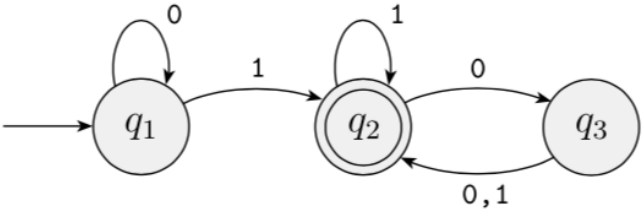
Administrivia

- HW1 is assigned and due Sept 7, 12:30pm
- Office hours: Tu/Wed 1:45-2:30pm
- Also on zoom: `https://arizona.zoom.us/j/85228664366` password: CS573)
- Questionnaires due by August 31 **in person**
- Videos of lectures and lecture notes on D2L (today)
- Reading: Chapter 0, Chapter 1
- Last time: we introduced finite automata and regular languages
- Today: properties of regular languages, nondeterministic FAs, regular expressions, grammars

Recall: Finite Automaton



A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_s, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $q_s \in Q$ is the start state,
- $F \subseteq Q$ is the set of accept states.

Computation of a Finite Automaton

Let $M = (Q, \Sigma, \delta, q_S, F)$ be a finite automaton and let $w = w_1 w_2 \ldots w_n$ be a string where each w_i is a member of the alphabet Σ. Then M accepts w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with three conditions:

- $r_0 = q_S$,
- $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, ..., n - 1$,
- $r_n \in F$.

A language is called a **regular language** if some finite automaton recognizes it.

Regular Operations

Let A and B be languages. We define the regular operations union, concatenation, and star as follows:

- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
- Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$.
- Star: $A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$.

A good idea to introduce:

- ϵ: the empty string
- ϵ: the empty language

Properties of Regular Languages

Theorem

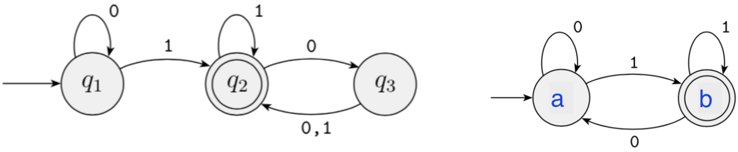
The class of regular languages is closed under the union operation.

Proof.

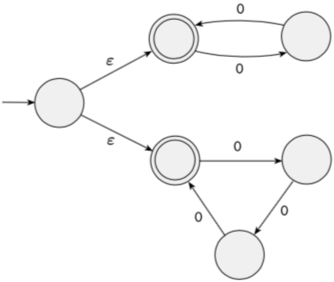
Given two regular languages A_1 and A_2 we want to show that $A_1 \cup A_2$ also is regular. Let M_1 be a finite automaton for A_1 and M_2 be a finite automaton for A_2. To prove that $A_1 \cup A_2$ is regular, we construct a finite automaton M that recognizes $A_1 \cup A_2$, using M_1 and M_2 as building blocks. M works by simultaneously simulating M_1 and M_2 and accepting if either of the simulations accept. M can keep track of the simulations by using as many states as the product of the states in M_1 and M_2. $\square$

Closure under Union

Example



Nondeterministic Finite Automata



A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_s, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma_\epsilon \to P(Q)$ is the transition function,
- $q_s \in Q$ is the start state,
- $F \subseteq Q$ is the set of accept states.

NFA Computation

Let $N = (Q, \Sigma, \delta, q_S, F)$ be a finite automaton and let $w = w_1 w_2 \ldots w_n$ be a string where each w_i is a member of the alphabet Σ. Then N accepts w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with three conditions:

- $r_0 = q_S$,
- $r_{i+1} \in \delta(r_i, w_{i+1})$, for $i = 0, ..., n-1$,
- $r_n \in F$.

An NFA accepts if any of its (possibly exponentially many) computation paths ends in an accept state.

Are Nondeterministic FA More Powerful?

What do we mean by *powerful*?

- the computational power of a machine is measured by the class of languages that it can recognize
- the computational power of a machine then is measured by the number of problems it can solve

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So, while it's easier to design NFAs (they can be smaller and simpler), it turns out that NFAs recognize exactly the same class of languages as DFAs

NFAs

Theorem

A language is regular if and only if some NFA recognizes it.

Proof.

$\Rightarrow$ Let L be a regular language. Then by definition there exists a DFA for L. Since a DFA is a special case of an NFA (an NFA without epsilon transitions), we are done.

