.3 Matrices

Determinant.h

**Description:** Calculates determinant of a matrix. Destroys the matrix.  
**Time:**  $\mathcal{O}(N^3)$

```
double det(vector<vector<double>> &a) {
    int n = SZ(a); double res = 1;
    FOR(i, 0, n) {
        int b = i;
        FOR(j, i+1, n) if(fabs(a[j][i]) > fabs(a[b][i])) b = j;
        if(i != b) swap(a[i], a[b]), res *= -1;
        res *= a[i][i];
        if(res == 0) return 0;
        FOR(j, i+1, n) {
            double v = a[j][i] / a[i][i];
            if(v != 0) FOR(k, i+1, n) a[j][k] -= v * a[i][k];
        }
    }
    return res;
}
```

### IntDeterminant.h

**Description:** Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.

**Time:**  $\mathcal{O}(N^3)$

```
46be95, 16 lines
const ll mod = 12345;
ll det(vector<vector<ll>> &a) {
    int n = SZ(a); ll ans = 1;
    FOR(i, 0, n) {
        FOR(j, i+1, n) while(a[j][i] != 0) { // gcd step
            ll t = a[i][i] / a[j][i];
            if(t) FOR(k, i, n) {
                a[i][k] = (a[i][k] - a[j][k] * t) % mod; }
            swap(a[i], a[j]);
            ans *= -1;
        }
        ans = ans * a[i][i] % mod;
        if(!ans) return 0;
    }
    return (ans + mod) % mod;
}
```

### SolveLinear.h

**Description:** Solves  $A*x=b$ . If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in  $A$  and  $b$  is lost.

**Time:**  $\mathcal{O}(n^2m)$

```
cfaaca, 33 lines
using vd = vector<double>;
const double eps = 1e-12;

int solveLinear(vector<vd> &A, vd &b, vd &x) {
    int n = SZ(A), m = SZ(x), rank = 0, br, bc;
    if(n) assert(SZ(A[0]) == m);
    vi col(m); iota(all(col), 0);
    FOR(i, 0, n) {
        double v, bv = 0;
        FOR(r, i, n) FOR(c, i, m) if((v = fabs(A[r][c])) > bv) {
            br = r, bc = c, bv = v; }
        if(bv <= eps) {
            FOR(j, i, n) if(fabs(b[j]) > eps) return -1;
            break;
        }
        swap(A[i], A[br]); swap(b[i], b[br]);
        swap(col[i], col[bc]);
        FOR(j, 0, n) swap(A[j][i], A[j][bc]);
        bv = 1 / A[i][i];
        FOR(j, i+1, n) {
            double fac = A[j][i] * bv;
            b[j] -= fac * b[i];
            FOR(k, i+1, m) A[j][k] -= fac*A[i][k];
        }
        rank++;
    }
    x.assign(m, 0);
    for(int i = rank; i--;) {
        b[i] /= A[i][i]; x[col[i]] = b[i];
        FOR(j,0,i) b[j] -= A[j][i] * b[i];
    }
    return rank; // (multiple solutions if rank < m)
}
```

### SolveLinear2.h

**Description:** To get all uniquely determined values of  $x$  back from SolveLinear, make the following changes:

```
a561f5, 7 lines
"SolveLinear.h"
FOR(j ,0, n) if(j != i) // instead of FOR(j, i+1, n)
// ... then at the end:
x.assign(m, undefined);
FOR(i, 0, rank) {
```

```
FOR(j, rank, m) if(fabs(A[i][j]) > eps) goto fail;
x[col[i]] = b[i] / A[i][i];
fail;; }
```

### SolveLinearBinary.h

**Description:** Solves  $Ax=b$  over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys  $A$  and  $b$ .

**Time:**  $\mathcal{O}(n^2m)$

```
038972, 26 lines
using bs = bitset<1000>;
int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
    int n = SZ(A), rank = 0, br;
    assert(m <= SZ(x));
    vi col(m); iota(all(col), 0);
    FOR(i, 0, n) {
        for(br = i; br < n; br++) if(A[br].any()) break;
        if(br == n) {
            FOR(j,i,n) if(b[j]) return -1;
            break; }
        int bc = (int)A[br]._Find_next(i-1);
        swap(A[i], A[br]); swap(b[i], b[br]);
        swap(col[i], col[bc]);
        FOR(j, 0, n) if(A[j][i] != A[j][bc]) {
            A[j].flip(i); A[j].flip(bc); }
        FOR(j, i+1, n) if(A[j][i]) b[j] ^= b[i], A[j] ^= A[i];
        rank++;
    }
    x = bs();
    for(int i = rank; i--;) {
        if(!b[i]) continue;
        x[col[i]] = 1;
        FOR(j, 0, i) b[j] ^= A[j][i];
    }
    return rank; // (multiple solutions if rank < m)
}
```

### MatrixInverse.h

**Description:** Invert matrix  $A$ . Returns rank; result is stored in  $A$  unless singular (rank < n). Can easily be extended to prime moduli; for prime powers, repeatedly set  $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$  where  $A^{-1}$  starts as the inverse of  $A \pmod{p}$ , and  $k$  is doubled in each step.

**Time:**  $\mathcal{O}(n^3)$

```
bc0e55, 31 lines
int matInv(vector<vector<double>> &A) {
    int n = SZ(A); vi col(n);
    vector<vector<double>> tmp(n, vector<double>(n));
    FOR(i, 0, n) tmp[i][i] = 1, col[i] = i;
    FOR(i, 0, n) {
        int r = i, c = i;
        FOR(j, i, n) FOR(k, i, n) {
            if(fabs(A[j][k]) > fabs(A[r][c])) r = j, c = k; }
        if(fabs(A[r][c]) < 1e-12) return i;
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        FOR(j, 0, n) {
            swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]); }
        swap(col[i], col[c]);
        double v = A[i][i];
        FOR(j, i+1, n) {
            double f = A[j][i] / v;
            A[j][i] = 0;
            FOR(k, i+1, n) A[j][k] -= f*A[i][k];
            FOR(k, 0, n) tmp[j][k] -= f*tmp[i][k];
        }
        FOR(j, i+1, n) A[i][j] /= v;
        FOR(j, 0, n) tmp[i][j] /= v;
        A[i][i] = 1;
    }
    per(i, 0, n-1) FOR(j, 0, i) {
        double v = A[j][i];
```

```
FOR(k, 0, n) tmp[j][k] -= v*tmp[i][k];
    }
    FOR(i, 0, n) FOR(j, 0, n) A[col[i]][col[j]] = tmp[i][j];
    return n;
}
```

### MatrixInverse-mod.h

**Description:** Invert matrix  $A$  modulo a prime. Returns rank; result is stored in  $A$  unless singular (rank < n). For prime powers, repeatedly set  $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$  where  $A^{-1}$  starts as the inverse of  $A \pmod{p}$ , and  $k$  is doubled in each step.

**Time:**  $\mathcal{O}(n^3)$

```
d36df0, 34 lines
int matInv(vector<vector<ll>>& A) {
    int n = SZ(A); vi col(n);
    vector<vector<ll>> tmp(n, vector<ll>(n));
    FOR(i, 0, n) tmp[i][i] = 1, col[i] = i;
    FOR(i, 0, n) {
        int r = i, c = i;
        FOR(j, i, n) FOR(k, i, n) if(A[j][k]) {
            r = j; c = k; goto found; }
        return i;
    found:
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        FOR(j, 0, n) {
            swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]); }
        swap(col[i], col[c]);
        ll v = modpow(A[i][i], mod - 2);
        FOR(j, i+1, n) {
            ll f = A[j][i] * v % mod;
            A[j][i] = 0;
            FOR(k, i+1, n) A[j][k] = (A[j][k] - f*A[i][k]) % mod;
            FOR(k,0,n) tmp[j][k] = (tmp[j][k] - f*tmp[i][k]) % mod;
        }
        FOR(j, i+1, n) A[i][j] = A[i][j] * v % mod;
        FOR(j, 0, n) tmp[i][j] = tmp[i][j] * v % mod;
        A[i][i] = 1;
    }
    per(i, 0, n-1) FOR(j, 0, i) {
        ll v = A[j][i];
        FOR(k, 0, n) tmp[j][k] = (tmp[j][k] - v*tmp[i][k]) % mod;
    }
    FOR(i, 0, n) FOR(j, 0, n) {
        A[col[i]][col[j]] = tmp[i][j] % mod +
            (tmp[i][j] < 0 ? mod : 0); }
    return n;
}
```

### Tridiagonal.h

**Description:**  $x = \text{tridiagonal}(d,p,q,b)$  solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, \, 1 \leq i \leq n,$$

where  $a_0, a_{n+1}, b_i, c_i$  and  $d_i$  are known.  $a$  can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, \dots, -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.

If  $|d_i| > |p_i| + |q_{i-1}|$  for all  $i$ , or  $|d_i| > |p_{i-1}| + |q_i|$ , or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.

<b>Time:</b> $\mathcal{O}(N)$	736a43, 24 lines
<pre>using T = double; vector&lt;T&gt; tridiagonal(vector&lt;T&gt; diag, const vector&lt;T&gt; &amp;super,     const vector&lt;T&gt; &amp;sub, vector&lt;T&gt; b) {     int n = SZ(b); vi tr(n);     FOR(i, 0, n-1) {         if(abs(diag[i]) &lt; 1e-9 * abs(super[i])) { // diag[i] == 0             b[i+1] -= b[i] * diag[i+1] / super[i];             if(i+2 &lt; n) b[i+2] -= b[i] * sub[i+1] / super[i];             diag[i+1] = sub[i]; tr[i+=1] = 1; }         else {             diag[i+1] -= super[i]*sub[i]/diag[i];             b[i+1] -= b[i]*sub[i]/diag[i]; }     }     for(int i = n; i--;) {         if(tr[i]) {             swap(b[i], b[i-1]);             diag[i-1] = diag[i];             b[i] /= super[i-1]; }         else {             b[i] /= diag[i];             if(i) b[i-1] -= b[i]*super[i-1]; }     }     return b; }</pre>	

### 3.4 Fourier transforms

**FastFourierTransform.h**  
**Description:** fft(a) computes  $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$  for all  $k$ .  $N$  must be a power of 2. Useful for convolution: conv(a, b) = c, where  $c[x] = \sum a[i]b[x-i]$ . For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by n, reverse(start+1, end), FFT back. Rounding is safe if  $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$  (in practice  $10^{16}$ ; higher for random inputs). Otherwise, use NTT/FFTMod.  
**Time:**  $\mathcal{O}(N \log N)$  with  $N = |A| + |B|$  ( $\sim 1$ s for  $N = 2^{22}$ )

<pre>using C = complex&lt;double&gt;; using vd = vector&lt;double&gt;; void fft(vector&lt;C&gt;&amp; a) {     int n = SZ(a), L = 31 - __builtin_clz(n);     static vector&lt;complex&lt;long double&gt;&gt; R(2, 1);     static vector&lt;C&gt; rt(2, 1); // (^ 10% faster if double)     for(static int k = 2; k &lt; n; k *= 2) {         R.resize(n); rt.resize(n);         auto x = polar(1.0L, acos(-1.0L) / k);         FOR(i, k, 2*k) rt[i] = R[i] = i&amp;1 ? R[i/2] * x : R[i/2];     }     vi rev(n);     FOR(i, 0, n) rev[i] = (rev[i / 2]   (i &amp; 1) &lt;&lt; L) / 2;     FOR(i, 0, n) if(i &lt; rev[i]) swap(a[i], a[rev[i]]);     for(int k = 1; k &lt; n; k *= 2)         for(int i = 0; i &lt; n; i += 2 * k) FOR(j,0,k) {             C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-rolled)             a[i + j + k] = a[i + j] - z; a[i + j] += z;         } } vd conv(const vd&amp; a, const vd&amp; b) {     if(a.empty()    b.empty()) return {};     vd res(SZ(a) + SZ(b) - 1);     int L = 32 - __builtin_clz(SZ(res)), n = 1 &lt;&lt; L;     vector&lt;C&gt; in(n), out(n); copy(all(a), begin(in));     FOR(i,0,SZ(b)) in[i].imag(b[i]);     fft(in); for(C &amp;x : in) x *= x;     FOR(i, 0, n) out[i] = in[-i &amp; (n - 1)] - conj(in[i]);     fft(out);     FOR(i, 0, SZ(res)) res[i] = imag(out[i]) / (4 * n);     return res; }</pre>	87d73c, 32 lines
---	------------------

<b>FastFourierTransformMod.h</b> <b>Description:</b> Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$ (in practice $10^{16}$ or higher). Inputs must be in $[0, \text{mod})$ . <b>Time:</b> $\mathcal{O}(N \log N)$ , where $N =  A  +  B $ (twice as slow as NTT or FFT) <i>"FastFourierTransform.h"</i>	db1041, 22 lines
<pre>using vll = vector&lt;ll&gt;; template&lt;int M&gt; vll convMod(const vll &amp;a, const vll &amp;b) {     if(a.empty()    b.empty()) return {};     vll res(SZ(a) + SZ(b) - 1);     int B=32-__builtin_clz(SZ(res)), n=1&lt;&lt;B, cut=int(sqrt(M));     vector&lt;C&gt; L(n), R(n), outs(n), outl(n);     FOR(i,0,SZ(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);     FOR(i,0,SZ(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);     fft(L), fft(R);     FOR(i,0,n) {         int j = -i &amp; (n - 1);         outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);         outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / li;     }     fft(outl), fft(outs);     FOR(i, 0, SZ(res)) {         ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);         ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);         res[i] = ((av % M * cut + bv) % M * cut + cv) % M;     }     return res; }</pre>	

#### NumberTheoreticTransform.h

**Description:** ntt(a) computes  $\hat{f}(k) = \sum_x a[x]g^{xk}$  for all  $k$ , where  $g = \text{root}^{(\text{mod}-1)/N}$ .  $N$  must be a power of 2. Useful for convolution modulo specific nice primes of the form  $2^a b + 1$ , where the convolution result has size at most  $2^a$ . For arbitrary modulo, see FFTMod. conv(a, b) = c, where  $c[x] = \sum a[i]b[x-i]$ . For manual convolution: NTT the inputs, multiply pointwise, divide by n, reverse(start+1, end), NTT back. Inputs must be in  $[0, \text{mod})$ .  
**Time:**  $\mathcal{O}(N \log N)$

<pre>const ll mod = (119 &lt;&lt; 23) + 1, root = 62; // = 998244353 // For p &lt; 2^30 there is also e.g. 5 &lt;&lt; 25, 7 &lt;&lt; 26, 479 &lt;&lt; 21 // and 483 &lt;&lt; 21 (same root). The last two are &gt; 10^9. using vll = vector&lt;ll&gt;; void ntt(vll &amp;a) {     int n = SZ(a), L = 31 - __builtin_clz(n);     static vll rt(2, 1);     for(static int k = 2, s = 2; k &lt; n; k *= 2, s++) {         rt.resize(n);         ll z[] = {1, modpow(root, mod &gt;&gt; s)};         FOR(i, k, 2*k) rt[i] = rt[i / 2] * z[i &amp; 1] % mod;     }     vi rev(n);     FOR(i, 0, n) rev[i] = (rev[i / 2]   (i &amp; 1) &lt;&lt; L) / 2;     FOR(i, 0, n) if(i &lt; rev[i]) swap(a[i], a[rev[i]]);     for(int k = 1; k &lt; n; k *= 2)         for(int i = 0; i &lt; n; i += 2 * k) FOR(j,0,k) {             ll z = rt[j + k] * a[i + j + k] % mod, &amp;ai = a[i + j];             a[i + j + k] = ai - z + (z &gt; ai ? mod : 0);             ai += (ai + z &gt;= mod ? z - mod : z);         } } vll conv(const vll &amp;a, const vll &amp;b) {     if(a.empty()    b.empty()) return {};     int s = SZ(a) + SZ(b) - 1, B = 32 - __builtin_clz(s);     int n = 1 &lt;&lt; B, inv = modpow(n, mod - 2);     vll L(a), R(b), out(n);     L.resize(n), R.resize(n);     ntt(L), ntt(R);     FOR(i,0,n) out[-i &amp; (n-1)] = (ll)L[i]*R[i] %mod * inv %mod;</pre>	040a10, 33 lines
---	------------------

