Multi-Level Graph Sketches

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Graph Sketching

A sketch of a large graph is a subgraph that preserves certain properties of the original graph

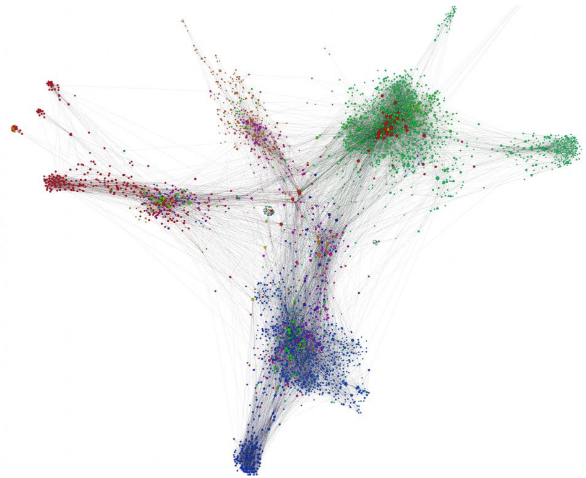
A technique for making polynomial time algorithms usable in practice

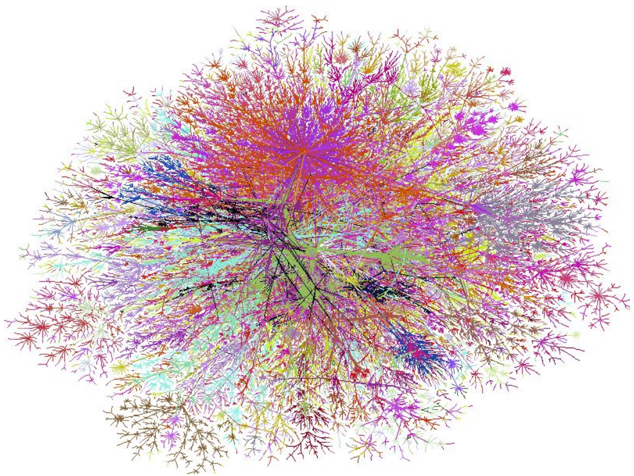
Examples of graph sketches:

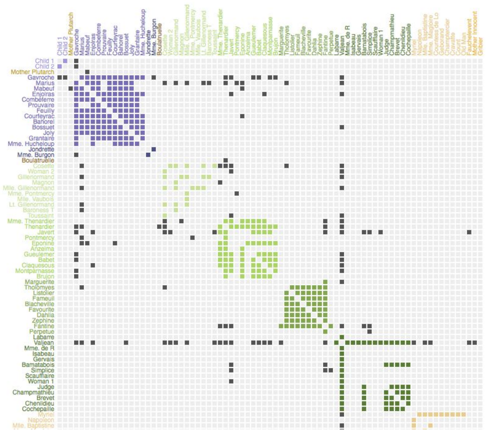
- Minimum spanning tree (preserves connectivity)
- Steiner tree (preserves weighted connectivity)
- graph spanner (preserves distances)

Motivation

- Many real-world graphs are large (millions of vertices, billions of edges)
- Analyzing such graphs is computationally expensive (even stats are superlinear)
- Interacting (query, search) with such graphs is not easy





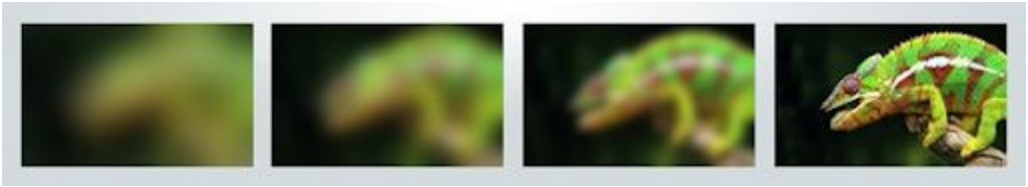


Multi-Level Graph Representation

A multi-level graph sketch preserving original graph properties at different scales

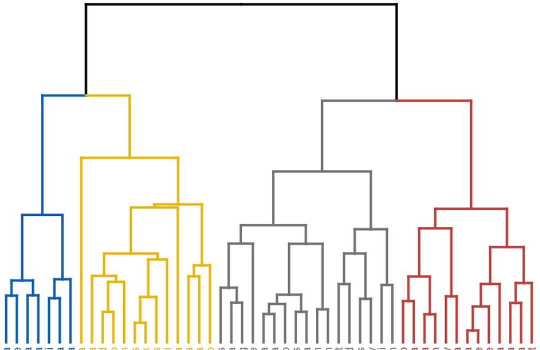
Example of multi-level representation:

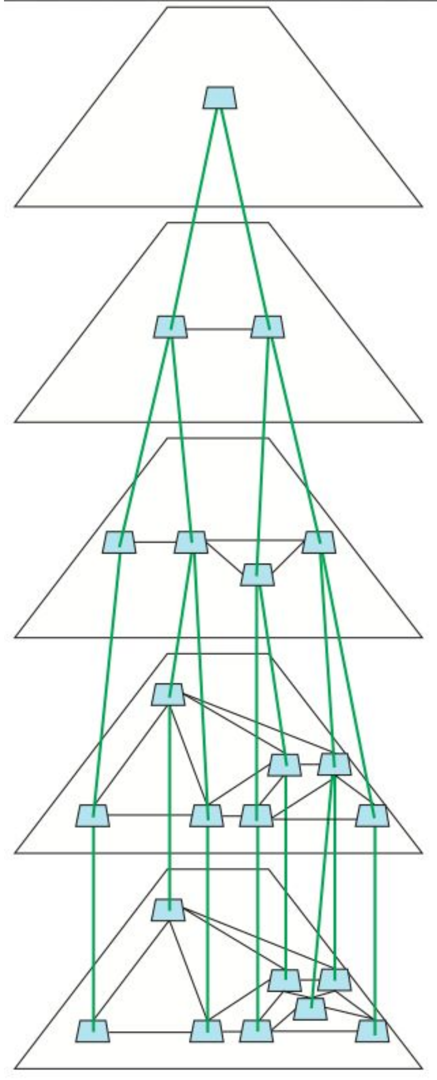
- Progressive image rendering
- Semantic zooming in geographic maps



Traditional Multi-Level Representation

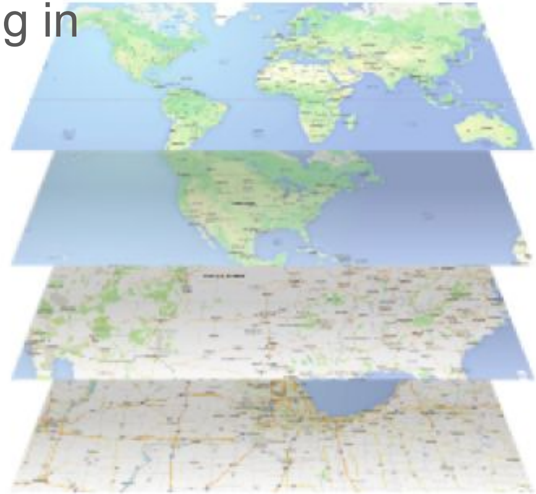
- Hierarchical clustering of the graph
- Abstract levels-of-detail: meta-nodes and meta-edges
- Some Issues
  - Interpreting clusters (meta-nodes)
  - Interpreting links (meta-edges)
  - Confusing visualizations
  - Difficult to interact with (search, zoom, pan)





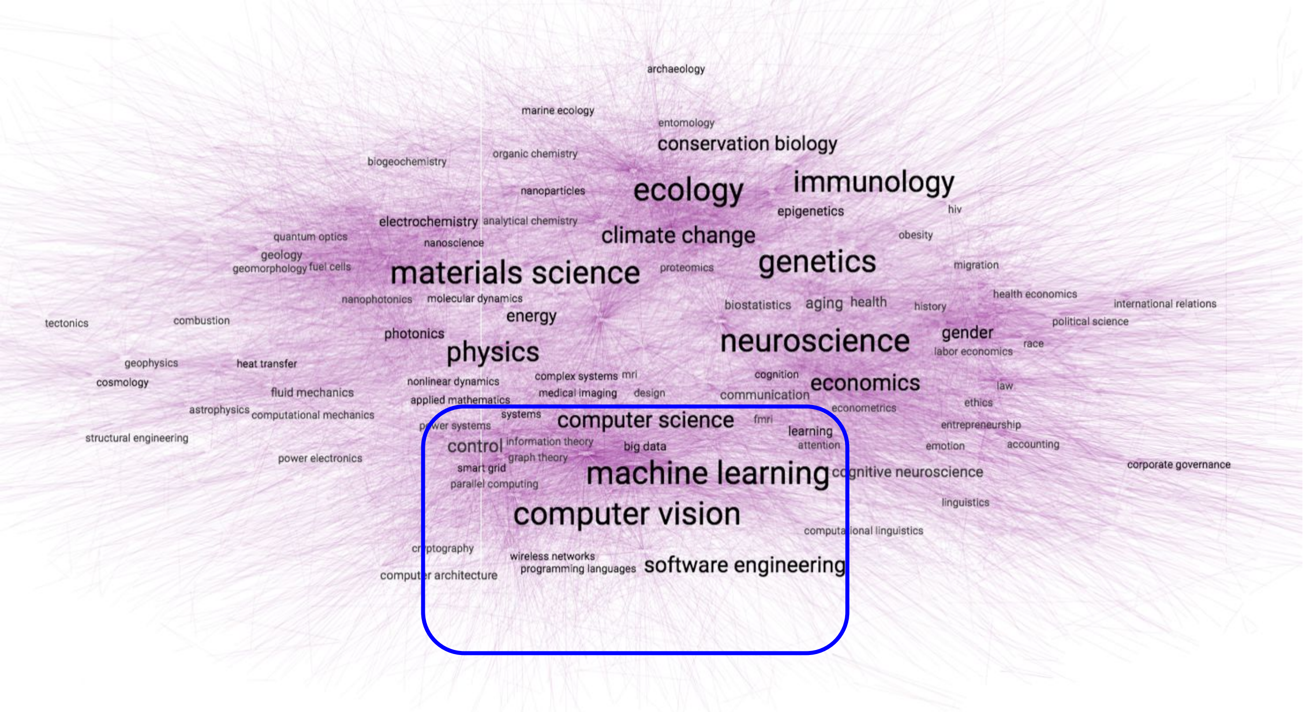
More Intuitive Multi-Level Representation

