SMALL SETS SUPPORTING FÁRY EMBEDDINGS OF PLANAR GRAPHS

HUBERT DE FRAYSSEIX*, JÁNOS PACH† AND RICHARD POLLACK‡

Abstract. Answering a question of Rosenstiehl and Tarjan, we show that every plane graph with n vertices has a Fáry embedding (i.e., straight-line embedding) on the $2n-4$ by $n-2$ grid and provide an $O(n)$ space, $O(n \log n)$ time algorithm to effect this embedding. The grid size is asymptotically optimal and it had been previously unknown whether one can always find a polynomial sized grid to support such an embedding. On the other hand we show that any set F, which can support a Fáry embedding of every planar graph of size n, has cardinality at least $n + (1 - o(1))\sqrt{n}$ which settles a problem of Mohar.

1. Introduction. The theorem of I. Fáry [F] shows that every plane graph has an embedding (drawing) in which the edges are straight line segments and the vertices are points in the plane. An embedding of this sort will be called a *Fáry embedding.* Starting with the paper of Tutte in 1963 there have been many algorithms offered for constructing a Fáry embedding ([T], [CYN], [Rd]). All present algorithms for Fáry embedding a plane graph exhibit several drawbacks. These drawbacks are: (1) that they require high precision real arithmetic relative to the size of the graph and (2) vertices tend to bunch together in the sense that the ratio of the smallest distance to the largest distance is unreasonably small. This means that: (1) for a graph of moderate size it is not possible to execute the algorithm and (2) even if it were, it would not be possible to view the resulting drawing on a terminal screen.

In fact, it has been an open question whether or not every planar graph of size n has a Fáry embedding on a grid of side length bounded by n^k for some fixed k [RT]. Questions , such as this, about how compactly graphs can be embedded on grids are related to the problems of VLSI layout design ([L], [U], [V]). Theorem 1 gives a positive answer (which is asymptotically sharp) to this question and its proof provides an algorithm consructing such an embedding.

THEOREM 1. *Any plane graph with n vertices has a Fáry embedding on the $2n-4$ by $n-2$ grid.*

It can be shown that any Fáry embedding of a nested sequence of $\frac{n}{3}$ triangles on a grid requires a grid of size at least $(\frac{2}{3}n - 1) \times (\frac{2}{3}n - 1)$.

Since the Hopcroft-Tarjan planarity testing algorithm [HT] outputs a topological embedding

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of a planar graph, we shall assume that any graph we look at has already passed that filter and comes equipped with such an embedding. A *maximal* plane graph is one which cannot have any additional edges without destroying its planarity. Such a graph is also called *triangulated* since all the faces are triangles. Since every planar graph can be triangulated by adding additional (dummy) edges, it suffices to prove Theorem 1 for maximal planar graphs.

We prove Theorem 1 in section 3 by presenting an algorithm which takes an arbitrary maximal plane graph and an arbitrary triangular face and then outputs the embedding with the given triangle as the exterior face. Our proof is based on a general construction called the *canonical representation of plane graphs* (section 2), which provides a suitable ordering of the vertices so that we can inductively Fáry embed the graph induced by the first k vertices on a grid and then by moving some of the vertices in this embedding in a controlled way we are able to add the next vertex and still have a Fáry embedding. On the face of it this algorithm has at least quadratic time complexity. The speedup to $O(n \log n)$ is obtained by not actually performing each embedding but instead storing all the information needed in a single permutation which can be constructed in time $O(n \log n)$. Then we will make $O(n)$ queries of this permutation of the form: for indices i and j, how many k, $i \leq k \leq j$ precede i in the permutation. The answers to the queries are then used to find the coordinates of the embedded vertices. These queries can be interpreted as rectangle range queries on a set of $2n$ points derived from the permutation and using a data structure of Chazelle [C] which uses linear space, $O(n \log n)$ preprocessing time, the queries can each be executed in time $O(\log n)$.

The canonical representation of plane graphs is a useful tool to establish the existence of Fáry embeddings with special geometric properties by easy inductional arguments. Some examples are propositions 1,2 and 3.