<pre>ntt(out); return {out.begin(), out.begin() + s}; }</pre>	
---	--

#### FastSubsetTransform.h

**Description:** Transform to a basis with fast convolutions of the form  $c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of  $a$  must be a power of two.  
**Time:**  $\mathcal{O}(N \log N)$

	846b5e, 16 lines
<pre>void FST(vi &amp;a, bool inv) {     for(int n = SZ(a), step = 1; step &lt; n; step *= 2) {         for(int i = 0; i &lt; n; i += 2 * step) FOR(j, i, i+step) {             int &amp;u = a[j], &amp;v = a[j + step]; tie(u, v) =                 inv ? pii(v - u, u) : pii(v, u + v); // AND                 inv ? pii(v, u - v) : pii(u + v, u); // OR                 pii(u + v, u - v); // XOR         }     }     if(inv) for(int &amp;x: a) x /= SZ(a); // XOR only } vi conv(vi a, vi b) {     FST(a, 0); FST(b, 0);     FOR(i, 0, SZ(a)) a[i] *= b[i];     FST(a, 1); return a; }</pre>	

#### PolynomialDivision.h

**Description:** Pseudocode for polynomial division.

	1066f2, 20 lines
<pre>vi polydiv(vi a, vi b) {     int n = SZ(a), m = SZ(b);     if(m &gt; n) return {0};     reverse(all(a)), reverse(all(b));     int s = moddiv(1, b[0], MOD);     for(int &amp;x : b) x = modmul(x, s, MOD);     int k = n - m + 1;     vi r, v, g = {1};     for(int w = 1; (1 &lt;&lt; (w - 1)) &lt; k; w++){         r = polymul(g, g); r.resize(1 &lt;&lt; w);         v = b; v.resize(1 &lt;&lt; w);         r = polymul(r, v);         r.resize(1 &lt;&lt; w); g.resize(1 &lt;&lt; w);         polyadd(g, g); polysub(g, r);     }     g = polymul(g, a); g.resize(k);     reverse(all(g));     for(int &amp;x : g) x = modmul(x, s, MOD);     return g; }</pre>	

## Number theory (4)

### 4.1 Modular arithmetic

<b>ModMulLL.h</b> <b>Description:</b> Calculate $a \cdot b \bmod c$ (or $a^b \bmod c$ ) for $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$ . <b>Time:</b> $\mathcal{O}(1)$ for modmul, $\mathcal{O}(\log b)$ for modpow	4c7c4a, 11 lines
--	------------------

<pre>using ull = unsigned long long; ull modmul(ull a, ull b, ull M) {     ll ret = a * b - M * ull(1.L / M * a * b);     return ret + M * (ret &lt; 0) - M * (ret &gt;= (ll)M); } ull modpow(ull b, ull e, ull mod) {     ull ans = 1;     for (; e; b = modmul(b, b, mod), e /= 2)         if (e &amp; 1) ans = modmul(ans, b, mod);</pre>	
--	--

```

11 calc (ll x) {
  if (x <= th) return p_f (x);
  auto d = mem.find (x);
  if (d != mem.end ()) return d -> second;
  ll ans = 0;
  for (ll i = 2, la; i <= x; i = la + 1) {
    la = x / (x / i);
    ans = ans + (p_g(la) - p_g(i - 1) + mod) * calc(x / i);
  }
  ans = p_c(x) - ans; ans = ans / inv;
  return mem[x] = ans;
}

```



```
ll solve (ll n, ll th) {
    if (n <= 0) return 0;
    prefix_mul::n = n; prefix_mul::th = th;
    inv = p_g(1);
    return calc(n);
}
};
```

4.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$

with  $m > n > 0, k > 0, m \perp n$ , and either  $m$  or  $n$  even.

4.6 Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$
$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$$
$$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m)g(\lfloor \frac{n}{m} \rfloor)$$

Combinatorial (5)

5.1 Permutations

5.1.1 Burnside’s lemma

Given a group  $G$  of symmetries and a set  $X$ , the number of elements of  $X$  up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by  $g$  ( $g.x = x$ ).

If  $f(n)$  counts “configurations” (of some sort) of length  $n$ , we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k)\phi(n/k).$$

5.2 General purpose numbers

5.2.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t - 1}$  (FFT-able).  
 $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^n n^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m)$$
$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

5.2.2 Stirling numbers of the first kind

Number of permutations on  $n$  items with  $k$  cycles.

$$c(n, k) = c(n - 1, k - 1) + (n - 1)c(n - 1, k), \quad c(0, 0) = 1$$
$$\sum_{k=0}^n c(n, k)x^k = x(x + 1) \dots (x + n - 1)$$

$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$   
 $c(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$

5.2.3 Eulerian numbers

Number of permutations  $\pi \in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k$   $j$ :s s.t.  $\pi(j) > \pi(j + 1)$ ,  $k + 1$   $j$ :s s.t.  $\pi(j) \geq j$ ,  $k$   $j$ :s s.t.  $\pi(j) > j$ .

$$E(n, k) = (n - k)E(n - 1, k - 1) + (k + 1)E(n - 1, k)$$
$$E(n, 0) = E(n, n - 1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

5.2.4 Stirling numbers of the second kind

Partitions of  $n$  distinct elements into exactly  $k$  groups.

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$
$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$$

5.2.5 Bell numbers

Total number of partitions of  $n$  distinct elements.  $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$  For  $p$  prime,

$$B(p^m + n) \equiv mB(n) + B(n + 1) \pmod{p}$$

5.2.6 Labeled unrooted trees

# on  $n$  vertices:  $n^{n-2}$   
# on  $k$  existing trees of size  $n_i$ :  $n_1 n_2 \dots n_k n^{k-2}$   
# with degrees  $d_i$ :  $(n - 2)! / ((d_1 - 1)! \dots (d_n - 1)!)$

MatroidIntersection.h

**Description:** Given two matroids, finds the largest common independent set. For the color and graph matroids, this would be the largest forest where no two edges are the same color. Pass the matroid with more expensive operations to M1.  
**Time:**  $\mathcal{O}(N^3)$ , in practice  $\mathcal{O}(N^2)$  oracle calls

858b54, 43 lines

```
template <class M1, class M2> struct MatroidIsect {
    M1 m1; M2 m2; int n; vi iset;
    MatroidIsect(M1 _m1, M2 _m2, int _n) :
        m1(_m1), m2(_m2), n(_n), iset(n + 1) {}
    vi solve() {
        per(i, 0, n-1) if(m1.check(i) && m2.check(i)) {
            iset[i] = 1, m1.add(i), m2.add(i);
        }
        while(augment());
        vi ans;
        FOR(i, 0, n) if(iset[i]) ans.pb(i);
        return ans;
    }
    bool augment() {
        vi frm(n, -1); queue<int> q({n}); // starts at dummy node
        auto fwdE = [&](int a) {
            vi ans; m1.clear();
            FOR(v, 0, n) if(iset[v] && v != a) m1.add(v);
            FOR(b, 0, n) if(!iset[b] && frm[b] == -1 && m1.check(b))
                ans.pb(b), frm[b] = a;
            return ans;
        };
        auto backE = [&](int b) {
            m2.clear();
            FOR(cas, 0, 2) FOR(v, 0, n)
                if((v == b || iset[v]) && (frm[v] == -1) == cas) {
                    if(!m2.check(v))
                        return cas ? q.push(v), frm[v] = b, v : -1;
                    m2.add(v);
                }
            return n;
        };
        while(!q.empty()) {
            int a = q.front(), c; q.pop();
            for(int b: fwdE(a))
                while((c = backE(b)) >= 0) if(c == n) {
                    while(b != n) iset[b] ^= 1, b = frm[b];
                    return 1;
                }
        }
        return 0;
    }
};
```

MatroidOracles.h

**Description:** An oracle has 3 functions: check(int x): returns if current matroid can add x without becoming dependent; add(int x): adds an element to the matroid (guaranteed to never make it dependent); clear(): sets the matroid to the empty matroid.  
**Time:**  $\mathcal{O}(\log N)$

7eef8c, 42 lines

```
struct ColorMat {
    vi clr, bound, cnt; int max_bound, offset = 0;
    ColorMat(vi _clr, vi _bound) : clr(_clr), bound(_bound) {
        cnt.resize(SZ(bound));
        max_bound = *max_element(all(bound));
    }
};
```

```
    }
    void fix(int x) { cnt[clr[x]] = max(cnt[clr[x]], offset); }
    bool check(int x) {
        fix(x); return cnt[clr[x]] - offset < bound[clr[x]];
    }
    void add(int x) { fix(x); cnt[clr[x]]++; }
    void clear() { offset += max_bound; }
};

struct GraphicMat { // use normal UF to reduce memory
    vector<pii> edg; UF uf; vi vis;
    GraphicMat(vector<pii> _edg) : edg(_edg), uf(0) {
        int max_n = 0;
        for(auto &[x, y]: edg) max_n = max({max_n, x, y});
        uf = UF(max_n + 1);
    }
    bool check(int x) {
        return uf.find(edg[x].st) != uf.find(edg[x].nd);
    }
    void add(int x) { uf.join(edg[x].st,edg[x].nd); vis.pb(x); }
    void clear() {
        for(int x: vis) uf.e[edg[x].st] = uf.e[edg[x].nd] = -1;
        vis.clear();
    }
};

struct LinearMat {
    vector<ll> vals; set<ll, greater<ll>> basis, start;
    LinearMat(vector<ll> _vals, vector<ll> _s) : vals(_vals) {
        for(ll x: _s) basis.insert(red(x));
        start = basis;
    }
    ll red(ll x) { for(ll y: basis) x = min(x, x^y); return x; }
    bool check(int x) { return red(vals[x]); }
    void add(int x) { basis.insert(red(vals[x])); }
    void clear() { basis = start; }
};
```

NimMultiplication.h  
Description: Nimber multiplication. To optimize use  $(a\oplus b)*c = a*c\oplus b*c$ .  
Time:  $\mathcal{O}((\log N)^{1.58})$  ddb872, 10 lines

```
using u64 = unsigned long long;
template<int L> inline u64 mulSlow(u64 a, u64 b) {
    static constexpr int l = L >> 1;
    const u64 a0 = a & ((1ull << l) - 1), a1 = a >> l;
    const u64 b0 = b & ((1ull << l) - 1), b1 = b >> l;
    const u64 a0b0 = mulSlow<l>(a0, b0);
    return (a0b0 ^ mulSlow<l>((1ull << (l-1), mulSlow<l>(a1,b1)))
        | (a0b0 ^ mulSlow<l>(a0 ^ a1, b0 ^ b1)) << l;
}
template<> inline u64 mulSlow<l>(u64 a, u64 b) {return a & b;}
```

## Graph (6)

### 6.1 Network flow

MinCostMaxFlow.h  
Description: Min-cost max-flow. If costs can be negative, call setpi before maxflow, but note that negative cost cycles are not supported. To obtain the actual flow, look at positive values only.  
Time:  $\mathcal{O}(FE\log(V))$  where F is max flow.  $\mathcal{O}(VE)$  for setpi. 580207, 68 lines

```
#include <bits/extc++.h>

const ll INF = numeric_limits<ll>::max() / 4;

struct MCMF {
```

```
    struct edge {
        int from, to, rev;
        ll cap, cost, flow;
    };
    int N;
    vector<vector<edge>> ed; vi seen;
    vector<ll> dist, pi; vector<edge*> par;

    MCMF(int N) : N(N), ed(N), seen(N), dist(N), pi(N), par(N) {}

    void addEdge(int from, int to, ll cap, ll cost) {
        if(from == to) return;
        ed[from].pb(edge{from, to, SZ(ed[to]), cap, cost, 0});
        ed[to].pb(edge{to, from, SZ(ed[from])-1, 0, -cost, 0});
    }

    void path(int s) {
        fill(all(seen), 0); fill(all(dist), INF);
        dist[s] = 0; ll di;
        __gnu_pbds::priority_queue<pair<ll, int>> q;
        vector<decltype(q)::point_iterator> its(N);
        q.push({0, s});
        while(!q.empty()) {
            s = q.top().nd; q.pop();
            seen[s] = 1; di = dist[s] + pi[s];
            for(edge &e: ed[s]) if(!seen[e.to]) {
                ll val = di - pi[e.to] + e.cost;
                if(e.cap - e.flow > 0 && val < dist[e.to]) {
                    dist[e.to] = val; par[e.to] = &e;
                    if(its[e.to] == q.end()) {
                        its[e.to] = q.push({-dist[e.to], e.to});
                    } else q.modify(its[e.to], {-dist[e.to], e.to});
                }
            }
        }
        FOR(i, 0, N) pi[i] = min(pi[i] + dist[i], INF);
    }

    pair<ll, ll> maxflow(int s, int t) {
        ll totflow = 0, totcost = 0;
        while(path(s), seen[t]) {
            ll fl = INF;
            for(edge* x = par[t]; x; x = par[x->from]) {
                fl = min(fl, x->cap - x->flow);
            }
            totflow += fl;
            for(edge* x = par[t]; x; x = par[x->from]) {
                x->flow += fl; ed[x->to][x->rev].flow -= fl;
            }
            FOR(i,0,N) for(edge &e: ed[i]) totcost += e.cost * e.flow;
            return {totflow, totcost / 2};
        }
    }

    // If some costs can be negative, call this before maxflow:
    void setpi(int s) { // (otherwise, leave this out)
        fill(all(pi), INF); pi[s] = 0;
        int it = N, ch = 1; ll v;
        while(ch-- && it--) FOR(i, 0, N) if(pi[i] != INF)
            for(edge &e: ed[i]) if(e.cap)
                if((v = pi[i] + e.cost) < pi[e.to]) {
                    pi[e.to] = v, ch = 1;
                }
        assert(it >= 0); // negative cost cycle
    }
};
```

```
pair<ll, ll> maxflow(int s, int t) {
    ll totflow = 0, totcost = 0;
    while(path(s), seen[t]) {
        ll fl = INF;
        for(edge* x = par[t]; x; x = par[x->from]) {
            fl = min(fl, x->cap - x->flow);
        }
        totflow += fl;
        for(edge* x = par[t]; x; x = par[x->from]) {
            x->flow += fl; ed[x->to][x->rev].flow -= fl;
        }
        FOR(i,0,N) for(edge &e: ed[i]) totcost += e.cost * e.flow;
        return {totflow, totcost / 2};
    }
}
```

```
// If some costs can be negative, call this before maxflow:
void setpi(int s) { // (otherwise, leave this out)
    fill(all(pi), INF); pi[s] = 0;
    int it = N, ch = 1; ll v;
    while(ch-- && it--) FOR(i, 0, N) if(pi[i] != INF)
        for(edge &e: ed[i]) if(e.cap)
            if((v = pi[i] + e.cost) < pi[e.to]) {
                pi[e.to] = v, ch = 1;
            }
    assert(it >= 0); // negative cost cycle
}
};
```

