Low Ply Graph Drawing

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***Abstract*—We consider the problem of characterizing graphs with low ply number and algorithms for creating layouts of graphs with low ply number. Informally, the *ply number* of a straight-line drawing of a graph is defined as the maximum number of overlapping disks, where each disk is associated with a vertex and has a radius that is half the length of the longest edge incident to that vertex. We show that internally triangulated biconnected planar graphs that admit a drawing with ply number 1 can be recognized in $O(n \log n)$ time, while the problem is in general NP-hard. We also show several classes of graphs that have 1-ply drawings. We then show that binary trees, stars, and caterpillars have 2-ply drawings, while general trees have $(h+1)$-ply drawings, where h is the height of the tree. Finally we discuss some generalizations of the notion of a ply number.**

I. Introduction

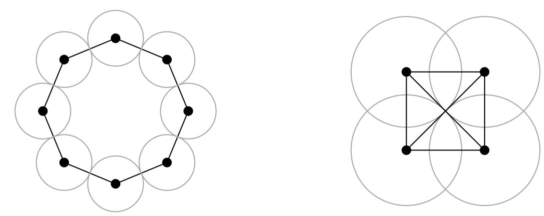
Graphs arise naturally in many domains: social networks, protein interaction networks, the world wide web, the power grid. They are usually visualized by node-link diagrams, with vertices as points and edges as line-segments connecting the corresponding points. Many different methods for drawing graphs have been developed and they typically aim to optimize *aesthetic criteria*, such as the number of edge crossings, the number of edge bends, the symmetry of the drawing, angular resolution, crossing angles, and vertex distribution [18].

In this paper we consider a new aesthetic criterion, called the *ply number*. Let D be a straight-line drawing of a graph. For each vertex $v \in D$, let C_v be the *open* disk centered at v whose radius r_v is half the length of the longest edge incident to v. Denote by S_q the set of disks sharing a point $q \in \mathbb{R}^2$, i.e., $S_q = \{C_v \mid \|v-q\| < r_v\}$. The *ply number* of D, denoted by $\mathsf{pn}(D)$, is defined as

$$\mathsf{pn}(D) = \max_{q \in \mathbb{R}^2} |S_q|.$$

In other words, the ply number of D is the maximum number of disks C_v mutually intersecting in D (see Figure 1 for an illustration). The ply number $\mathsf{pn}(G)$ of a graph G is the minimum ply number over all straight-line drawings of G. A k-ply drawing of a graph is one with ply number k.

The inspiration for the definition of this new aesthetic parameter comes from the study of road networks. Road networks can be seen as embedded graphs and even though roads are embedded on the surface of a sphere, the graphs they form are not necessarily planar. Eppstein and Goodrich point out that in the road network of the state of California there are over 6,000 edge crossings and propose characterizing road networks as subgraphs of disk intersection graphs [4]. Specifically, although road networks are not planar, they have low *ply number*.



(a) (b)

Fig. 1. (a) A drawing with ply number 1. (b) A drawing with ply number 2.

While the ply number of a given embedded graph is fixed, it is a natural question to ask whether certain graphs allow for layouts that have low (constant) ply number. In particular, does every planar graph have a drawing with low ply number? What non-planar graphs have low ply number drawings?

The study of the ply number is also motivated by force-directed algorithms. Classical force-directed methods such as Fruchterman-Reingold [5] and Kamada-Kawai [13], and more recent multi-scale variants [7], [11], define and minimize the "energy" of the layout; layouts with minimal energy tend to be aesthetically pleasing. Similarly, methods based on multi-dimensional scaling (MDS) minimize a particular energy function of the layout, defined as the variance of edge lengths in the drawing, known as *stress* [6], [16]. Assume a graph $G = (V, E)$ is drawn with p_i being the position of vertex $i \in V$. Denote the distance between two vertices $i, j \in V$ by $||p_i - p_j||$. The energy of the graph layout is measured by

$$\sum_{i,j \in V} w_{ij}(||p_i - p_j|| - d_{ij})^2,$$

where d_{ij} is the ideal distance between vertices i and j, and w_{ij} is a weight factor. Typically an ideal distance d_{ij} is defined as the length of the shortest path in G between i and j and $w_{i,j} = 1$ when the graph is unweighted. Lower stress values correspond to a better layout.