PROPOSITION 1. *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices, $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$ and a Fáry embedding f such that the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots f(v_k)\}$ is the same as the convex hull of $\{f(v_1), f(v_2), f(v_k)\}$ for $k = 4, \ldots, n$.*

PROPOSITION 2. *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices, $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$ and a Fáry embedding f such that the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots f(v_k)\}$ is the same as the convex hull of $\{f(v_{k-2}), f(v_{k-1}), f(v_k)\}$ for $k = 4, \ldots, n$.*

PROPOSITION 3. *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices, $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$ and a Fáry embedding f such that the boundary of the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots f(v_k)\}$ is a cycle in G and $f(v_{k+1})$ is not contained in the convex hull of $\{f(v_1), f(v_2), f(v_3), \ldots f(v_k)\}$.*

Theorem 1 gives an asymptotically sharp bound on the size of the smallest grid that will support all planar graphs of size n. Even though a grid is a natural set on which to embed graphs it also makes sense to drop the restriction to a grid and ask for bounds on the size of a set which supports all planar graphs of size n. Last year Bojan Mohar [M] asked whether or not there exists a set F of n points in the plane which supports every planar graph with n vertices (a set F *supports* a simple planar graph if there exists an injective map $f : V(G)) \rightarrow F$ such that the segments $[f(a), f(b)]$ and $[f(c), f(d)]$ are openly disjoint if $[a, b]$ and $[c, d]$ are edges of G). F is called *universal* for a set of planar graphs if it supports all graphs in the set. Thus $\mathbb{R}^2$ is universal for all planar graphs by the theorem of Fáry and by Theorem 1 the $n-2$ by $2n-4$ grid is universal for all planar graphs with n vertices and Mohar is asking whether there is a universal set of size n for all planar graphs with n vertices.

We give a negative answer to this in section 5 by proving;

THEOREM 2. *If F is universal for planar graphs with n vertices then $|F| > n+(1-o(1))\sqrt{n}$.*

2. The canonical representation of plane graphs.

The aim of this section is to describe a canonical way of constructing a plane graph, which will be a basic tool of our investigations in the rest of this paper, and also provides easy proofs of generalizations of Fáry's theorem (Propositions 1.2 and 3).

The following simple observation will be essential for our purposes.

LEMMA. *Let G be a simple planar graph embedded in the plane and $u = u_1, u_2, \ldots, u_k = v$ be a cycle of G. Then there exists a vertex w' (resp. w'') on the cycle different from u and v and not adjacent to any inside chord (resp. outside chord).*

Proof. If the cycle has no inside (resp. outside) chords, then there is nothing to prove. Otherwise, let (u_i, u_j), $j > i+1$ be an inside (outside) chord such that $j-i$ is minimal. Then u_{i+1} cannot be adjacent to any inside chord of the cycle $u_i, u_{i+1}, \ldots, u_j$, by minimality. Nor can it be adjacent to any other inside (outside) chord of the original cycle, by planarity. □

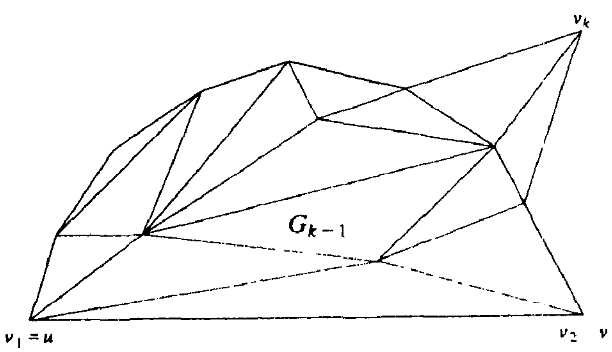

Figure 1.

Now we are in the position to establish the following

CANONICAL REPRESENTATION LEMMA FOR PLANE GRAPHS. *Let G be a maximal planar graph embedded in the plane with exterior face u, v, w. Then there exists a labelling of the vertices $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ meeting the following requirements for every $4 \le k \le n$.*

(i) *The subgraph $G_{k-1} \subseteq G$ induced by $v_1, v_2, \ldots, v_{k-1}$ is 2-connected, and the boundary of its exterior face is a cycle C_{k-1} containing the edge uv;*

(ii) *v_k is in the exterior face of G_{k-1}, and its neighbors in G_{k-1} form an (at least 2-element) subinterval of the path $C_{k-1} - uv$. (See Figure 1.)*

Proof. The vertices $v_n, v_{n-1}, \ldots, v_3$ will be defined by reverse induction.