Dinic.h  
Description: Flow algorithm with complexity  $\mathcal{O}(VE\log U)$  where  $U = \max|cap|$ .  $\mathcal{O}(\min(E^{1/2}, V^{2/3})E)$  if  $U = 1$ ;  $\mathcal{O}(\sqrt{VE})$  for bipartite matching. 995644, 42 lines

```
struct Dinic {
    struct Edge {
        int to, rev; ll c, oc;
        ll flow() { return max(oc - c, 0ll); } // if flows needed
    };
    vi lvl, ptr, q;
    vector<vector<Edge>> adj;
    Dinic(int n) : lvl(n), ptr(n), q(n), adj(n) {}
    void addEdge(int a, int b, ll c, ll rcap = 0) {
        adj[a].pb({b, SZ(adj[b]), c, c});
        adj[b].pb({a, SZ(adj[a]) - 1, rcap, rcap});
    }
    ll dfs(int v, int t, ll f) {
        if(v == t || !f) return f;
        for(int &i = ptr[v]; i < SZ(adj[v]); i++) {
            Edge &e = adj[v][i];
            if(lvl[e.to] == lvl[v] + 1) {
                if(ll p = dfs(e.to, t, min(f, e.c))) {
                    e.c -= p, adj[e.to][e.rev].c += p;
                    return p;
                }
            }
        }
        return 0;
    }
    ll calc(int s, int t) {
        ll flow = 0; q[0] = s;
        FOR(L, 0, 31) do { // 'int L=30' maybe faster for random data
            lvl = ptr = vi(SZ(q));
            int qi = 0, qe = lvl[s] = 1;
            while(qi < qe && !lvl[t]) {
                int v = q[qi++];
                for(Edge e: adj[v])
                    if(!lvl[e.to] && e.c >> (30 - L)) {
                        q[qi++] = e.to, lvl[e.to] = lvl[v] + 1;
                    }
                while(ll p = dfs(s, t, LLONG_MAX)) flow += p;
            } while(lvl[t]);
            return flow;
        }
        bool leftOfMinCut(int a) { return lvl[a] != 0; }
    };
};
```

GlobalMinCut.h  
Description: Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.  
Time:  $\mathcal{O}(V^3)$  9ebe69, 19 lines

```
pair<int, vi> globalMinCut(vector<vi> mat) {
    pair<int, vi> best = {INT_MAX, {}};
    int n = SZ(mat); vector<vi> co(n);
    FOR(i, 0, n) co[i] = {i};
    FOR(ph, 1, n) {
        vi w = mat[0]; int s = 0, t = 0;
        FOR(it,0,n-ph) { //  $\mathcal{O}(V^2) \rightarrow \mathcal{O}(E\log V)$  with prio. queue
            w[t] = INT_MIN;
            s = t, t = max_element(all(w)) - w.begin();
            FOR(i, 0, n) w[i] += mat[t][i];
        }
        best = min(best, {w[t] - mat[t][t], co[t]});
        co[s].insert(co[s].end(), all(co[t]));
        FOR(i, 0, n) mat[s][i] += mat[t][i];
        FOR(i, 0, n) mat[i][s] = mat[s][i];
        mat[0][t] = INT_MIN;
    }
    return best;
};
```

### GomoryHu.h

**Description:** Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.

**Time:**  $\mathcal{O}(V)$  Flow Computations

"Dinic.h"ce350b, 12 lines

```
using Edge = array<ll, 3>;
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
    vector<Edge> tree; vi par(N);
    FOR(i, 1, N) {
        Dinic D(N); // Any flow works
        for(Edge t: ed) D.addEdge(t[0], t[1], t[2], t[2]);
        tree.pb({i, par[i], D.calc(i, par[i])});
        FOR(j, i+1, N)
            if(par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i;
    }
    return tree;
}
```

## 6.2 Matching

### hopcroftKarp.h

**Description:** Fast bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of -1's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or  $-1$  if it's not matched.

**Usage:** vi btoa(m, -1); hopcroftKarp(g, btoa);

**Time:**  $\mathcal{O}(\sqrt{VE})$

05aa1d, 33 lines

```
bool dfs(int a, int L, vector<vi> &g, vi &btoa, vi &A, vi &B) {
    if(A[a] != L) return 0;
    A[a] = -1;
    for(int b: g[a]) if(B[b] == L + 1) {
        B[b] = 0;
        if(btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B)) {
            return btoa[b] = a, 1; }
    }
    return 0;
}
```

```
int hopcroftKarp(vector<vi> &g, vi &btoa) {
    int res = 0; vi A(SZ(g)), B(SZ(btoa)), cur, next;
    for(;;) {
        fill(all(A), 0); fill(all(B), 0);
        cur.clear();
        for(int a: btoa) if(a != -1) A[a] = -1;
        FOR(a, 0, SZ(g)) if(A[a] == 0) cur.pb(a);
        for(int lay = 1;; lay++) {
            bool islast = 0; next.clear();
            for(int a: cur) for(int b: g[a]) {
                if(btoa[b] == -1) { B[b] = lay; islast = 1; }
                else if (btoa[b] != a && !B[b]) {
                    B[b] = lay; next.pb(btoa[b]); }
            }
            if(islast) break;
            if(next.empty()) return res;
            for(int a: next) A[a] = lay;
            cur.swap(next);
        }
        FOR(a, 0, SZ(g)) res += dfs(a, 0, g, btoa, A, B);
    }
}
```

### MinimumVertexCover.h

**Description:** Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

"hopcroftKarp.h"e38bb1, 17 lines

```
vi cover(vector<vi>& g, int n, int m) {
```

```
vi match(m, -1);
int res = hopcroftKarp(g, match);
vector<bool> lfound(n, 1), seen(m);
for(int it: match) if(it != -1) lfound[it] = 0;
vi q, cover;
FOR(i, 0, n) if(lfound[i]) q.pb(i);
while(!q.empty()) {
    int i = q.back(); q.pop_back(); lfound[i] = 1;
    for(int e: g[i]) if(!seen[e] && match[e] != -1) {
        seen[e] = true; q.pb(match[e]); }
}
FOR(i, 0, n) if(!lfound[i]) cover.pb(i);
FOR(i, 0, m) if(seen[i]) cover.pb(n + i);
assert(SZ(cover) == res);
return cover;
}
```

### WeightedMatching.h

**Description:** Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost. Requires  $N \leq M$ .

**Time:**  $\mathcal{O}(N^2M)$

ff486d, 27 lines

```
pair<int, vi> hungarian(const vector<vi> &a) {
    if(a.empty()) return {0, {}};
    int n = SZ(a) + 1, m = SZ(a[0]) + 1;
    vi u(n), v(m), p(m), ans(n - 1);
    FOR(i, 1, n) {
        p[0] = i; int j0 = 0; // add "dummy" worker 0
        vi dist(m, INT_MAX), pre(m, -1);
        vector<bool> done(m + 1);
        do { // dijkstra
            done[j0] = 1; int i0 = p[j0], j1, delta = INT_MAX;
            FOR(j, 1, m) if(!done[j]) {
                auto cur = a[i0 - 1][j - 1] - u[i0] - v[j];
                if(cur < dist[j]) dist[j] = cur, pre[j] = j0;
                if(dist[j] < delta) delta = dist[j], j1 = j;
            }
            FOR(j, 0, m) {
                if(done[j]) u[p[j]] += delta, v[j] -= delta;
                else dist[j] -= delta;
            }
            j0 = j1;
        } while(p[j0]);
        while(j0) { // update alternating path
            int j1 = pre[j0]; p[j0] = p[j1], j0 = j1; }
    }
    FOR(j, 1, m) if(p[j]) ans[p[j] - 1] = j - 1;
    return {-v[0], ans}; // min cost
}
```

### GeneralMatching.h

**Description:** Unweighted matching for general graphs. If white[v] = 0 at the end, v is part of every max matching.

**Time:**  $\mathcal{O}(NM)$ , faster in practice

28552a, 48 lines

```
struct MaxMatching { // 1-indexed
    vector<vi> G; int n;
    vi mate, par, white; vector<pii> l;
    MaxMatching(vector<vi> _G) : G(_G), n(SZ(G)), mate(n),
        par(n), white(n), l(n) {}
    int group(int x) {
        return par[x] = (white[par[x]] ? group(par[x]) : par[x]);
    }
    void match(int p, int b) {
        swap(mate[p], b); if(mate[b] != p) return;
        if(!l[p].nd) mate[b] = l[p].st, match(l[p].st, b);
```

```
    else match(l[p].st, l[p].nd), match(l[p].nd, l[p].st);
}
bool augment(int a) {
    white[a] = 1; par[a] = 0; l[a] = {0, 0};
    queue<int> q; q.push(a);
    while(!q.empty()) {
        a = q.front(); q.pop();
        for(int b: G[a]) {
            if(white[b]) {
                int x = group(a), y = group(b), lca = 0;
                while(x || y) {
                    if(y) swap(x, y);
                    if(l[x] == mp(a, b)) { lca = x; break; }
                    l[x] = {a, b}; x = group(l[mate[x]].st);
                }
                for(int v: {group(a), group(b)}) while(v != lca) {
                    q.push(v); white[v] = 1; par[v] = lca;
                    v = group(l[mate[v]].st);
                }
            }
            else if(!mate[b]) {
                mate[b] = a; match(a, b); fill(all(white), 0);
                return 1;
            }
            else if(!white[mate[b]]) {
                white[mate[b]] = 1; par[mate[b]] = b;
                l[b] = {0, 0}; l[mate[b]] = {a, 0};
                q.push(mate[b]);
            }
        }
    }
    return 0;
}
int max_matching() {
    int ans = 0;
    FOR(v, 1, n) if(!mate[v]) ans += augment(v);
    return ans;
}
```

## 6.3 DFS algorithms

### BiconnectedComponents.h

**Description:** Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.

**Usage:** int eid = 0; ed.resize(N);

for each edge (a,b) {  
ed[a].emplace\_back(b, eid);  
ed[b].emplace\_back(a, eid++); }  
bicomps[&] (const vi& edgelist) {...});

**Time:**  $\mathcal{O}(E + V)$

43b75d, 30 lines

```
vi num, st;
vector<vector<pii>> ed;
int Time;
template<class F>
int dfs(int at, int par, F& f) {
    int me = num[at] = ++Time, top = me;
    for(auto [y, e]: ed[at]) if(e != par) {
        if(num[y]) {
            top = min(top, num[y]);
            if(num[y] < me) st.pb(e); }
        else {
            int si = SZ(st), up = dfs(y, e, f);
            top = min(top, up);
            if(up == me) {
                st.pb(e);
                f(vi(st.begin() + si, st.end()));
                st.resize(si);
```

```
UW:
}
else if(up < me) st.pb(e);
else { /* e is a bridge */
}
}
return top;
}

template<class F>
void bicomps(F f) {
    num.assign(SZ(ed), 0);
    FOR(i, 0, SZ(ed)) if(!num[i]) dfs(i, -1, f);
}
```

2sat.h  
**Description:** Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type  $(a|||b)&\&(!a|||c)&\&(d|||b)&\&...$  becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions (~x).  
**Usage:** TwoSat ts(number of boolean variables);  
ts.either(0, ~3); // Var 0 is true or var 3 is false  
ts.setValue(2); // Var 2 is true  
ts.atMostOne({0,~1,2}); // <= 1 of vars 0, ~1 and 2 are true  
ts.solve(); // Returns true iff it is solvable  
ts.values[0..N-1] holds the assigned values to the vars  
**Time:**  $\mathcal{O}(N + E)$ , where N is the number of boolean variables, and E is the number of clauses.

```
struct TwoSat {
    int n; vector<vi> gr; vi values; // 0 = false, 1 = true
    TwoSat(int _n = 0) : n(_n), gr(n * 2) {}

    int addVar() { FOR(i, 0, 2) gr.pb({}); return n++; }

    void either(int f, int j) {
        f = max(2*f, -1-2*f); j = max(2*j, -1-2*j);
        gr[f].pb(j^1); gr[j].pb(f^1);
    }

    void setValue(int x) { either(x, x); }

    void atMostOne(const vi& li) { // (optional)
        if(SZ(li) <= 1) return;
        int cur = ~li[0];
        FOR(i, 2, SZ(li)) {
            int next = addVar();
            either(cur, ~li[i]); either(cur, next);
            either(~li[i], next); cur = ~next;
        }
        either(cur, ~li[1]);
    }

    vi val, comp, z; int time = 0;
    int dfs(int i) {
        int low = val[i] = ++time, x; z.pb(i);
        for(int e: gr[i]) if(!comp[e]) {
            low = min(low, val[e] ? dfs(e));
            if(low == val[i]) do {
                x = z.back(); z.pop_back(); comp[x] = low;
                if(values[x >> 1] == -1) values[x >> 1] = x & 1;
            } while(x != i);
            return val[i] = low;
        }

        bool solve() {
            values.assign(n, -1); val.assign(n * 2, 0); comp = val;
            FOR(i, 0, n * 2) if(!comp[i]) dfs(i);
            FOR(i, 0, n) if(comp[2*i] == comp[2*i+1]) return 0;
            return 1;
        }
    }
};
```

2sat EulerWalk HLD CentroidTree LinkCutTree

EulerWalk.h  
**Description:** Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.  
**Time:**  $\mathcal{O}(V + E)$

```
vi eulerWalk(vector<vector<pii>> &gr, int nedges, int src=0) {
    int n = SZ(gr); vi D(n), its(n), eu(nedges), ret, s = {src};
    D[src]++; // to allow Euler paths, not just cycles
    while(!s.empty()) {
        int x = s.back(), y, e, &it = its[x], end = SZ(gr[x]);
        if(it == end) { ret.pb(x); s.pop_back(); continue; }
        tie(y, e) = gr[x][it++];
        if(!eu[e]) D[x]--, D[y]++, eu[e] = 1, s.pb(y);
    }
    for(int x: D) if(x < 0 || SZ(ret) != nedges + 1) return {};
    return {ret.rbegin(), ret.rend()};
}
```