- Borrow from the details-on-demand *map metaphor*
- On every level of the hierarchy we have a graph with real vertices and edges
- The graph on each level is "*similar*" to the original graph
- Important vertices and paths are shown on high levels
- Less important vertices and paths appear when zooming in
    - What is the notion of *importance*?
    - Do vertices and edges come with importance values?



Using the Map Metaphor

- How can we define a natural “semantic zoom” representation?



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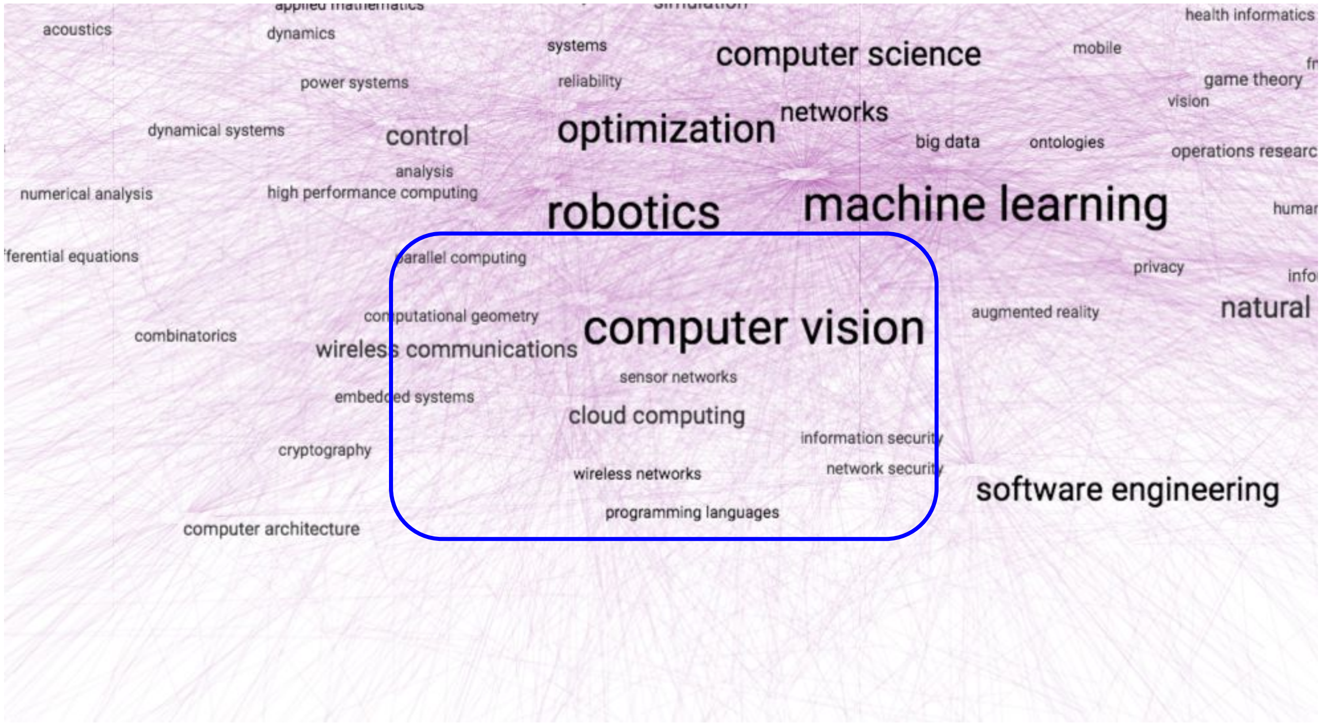