Set $v_n = w$, and let G_{n-1} denote the subgraph of G after deleting v_n. By the maximality of G, the neighbors of w form a cycle C_{n-1} containing uv, and this cycle is the boundary of the exterior face of G_{n-1}.

Let $i < n$ be fixed, and assume that v_k has already been determined for every $k > i$ such that the subgraph G_{k-1} induced by $V(G) \setminus \{v_k, v_{k+1}, \ldots, v_n\}$ satisfies conditions (i) and (ii). Let C_{k-1} denote the boundary of the exterior face of G_{k-1}. Applying the Lemma to the cycle C_i in G_i, we obtain that there is a vertex $w' \in C_i$ different from u and v and not adjacent to any chord of C_i. (Observe that C_i has no exterior chords.) Letting $v_i = w'$, the subgraph G_{i-1} induced by $V(G) \setminus \{v_i, v_{i+1}, \ldots, v_n\}$ obviously meets the requirements. □

Proposition 1 now follows almost immediately.

Proof of Proposition 1. Let $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ be the canonical labelling of the vertices of G, as described above. We will define $f(v_k)$, $1 \le k \le n$ by induction on k.

Set $f(v_1) = (0,0), f(v_2) = (2,0), f(v_3) = (1,1)$. Assume that $f(v_i)$ has already been determined for $i = 1, 2, \ldots, k-1$ such that f is a Fáry embedding of G_{k-1} with

$$\begin{aligned} &\mathrm{conv}\{f(v_1), f(v_2), \ldots, f(v_i)\} \\ &= \mathrm{conv}\{f(v_1), f(v_2), f(v_i)\}, 3 \le i \le k-1. \end{aligned}$$

We want to extend it to an embedding of G_k.

Let $u = w_1, w_2, \ldots, w_m = v$ denote the vertices of C_{k-1} in the order as they appear along the boundary of the exterior face of G_{k-1}. Let $x(w_j)$ and $y(w_j)$ denote the x-coordinate and y-coordinate of $f(w_j)$, respectively. Suppose, by induction,

$$\text{(iii)} \qquad \begin{aligned} &x(w_1) < x(w_2) < \cdots < x(w_m), \\ &y(w_j) > 0 \text{ for } 3 \le j \le m. \end{aligned}$$

By property (ii) of the canonical labelling, v_k is connected to $w_p, w_{p+1}, \ldots, w_q$ for some $1 \le p < q \le m$.

Let us fix any number x^* between $x(w_p)$ and $x(w_q)$, and set $f(v_k) = (x^*, y^*)$ for some $y^* > 0$. It is now clear that, if y^* is sufficiently large, then we obtain a Fáry embedding of G_k with the desired properties. Furthermore, our auxiliary hypothesis (iii) will remain true for the points of C_k. □

Propositions 2 and 3 can be proved in a similar way.

3. Drawing a plane graph on a grid. It suffices to prove Theorem 1 for maximal plane graphs. Let G be such a graph with exterior face u, v, w, and let $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$ be the canonical labelling of its vertices.

The idea of the proof is the following. Suppose that at step k of our algorithm G_k has already been Fáry embedded on the grid such that

(1) v_1 is at $(0,0)$, v_2 is at $(2k-4, 0)$;

(2) If $v_1 = w_1, w_2, \ldots, w_m = v_2$ denote the vertices on the exterior face of G_k (in the order of their appearance), and $x(w_i)$ denotes the x-coordinate of w_i, then

$$x(w_1) < x(w_2) < \cdots < x(w_m);$$

(3) The edges $[w_i, w_{i+1}]$, $1 \le i < m$, all have slopes $+1$ or -1.

Note that (3) implies that the Manhattan distance between any two vertices w_i and w_j of the exterior face is *even*. (The Manhattan distance of (x, y) and (x', y') is $|x - x'| + |y - y'|$.) Hence, if $i < j$, then the intersection of the line with slope $+1$ through w_i and the line with slope -1 through w_j is a *lattice point* $P(w_i, w_j)$.