6.4 Trees  
HLD.h  
**Description:** Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most log(n) light edges. Code does additive modifications and max queries, but can support commutative segtree modifications/queries on paths and subtrees. Takes as input the full adjacency list. VALS.EDGES being true means that values are stored in the edges, as opposed to the nodes. All values initialized to the segtree default. Root must be 0.  
**Time:**  $\mathcal{O}((\log N)^2)$

```
struct HLD {
    int N, tim = 0, VALS_EDGES = 0; // change to 1 if needed
    vector<vi> adj; vi par, sz, depth, rt, pos;
    HLD(vector<vi> _adj) : N(SZ(_adj)), adj(_adj), par(N, -1),
        sz(N, 1), depth(N), rt(N), pos(N) { dfsSz(0); dfsHld(0); }
    void dfsSz(int v) {
        if(par[v] != -1) adj[v].erase(find(all(adj[v]), par[v]));
        for(int &u: adj[v]) {
            par[u] = v, depth[u] = depth[v] + 1;
            dfsSz(u); sz[v] += sz[u];
            if(sz[u] > sz[adj[v][0]]) swap(u, adj[v][0]);
        }
    }
    void dfsHld(int v) {
        pos[v] = tim++;
        for(int u: adj[v]) {
            rt[u] = (u == adj[v][0] ? rt[v] : u); dfsHld(u);
        }
    }
    vector<pii> path(int u, int v) {
        vector<pii> paths;
        for(; rt[u] != rt[v]; v = par[rt[v]]) {
            if(depth[rt[u]] > depth[rt[v]]) swap(u, v);
            paths.pb({pos[rt[v]], pos[v]});
        }
        if(depth[u] > depth[v]) swap(u, v);
        paths.pb({pos[u] + VALS_EDGES, pos[v]});
        return paths;
    }
    pii subtree(int v) {
        return {pos[v] + VALS_EDGES, pos[v] + sz[v] - 1};
    }
};
```

CentroidTree.h  
**Description:** Builds a centroid tree.  
**Time:**  $\mathcal{O}(N \log N)$

```
struct CentroidTree { // 0-indexed
    vector<vector<pii>> G; // {neighbour, distance}
    constexpr int LOG = 20; vector<ll> dist[LOG];
    vi sz, block, par, dpth;

    CentroidTree(vector<vector<pii>> &_G) : G(_G) {
        int n = SZ(G); sz = block = par = dpth = vi(n);
        FOR(i, 0, LOG) dist[i].resize(n);
        decomp(0, -1);
    }

    void sz_dfs(int v, int p) {
        sz[v] = 1;
        for(auto &[u, _]: G[v])
            if(u != p && !block[u]) sz_dfs(u, v), sz[v] += sz[u];
    }
    int find_centr(int v, int p, int sum) {
        for(auto &[u, _]: G[v])
            if(u != p && !block[u] && sz[u] > sum / 2) {
                return find_centr(u, v, sum);
            }
        return v;
    }
    void dist_dfs(int v, int p, int gleb, ll akt) {
        dist[gleb][v] = akt;
        for(auto &[u, d]: G[v])
            if(u != p && !block[u]) dist_dfs(u, v, gleb, akt + d);
    }
    void decomp(int v, int p) {
        sz_dfs(v, -1); v = find_centr(v, -1, sz[v]);
        par[v] = p;
        if(p != -1) dpth[v] = dpth[p] + 1;
        dist_dfs(v, -1, dpth[v], 0); block[v] = 1;
        for(auto &[u, _]: G[v]) if(!block[u]) decomp(u, v);
    }
};
```

LinkCutTree.h  
**Description:** One-indexed. Represents a forest of unrooted trees. You can add and remove edges (as long as the result is still a forest), and check whether two nodes are in the same tree. Also you can update a value in a vertex, and do standard path and subtree queries. Operation on subtrees must be invertible.  
**Time:** All operations take amortized  $\mathcal{O}(\log N)$ .

```
struct SplayTree {
    struct Node {
        int p = 0, ch[2] = {0, 0};
        ll self = 0, path = 0; // Path aggregates
        ll sub = 0, vir = 0; // Subtree aggregates
        bool flip = 0; // Lazy tags
    };
    vector<Node> t;
    SplayTree(int n) : t(n + 1) {}

    void push(int v) {
        if(!v || !t[v].flip) return;
        auto &[l, r] = t[v].ch;
        t[l].flip ^= 1, t[r].flip ^= 1;
        swap(l, r), t[v].flip = 0;
    }

    void pull(int v) {
        auto [l, r] = t[v].ch; push(l), push(r);
        t[v].path = t[l].path + t[v].self + t[r].path;
        t[v].sub = t[v].vir + t[l].sub + t[v].self + t[r].sub;
    }
};
```

```
void set(int u, int d, int v) {
    t[u].ch[d] = v, t[v].p = u, pull(u);
}

void splay(int v) {
    auto dir = [&](int x) {
        int u = t[x].p;
        return t[u].ch[0] == x ? 0 : t[u].ch[1] == x ? 1 : -1;
    };
    auto rotate = [&](int x) {
        int y = t[x].p, z = t[y].p, dx = dir(x), dy = dir(y);
        set(y, dx, t[x].ch[!dx]), set(x, !dx, y);
        if(~dy) set(z, dy, x);
        t[x].p = z;
    };
    for(push(v); ~dir(v); ) {
        int y = t[v].p, z = t[y].p;
        push(z), push(y), push(v);
        int dv = dir(v), dy = dir(y);
        if(~dy) rotate(dv == dy ? y : v);
        rotate(v);
    }
}

struct LinkCut : SplayTree { // 1-indexed
    LinkCut(int n) : SplayTree(n) {}
    int access(int v) {
        int u = v, x = 0;
        for(; u; x = u, u = t[u].p) {
            splay(u);
            int &ox = t[u].ch[1];
            t[u].vir += t[ox].sub;
            t[u].vir -= t[x].sub;
            ox = x, pull(u);
        }
        return splay(v), x;
    }
    void reroot(int v) { access(v), t[v].flip ^= 1, push(v); }
    void link(int u, int v) {
        reroot(u), access(v);
        t[v].vir += t[u].sub; t[u].p = v, pull(v);
    }
    void cut(int u, int v) {
        reroot(u), access(v); t[v].ch[0] = t[u].p = 0, pull(v);
    }
    // Rooted tree LCA. Returns 0 if u and v are not connected.
    int lca(int u, int v) {
        if(u == v) return u;
        access(u); int ret = access(v); return t[u].p ? ret : 0;
    }
    // Query subtree of u where v is outside the subtree.
    ll getSub(int u, int v) {
        reroot(v), access(u); return t[u].vir + t[u].self;
    }
    ll getPath(int u, int v) {
        reroot(u), access(v); return t[v].path;
    }
    // Update vertex u with value val
    void update(int u, ll val) {
        access(u), t[u].self = val, pull(u);
    }
};
```

DirectedMST.h

**Description:** Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

```
Time:  $\mathcal{O}(E \log V)$ 
"../data-structures/UnionFindRollback.h" 76f859, 56 lines

struct Edge { int a, b; ll w; };
struct Node {
    Edge key; Node *l, *r; ll delta;
    void prop() {
        key.w += delta;
        if(l) l->delta += delta;
        if(r) r->delta += delta;
        delta = 0;
    }
    Edge top() { prop(); return key; }
};

Node *merge(Node *a, Node *b) {
    if(!a || !b) return a ? b;
    a->prop(), b->prop();
    if(a->key.w > b->key.w) swap(a, b);
    swap(a->l, (a->r = merge(b, a->r)));
    return a;
}

void pop(Node& a) { a->prop(); a = merge(a->l, a->r); }

pair<ll, vi> dmst(int n, int r, vector<Edge> &g) {
    RollbackUF uf(n);
    vector<Node*> heap(n);
    for(Edge e: g) heap[e.b] = merge(heap[e.b], new Node{e});
    ll res = 0; vi seen(n, -1), path(n), par(n);
    seen[r] = r;
    vector<Edge> Q(n), in(n, {-1,-1}), comp;
    deque<tuple<int, int, vector<Edge>>> cyps;
    FOR(s, 0, n) {
        int u = s, qi = 0, w;
        while(seen[u] < 0) {
            if(!heap[u]) return {-1, {}};
            Edge e = heap[u]->top();
            heap[u]->delta -= e.w, pop(heap[u]);
            Q[qi] = e, path[qi++] = u, seen[u] = s;
            res += e.w, u = uf.find(e.a);
            if(seen[u] == s) {
                Node* cyc = 0;
                int end = qi, time = uf.time();
                do { cyc = merge(cyc, heap[w = path[--qi]]);
                    } while(uf.join(u, w));
                u = uf.find(u), heap[u] = cyc, seen[u] = -1;
                cyps.push_front({u, time, {&Q[qi], &Q[end]}});
            }
        }
        FOR(i, 0, qi) in[uf.find(Q[i].b)] = Q[i];
    }

    for(auto &[u, t, comp]: cyps) { // restore sol (optional)
        uf.rollback(t); Edge inEdge = in[u];
        for(auto &e: comp) in[uf.find(e.b)] = e;
        in[uf.find(inEdge.b)] = inEdge;
    }
    FOR(i, 0, n) par[i] = in[i].a;
    return {res, par};
}
```

DominatorTree.h

**Description:** Computes Dominator tree.

```
Time:  $\mathcal{O}(V + E)$ 
46af9c, 43 lines

struct DominatorTree { // 0-indexed
    int n, t;
    vector<basic_string<int>> g, rg, bucket;
    basic_string<int> arr, par, rev, sdom, dom, dsu, label;
    DominatorTree(int _n) : n(_n), t(0), g(n), rg(n),
        bucket(n), arr(n, -1), par(n, -1), rev(n, -1),
```

```
    sdom(n, -1), dom(n, -1), dsu(n, 0), label(n, 0) {}
    void add_edge(int u, int v) { g[u] += v; }
    void dfs(int u) {
        arr[u] = t; rev[t] = u;
        label[t] = sdom[t] = dsu[t] = t; t++;
        for(int w: g[u]) {
            if(arr[w] == -1) dfs(w), par[arr[w]] = arr[u];
            rg[arr[w]] += arr[u];
        }
    }
    int find(int u, int x = 0) {
        if(u == dsu[u]) return x ? -1 : u;
        int v = find(dsu[u], x + 1);
        if(v < 0) return u;
        if(sdom[label[dsu[u]]] < sdom[label[u]]) {
            label[u] = label[dsu[u]];
            dsu[u] = v;
            return x ? v : label[u];
        } // returns -1 for unreachable
    }
    vi run(int root) {
        dfs(root); iota(all(dom), 0);
        per(i, 0, t-1) {
            for(int w: rg[i]) sdom[i] = min(sdom[i], sdom[find(w)]);
            if(i) bucket[sdom[i]] += i;
            for(int w: bucket[i]) {
                int v = find(w);
                if(sdom[v] == sdom[w]) dom[w] = sdom[w];
                else dom[w] = v;
            }
            if(i > 1) dsu[i] = par[i];
        }
        FOR(i, 1, t) if(dom[i] != sdom[i]) dom[i] = dom[dom[i]];
        vi outside_dom(n, -1);
        FOR(i, 1, t) outside_dom[rev[i]] = rev[dom[i]];
        return outside_dom;
    }
};
```

6.5 Various

EdgeColoring.h

**Description:** Given a simple, undirected graph with max degree  $D$ , computes a  $(D + 1)$ -coloring of the edges such that no neighboring edges share a color. ( $D$ -coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

```
Time:  $\mathcal{O}(NM)$ 
e1c352, 26 lines

vi edgeColoring(int N, vector<pii> eds) {
    vi cc(N + 1), ret(SZ(eds)), fan(N), free(N), loc;
    for(pii e : eds) ++cc[e.first], ++cc[e.second];
    int u, v, ncols = *max_element(all(cc)) + 1;
    vector<vi> adj(N, vi(ncols, -1));
    for(pii e: eds) {
        tie(u, v) = e; fan[0] = v; loc.assign(ncols, 0);
        int at = u, end = u, d, c = free[u], ind = 0, i = 0;
        while(d = free[v], !loc[d] && (v = adj[u][d]) != -1) {
            loc[d] = ++ind, cc[ind] = d, fan[ind] = v; }
        cc[loc[d]] = c;
        for(int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd]) {
            swap(adj[at][cd], adj[end = at][cd ^ c ^ d]); }
        while(adj[fan[i]][d] != -1) {
            int left = fan[i], right = fan[++i], e = cc[i];
            adj[u][e] = left; adj[left][e] = u;
            adj[right][e] = -1; free[right] = e;
        }
        adj[u][d] = fan[i]; adj[fan[i]][d] = u;
        for(int y : {fan[0], u, end})
            for(int &z = free[y] = 0; adj[y][z] != -1; z++);
    }
    FOR(i, 0, SZ(eds))
```

```
    for(tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
    return ret;
}
```

### MaximumClique.h

**Description:** Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.  
**Time:** Runs in about 1s for n=155 and worst case random graphs (p=.90). Runs faster for sparse graphs.

e1d2b4, 46 lines

```
using vb = vector<bitset<200>>;
struct Maxclique {
    double limit = 0.025, pk = 0;
    struct Vertex { int i, d=0; };
    using vv = vector<Vertex>;
    vb e; vv V; vector<vi> C;
    vi qmax, q, S, old;
    void init(vv &r) {
        for(auto &v : r) v.d = 0;
        for(auto &v : r) for(auto j: r) v.d += e[v.i][j.i];
        sort(all(r), [](auto a, auto b) { return a.d > b.d; });
        int mxD = r[0].d;
        FOR(i, 0, SZ(r)) r[i].d = min(i, mxD) + 1;
    }
    void expand(vv &R, int lev = 1) {
        S[lev] += S[lev - 1] - old[lev];
        old[lev] = S[lev - 1];
        while(SZ(R)) {
            if(SZ(q) + R.back().d <= SZ(qmax)) return;
            q.pb(R.back().i); vv T;
            for(auto v: R) if (e[R.back().i][v.i]) T.pb({v.i});
            if(SZ(T)) {
                if(S[lev]++ / ++pk < limit) init(T);
                int j = 0, mxk = 1, mnk = max(SZ(qmax) - SZ(q)+1, 1);
                C[1].clear(), C[2].clear();
                for(auto v: T) {
                    int k = 1;
                    auto f = [&](int i) { return e[v.i][i]; };
                    while(any_of(all(C[k]), f)) k++;
                    if(k > mxk) mxk = k, C[mxk + 1].clear();
                    if(k < mnk) T[j++] .i = v.i;
                    C[k].pb(v.i);
                }
                if(j > 0) T[j - 1].d = 0;
                FOR(k, mnk, mxk + 1) for(int i: C[k]) {
                    T[j].i = i, T[j++].d = k; }
                expand(T, lev + 1);
            } else if(SZ(q) > SZ(qmax)) qmax = q;
            q.pop_back(), R.pop_back();
        }
    }
    vi maxClique() { init(V), expand(V); return qmax; }
    Maxclique(vb conn) : e(conn), C(SZ(e)+1), S(SZ(C)), old(S) {
        FOR(i, 0, SZ(e)) V.pb({i});
    }
};
```

### MaximalCliques.h

**Description:** Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.

**Time:**  $\mathcal{O}\left(3^{n/3}\right)$ , much faster for sparse graphs

745132, 11 lines

```
using B = bitset<128>;
template<class F>
void cliques(vector<B> &eds, F f, B P= ~B(), B X={}, B R={}) {
    if(!P.any()) { if(!X.any()) f(R); return; }
    auto q = (P | X)._Find_first();
```

```
    auto cands = P & ~eds[q];
    FOR(i, 0, SZ(eds)) if(cands[i]) {
        R[i] = 1; cliques(eds, f, P & eds[i], X & eds[i], R);
        R[i] = P[i] = 0; X[i] = 1;
    }
}
```

## 6.6 Math

### 6.6.1 Number of Spanning Trees

Create an  $N \times N$  matrix  $\mat$ , and for each edge  $a \rightarrow b \in G$ , do  $\mat[a][b]--$ ,  $\mat[b][b]++$  (and  $\mat[b][a]--$ ,  $\mat[a][a]++$  if  $G$  is undirected). Remove the  $i$ th row and column and take the determinant; this yields the number of directed spanning trees rooted at  $i$  (if  $G$  is undirected, remove any row/column).