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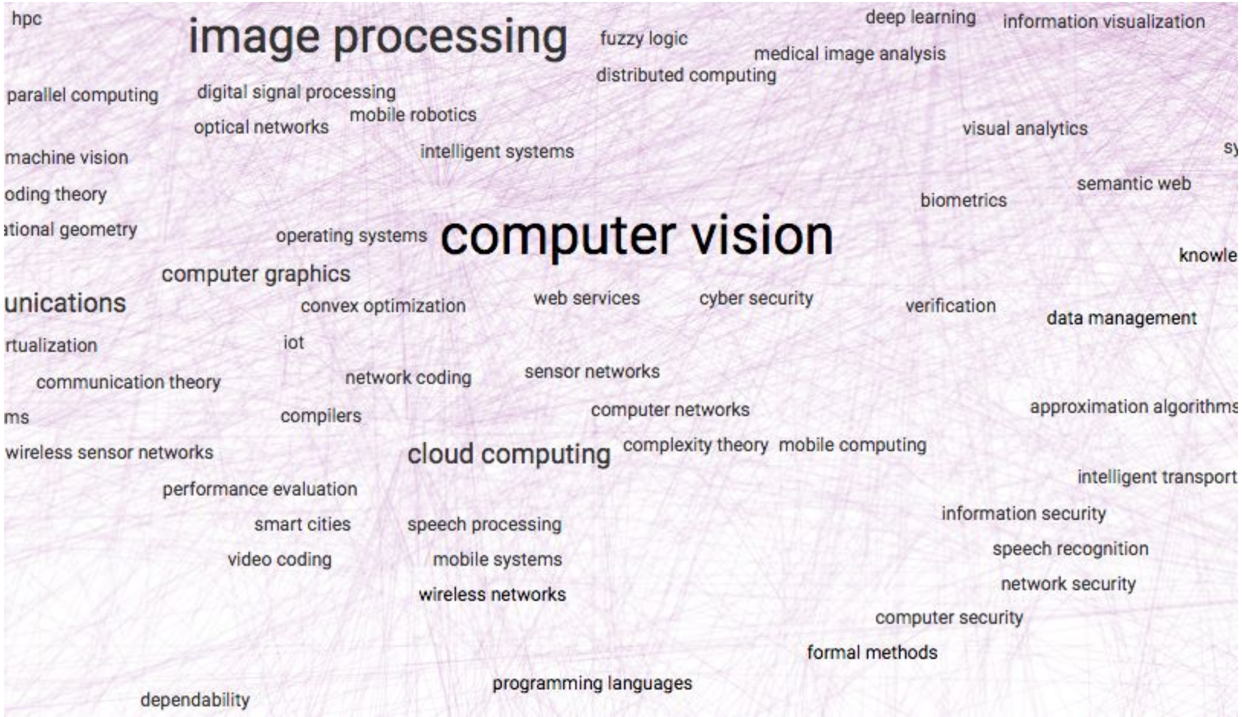
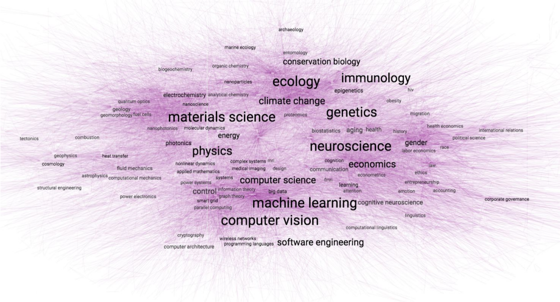
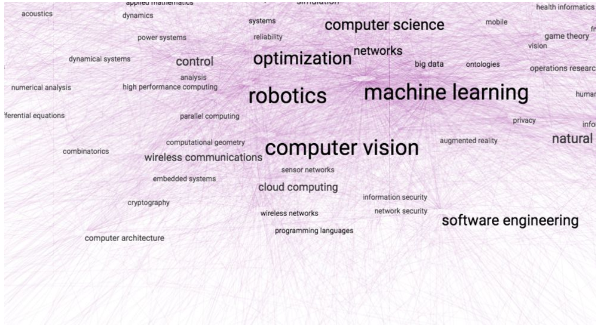


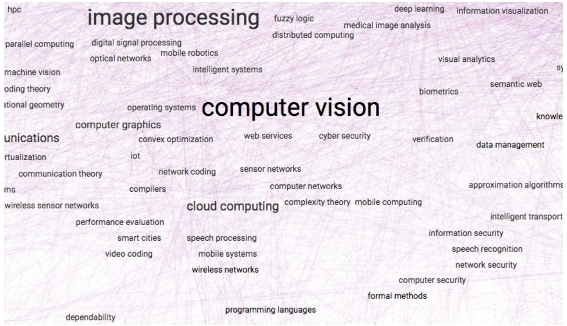
Using the Map Metaphor

How can we define a natural “semantic zoom” representation?

- We can often group vertices by importance (at what level of detail do they show)
- But what about the edges?