Let $w_p, w_{p+1}, \ldots, w_q$ be the neighbors of v_{k+1} in G_{k+1} ($1 \le p < q \le m$), (cf. part (ii) of the Canonical Representation Lemma). Then $P(w_p, w_q)$ is a good candidate for placing v_{k+1}, except that it may fail to see e.g., w_p (see fig. 2).

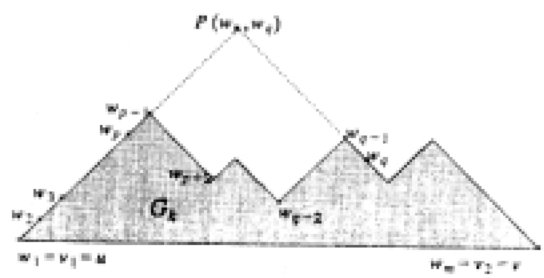

Figure 2.

To make sure that $P(w_p, w_q)$ sees all the points $w_p, w_{p+1}, \ldots, w_q$, we shall deform the embedding (drawing) to guarantee that the slope of the edge

$[w_p, w_{p+1}]$ is < 1 and the slope of $[w_{q-1}, w_q]$ is > -1 while the slopes of all other edges of the exterior face of G_k remain the same. One way to ensure this is to move $w_{p+1}, w_{p+2}, \ldots, w_m$ one unit to the right and then to move $w_q, w_{q+1}, \ldots, w_m$ an additional unit to the right. However, to keep a Fáry embedding, it may be necessary to move some other vertices (not on the exterior face) as well. Though it is difficult to know globally which set of points has to move together with a given exterior vertex, there is an elegant way to define such sets recursively at each step of our algorithm.

To realize this goal, assume that for each vertex w_i on the exterior face of G_k we have already defined a subset $M(k, w_i) \subseteq V(G_k)$ such that

(a) $w_j \in M(k, w_i)$ iff $j \geq i$;

(b) $M(k, w_1) \supset M(k, w_2) \supset \cdots \supset M(k, w_m)$;

(c) For any nonnegative numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, if we sequentially translate all vertices in $M(k, w_i)$ with distance α_i to the right ($i = 1, 2, \ldots, m$), then the embedding of G_k remains a Fáry embedding. (Note that many vertices will move several times; e.g., all points in $M(k, w_i) \setminus M(k, w_{i+1})$ will be translated by $\alpha_1 + \alpha_2 + \cdots + \alpha_i$.)

For $k = 3$ these condition are met by the Fáry embedding $v_1 \rightarrow (0,0)$, $v_2 \rightarrow (2,0)$, $v_3 \rightarrow (1,1)$ and by the sets $M(3, v_1) = \{v_1, v_2, v_3\}$, $M(3, v_2) = \{v_2, v_3\}$, $M(3, v_3) = \{v_3\}$.

Apply condition (c) with $\alpha_{p+1} = \alpha_q = 1$ and all other $\alpha_i = 0$ to find a new Fáry embedding of G_k. The Manhattan distance between w_p and the new location of w_q is still even, thus we can place v_{k+1} at the intersection of the lines with slope $+1$ and -1 through w_p and the new location of w_q, respectively. Conditions (1), (2) and (3) will trivially remain true for this new embedding of G_{k+1}. (See fig. 3)

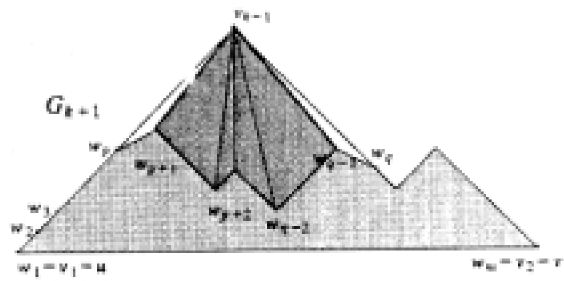

Figure 3.

The vertices of the exterior face of G_{k+1} are

$$u = w_1, w_2, \ldots, w_p, v_{k+1}, w_q, \ldots, w_m = v.$$

For each member z of this sequence we have to define a set $M(k+1, z) \subseteq V(G_{k+1})$. Let

$$M(k+1, w_i) = M(k, w_i) \cup \{v_{k+1}\} \text{ for } i \leq p,$$
$$M(k+1, v_{k+1}) = M(k, w_{p+1}) \cup \{v_{k+1}\},$$
$$M(k+1, w_j) = M(k, w_j) \quad \text{for } j \geq q.$$

It is obvious, by induction, that these sets have properties (a) and (b).