### 6.6.2 Erdős–Gallai theorem

A simple graph with node degrees  $d_1 \geq \dots \geq d_n$  exists iff  $d_1 + \dots + d_n$  is even and for every  $k = 1 \dots n$ ,

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k).$$

### 6.6.3 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j / \pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1).  $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1P}$ .

A Markov chain is an A-chain if the states can be partitioned into two sets  $\mathbf{A}$  and  $\mathbf{G}$ , such that all states in  $\mathbf{A}$  are absorbing ( $p_{ii} = 1$ ), and all states in  $\mathbf{G}$  leads to an absorbing state in  $\mathbf{A}$ . The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is  $j$ , is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is  $i$ , is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

## Geometry (7)

### 7.1 Geometric primitives

#### Point.h

**Description:** Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

b1a082, 29 lines

```
using D = double;
template<class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
    using P = Point;
    T x, y;
    explicit Point(T _x=0, T _y=0) : x(_x), y(_y) {}
    bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
    bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
    P operator+(P p) const { return P(x+p.x, y+p.y); }
    P operator-(P p) const { return P(x-p.x, y-p.y); }
    P operator*(T d) const { return P(x*d, y*d); }
    P operator/(T d) const { return P(x/d, y/d); }
    T dot(P p) const { return x*p.x + y*p.y; }
    T cross(P p) const { return x*p.y - y*p.x; }
    T cross(P a, P b) const { return (a-*this).cross(b-*this); }
    T dist2() const { return x*x + y*y; }
    D dist() const { return sqrt((D)dist2()); }
    // angle to x-axis in interval [-pi, pi]
    D angle() const { return atan2(y, x); }
    P unit() const { return *this/dist(); } // makes dist()=1
    P perp() const { return P(-y, x); } // rotates +90 degrees
    P normal() const { return perp().unit(); }
    // returns point rotated 'a' radians ccw around the origin
    P rotate(D a) const {
        return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
    friend ostream& operator<<(ostream& os, P p) {
        return os << "(" << p.x << ", " << p.y << ")"; }
};
```

#### lineDistance.h

**Description:**

Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.

"Point.h"0052e6, 4 lines

```
template<class P>
D lineDist(const P &a, const P &b, const P &p) {
    return (D) (b-a).cross(p-a) / (b-a).dist();
}
```

#### SegmentDistance.h

**Description:**

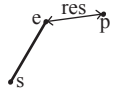
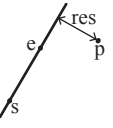
Returns the shortest distance between point p and the line segment from point s to e.

**Usage:** Point<D> a, b(2,2), p(1,1);

bool onSegment = segDist(a,b,p) < 1e-10;


"Point.h"677ad0, 6 lines

```
using P = Point<D>;
D segDist(P &s, P &e, P &p) {
    if(s == e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d, fmax(0, (p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist() / d;
}
```



### SegmentIntersection.h

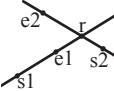
**Description:**  
If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.  
**Usage:** vector<P> inter = segInter(s1,e1,s2,e2);  
if (sz(inter)==1)  
cout << "segments intersect at " << inter[0] << endl;  
"Point.h", "OnSegment.h" 9d57f2, 13 lines



**template<class P> vector<P> segInter(P a, P b, P c, P d) {**  
  **auto** oa = c.cross(d, a), ob = c.cross(d, b),  
  oc = a.cross(b, c), od = a.cross(b, d);  
  *// Checks if intersection is single non-endpoint point.*  
  **if** (sgn(oa) \* sgn(ob) < 0 && sgn(oc) \* sgn(od) < 0)  
    **return** {(a \* ob - b \* oa) / (ob - oa)};  
  set<P> s;  
  **if** (onSegment(c, d, a)) s.insert(a);  
  **if** (onSegment(c, d, b)) s.insert(b);  
  **if** (onSegment(a, b, c)) s.insert(c);  
  **if** (onSegment(a, b, d)) s.insert(d);  
  **return** {all(s)};  
}

### lineIntersection.h

**Description:**  
If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.  
**Usage:** auto res = lineInter(s1,e1,s2,e2);  
if (res.first == 1)  
cout << "intersection point at " << res.second << endl;  
"Point.h" a01f81, 8 lines



**template<class P>**  
**pair<int, P> lineInter(P s1, P e1, P s2, P e2) {**  
  **auto** d = (e1 - s1).cross(e2 - s2);  
  **if** (d == 0) *// if parallel*  
    **return** {-(s1.cross(e1, s2) == 0), P(0, 0)};  
  **auto** p = s2.cross(e1, e2), q = s2.cross(e2, s1);  
  **return** {1, (s1 \* p + e1 \* q) / d};  
}

### sideOf.h

**Description:** Returns where *p* is as seen from *s* towards *e*. 1/0/-1 ⇔ left/on line/right. If the optional argument *eps* is given 0 is returned if *p* is within distance *eps* from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.  
**Usage:** bool left = sideOf(p1,p2,q)==1;  
"Point.h" c2efb9, 8 lines

**template<class P>**  
**int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }**

**template<class P>**  
**int sideOf(const P& s, const P& e, const P& p, D eps) {**  
  **auto** a = (e-s).cross(p-s); D l = (e-s).dist()\*eps;  
  **return** (a > l) - (a < -l);  
}

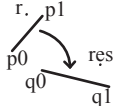
### OnSegment.h

**Description:** Returns true iff p lies on the line segment from s to e. Use (segDist(s,e,p)<=epsilon) instead when using Point<D>.  
"Point.h" c597e8, 3 lines

**template<class P> bool onSegment(P s, P e, P p) {**  
  **return** p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;  
}

### linearTransformation.h

**Description:**  
Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.  
"Point.h" 1ef732, 6 lines



**using** P = Point<D>;  
P linearTransformation(const P &p0, const P &p1,  
  const P &q0, const P &q1, const P &r) {  
  P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));  
  **return** q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();  
}

### LineProjectionReflection.h

**Description:** Projects point p onto line ab. Set refl=true to get reflection of point p across line ab instead. The wrong point will be returned if P is an integer point and the desired point doesn't have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow.  
"Point.h" b5562d, 5 lines

**template<class P>**  
P lineProj(P a, P b, P p, bool refl = false) {  
  P v = b - a;  
  **return** p - v.perp()\* (1+refl)\*v.cross(p-a)/v.dist2();  
}

## 7.2 Circles

### CircleIntersection.h

**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.  
"Point.h" 9dd9e4, 11 lines

**using** P = Point<D>;  
**bool** circleInter(P a, P b, D r1, D r2, pair<P, P>\* out) {  
  **if** (a == b) { assert(r1 != r2); **return** 0; }  
  P vec = b - a;  
  D d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,  
  p = (d2 + r1\*r1 - r2\*r2)/(d2\*2), h2 = r1\*r1 - p\*p\*d2;  
  **if** (sum\*sum < d2 || dif\*dif > d2) **return** 0;  
  P mid = a + vec\*p, per = vec.perp() \* sqrt(fmax(0, h2) / d2);  
  \*out = {mid - per, mid + per};  
  **return** 1;  
}

### CircleTangents.h

**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.  
"Point.h" c6361c, 13 lines

**template<class P>**  
**vector<pair<P, P>> tangents(P c1, D r1, P c2, D r2) {**  
  P d = c2 - c1;  
  D dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr \* dr;  
  **if** (d2 == 0 || h2 < 0) **return** {};  
  vector<pair<P, P>> out;  
  **for** (D sign: {-1, 1}) {

**P v = (d \* dr + d.perp() \* sqrt(h2) \* sign) / d2;**  
  out.pb({c1 + v \* r1, c2 + v \* r2});  
}  
**if** (h2 == 0) out.pop\_back();  
**return** out;  
}

### CircleLine.h

**Description:** Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<D>.  
"Point.h" f4015d, 9 lines

**template<class P>**  
**vector<P> circleLine(P c, D r, P a, P b) {**  
  P ab = b - a, p = a + ab \* (c-a).dot(ab) / ab.dist2();  
  D s = a.cross(b, c), h2 = r\*r - s\*s / ab.dist2();  
  **if** (h2 < 0) **return** {};  
  **if** (h2 == 0) **return** {p};  
  P h = ab.unit() \* sqrt(h2);  
  **return** {p - h, p + h};  
}

### CirclePolygonIntersection.h

**Description:** Returns the area of the intersection of a circle with a ccw polygon.  
**Time:**  $\mathcal{O}(n)$   
"Point.h" de735b, 19 lines

**using** P = Point<D>;  
**#define** arg(p, q) atan2(p.cross(q), p.dot(q))  
D circlePoly(P c, D r, vector<P> ps) {  
  **auto** tri = [&](P p, P q) {  
    **auto** r2 = r \* r / 2;  
    P d = q - p;  
    **auto** a = d.dot(p)/d.dist2(), b = (p.dist2()-r\*r)/d.dist2();  
    **auto** det = a \* a - b;  
    **if** (det <= 0) **return** arg(p, q) \* r2;  
    **auto** s = fmax(0, -a-sqrt(det)), t = fmin(1, -a+sqrt(det));  
    **if** (t < 0 || 1 <= s) **return** arg(p, q) \* r2;  
    P u = p + d \* s, v = p + d \* t;  
    **return** arg(p,u) \* r2 + u.cross(v)/2 + arg(v,q) \* r2;  
  };  
  D sum = 0;  
  **FOR** (i, 0, SZ(ps))  
    sum += tri(ps[i] - c, ps[(i + 1) % SZ(ps)] - c);  
  **return** sum;  
}

### CirclesUnionArea.h

**Description:** Returns the area of the sum of circles.  
**Time:**  $\mathcal{O}(n^2 \log n)$   
"CircleIntersection.h" 243537, 33 lines

D circlesArea(vector<pair<P, D>> c) {  
  const D PI = acos((D)-1);  
  sort(all(c)); c.erase(unique(all(c)), c.end());  
  D res = 0;  
  **for** (auto &[p, r]: c) {  
    **int** cnt = 0, cover = 0;  
    vector<pair<D, **int**>> eve = {{-PI, 0}};  
    **for** (auto &[q, s]: c) **if** (mp(p, r) != mp(q, s)) {  
      D dst = (p - q).dist();  
      **if** (r + dst <= s) { cover = 1; **break**; }  
      pair<P, P> inters;  
      **if** (!circleInter(p, q, r, s, &inters)) **continue**;  
      D le = (inters.st - p).angle();  
      D re = (inters.nd - p).angle();  
      cnt += le > re;  
      eve.pb({le, 1}); eve.pb({re, -1});  
    }  
  }

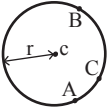


```
if(cover) continue;
sort(eve.begin() + 1, eve.end());
eve.pb({PI, 0});
D loc = 0;
FOR(i, 1, SZ(eve)) {
    if(!cnt) {
        D a = eve[i-1].st, b = eve[i].st;
        loc += r * (b - a) +
            p.cross(P(cos(b)-cos(a), sin(b)-sin(a)));
    }
    cnt += eve[i].nd;
}
res += r * loc;
}
return res / 2;
}
```

circumcircle.h

Description:

The circumcirle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.



"Point.h"	162089, 9 lines
<pre>using P = Point&lt;D&gt;; D ccRadius(const P &amp;A, const P &amp;B, const P &amp;C) {     return (B-A).dist()*(C-B).dist()*(A-C).dist()/         abs((B-A).cross(C-A))/2; } P ccCenter(const P &amp;A, const P &amp;B, const P &amp;C) {     P b = C-A, c = B-A;     return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2; }</pre>	

MinimumEnclosingCircle.h

Description: Computes the minimum circle that encloses a set of points. Time: expected  $\mathcal{O}(n)$

"circumcircle.h"	81b8ec, 16 lines
<pre>pair&lt;P, D&gt; mec(vector&lt;P&gt; ps) {     shuffle(all(ps), mt19937(1337));     P o = ps[0]; D r = 0, EPS = 1 + 1e-8;     FOR(i, 0, SZ(ps)) if((o - ps[i]).dist() &gt; r * EPS) {         o = ps[i], r = 0;         FOR(j, 0, i) if((o - ps[j]).dist() &gt; r * EPS) {             o = (ps[i] + ps[j]) / 2;             r = (o - ps[i]).dist();             FOR(k, 0, j) if((o - ps[k]).dist() &gt; r * EPS) {                 o = ccCenter(ps[i], ps[j], ps[k]);                 r = (o - ps[i]).dist();             }         }     }     return {o, r}; }</pre>	

7.3 Polygons

InsidePolygon.h

Description: Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow. Usage: vector<P> v = {P{4,4}, P{1,2}, P{2,1}}; bool in = inPolygon(v, P{3, 3}, false); Time:  $\mathcal{O}(n)$

"Point.h", "OnSegment.h", "SegmentDistance.h"	da435c, 11 lines
<pre>template&lt;class P&gt; bool inPolygon(vector&lt;P&gt; &amp;p, P a, bool strict = true) {     int cnt = 0, n = SZ(p);</pre>	

FOR(i, 0, n) {     P q = p[(i + 1) % n];     if(onSegment(p[i], q, a)) return !strict;     //or: if(segDist(p[i], q, a) <= eps) return !strict;     cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) > 0; } return cnt; }	362a7f, 6 lines
---	-----------------

PolygonArea.h

Description: Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h"	362a7f, 6 lines
<pre>template&lt;class T&gt; T polygonArea2(vector&lt;Point&lt;T&gt;&gt; &amp;v) {     T a = v.back().cross(v[0]);     FOR(i, 0, SZ(v)-1) a += v[i].cross(v[i+1]);     return a; }</pre>	

PolygonCenter.h

Description: Returns the center of mass for a polygon. Time:  $\mathcal{O}(n)$

"Point.h"	619437, 9 lines
<pre>using P = Point&lt;D&gt;; P polygonCenter(const vector&lt;P&gt;&amp; v) {     P res(0, 0); D A = 0;     for(int i = 0, j = SZ(v) - 1; i &lt; SZ(v); j = i++) {         res = res + (v[i] + v[j]) * v[j].cross(v[i]);         A += v[j].cross(v[i]);     }     return res / A / 3; }</pre>	

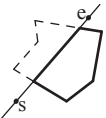
PolygonCut.h

Description:

Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.