Note that these layout methods are not designed to directly optimize a specific graph drawing aesthetic criterion. Yet these methods tend to produce aesthetically pleasing drawings and there is evidence that reducing the stress of a graph layout is correlated with improved aesthetics [14]. Even though minimizing edge crossings is the most cited and the most commonly used aesthetic [18], [19], [12], one drawback of force-directed and stress-based algorithms is that they do not guarantee crossing-free layouts for planar graphs. In fact, since edge crossings are not explicitly considered, many planar graphs (e.g., trees, nested triangles, nested squares, skeletons of 3D polyhedra such as the cube, the octahedron, the dodecahedron, etc.) invariably have crossings in force-directed layouts. At the same time, it can be observed that such layouts, have low ply number. Is it possible that force-directed algorithms directly or indirectly minimize the ply number of a layout?

Work on graph drawings with low ply number is related to circle-contact representation and disk-intersection representation of graphs. A *circle-contact representation* for a graph $G = (V, E)$ is a collection $\mathcal{C}$ of interior-disjoint circles in the plane corresponding to the vertices of G, such that two vertices are adjacent if and only if their corresponding circles are tangent to each other [8], [9]. Every triangulated planar graph has a circle-contact representation [15]. One of the drawbacks of such representations is that the sizes of the circles may vary exponentially, making the resulting drawings difficult to read. In *balanced* circle packings and circle-contact representations for planar graphs, the ratio of the maximum and minimum diameters for the set of circles is polynomial in the number of vertices in the graph. Such drawings could be drawn with polynomial area, for instance, where the smallest circle determines the minimum resolution. It is known that trees and planar graphs with bounded tree-depth have balanced circle-contact representation [1]. Breu and Kirkpatrick [3] show that it is NP-complete to test whether a graph has a perfectly-balanced circle-contact representation, in which every circle is the same size. Circle-contact graphs are a special case of *disk-intersection graphs* [10], which represent a graph by intersecting disks, where the interiors of the disks are not required to be disjoint. McDiarmid and Müller [17] show that there are n-vertex graphs such that in every realization by disks with integer radii, at least one coordinate or radius is $2^{2^{\Omega(n)}}$, and they also show that every disk graph can be realized by disks with integer coordinates and radii that are at most $2^{2^{O(n)}}$.

In this paper we study graphs with low ply number. Our contributions are the following:

- In Section II we study graphs with ply number 1. We show that testing whether an internally triangulated biconnected planar graphs has ply number 1 can be done in $O(n \log n)$ time. We also prove that the class of 1-ply graphs coincides with the class of graphs that have a contact representation with unit disks, which makes the recognition problem NP-hard for general graphs. Finally we consider some classes of planar graphs that have 1-ply drawings.
- In Section III we show that any binary tree, star or caterpillar has ply number 2. We then use these results to show that any rooted tree has ply number $h + 1$, where h is the height of the tree.
- In Section IV we describe several open problems related to our results and consider generalizations of the notion of a ply number.

II. Drawings with ply number 1

We begin by proving some necessary properties of drawings with ply number 1. These properties imply that the drawings have uniform edge length and large *vertex angular resolution*. Recall that the vertex angular resolution of a drawing D is the minimum angle formed by any two adjacent edges at their common end-vertex.

Lemma 1: Let D be a drawing of a graph such that $\mathsf{pn}(D) = 1$. Then all edges of D have the same length.

Proof: Suppose for a contradiction that D contains two edges with different lengths. Since we are assuming that D is a drawing of a connected graph, this implies that there are two adjacent edges $e_1 = (u, v)$ and $e_2 = (w, v)$ with different lengths; without loss of generality, let e_1 be the longer edge. By definition, the disk C_v centered at v has a radius that is at least half the length of e_1 and therefore more than half the length of e_2. On the other hand, the disk C_w centered at w has radius that is at least half the length of e_2. It follows that C_v and C_w have a non-empty intersection, a contradiction. ■

An immediate consequence of Lemma 1 is that all disks C_v of a drawing with ply number 1 have the same radius and their boundaries touch in the middle of each edge, which implies the following.

Lemma 2: Let D be a 1-ply drawing of a graph. Then D has no crossings.

Proof: Consider any two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$. By Lemma 1 both edges have the same length ℓ and the disks C_{u_1}, C_{u_2}, C_{v_1}, and C_{v_2} all have radius $\frac{\ell}{2}$. If e_1 and e_2 cross each other, then at least one of C_{u_1} and C_{v_1} intersects at least one of C_{u_2} and C_{v_2}. ■

Lemma 3: Let D be a 1-ply drawing of a graph. Then the vertex angular resolution of D is at least $60°$.

Proof: Let $e_1 = (u, v)$ and $e_2 = (w, v)$ be two adjacent edges of D. Suppose for a contradiction that they form an angle smaller than $60°$ at v. By Lemma 1, e_1 and e_2 have the same length ℓ. Thus, the triangle with vertices u, v, and w is an isosceles triangle and, since the angle at v in this triangle is less than $60°$, the side connecting u and w is shorter than ℓ. Since both C_u and C_w have radius $\frac{\ell}{2}$, it follows that they have a non-empty intersection, a contradiction. ■

An immediate consequence of Lemma 3 is that a graph G with vertex degree greater than 6 does not admit a drawing with ply number 1. On the positive side, we can prove the following result.