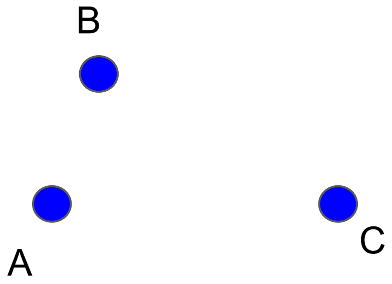


Starting Simple: Steiner Trees

"*A very simple but instructive problem was treated by Jacob Steiner, the famous representative of geometry at the University of Berlin in the early nineteenth century. Three villages A,B,C are to be joined by a system of roads of minimum length.*" Courant and Robbins, 1941

Fermat in 1679 already mentions the problem of "*finding for three given points in the plane a fourth one in such a way that the sum of its distances to the given points is minimized.*"

Torricelli found a geometrical solution for this problem in 1640.



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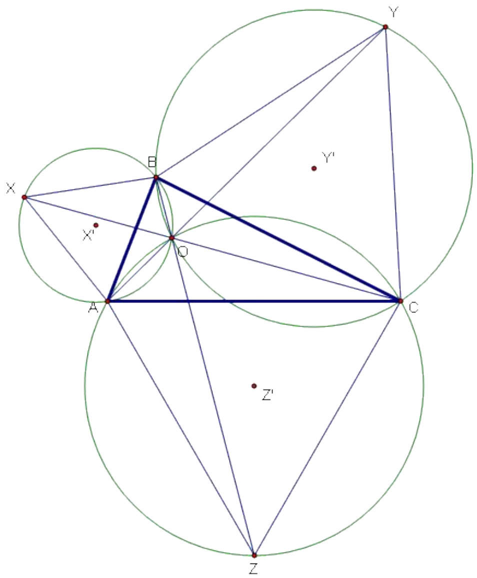


Steiner Problem in Graphs

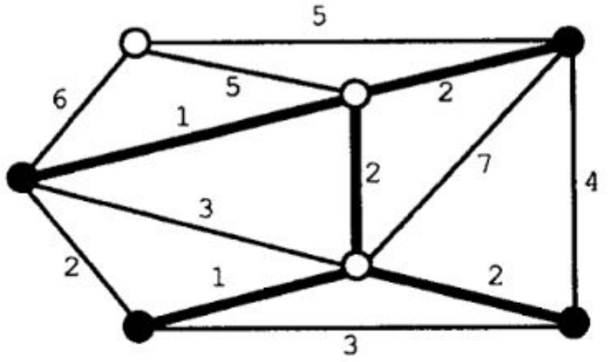
Given: A connected graph $G=(V,E)$ and set $K \subseteq V$ of terminals.

Find: A Steiner minimum tree for K in G, that is, a Steiner tree T for K such that $|E(T)|=min\{|E(T')|$, where T' is a Steiner tree for K in $G\}$.

In the weighted variant edges have weights and the goal is to minimize the sum of edges in the tree (rather than just the number of edges).