Usage: vector<P> p = ...; p = polygonCut(p, P(0,0), P(1,0));

"Point.h", "lineIntersection.h"	ad7124, 12 lines
<pre>using P = Point&lt;D&gt;; vector&lt;P&gt; polygonCut(const vector&lt;P&gt; &amp;poly, P s, P e) {     vector&lt;P&gt; res;     FOR(i, 0, SZ(poly)) {         P cur = poly[i], prev = i ? poly[i-1] : poly.back();         bool side = s.cross(e, cur) &lt; 0;         if(side != (s.cross(e, prev) &lt; 0))             res.pb(lineInter(s, e, cur, prev).nd);         if(side) res.pb(cur);     }     return res; }</pre>	



vector<pair<D, int>> segs = {{0, 0}, {1, 0}}; FOR(j,0,SZ(poly)) if(i != j) FOR(u, 0, SZ(poly[j])) {     P C = poly[j][u], E = poly[j][(u + 1) % SZ(poly[j])];     int sc = sideOf(A, B, C), sd = sideOf(A, B, E);     if(sc != sd) {         D sa = C.cross(E, A), sb = C.cross(E, B);         if(min(sc, sd) < 0)             segs.emplace_back(sa / (sa - sb), sgn(sc - sd));     } else if(!sc && !sd && j<i && sgn((B-A).dot(E-C))>0) {         segs.emplace_back(rat(C - A, B - A), 1);         segs.emplace_back(rat(E - A, B - A), -1);     } } sort(all(segs)); for(auto &s: segs) s.st = min(max(s.st, 0.0), 1.0); D sum = 0; int cnt = segs[0].nd; FOR(j, 1, SZ(segs)) {     if(!cnt) sum += segs[j].st - segs[j - 1].st;     cnt += segs[j].nd; } ret += A.cross(B) * sum; } return ret / 2; }	
---	--

ConvexHull.h

Description:

Returns a vector of the points of the convex hull in counter-clockwise order. Points on the edge of the hull between two other points are not considered part of the hull.

Time:  $\mathcal{O}(n \log n)$

"Point.h"	c06f0e, 12 lines
<pre>using P = Point&lt;ll&gt;; vector&lt;P&gt; convexHull(vector&lt;P&gt; pts) {     if(SZ(pts) &lt;= 1) return pts;     sort(all(pts)); vector&lt;P&gt; h(SZ(pts)+1);     int s = 0, t = 0;     for(int it = 2; it--; s = --t, reverse(all(pts)))         for(P p: pts) {             while(t &gt;= s + 2 &amp;&amp; h[t-2].cross(h[t-1], p) &lt;= 0) t--;             h[t++] = p;         }     return {h.begin(), h.begin() + t - (t == 2 &amp;&amp; h[0] == h[1])}; }</pre>	



HullDiameter.h

Description: Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).

Time:  $\mathcal{O}(n)$

"Point.h"	2aeea8, 11 lines
<pre>using P = Point&lt;ll&gt;; array&lt;P, 2&gt; hullDiameter(vector&lt;P&gt; S) {     int n = SZ(S), j = n &lt; 2 ? 0 : 1;     pair&lt;ll, array&lt;P, 2&gt;&gt; res({0, {S[0], S[0]}});     FOR(i, 0, j) for(;; j = (j + 1) % n) {         res = max(res, {{S[i] - S[j]}.dist2(), {S[i], S[j]}});         if((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) &gt;= 0)             break;     }     return res.nd; }</pre>	

MinkowskiSum.h

Description: Computes the Minkowski sum of two convex polygons. Distance between two convex polygons can be computed by finding the closest edge of P - Q to the origin.

Time:  $\mathcal{O}(N)$

"Point.h"	d461a3, 14 lines
-----------	------------------



```
template<class P>
vector<P> minkowski(vector<P> a, vector<P> b) {
    rotate(a.begin(), min_element(all(a)), a.end());
    rotate(b.begin(), min_element(all(b)), b.end());
    FOR(i, 0, 2) a.pb(a[i]), b.pb(b[i]);
    vector<P> res;
    for(int i = 0, j = 0; i < SZ(a) - 2 || j < SZ(b) - 2; ) {
        res.pb(a[i] + b[j]);
        auto cross = (a[i + 1] - a[i]).cross(b[j + 1] - b[j]);
        if(cross >= 0 && i < SZ(a)) i++;
        if(cross <= 0 && j < SZ(b)) j++;
    }
    return res;
}
```

**PointInsideHull.h**  
**Description:** Determine whether a point  $t$  lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.  
**Time:**  $\mathcal{O}(\log N)$

"Point.h", "sideOf.h", "OnSegment.h"	550f0d, 13 lines
--------------------------------------	------------------

```
using P = Point<ll>;
bool inHull(const vector<P> &l, P p, bool strict = true) {
    int a = 1, b = SZ(l) - 1, r = !strict;
    if(SZ(l) < 3) return r && onSegment(l[0], l.back(), p);
    if(sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
    if(sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p)<= -r)
        return 0;
    while(abs(a - b) > 1) {
        int c = (a + b) / 2;
        (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
    }
    return sgn(l[a].cross(l[b], p)) < r;
}
```

**LineHullIntersection.h**  
**Description:** Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon:  $\bullet(-1, -1)$  if no collision,  $\bullet(i, -1)$  if touching the corner  $i$ ,  $\bullet(i, i)$  if along side  $(i, i + 1)$ ,  $\bullet(i, j)$  if crossing sides  $(i, i + 1)$  and  $(j, j + 1)$ . In the last case, if a corner  $i$  is crossed, this is treated as happening on side  $(i, i + 1)$ . The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.  
**Time:**  $\mathcal{O}(\log n)$

"Point.h"	e112e1, 39 lines
-----------	------------------

```
#define cmp(i, j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P> &poly, P dir) {
    int n = SZ(poly), lo = 0, hi = n;
    if(extr(0)) return 0;
    while(lo + 1 < hi) {
        int m = (lo + hi) / 2;
        if(extr(m)) return m;
        int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
        (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
    }
    return lo;
}

#define cmpL(i) sgn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P> &poly) {
    int endA = extrVertex(poly, (a - b).perp());
    int endB = extrVertex(poly, (b - a).perp());
    if(cmpL(endA) < 0 || cmpL(endB) > 0) return {-1, -1};
    array<int, 2> res;
```

```
FOR(i, 0, 2) {
    int lo = endB, hi = endA, n = SZ(poly);
    while((lo + 1) % n != hi) {
        int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
        (cmpL(m) == cmpL(endB) ? lo : hi) = m;
    }
    res[i] = (lo + !cmpL(hi)) % n;
    swap(endA, endB);
}
if(res[0] == res[1]) return {res[0], -1};
if(!cmpL(res[0]) && !cmpL(res[1])) {
    switch ((res[0] - res[1] + SZ(poly) + 1) % SZ(poly)) {
        case 0: return {res[0], res[0]};
        case 2: return {res[1], res[1]};
    }
}
return res;
}
```

**PolygonCutter.h**  
**Description:** Returns area of polygon halfplane intersection.

"LineHullIntersection.h", "lineIntersection.h"	8f8a19, 18 lines
--	------------------

```
using P = Point<D>;
struct PolyCutter {
    vector<D> pArea = {}; vector<P> poly; int n;
    PolyCutter(vector<P> &_poly) : poly(_poly), n(SZ(poly)) {
        FOR(i, 0, n)
            pArea.pb(pArea.back() + poly[i].cross(poly[(i+1)%n]));
    }
    D cutArea(P a, P b) {
        auto [i, j] = lineHull(a, b, poly);
        if(j == -1 || i == j)
            return cmpL((i + 2) % n) < 0 ? pArea[n] / 2 : 0;
        D res = pArea[i] - pArea[j + 1] + (i < j ? pArea[n] : 0);
        P p = lineInter(poly[i], poly[(i + 1) % n], a, b).nd;
        P q = lineInter(poly[j], poly[(j + 1) % n], a, b).nd;
        res += poly[i].cross(p)+p.cross(q)+q.cross(poly[(j+1)%n]);
        return res / 2;
    }
};
```

**PolygonTangents.h**  
**Description:** Polygon tangents from a given point. The polygon must be ccw and have no collinear points. Returns a pair of indices of the given polygon. Should work for a point on border (for a point being polygon vertex returns previous and next one).  
**Time:**  $\mathcal{O}(\log n)$

"Point.h"	4d1a00, 21 lines
-----------	------------------

```
#define pdir(i) (ph ? p - poly[(i)%n] : poly[(i)%n] - p)
#define cmp(i, j) sgn(pdir(i).cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P>
array<int, 2> polygonTangents(vector<P> &poly, P p) {
    auto bs = [&](int ph) {
        int n = sz(poly), lo = 0, hi = n;
        if(extr(0)) return 0;
        while(lo + 1 < hi) {
            int m = (lo + hi) / 2;
            if(extr(m)) return m;
            int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
            (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
        }
        return lo;
    };
    array<int, 2> res = {bs(0), bs(1)};
    if(res[0] == res[1]) res[0] = (res[0] + 1) % SZ(poly);
    if(poly[res[0]] == p) res[0] = (res[0] + 1) % SZ(poly);
    return res;
}
```

```
}

HalfplaneIntersectionDynamic.h
Description: Data structure that dynamically keeps track of the intersection of half-planes.
Time: amortized  $\mathcal{O}(\log n)$ 

"lineIntersection.h"
6d30ae, 57 lines

using P = Point<D>;
int hf(P a) { return a.y < 0 || (a.y == 0 && a.x < 0); }
struct polarCmp {
    bool operator()(const P &a, const P &b) const {
        return hf(a) == hf(b) ? a.cross(b) > 0 : hf(a) < hf(b);
    }
};

struct HalfplaneSet : map<P, P, polarCmp> {
    D INF = 1e6, area = 8 * INF * INF;
    HalfplaneSet() {
        P p(-INF, -INF), d(1, 0);
        FOR(k, 0, 4) {
            insert({d, p}); p = p + d * 2 * INF; d = d.perp(); }
    }
    auto fix(auto it) { return it == end() ? begin() : it; }
    auto getNext(auto it) { return fix(next(it)); }
    auto getPrev(auto it) {
        return it == begin() ? prev(end()) : prev(it);
    }
    auto uSide(auto it, int change) { // 1 - add, -1 - del
        area += change * it->nd.cross(getNext(it)->nd);
    }
    auto del(auto it) {
        uSide(getPrev(it), -1), uSide(it, -1);
        it = fix(erase(it));
        if(size()) uSide(getPrev(it), 1);
        return it;
    }
    void add(P s, P e) {
        auto eval = [&](auto it) {
            return sgn(s.cross(e, it->nd));
        };
        auto intersect = [&](auto it) {
            return lineInter(s, e, it->nd, it->st + it->nd).nd;
        };
        auto it = fix(lower_bound(e - s));
        if(empty() || eval(it) >= 0) return;
        while(size() && eval(getPrev(it)) < 0) del(getPrev(it));
        while(size() && eval(getNext(it)) < 0) it = del(it);
        if(empty()) return;
        if(eval(getNext(it)) > 0) {
            uSide(getPrev(it), -1), uSide(it, -1);
            it->nd = intersect(it);
            uSide(getPrev(it), 1), uSide(it, 1);
        }
        else it = del(it);
        it = getPrev(it);
        uSide(it, -1); insert(it, {e - s, intersect(it)});
        uSide(it, 1), uSide(getNext(it), 1);
        if(eval(it) == 0) del(it);
    }
}
D maxDot(P a) {
    return a.dot(fix(lower_bound(a.perp())->nd);
}
D getArea() { return area / 2; }
};
```

## 7.4 Misc. Point Set Problems

### ClosestPair.h

**Description:** Finds the closest pair of points.

**Time:**  $\mathcal{O}(n \log n)$

"Point.h"	5c381c, 16 lines
<pre>using P = Point&lt;ll&gt;; pair&lt;P, P&gt; closest(vector&lt;P&gt; v) {     assert(SZ(v) &gt; 1); set&lt;P&gt; S;     sort(all(v), [](P a, P b) { return a.y &lt; b.y; });     pair&lt;ll, pair&lt;P, P&gt;&gt; ret{LLONG_MAX, {P(), P()}};     int j = 0;     for(P p: v) {         P d{1 + (ll)sqrt(ret.first), 0};         while(v[j].y &lt;= p.y - d.x) S.erase(v[j++]);         auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);         for(; lo != hi; ++lo) {             ret = min(ret, {( *lo - p).dist2(), { *lo, p } });             S.insert(p);         }         return ret.nd;     }</pre>	

### ManhattanMST.h

**Description:** Given N points, returns up to 4\*N edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights  $w(p, q) = -p.x - q.x + -p.y - q.y$ . Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.

**Time:**  $\mathcal{O}(N \log N)$

"Point.h"	3dd86e, 22 lines
<pre>using P = Point&lt;int&gt;; vector&lt;array&lt;int, 3&gt;&gt; manhattanMST(vector&lt;P&gt; ps) {     vi id(SZ(ps)); iota(all(id), 0);     vector&lt;array&lt;int, 3&gt;&gt; edges;     FOR(k, 0, 4) {         sort(all(id), [&amp;](int i, int j) {             return (ps[i]-ps[j]).x &lt; (ps[j]-ps[i]).y; });         map&lt;int, int&gt; sweep;         for(int i: id) {             for(auto it = sweep.lower_bound(-ps[i].y);                 it != sweep.end(); sweep.erase(it++)) {                 int j = it-&gt;second;                 P d = ps[i] - ps[j];                 if(d.y &gt; d.x) break;                 edges.pb({d.y + d.x, i, j});             }             sweep[-ps[i].y] = i;         }         for(P &amp;p : ps) if(k &amp; 1) p.x = -p.x; else swap(p.x, p.y);     }     return edges; }</pre>	

### kDTree.h

**Description:** KD-tree (2d, can be extended to 3d)

"Point.h"	ee5448, 57 lines
<pre>using T = ll; using P = Point&lt;T&gt;; const T INF = numeric_limits&lt;T&gt;::max();  bool on_x(const P &amp;a, const P &amp;b) { return a.x &lt; b.x; } bool on_y(const P &amp;a, const P &amp;b) { return a.y &lt; b.y; }  struct Node {     P pt; // if this is a leaf, the single point in it     T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds     Node *first = 0, *second = 0;</pre>	

<pre>T distance(const P&amp; p) { // min squared distance to a point     T x = (p.x &lt; x0 ? x0 : p.x &gt; x1 ? x1 : p.x);     T y = (p.y &lt; y0 ? y0 : p.y &gt; y1 ? y1 : p.y);     return (P(x,y) - p).dist2(); }  Node(vector&lt;P&gt;&amp;&amp; vp) : pt(vp[0]) {     for(P p : vp) {         x0 = min(x0, p.x); x1 = max(x1, p.x);         y0 = min(y0, p.y); y1 = max(y1, p.y);     }     if(SZ(vp) &gt; 1) {         // split on x if width &gt;= height (not ideal...)         sort(all(vp), x1 - x0 &gt;= y1 - y0 ? on_x : on_y);         // divide by taking half the array for each child (not         // best performance with many duplicates in the middle)         int half = SZ(vp) / 2;         first = new Node({vp.begin(), vp.begin() + half});         second = new Node({vp.begin() + half, vp.end()});     } }  };  struct KDTree {     Node* root;     KDTree(const vector&lt;P&gt;&amp; vp) : root(new Node({all(vp)})) {}      pair&lt;T, P&gt; search(Node *node, const P&amp; p) {         if(!node-&gt;first) {             // uncommment if we should not find the point itself:             // if (p == node-&gt;pt) return {INF, P()};             return mp((p - node-&gt;pt).dist2(), node-&gt;pt);         }         Node *f = node-&gt;first, *s = node-&gt;second;         T bfirst = f-&gt;distance(p), bsec = s-&gt;distance(p);         if(bfirst &gt; bsec) swap(bsec, bfirst), swap(f, s);         // search closest side first, other side if needed         auto best = search(f, p);         if(bsec &lt; best.st) best = min(best, search(s, p));         return best;     }     // find nearest point to a point, and its squared distance     // (requires an arbitrary operator&lt; for Point)     pair&lt;T, P&gt; nearest(const P &amp;p) { return search(root, p); } };</pre>	
---	--

### DelaunayTriangulation.h

**Description:** Computes the Delaunay triangulation of a set of points. Each circumcircle contains none of the input points. If any three points are collinear or any four are on the same circle, behavior is undefined.