NFAs

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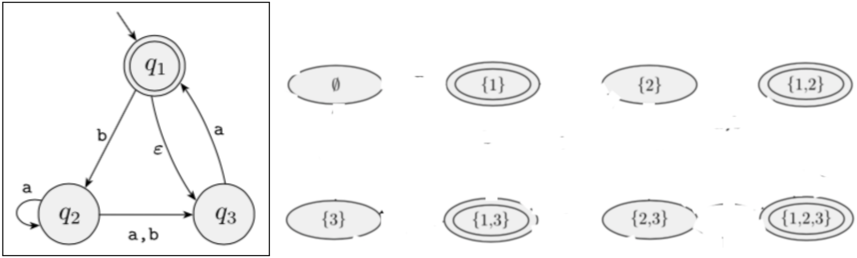
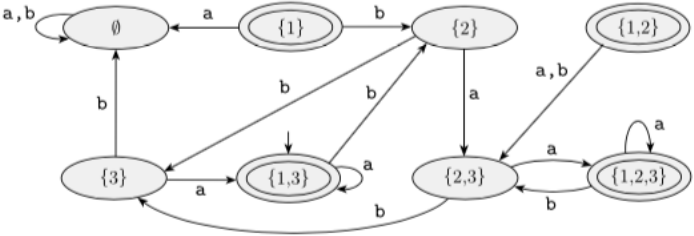
Proof.

$\Rightarrow$ Let L be a regular language. Then by definition there exists a DFA for L. Since a DFA is a special case of an NFA (an NFA without epsilon transitions), we are done.
$\Leftarrow$ Let N be an NFA. We will show how to create an equivalent DFA M. Then $L(N) = L(M)$ and by definition, this language is regular. $\square$

NFA to DFA Example



NFA to DFA Example



Equivalence of NFAs and DFAs (cont.)

Theorem

Every NFA has an equivalent DFA

Proof.

Let $N = (Q, \Sigma, \delta, q_s, F)$ be an NFA that recognizes language A. We will construct DFA $M = (Q', \Sigma, \delta', q'_s, F')$ that recognizes A.

- $Q' = P(Q)$
- for $R \in Q'$ and $a \in \Sigma$ let $\delta'(R, a) = \{q \in Q | q \in E(\delta(r, a))$ for some $r \in R\}$
- $q'_S = E(q_s)$
- $F' = \{R \in Q' | R$ contains an accept state of $N\}$

where $E(R) = \{q | q$ can be reached from R by traveling along 0 or more ϵ arrows. □

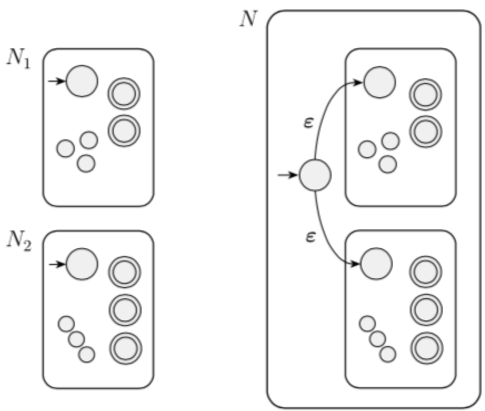
Closure under Union

Theorem

The class of regular languages is closed under the union operation.

Proof.

We proved this with DFAs; now we redo it with NFAs.



Closure under Union

Theorem

The class of regular languages is closed under the union operation.

Proof.

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

- $Q = \{q_0\} \cup Q_1 \cup Q_2$.
- start state q_0
- $F = F_1 \cup F_2$.
- $$\delta(q, a) = \begin{cases} \delta_1(q, a) : q \in Q_1 \\ \delta_2(q, a) : q \in Q_2 \\ \{q_1, q_2\} : q = q_0, a = \epsilon \\ \emptyset : q = q_0, a \neq \epsilon \end{cases}$$

$\square$

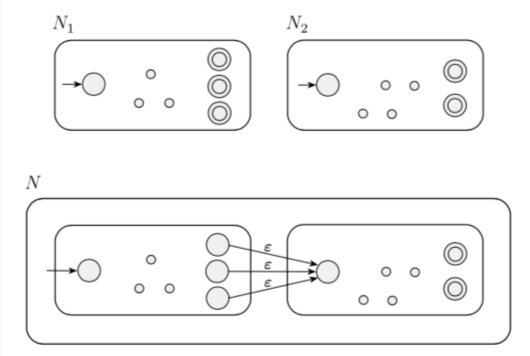
Closure under Concatenation

Recall concatenation: $A_1 \circ A_2 = \{xy | x \in A_1 \text{ and } y \in A_2\}$.

Theorem

The class of regular languages is closed under the concatenation operation.

Proof.



□

Closure under Concatenation

Theorem

The class of regular languages is closed under the concatenation operation.

Proof.

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2. Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$.

- $Q = Q_1 \cup Q_2$.
- start state q_1
- accept states F_2
- $$\delta(q, a) = \begin{cases} \delta_1(q, a) : q \in Q_1, q \neq F_1 \\ \delta_1(q, a) : q \in F_1, a \neq \epsilon \\ \delta_1(q, a) \cup \{q_2\} : q \in F_1, a = \epsilon \\ \delta_2(q, a) : q \in Q_2 \end{cases}$$

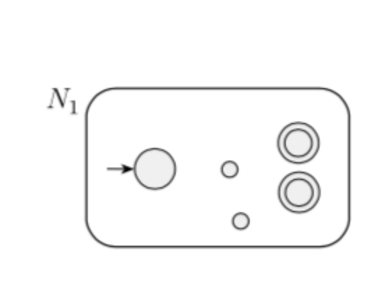
Closure under Star

Recall star: $A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$.