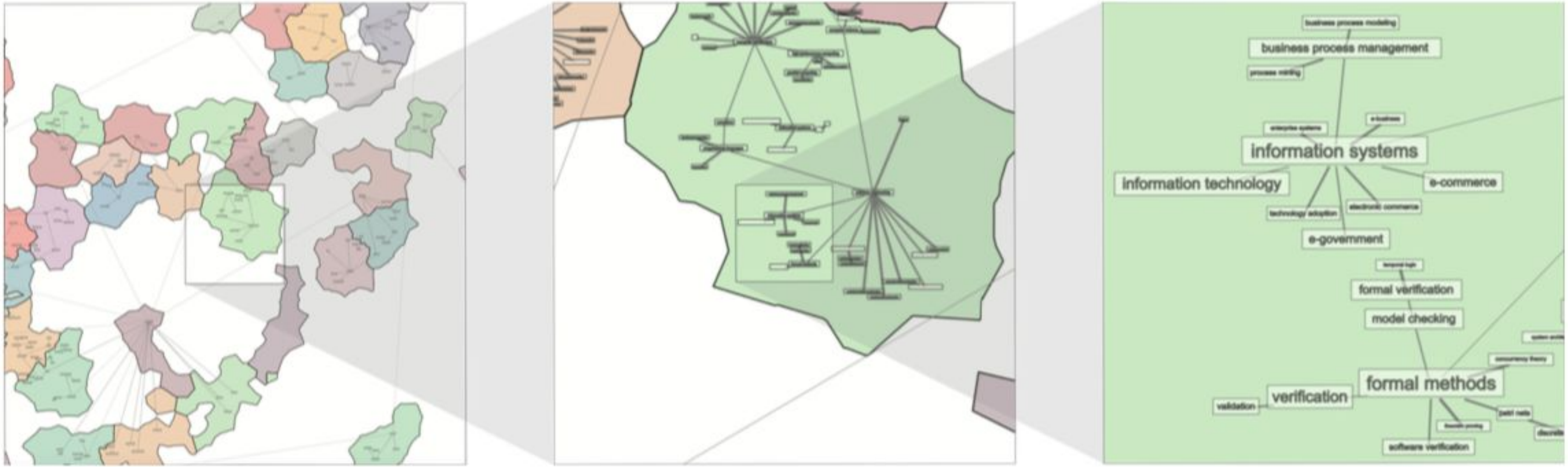
Both variants are NP-hard…

but there are good approximation algorithms



Multi-level Steiner Tree (MLST) for Multi-level Layout

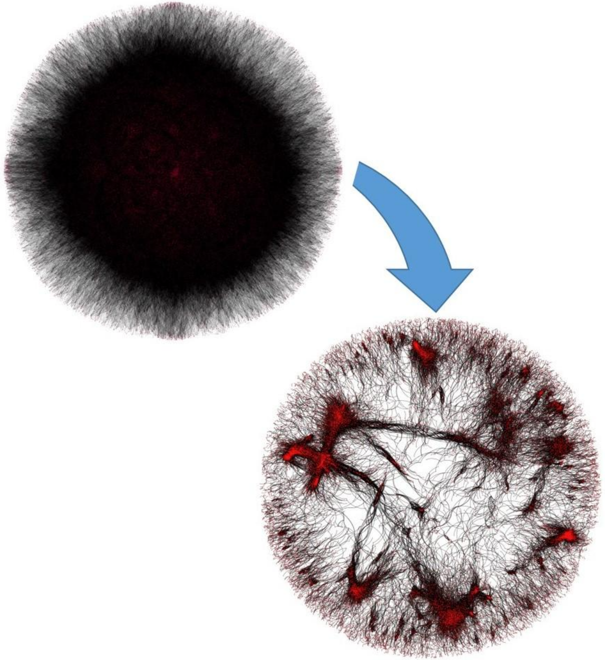
- Given: graph $G=(V,E)$, along with a vertex filtration $V \supset V_1 \supset V_2 \supset \ldots \supset V_i$
- Find: a set of Steiner trees $T \supset T_1 \supset T_2 \supset \ldots \supset T_i$ so T_i spans V_i edge cost sum is minimized
- Layout: use the trees to obtain a multi-level layout



Graph Spanners

Given a connected graph $G=(V,E)$ and integer $t \geq 1$, $S=(V,E')$ is a *t-spanner* of G, if for every pair of vertices $u,v \in V$, the distance between u,v in S is at most t times the distance between u,v in G.

Goal: remove as many edges as possible from G while still approximately preserving distances between all pairs of vertices.

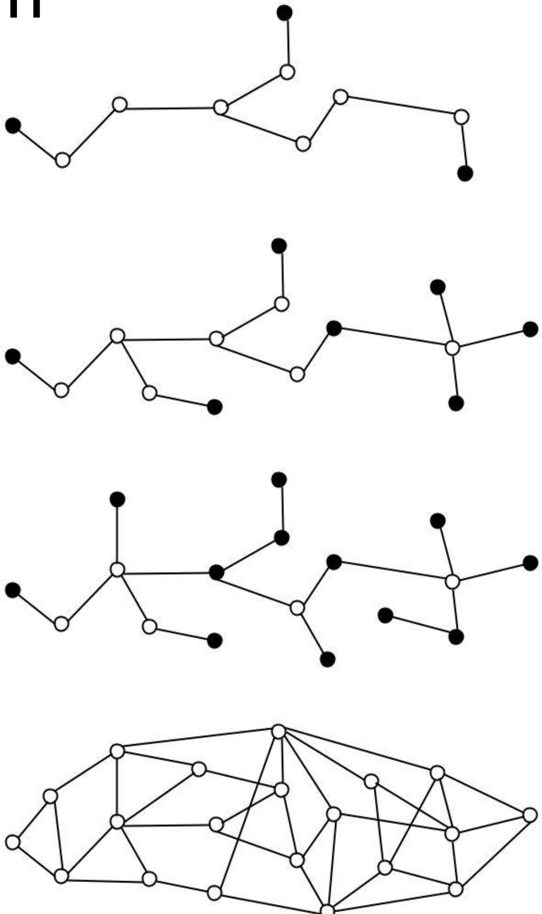


The Multi-Level Graph Spanner Problem

Given: graph $G=(V,E)$, along with a vertex filtration $V \supset V_1 \supset V_2 \supset \ldots \supset V_i$ and $t_1 \leq t_2 \leq \ldots \leq t_i$

Find: a set of spanners $S_1 \supset S_2 \supset \ldots \supset S_i$ such that S_i is a t_i*-spanner* for V_i and the total number of edges is minimized.