To check that property (c) remains true as well, fix a sequence of nonnegative numbers $\alpha(w_1), \ldots, \alpha(w_p), \alpha(v_{k+1}), \alpha(w_q), \ldots, \alpha(w_m)$. For all z on the exterior face of G_{k+1} translate the sets $M(k+1, z)$ with distance $\alpha(z)$ to the right. Observe that after this motion the part of G_{k+1} below the polygon $w_1 w_2 \ldots w_m$ (i.e., G_k) remains Fáry embedded (by condition (c) applied to G_k with $\alpha_1 = \alpha(w_1), \ldots, \alpha_p = \alpha(w_p)$, $\alpha_{p+1} = 1 + \alpha(v_{k+1})$, $\alpha_q = 1 + \alpha(w_q)$, $\alpha_{q+1} = \alpha(w_{q+1}), \ldots, \alpha_m = \alpha(w_m)$ and every other $\alpha_i = 0$). On the other hand, it is easy to see that the part of G_{k+1} above $w_1 w_2 \ldots w_m$ (i.e., v_{k+1} and the upper contour of G_k) remains Fáry embedded, too, since during the motion the (darkly shaded) subgraph induced by $w_{p+1}, w_{p+2}, \ldots, w_{q-1}$ and v_{k+1} moves rigidly (to a distance $\alpha(w_1)+\cdots+\alpha(w_p)+\alpha(v_{k+1})$).

The final output of our algorithm is a Fáry embedding f of $G_n = G$ satisfying conditions (1), (2) and (3) with $k = n$. This immediately implies that every point of G is embedded in some lattice point of the triangle determined by $f(v_1) = f(u) = (0,0)$, $f(v_2) = f(v) = (2n-4, 0)$ and $f(v_n) = f(w) = (n-2, n-2)$. This completes the proof of Theorem 1. ||

4. Outline of an $O(n \log n)$ algorithm for drawing a maximal planar graph on a grid. Given a planar graph G we might as well assume that G is maximal, since, in linear time dummy edges can be added to make it so [Rd]. We further label all vertices as (a) not yet visited, (b) visited once or (i) visited more than once and the visited edges form i connected components in the circular order of all edges adjacent to this vertex. These labels are updated after we choose vertex v_{k+1}. We visit each neighbor of v_{k+1}. Suppose that v is such a vertex. If v has label (a), label (b) replaces label (a). If v has label (b) and the edge from v_{k+1} is adjacent to a previous edge, in the circular order of edges at v, along which v was visited, label (1) replaces label (b) and if not, label (2) replaces label (b). Finally if v has label (j) and the left and right neighbors of the edge from v_{k+1} to v have already been traversed then label $(j-1)$ replaces label j. If only one of these edges has been traversed then the label is unchanged and if neither has been traversed then label $(j+1)$ replaces label (j). It is clear that the label (j) on v means that the edges already traversed and incident to v are composed of j intervals in the circular order of edges at v. It is easy to see that any vertex with label (1) can be chosen as v_{k+2}. Since there are only a linear number of edges we find the order to insert vertices in linear time.

We define a sequence of permutations inductively as follows. $\pi_2 = (1,2)$ Suppose π_k is defined and the vertex v_{k+1} is adjacent (in counterclockwise order) to the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ in G_k. Then we generate π_{k+1} by inserting $k+1$ just to the left of i_2 and $n+k+1$ just to the left of i_j in the permutation π_k. Clearly π_n can be constructed in time $O(n \log n)$. It is clear, identifying the vertex v_j with the index j, that

$$\begin{aligned} M(k, v_i) &= \{j \mid j \le n, j \text{ does not precede } i \text{ in } \pi_k\} \\ &= \{j \mid i \le j \le n, j \text{ does not precede } i \text{ in } \pi_n\}. \end{aligned}$$

Suppose that v_k is placed at $(x_k(j), y_k(j))$ when v_j is placed on the grid. Then $y_k(j) = y_k(k)$ and $x_k(j) = x_k(k) + \sigma(k,j)$ where

$$\begin{aligned} \sigma(k,j) &= |\{i \mid k < i, i \text{ precedes } k \text{ in } \pi_j\}| \\ &= |\{i \mid k < i \le j, i \text{ precedes } k \text{ in } \pi_n\}|. \end{aligned}$$