**Time:**  $\mathcal{O}(n^2)$

"Point.h", "3DHull.h"	80635c, 10 lines
<pre>template&lt;class P, class F&gt; void delaunay(vector&lt;P&gt; &amp;ps, F trifun) {     if(SZ(ps) == 3) { int d = (ps[0].cross(ps[1], ps[2]) &lt; 0);         trifun(0,1+d,2-d); }     vector&lt;P3&gt; p3;     for(P p: ps) p3.emplace_back(p.x, p.y, p.dist2());     if(SZ(ps) &gt; 3) for(auto t: hull3d(p3)) if((p3[t.b]-p3[t.a]).         cross(p3[t.c]-p3[t.a]).dot(P3(0,0,1)) &lt; 0)         trifun(t.a, t.c, t.b); }</pre>	

### FastDelaunay.h

**Description:** Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.

**Time:**  $\mathcal{O}(n \log n)$

"Point.h"	38ce5d, 86 lines
<pre>using P = Point&lt;ll&gt;; using Q = struct Quad*; using lll = __int128_t; // (can be ll if coords are &lt; 2e4) P arb(LLONG_MAX, LLONG_MAX); // not equal to any other point  struct Quad {     Q rot, o; P p = arb; bool mark;     P&amp; F() { return r()-&gt;p; }     Q&amp; r() { return rot-&gt;rot; }     Q prev() { return rot-&gt;o-&gt;rot; }     Q next() { return r()-&gt;prev(); } } *H;  bool circ(P p, P a, P b, P c) { // is p in the circumcircle?     lll p2 = p.dist2(), A = a.dist2()-p2,         B = b.dist2()-p2, C = c.dist2()-p2;     return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B &gt; 0; }  Q makeEdge(P orig, P dest) {     Q r = H ? H : new Quad{new Quad{new Quad{new Quad{0}}}};     H = r-&gt;o; r-&gt;r()-&gt;r() = r;     FOR(i,0,4) r = r-&gt;rot, r-&gt;p = arb, r-&gt;o = i &amp; 1 ? r : r-&gt;r();     r-&gt;p = orig; r-&gt;F() = dest;     return r; }  void splice(Q a, Q b) {     swap(a-&gt;o-&gt;rot-&gt;o, b-&gt;o-&gt;rot-&gt;o); swap(a-&gt;o, b-&gt;o); }  Q connect(Q a, Q b) {     Q q = makeEdge(a-&gt;F(), b-&gt;p);     splice(q, a-&gt;next()); splice(q-&gt;r(), b);     return q; }  pair&lt;Q,Q&gt; rec(const vector&lt;P&gt;&amp; s) {     if(SZ(s) &lt;= 3) {         Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());         if(SZ(s) == 2) return { a, a-&gt;r() };         splice(a-&gt;r(), b);         auto side = s[0].cross(s[1], s[2]);         Q c = side ? connect(b, a) : 0;         return {side &lt; 0 ? c-&gt;r() : a, side &lt; 0 ? c : b-&gt;r() };     } }</pre>	

<pre>#define H(e) e-&gt;F(), e-&gt;p #define valid(e) (e-&gt;F().cross(H(base)) &gt; 0) Q A, B, ra, rb; int half = SZ(s) / 2; tie(ra, A) = rec({all(s) - half}); tie(B, rb) = rec({SZ(s) - half + all(s)}); while((B-&gt;p.cross(H(A)) &lt; 0 &amp;&amp; (A = A-&gt;next()))        (A-&gt;p.cross(H(B)) &gt; 0 &amp;&amp; (B = B-&gt;r()-&gt;o))); Q base = connect(B-&gt;r(), A); if(A-&gt;p == ra-&gt;p) ra = base-&gt;r(); if(B-&gt;p == rb-&gt;p) rb = base;  #define DEL(e, init, dir) Q e = init-&gt;dir; if (valid(e)) \     while(circ(e-&gt;dir-&gt;F(), H(base), e-&gt;F())) { \         Q t = e-&gt;dir; \         splice(e, e-&gt;prev()); \         splice(e-&gt;r(), e-&gt;r()-&gt;prev()); \         e-&gt;o = H; H = e; e = t; \</pre>	
---	--

```

    }
    for(;;) {
        DEL(LC, base->r(), o); DEL(RC, base, prev());
        if(!valid(LC) && !valid(RC)) break;
        if(!valid(LC) || (valid(RC) && circ(H(RC), H(LC)))) {
            base = connect(RC, base->r()); }
        else base = connect(base->r(), LC->r());
    }
    return { ra, rb };
}

```

```

vector<P> triangulate(vector<P> pts) {
    sort(all(pts)); //assert(unique(all(pts)) == pts.end());
    if(SZ(pts) < 2) return {};
    Q e = rec(pts).st;
    vector<Q> q = {e};
    int qi = 0;
    while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.pb(c->p); \
    q.pb(c->r()); c = c->next(); } while (c != e); }
    ADD; pts.clear();
    while(qi < SZ(q)) if (!(e = q[qi++])>mark) ADD;
    return pts;
}

```

## directedSegment.h

**Description:** Segment representation usefull for sweeping. Compares two disjoint (can touch) segments on sweep line (OY projection). Transitivity breaks if three segments touch in one point, fix it when needed.

"Point.h" d144f8, 15 lines

```

template<class P>
struct dirSeg {
    P s, e; int rev;
    dirSeg(P _s, P _e) : s(_s), e(_e), rev(0) {
        if(e < s) swap(s, e), rev = 1;
    }
    P getY(P X) { // takes x * 2, returns y * 2 as a fraction
        P d = (e - s);
        return !sgn(d.x) ? P(s.y+e.y, 1) : P(d.cross(s*2-X), d.x);
    }
    int cmp(dirSeg b) { // needs ~64 * M^3 !
        P X(max(s.x, b.s.x) + min(e.x, b.e.x), 0);
        return sgn(getY(X).cross(b.getY(X)));
    }
};

```

## PointLocation.h

**Description:** Computes (not necessarily convex) polygon tree structure. Also for each query point computes its location (including boundaries).

**Time:**  $\mathcal{O}(n \log n)$

"directedSegment.h" edc289, 47 lines

```

template<class P>
pair<vi,vi> pointLoc(vector<vector<P>> polys, vector<P> pts) {
    vector<tuple<P, int, int>> eve; // {point, event_type, id}
    vector<pair<dirSeg<P>, int>> segs; // {s, e, poly_id}
    FOR(i, 0, SZ(polys)) FOR(j, 0, SZ(polys[i])) {
        dirSeg<P> seg(polys[i][j], polys[i][(j+1)%SZ(polys[i])]);
        eve.pb({seg.s,0,SZ(segs)}), eve.pb({seg.e,2,SZ(segs)});
        segs.pb({seg, i});
    }
    FOR(i, 0, SZ(pts)) eve.pb({pts[i], 1, i});
    sort(all(eve));
    vi par(SZ(polys), -2), ans(SZ(pts), -1);
    auto cmp = [](auto a, auto b) {
        return mp(a.st.cmp(b.st), a.nd) < mp(0, b.nd);
    };
    set<pair<dirSeg<P>, int>, decltype(cmp)> s(cmp);
    for(auto &[_ , eve_tp, id]: eve) {

```

```

        if(eve_tp == 1) { // point query
            P p = pts[id];
            auto it = s.lower_bound({dirSeg(p, p), 0});
            if(it != s.begin()) { // on vertical segment?
                auto prv = prev(it);
                if(!sgn(p.cross(prv->st.s, prv->st.e))) it--;
            }
            if(it == s.end()) ans[id] = -1;
            else {
                auto [seg, seg_id] = *it;
                int poly_id = segs[seg_id].nd; // strictness there!
                ans[id] = !seg.rev && sgn(p.cross(seg.s, seg.e))
                    ? par[poly_id] : poly_id;
            }
        }
        if(eve_tp == 0) { // add segment
            auto it = next(s.insert({segs[id].st, id}).st);
            int poly_id = segs[id].nd;
            if(par[poly_id] == -2) {
                if(it == s.end()) par[poly_id] = -1;
                else {
                    int up_rev = it->st.rev, up_id = segs[it->nd].nd;
                    par[poly_id] = !up_rev ? par[up_id] : up_id;
                }
            }
        }
        if(eve_tp == 2) s.erase({segs[id].st, id}); // del segment
    }
    return {par, ans};
}

```

## SegmentsIntersection.h

**Description:** Finds one of segments intersections. You should change dirSeg's comparator, to compare segments at their left end.

**Time:**  $\mathcal{O}(N \log N)$

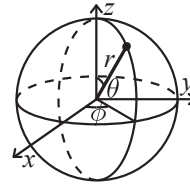
"SegmentIntersection.h", "directedSegment.h" 79d27e, 33 lines

```

template<class P>
pii segmentsIntersect(vector<pair<P, P>> segments) {
    vector<tuple<P, int, int>> eve; // {point, event_type, id}
    vector<dirSeg<P>> segs;
    for(auto &[s, e]: segments) {
        dirSeg<P> seg(s, e);
        eve.pb({seg.s,0,SZ(segs)}), eve.pb({seg.e,1,SZ(segs)});
        segs.pb(seg);
    }
    sort(all(eve));
    auto inter = [](auto a, auto b) {
        return SZ(segInter(a->st.s, a->st.e, b->st.s, b->st.e));
    };
    auto cmp = [](auto a, auto b) {
        return mp(a.st.cmp(b.st), a.nd) < mp(0, b.nd);
    };
    set<pair<dirSeg<P>, int>, decltype(cmp)> s(cmp);
    for(auto &[_ , eve_tp, id]: eve) {
        if(eve_tp == 0) { // add segment
            auto it = s.insert({segs[id], id}).st;
            if(next(it) != s.end() && inter(it, next(it)))
                return {it->nd, next(it)->nd};
            if(it != s.begin() && inter(it, prev(it)))
                return {it->nd, prev(it)->nd};
        }
        if(eve_tp == 1) { // del segment
            auto it = s.erase(s.find({segs[id], id}));
            if(it!=s.begin() && it!=s.end() && inter(it, prev(it)))
                return {it->nd, prev(it)->nd};
        }
    }
    return {-1, -1};
}

```

## 7.5 Spherical coordinates



$$\begin{aligned}
 x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\
 y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\
 z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x)
 \end{aligned}$$

## 7.6 3D

### PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.

3058c3, 6 lines

```

template<class V, class L>
double signedPolyVolume(const V &p, const L &trilist) {
    double v = 0;
    for(auto i: trilist) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
    return v / 6;
}

```

### Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.

e193d4, 34 lines

```

using D = double;
template<class T> struct Point3D {
    using P = Point3D;
    using R = const P&;
    T x, y, z;
    explicit Point3D(T _x=0, T _y=0, T _z=0)
        : x(_x), y(_y), z(_z) {}
    bool operator<(R p) const {
        return tie(x, y, z) < tie(p.x, p.y, p.z); }
    bool operator==(R p) const {
        return tie(x, y, z) == tie(p.x, p.y, p.z); }
    P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
    P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
    P operator*(T d) const { return P(x*d, y*d, z*d); }
    P operator/(T d) const { return P(x/d, y/d, z/d); }
    T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
    P cross(R p) const {
        return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
    }
    T dist2() const { return x*x + y*y + z*z; }
    D dist() const { return sqrt((D)dist2()); }
    //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
    D phi() const { return atan2(y, x); }
    //Zenith angle (latitude) to the z-axis in interval [0, pi]
    D theta() const { return atan2(sqrt(x*x+y*y),z); }
    P unit() const { return *this/(T)dist(); } //makes dist()==1
    //returns unit vector normal to *this and p
    P normal(P p) const { return cross(p).unit(); }
    //returns point rotated 'angle' radians ccw around axis
    P rotate(D angle, P axis) const {
        D s = sin(angle), c = cos(angle); P u = axis.unit();
        return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
    }
};

```

### 3DHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

**Time:**  $\mathcal{O}(n^2)$

"Point3D.h"	892282, 41 lines
<pre>using P3 = Point3D&lt;D&gt;; struct PR {     void ins(int x) { (a == -1 ? a : b) = x; }     void rem(int x) { (a == x ? a : b) = -1; }     int cnt() { return (a != -1) + (b != -1); }     int a, b; };  struct F { P3 q; int a, b, c; }; vector&lt;F&gt; hull3d(const vector&lt;P3&gt; &amp;A) {     assert(SZ(A) &gt;= 4);     vector&lt;vector&lt;PR&gt;&gt; E(SZ(A), vector&lt;PR&gt;(SZ(A), {-1, -1})); #define E(x,y) E[f.x][f.y]     vector&lt;F&gt; FS;     auto mf = [&amp;](int i, int j, int k, int l) {         P3 q = (A[j] - A[i]).cross((A[k] - A[i]));         if(q.dot(A[l]) &gt; q.dot(A[i])) q = q * -1;         F f{q, i, j, k};         E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);         FS.pb(f);     };     FOR(i, 0, 4) FOR(j, i+1, 4) FOR(k, j+1, 4) {         mf(i, j, k, 6 - i - j - k); }     FOR(i, 4, SZ(A)) {         FOR(j, 0, SZ(FS)) {             F f = FS[j];             if(f.q.dot(A[i]) &gt; f.q.dot(A[f.a])) {                 E(a,b).rem(f.c); E(a,c).rem(f.b); E(b,c).rem(f.a);                 swap(FS[j--], FS.back()); FS.pop_back();             }         }         int nw = SZ(FS);         FOR(j,0,nw) {             F f = FS[j]; #define C(a, b, c) if(E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);             C(a, b, c); C(a, c, b); C(b, c, a);         }         for (F &amp;it: FS) if((A[it.b] - A[it.a]).cross(             A[it.c] - A[it.a]).dot(it.q) &lt;= 0) swap(it.c, it.b);         return FS;     }; };</pre>	

### sphericalDistance.h

**Description:** Returns the shortest distance on the sphere with radius *radius* between the points with azimuthal angles (longitude) *f1* ( $\phi_1$ ) and *f2* ( $\phi_2$ ) from *x* axis and zenith angles (latitude) *t1* ( $\theta_1$ ) and *t2* ( $\theta_2$ ) from *z* axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. *dx\*radius* is then the difference between the two points in the *x* direction and *d\*radius* is the total distance between the points.

232b64, 8 lines
<pre>using D = double; D sphericalDistance(D f1, D t1, D f2, D t2, D radius) {     D dx = sin(t2) * cos(f2) - sin(t1) * cos(f1);     D dy = sin(t2) * sin(f2) - sin(t1) * sin(f1);     D dz = cos(t2) - cos(t1);     D d = sqrt(dx * dx + dy * dy + dz * dz);     return radius * 2 * asin(d/2); }</pre>

### SpheresIntersection.h

**Description:** Computes the intersection points of 3 spheres.

"Point3D.h"	738a7f, 17 lines
<pre>using P = Point3D&lt;D&gt;; vector&lt;P&gt; trilaterate(P a, P b, P c, D p, D q, D r) {     b = b - a, c = c - a;     P e_x = b.unit();     D i = e_x.dot(c);     P e_y = (c - e_x * i).unit(), e_z = e_x.cross(e_y);     D d = b.dist(), j = e_y.dot(c);     D x = (p * p - q * q + d * d) / 2 / d;     D y = (p * p - r * r - 2 * i * x + i * i + j * j) / 2 / j;     D z2 = p * p - x * x - y * y;     const D EPS = 1e-8; // take care!     if(z2 &lt; -EPS) return {};     D z = sqrt(fmax(z2, 0));     P sol = a + e_x * x + e_y * y;     if(z2 &lt; EPS) return {sol};     return {sol - e_z * z, sol + e_z * z}; }</pre>	

## Strings (8)

### KMP.h

**Description:** *pi*[*x*] computes the length of the longest prefix of *s* that ends at *x*, other than *s*[0..*x*] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.