Theorem

The class of regular languages is closed under the star operation.

Proof.



Closure under Star

Recall star: $A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$.

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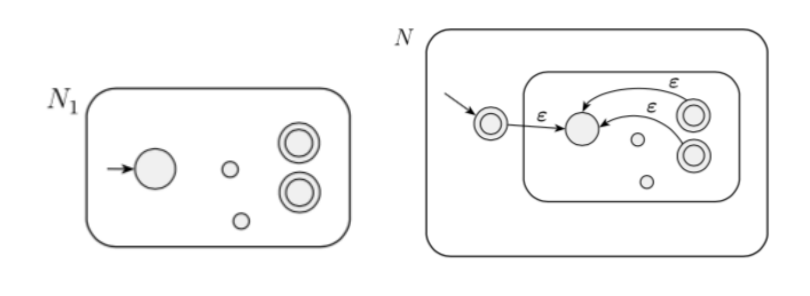


□

Closure under Star

Theorem

The class of regular languages is closed under the star operation.

Proof.

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

- $Q = \{q_0\} \cup Q_1$
- start state q_0
- $F = \{q_0\} \cup F_1$.

$$\delta(q, a) = \begin{cases} \delta_1(q, a) : q \in Q_1, q \neq F_1 \\ \delta_1(q, a) : q \in F_1, a \neq \epsilon \\ \delta_1(q, a) \cup \{q_1\} : q \in F_1, a = \epsilon \\ \{q_1\} : q = q_0, a = \epsilon \\ \emptyset : q = q_0, a \neq \epsilon \end{cases}$$

□

Regular Expressions

R is a regular expression if R is

1. a for some a in the alphabet Σ,
2. ϵ,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

RE Examples

Let $\Sigma = \{0, 1\}$

1. $0^*10^* = \{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^* = \{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^* = \{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$.
4. $1^*(01^+)^* = \{w \mid \text{every } 0 \text{ in } w \text{ is followed by at least one } 1\}$.
5. $(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}$.[5]
6. $(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.
7. $01 \cup 10 = \{01, 10\}$.

RE $\Longleftrightarrow$ FAs

Theorem

A language is regular if and only if some regular expression describes it.

Proof.

$\Leftarrow$ Consider some regular expression R that describes language A. We show how to convert R into an NFA recognizing A.

1. $R = a$ for some $a \in \Sigma$. Then $L(R) = \{a\}$, and the following NFA recognizes $L(R)$
2. $R = \epsilon$. Then $L(R) = \{\epsilon\}$, and the following NFA recognizes $L(R)$
3. $R = \emptyset$. Then $L(R) = \emptyset$, and the following NFA recognizes $L(R)$
4. $R = R_1 \cup R_2$
5. $R = R_1 \circ R_2$
6. $R = R_1^*$

RE $\iff$ FAs

Theorem

A language is regular if and only if some regular expression describes it.

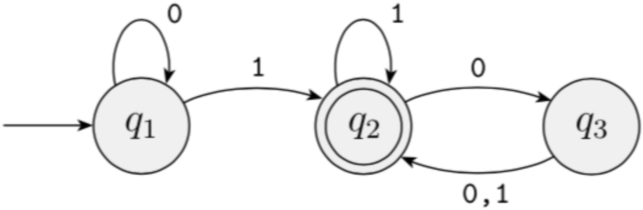
Proof.

$\Rightarrow$ Given some regular language A, there exists a DFA that recognizes A. We show how to extract an equivalent regular expression, by means a generalized nondeterministic finite automaton (GNFA). See textbook for complete proof. $\square$

Grammars

- We have seen two different, though equivalent, methods of describing languages: FAs and REs
- A grammar provides yet another way to describe a language
- While a FA *recognizes* a language, a grammar *generates* the language.
- We start with grammars for regular languages, also known as *type 3 grammars*

Type 3 Grammars: Example



$S \to 0S$
$S \to 1A$
$A \to 1A$
$A \to 0B$
$A \to \epsilon$
$B \to 0A$
$B \to 1A$

$S \to 0S|1A$
$A \to 1A|0B|\epsilon$
$B \to 0A|1A$

Type 3 Grammars

What is a grammar?

- substitution rules
- variables
- terminals
- start variable

$$S \to 0S|1A$$
$$A \to 1A|0B|\epsilon$$
$$B \to 0A|1A$$

What makes this a type 3 grammar?

- Chomsky's hierarchy: type 0, type 1, type 2, type 3
- the most general rule is $\alpha \to \beta$, where α and β are strings of terminals and variables
- rule restrictions: $\alpha \in V$ $|\alpha| = 1$, $|\beta| \leq 2$, etc.
- e.g., the grammar above it not only type 3 but also a right linear grammar