It follows that we can find $(x_k(k), y_k(k))$ in constant time given $(x_j(j), y_j(j))$ and $\sigma(j,k)$ for v_j the left-most neighbor of v_k and for v_j the right-most neighbor of v_k in G_{k-1}. Then in constant time we find the embedded coordinates of v_k, $(x_k(n), y_k(n))$. Thus, the entire algorithm runs in time $O(nT)$, where T is the time to calculate any $\sigma(j,k)$. Finally, let S be the set of points

$$\begin{aligned} &\{(1,1),(2,2n-3)\} \cup \\ &\qquad \{(k, \pi_n^{-1}(k)), (k, \pi_n^{-1}(n+k)) \mid 3 \le k \le n\}. \end{aligned}$$

It is evident that $\sigma(k,j) = |S \cap R(k,j)|$ where $R(k,j)$ is the rectangle,

$$R(k,j) = \{(x,y) \mid k+1 \le x \le j, 1 \le y \le \pi_n^{-1}(k)\}.$$

Hence it follows from [C] that $T = \log n$. □

5. Lower bounds on the size of a set which supports all planar graphs. Let G_k be a fixed maximal planar graph with k vertices and $2k-4$ triangular faces. Given any natural numbers n_i, $1 \le i \le 2k-4$ with $\sum_{i=1}^{2k-4} n_i = n-k$, let $G_k(n_1, n_2, \ldots, n_{2k-4})$ be a fixed maximal planar graph on the vertices $\{P_1, P_2, \ldots, P_n\}$

whose restriction to $\{P_1, P_2, \ldots, P_k\}$ is G_k and has n_i vertices in the face f_i. Since G_k and $G_k(n_1, n_2 \ldots, n_{2k-4})$ are maximal, Steinitz's theorem shows that they have unique planar maps. There are $\binom{n+k-5}{2k-5}$ of these graphs. Now suppose that F, a subset of $\mathbb{R}^2$ supports all these graphs, and fix an embedding $f_G : V(G) \to F$ for each of them. There are at most $|F|^k$ ways to embed $(P_1, P_2, \ldots, P_k)$, hence the embedding of at least $\binom{n+k-5}{2k-5}/|F|^k$ of these graphs is the same on the vertices $\{P_1, P_2, \ldots, P_k\}$. On the other hand a given embedding of G_k on F can be extended to at most $\binom{|F|-n+2k-5}{2k-5}$ of our graphs. Hence,

$$\binom{|F| - n + 2k - 5}{2k - 5} > \binom{n + k - 5}{2k - 5}/|F|^k.$$

From this, by elementary calculations we obtain

$$\left(\frac{(|F| - n + 2k)^2}{n}\right)^k \left(\frac{|F|}{n}\right)^k > n^{-5}\left(1 - \frac{k}{n}\right)^{2k}$$

Now choosing $k = \frac{\varepsilon}{4}\sqrt{n}$ we contradict $|F| < n + (1 - \varepsilon)\sqrt{n}$ for n sufficiently large, and Theorem 2 follows. □

6. Remarks. The algorithm implicit in the proof of theorem 1, as well as another which introduces new edges on the exterior face which may be horizontal at the time of insertion rather than insisting that they have slopes 1 or -1 has been implemented by Nejia Assila. This second version, has the advantage that the graph may have a Fáry embedding on a considerably smaller grid. Figures 4a and 4b show the output of each algorithm on the same graph.

For $f(n)$, the smallest cardinality of a set which supports all planar graphs, theorems 1 and 2 show that

$$n + (1 - o(1))\sqrt{n} \leq f(n) < n^2.$$

An interesting open problem is to tighten these bounds.

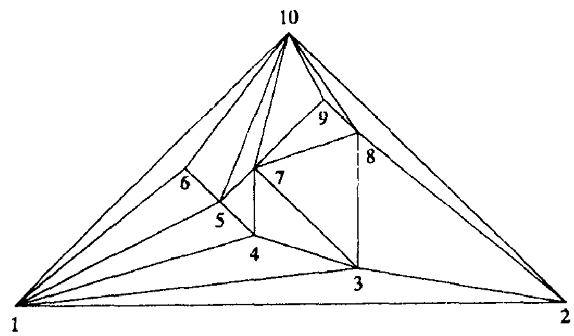

Figure 4a.