**Time:**  $\mathcal{O}(n)$

e144f7, 16 lines
<pre>vi pi(const string &amp;s) {     vi p(SZ(s));     FOR(i, 1, SZ(s)) {         int g = p[i-1];         while (g &amp;&amp; s[i] != s[g]) g = p[g-1];         p[i] = g + (s[i] == s[g]);     }     return p; }  vi match(const string &amp;s, const string &amp;pat) {     vi p = pi(pat + '\0' + s), res;     FOR(i, SZ(p)-SZ(s), SZ(p))         if(p[i] == SZ(pat)) res.pb(i - 2 * SZ(pat));     return res; }</pre>

### Zfunc.h

**Description:** *z*[*x*] computes the length of the longest common prefix of *s*[*i*:] and *s*, except *z*[0] = 0. (abacaba -> 0010301)

**Time:**  $\mathcal{O}(n)$

19b0b9, 9 lines
<pre>vi Z(const string &amp;S) {     vi z(SZ(S)); int l = -1, r = -1;     FOR(i, 1, SZ(S)) {         z[i] = i &gt;= r ? 0 : min(r - i, z[i - 1]);         while(i + z[i] &lt; SZ(S) &amp;&amp; S[i + z[i]] == S[z[i]]) z[i]++;         if(i + z[i] &gt; r) l = i, r = i + z[i];     }     return z; }</pre>

### Manacher.h

**Description:** Computes palindromic ranges.

s: a b a a b a a c a a b b b b a a  
r: 0 0 1 0 0 3 0 0 2 0 0 1 0 0 3 0 0 1 0 0 0 1 1 4 1 1 0 0 0 1 0  
**Time:**  $\mathcal{O}(N)$

vi Manacher(string s) {

```
vi r(SZ(s) * 2 - 1);
for(int i=0, m=0, k=0, p=0; i < 2*SZ(s) - 1; m = i++ - 1) {
    while(p < k && i / 2 + r[m] != k) {
        r[i++] = min(r[m--], (k + 1 - p++) / 2); }
    while(k+1 < SZ(s) && p > 0 && s[k+1] == s[p-1]) k++, p--;
    r[i] = (k + 1 - p++) / 2;
}
return r;
}
```

### SuffixArray.h

**Description:** Builds suffix array for a string. *sa*[*i*] is the starting index of the suffix which is *i*'th in the sorted suffix array. The returned vector is of size *n* + 1, and *sa*[0] = *n*. The *lcp* array contains longest common prefixes for neighbouring strings in the suffix array.

**Time:**  $\mathcal{O}(n \log n)$

d987b7, 34 lines
<pre>int s[N]; /* s[i] &gt; 0, n = len(s), A = sigma(s) */ int n, A, sa[N], lcp[N], cnt[N]; vi x, y;  bool dif(int a, int b, int k) {     return y[a] != y[b]            (a + k &lt;= n ? y[a + k] : -1) != (b + k &lt;= n ? y[b + k] : -1); } /* 1-indexed */ void build() {     x.clear(); x.resize(max(A, n) + 2); y = x; // +2 enough?     int j = 0;     rep(i, 1, n) cnt[x[i] = s[i]]++;     rep(i, 1, A) cnt[i] += cnt[i - 1];     per(i, 1, n) sa[cnt[x[i]]--] = i;     for(int k = 1; k &lt; n; k *= 2) {         int p = 0;         rep(i, n - k + 1, n) y[++p] = i;         rep(i, 1, n) if(sa[i] &gt; k) y[++p] = sa[i] - k;         rep(i, 1, A) cnt[i] = 0;         rep(i, 1, n) cnt[x[i]]++;         rep(i, 1, A) cnt[i] += cnt[i - 1];         per(i, 1, n) sa[cnt[x[y[i]]]--] = y[i];         swap(x, y); A = x[sa[1]] = 1;         rep(i, 2, n) x[sa[i]] = (A += dif(sa[i - 1], sa[i], k));         if(n == A) break;     }     rep(i, 1, n) {         if(x[i] == n) { lcp[x[i]] = 0; continue; }         int nxt = sa[x[i] + 1];         while(max(i, nxt) + j &lt;= n &amp;&amp; s[i + j] == s[nxt + j]) j++;         lcp[x[i]] = j; j = max(j - 1, 0);     } }</pre>

### TandemRepeats.h

**Description:** Find all (*i*,*p*) such that *s*.substr(*i*,*p*) == *s*.substr(*i*+*p*,*p*). No two intervals with the same period intersect or touch.

**Usage:** solve("aaabababab") // {{0, 1, 1}, {2, 5, 2}}

**Time:**  $\mathcal{O}(N \log N)$

3e559b, 13 lines
<pre>vector&lt;array&lt;int, 3&gt;&gt; solve(string s) {     int N = SZ(s); SuffixArray A, B;     A.init(s); reverse(all(s)); B.init(s);     vector&lt;array&lt;int, 3&gt;&gt; runs;     for(int p = 1; 2*p &lt;= N; ++p) { // do in O(N/p) for period p         for(int i = 0, lst = -1; i+p &lt;= N; i += p) {             int l = i-B.getLCP(N-i-p,N-i), r = i-p+A.getLCP(i,i+p);             if (l &gt; r    l == lst) continue;             runs.pb({lst = l,r,p}); // for each i in [l,r],         } // s.substr(i,p) == s.substr(i+p,p)     }</pre>

```
    }
    return runs;
}
```

### Hasher.h

**Description:** Self-explanatory methods for string hashing.

19f1f5, 22 lines

```
const __int128 C = 311;
const ll mod = 1l(1e18) + 31;

struct Hasher {
    vector<ll> ha, pw;
    Hasher(string &str) : ha(SZ(str)+1), pw(SZ(str)+1, 1) {
        FOR(i, 0, SZ(str)) {
            ha[i+1] = (ha[i] * C + str[i]) % mod;
            pw[i+1] = pw[i] * C % mod;
        }
    }
    ll hashInterval(int a, int b) { // hash [a, b]
        ll res = (ha[b+1] - __int128(ha[a]) * pw[b+1 - a]) % mod;
        return res < 0 ? res + mod : res;
    }
};

ll hashString(string &s) {
    ll h = 0;
    for(char c: s) h = (h * C + c) % mod;
    return h;
}
```

### FastHasher.h

**Description:** Self-explanatory methods for string hashing.

602bc4, 17 lines

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "using H = ull;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
using ull = unsigned long long;
struct H {
    ull x; H(ull _x=0) : x(_x) {}
    H operator+(H o) { return x + o.x + (x + o.x < x); }
    H operator-(H o) { return *this + ~o.x; }
    H operator*(H o) { auto m = (__uint128_t)x * o.x;
        return H((ull)m) + (ull)(m >> 64); }
    ull get() const { return x + !~x; }
    bool operator==(H o) const { return get() == o.get(); }
    bool operator<(H o) const { return get() < o.get(); }
};
static const H C = (1l)1e11+3; // (order ~ 3e9; random also ok)
```

### AhoCorasick.h

**Description:** Famous Bohun's Aho

f89aac, 26 lines

```
const int N = 1e6 + 5;
int cnt, fail[N], go[N][26];

void add(string s) {
    int u = 0;
    for(auto ch : s) {
        int c = ch - 'a';
        if(!go[u][c]) go[u][c] = ++cnt;
        u = go[u][c];
    }
}

void build() {
    queue<int> q; q.push(0);
    while(!q.empty()) {
```

```
        int u = q.front(); q.pop();
        FOR(c, 0, 26) {
            int &v = go[u][c];
            if(v == 0) v = go[fail[u]][c];
            else {
                fail[v] = u == 0 ? 0 : go[fail[u]][c];
                q.push(v);
            }
        }
    }
}
```

### Eertree.h

**Description:** TODO

26a87b, 24 lines

```
const int N = 1e6 + 5, A = 26;
int nxt[N][A], fail[N], last[N], len[N], cnt, par[N];
char s[N];

void prepare(int n) { // 0 <= i <= n + 1
    rep(i, 0, n + 1) FOR(j, 0, A) nxt[i][j] = 0;
    s[0] = '#'; // CUSTOM
    last[0] = cnt = fail[0] = fail[1] = 1;
    len[1] = -1;
}

void add(int n){
    int c = s[n] - 'a'; // CUSTOM
    int v = last[n - 1];
    while(s[n - len[v] - 1] != s[n]) v = fail[v];
    if(!nxt[v][c]){
        int now = ++cnt, k = fail[v];
        len[now] = len[v] + 2;
        while(s[n - len[k] - 1] != s[n]) k = fail[k];
        fail[now] = nxt[k][c]; nxt[v][c] = now;
        par[now] = v;
    }
    last[n] = nxt[v][c];
}
```

### LyndonFactor.h

**Description:** A string is "simple" if it is strictly smaller than any of its own nontrivial suffixes. The Lyndon factorization of the string  $s$  is a factorization  $s = w_1w_2\dots w_k$  where all strings  $w_i$  are simple and  $w_1 \geq w_2 \geq \dots \geq w_k$ . Min rotation gets min index  $i$  such that cyclic shift of  $s$  starting at  $i$  is minimum.

be259a, 19 lines

```
vector<string> duval(string s) {
    vector<string> facts;
    for(int i = 0; i < SZ(s); ) {
        int j = i + 1, k = i;
        while(j < SZ(s) && s[k] <= s[j]) {
            if(s[k] < s[j++]) k = i;
            else k++;
        }
        for(; i <= k; i += j-k) facts.pb(s.substr(i, j-k));
    }
    return facts;
}

int minRotation(string s) {
    vector<string> d = duval(s + s); int ind = 0, ans = 0;
    while(ans + SZ(d[ind]) < SZ(s)) ans += SZ(d[ind++]);
    while(ind && d[ind] == d[ind-1]) ans -= SZ(d[ind--]);
    return ans;
}
```

### CyclicLCS.h

**Description:** Cyclic Longest Common Subsequence. maximum of lcs(any cyclic shift of  $s$ , any cyclic shift of  $t$ )

bd8b93, 39 lines

```
Time:  $\mathcal{O}(NM)$ 

const int N = 2010;

int dp[N * 2][N], from[N * 2][N];
int cyclic_lcs(string s, string t) {
    int n = SZ(s), m = SZ(t);
    auto eq = [&](int a, int b) {
        return s[(a - 1) % n] == t[(b - 1) % m]; };
    dp[0][0] = from[0][0] = 0;
    rep(i, 0, n * 2) rep(j, 0, m) {
        dp[i][j] = 0;
        if(j && dp[i][j - 1] > dp[i][j]) {
            dp[i][j] = dp[i][j - 1]; from[i][j] = 0;
        }
        if(i && j && eq(i, j) && dp[i-1][j-1] + 1 > dp[i][j]) {
            dp[i][j] = dp[i - 1][j - 1] + 1; from[i][j] = 1;
        }
        if(i && dp[i - 1][j] > dp[i][j]) {
            dp[i][j] = dp[i - 1][j]; from[i][j] = 2;
        }
    }
    int ret = 0;
    FOR(i, 0, n) {
        ret = max(ret, dp[i + n][m]);
        // re-root
        int x = i + 1, y = 0;
        while(y <= m && from[x][y] == 0) ++y;
        for(; y <= m && x <= n * 2; ++x) {
            from[x][y] = 0, --dp[x][m];
            if(x == n * 2) break;
            for(; y <= m; ++y) {
                if(from[x + 1][y] == 2) break;
                if(y + 1 <= m && from[x + 1][y + 1] == 1) {
                    ++y; break;
                }
            }
        }
    }
    return ret;
}
```

### AllSubstringLCS.h

**Description:** Computes LCS of  $s$  and  $t[i, j]$

23ce75, 17 lines

```
Time:  $\mathcal{O}(NM)$ 

const int N = 2002;
int f[N][N], g[N][N], ans[N][N];
void all_substring_lcs(string s, string t) {
    int n = SZ(s), m = SZ(t);
    s = "#" + s; t = "#" + t;
    rep(i, 1, m) f[0][i] = i;
    rep(i, 1, n) rep(j, 1, m) {
        if(s[i] == t[j]) {
            f[i][j] = g[i][j - 1]; g[i][j] = f[i - 1][j];
        } else {
            f[i][j] = max(f[i - 1][j], g[i][j - 1]);
            g[i][j] = min(g[i][j - 1], f[i - 1][j]);
        }
    }
    rep(i, 1, m) for(int j = i, a = 0; j <= m; ++j) {
        a += i > f[n][j]; ans[i][j] = a; }
}
```

## Various (9)

### 9.1 Misc. algorithms

#### FastKnapsack.h

**Description:** Given N non-negative integer weights w and a non-negative target t, computes the maximum  $S \leq t$  such that S is the sum of some subset of the weights.  
**Time:**  $\mathcal{O}(N \max(w_i))$

b21633, 15 lines

```
int knapsack(vi w, int t) {
    int a = 0, b = 0, x;
    while(b < SZ(w) && a + w[b] <= t) a += w[b++];
    if(b == SZ(w)) return a;
    int m = *max_element(all(w));
    vi u, v(2*m, -1); v[a + m - t] = b;
    FOR(i, b, SZ(w)) {
        u = v;
        FOR(x, 0, m) v[x + w[i]] = max(v[x + w[i]], u[x]);
        for(x = 2*m; --x > m;) FOR(j, max(0, u[x]), v[x]) {
            v[x-w[j]] = max(v[x-w[j]], j);
        }
        for(a = t; v[a + m - t] < 0; a--);
        return a;
    }
}
```

#### KnightMoves.h

**Description:** Finds minimum number of knight moves from (x,y) to (0,0) in non-negative infinite chessboard.  
**Time:**  $\mathcal{O}(1)$

cc32ee, 8 lines

```
ll knightMoves(ll x, ll y) {
    ll cnt = max({(x + 1) / 2, (y + 1) / 2, (x + y + 2) / 3});
    while((cnt % 2) != (x + y) % 2) cnt++;
    if(x == 1 && !y) return 3;
    if(y == 1 && !x) return 3;
    if(x == y && x == 2) return 4;
    return cnt;
}
```

### 9.2 Optimization tricks

#### 9.2.1 Bit hacks

- $x \& -x$  is the least bit in  $x$ .
- `for(int x = m; x; ) { --x &= m; ... }` loops over all subset masks of  $m$  (except  $m$  itself).
- `c = x&-x, r = x+c; ((r^x) >> 2)/c | r` is the next number after  $x$  with the same number of bits set.
- `FOR(b,0,K) FOR(i,0,(1 << K)) if(i & 1 << b) D[i] += D[i^(1 << b)];` computes all sums of subsets.