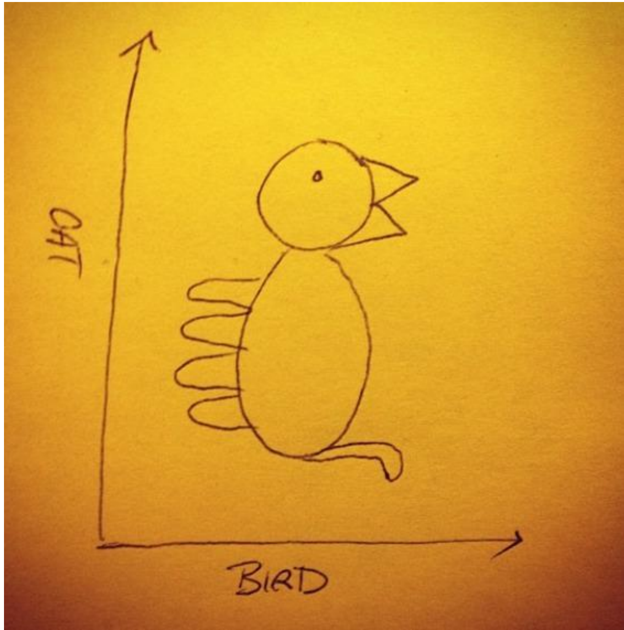
Note: when the t-values are large, we get the MLST problem



What Was the Problem?

We are looking for multi-level graph sketches:

- On every level we have a graph with real vertices and edges
- Important vertices and paths shown on high levels
- Less important vertices and paths appear lower
- The graph on each level is *similar* to the input graph



What are we doing instead?

- two new problems: MLST and MLGS

Multi-Layer Networks

The same questions can be asked for a multi-layer network, rather that a graph

Multilayer networks
M Kivelä, A Arenas, M Barthelemy… - … of complex **networks**, 2014 - academic.oup.com
In most natural and engineered systems, a set of entities interact with each other in complicated patterns that can encompass multiple types of relationships, change in time and include other types of complications. Such systems include multiple subsystems and layers …
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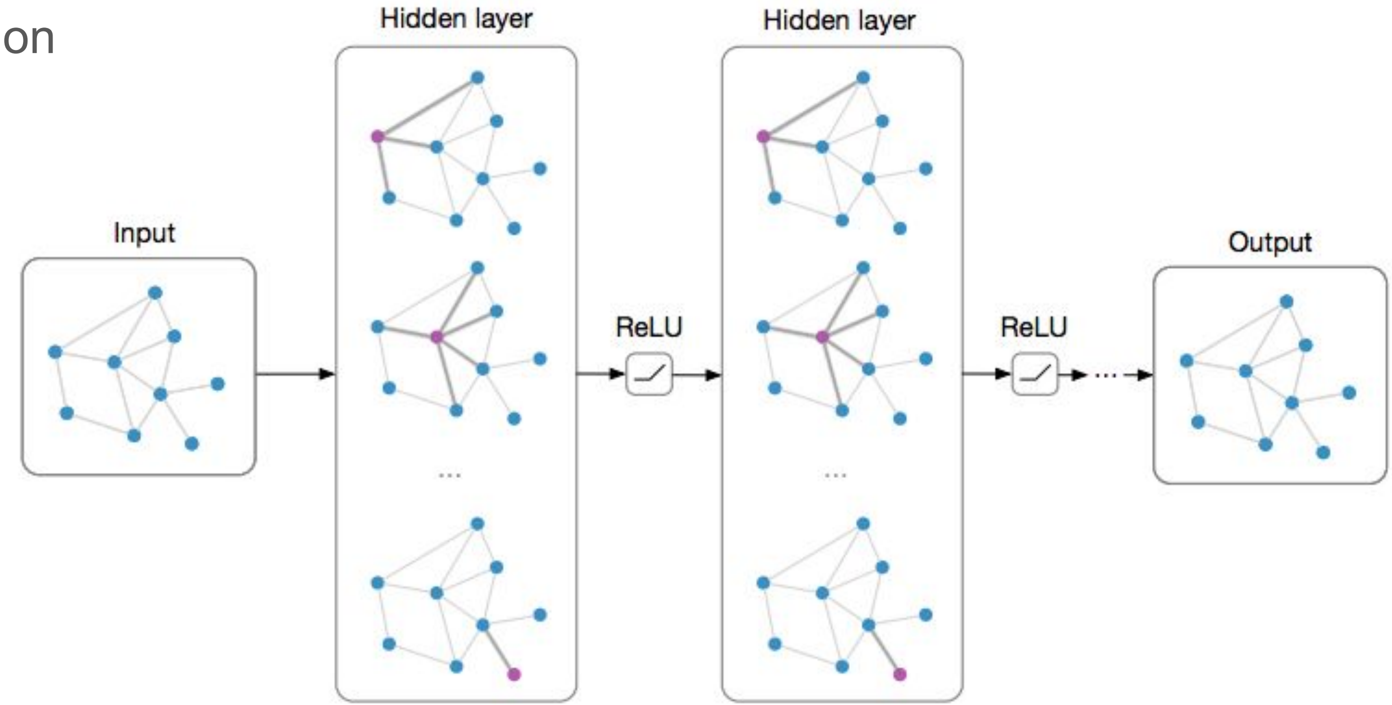
Name	Aligned	Disj.	Eq. Size	Diag.	Lcoup.	Cat.	$\|L\|$
Multilayer network				✓	✓	✓	Any
	✓†		✓†				Any
Multiplex network	✓†		✓†	✓			Any
	✓		✓	✓	✓	✓	Any
	✓		✓	✓	✓	✓	2
				✓	✓	✓	Any
	✓		✓	✓	✓		Any
Multivariate network	✓		✓	✓	✓	✓	Any
Multinetwork	✓		✓	✓	✓	✓	Any
	✓		✓	✓	✓	✓	Any
Multirelational network	✓		✓	✓	✓	✓	Any
Multirelational data	✓		✓	✓	✓	✓	Any