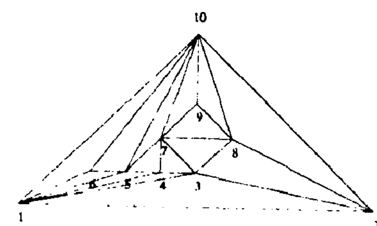

Figure 4b.

Finally, we suspect that a linear time algorithm exists to Fáry embed any planar graph on a linear sized grid. A weaker, but equally important, question is whether the algorithm can be improved dynamically. Is it possible, given an embedding, to insert a new vertex in linear time?

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REFERENCES

[C] B. Chazelle, *Slimming down search structures: a functional approach to algorithm design*, in *Proceedings of the Twenty Sixth Annual Symposium on the Foundations of Computer Science*, 1985, pp. 165-174.

[CYN] N. CHIBA, T. YAMANOUCHI, AND T. NISHIZEKI, *Linear algorithms for convex drawings of planar graphs*, in *Progress in graph theory*, 1982, pp. 153-173.

[Deal] P. DUCHET, Y. HAMIDOUNE, M. LAS VERGNAS, AND H. MEYNIEL, *Representing a planar graph by vertical lines joining different intervals*, Discrete Math., 46 (1983), pp. 319–321.

[F] I. FÁRY, *On straight line representing of planar graphs*, Acta. Sci. Math. (Szeged), 11 (1948), pp. 229–233.

[dF] H. DE FRAYSSEIX, *Drawing planar and non planar graphs from the half-edge code*, (to appear).

[dFR1] H. DE FRAYSSEIX AND P. ROSENSTIEHL, *Structures combinatoires pour traces automatiques de resaux*, in *Proceedings of the Third European Conference on CAD CAM and Computer Graphics*, 1984, pp. 332-337.

[dFR2] H. DE FRAYSSEIX AND P. ROSENSTIEHL, *L'algorithme Gauche-droite pour le plongement des graphes dans le plan*, (to appear).

[G] D. H. GREENE, *Efficient coding and drawing of planar graphs*, Xerox Palo Alto Research Center, Palo Alto, CA.

[Gr] G. GRÜNBAUM, *Convex Polytopes*, John Wiley.

[HT] J. HOPCROFT AND R. TARJAN, *Efficient planarity testing*, J. Comput. Mach., 21 (1974), pp. 549–568.

[L] F. T. LEIGHTON, *Complexity Issues in VLSI*, The MIT Press.

[M] B. MOHAR, private communication.

[OvW] R. H. J. M. OTTEN AND J. G. VAN WIJK, *Graph representation in interactive layout design*, in *Proceedings of the IEEE International Symposium on Circuits and Systems*, 1978, pp. 914–918.

[R] P. ROSENSTIEHL, *Embedding in the plane with orientation constraints*, Ann. N. Y. Acad. Sci. (to appear).

[Rd] R. C. READ, *A new method for drawing a planar graph given the cyclic order of the edges at each vertex*, Congresus Numerantium, 56, pp. 31-44.

[RT] P. ROSENSTIEHL AND R. E. TARJAN, *Rectilinear planar layouts and bipolar orientations of planar graphs*, Disc. Comp. Geom., 1 (1986), pp. 343–353.

[TT] R. TOMMASIA AND I. G. TOLLIS, *A unified approach to visibility representations of planar graphs*, Disc. Comp. Geom., 1 (1986), pp. 321–341.

[T] W. T. TUTTE, *How to draw a graph*, Proc. London Math. Soc., 13 (1963), pp. 743–768.

[U] J. D. ULLMAN, *Computational Aspects of VLSI*, Computer Science Press.

[V] L. G. VALIANT, *Universality considerations in VLSI circuits*, IEEE Trans. on Computers, C-30, pp. 135–140.

[W] D. R. WOODS, *Drawing planar graphs*, in Report N. STAN-CS-82-943 , Computer Science Department, Stanford University, CA, 1981.