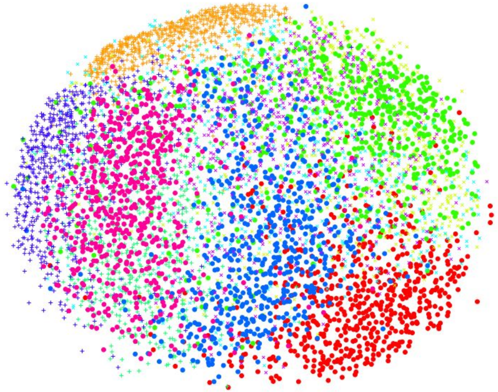
Graph Neural Networks

Node classification

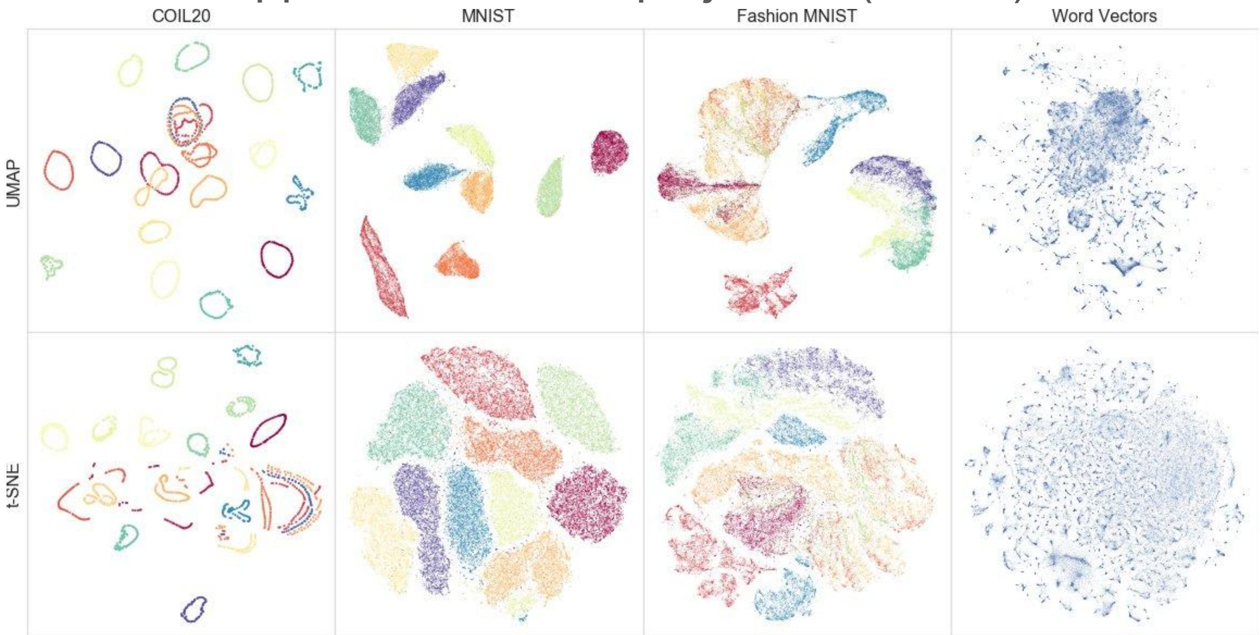
Graph classification



Dimensionality Reduction



1. Multi-dimensional scaling (MDS)
2. T-stochastic neighborhood embedding (t-SNE)
3. Uniform manifold approximation and projection (UMAP)



Questions

1. Graph sketches for graph visualization
2. Multi-level graph sketches and multi-level graph representation
3. Multi-layer networks, sketches, and visualization
4. Graph neural networks for graph visualization
5. Dimensionality reduction for graph visualization

References

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