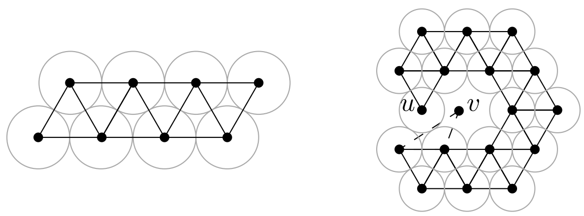
Theorem 1: Every simple cycle has ply number 1.

Proof: Let C be a simple cycle with k vertices. It is sufficient to realize C as a regular k-gon to obtain a drawing with ply number 1 (see Figure 1(a) for an illustration). Observe that the minimum value of k is three; in this case, according to Lemmas 1 and 3, the only drawing of C with ply number 1 is an equilateral triangle. ■

Since the ply number of a drawing does not increase by removing edges, Theorem 1 implies that every path admits a drawing with ply number 1.

Theorem 2: Any internally triangulated biconnected outerplanar graph with maximum vertex-degree 4 has ply number 1.

Proof: Let G be an internally triangulated biconnected outerplanar graph with vertex-degree at most 4. By Lemmas 1 and 3, in any drawing of G with ply number 1 every internal face of G must be an equilateral triangle. Let G^* be its weak dual[1]. It is easy to see that G^* is either a path or a star with three leaves. Namely, each vertex of G^* has degree at most three (because G is triangulated). On the other hand if v^* is a vertex of G^* of degree three, then the three vertices adjacent to v^* must have degree one, because otherwise, at least one of the vertices of the face corresponding to v^* would have degree larger than four. Both in the case when G^* is a path and when it is a star, it is easy to construct a drawing where all faces are equilateral triangles (see Figure 2(a)). ■



(a) (b)

Fig. 2. (a) A 1-ply drawing of an internally triangulated biconnected outerplanar graph of maximum vertex-degree 4. (b) An internally triangulated biconnected outerplanar graph of maximum vertex-degree 5 that does not admit a 1-ply drawing: in order to complete the drawing vertex v must be placed on top of u.

Not all triangulated planar graphs admit drawings with ply number 1 (even if the degree is at most 6). For example, there exist internally triangulated biconnected outerplanar graphs with maximum vertex-degree 5 that do not admit a drawing with ply number 1 (see Figure 2(b)). The following general result holds.

Theorem 3: Let G be an internally triangulated biconnected planar graph with n vertices. There exists an $O(n \log n)$-time algorithm that decides whether G has ply number 1.

Proof: By Lemmas 1 and 3 each face of G must be drawn as an equilateral triangle. Thus a drawing of G with ply number 1, if it exists, is unique (up to translations, rotations, and reflections) and can be constructed by adding one face at a time. Specifically, we begin with an arbitrary face and draw it as an equilateral triangle. Then, at each step, we choose an edge e of the external boundary of the drawing computed so far and that is shared by two internal faces of G. One of these two faces is already drawn and the other one, call it f, is not. If one vertex v of f is not drawn (the two end-vertices of e are already drawn), then its position is uniquely defined by the positions of the other two vertices of f. If all the vertices of f are already drawn, then one edge e' of f has not been added to the drawing (while its end-vertices are already drawn). It is easy to see that the drawing can be computed in $O(n)$ time.

When the drawing is completed, we need to check whether it is planar or not. This can be done in $O(n \log n)$ using a plane sweep algorithm [2] to compute intersections between a set of segments. ■

While the existence of drawings with ply number 1 can be tested efficiently for triangulated graphs, the problem is NP-hard in the general case (see Corollary 1). This is a consequence of the next characterization (Theorem 4) and of the fact that recognizing graphs that can be represented with contacts of unit disks is NP-hard [3].

Theorem 4: A graph has ply number 1 if and only if it has a contact representation with unit disks.

Proof: In a contact representation with unit disks each vertex is represented by a disk of the same radius, say 1, and two vertices are adjacent if and only if the two corresponding disks touch each other. If we connect the centers of the disks of a unit disk representation of a graph G, then we get a drawing of G with ply number 1. Conversely, suppose that D is a drawing of G such that $\mathsf{pn}(D) = 1$. By Lemma 1, all disks C_v have the same radius and their boundaries touch in the middle of each edge. Thus, the drawing consisting of all disks C_v is a unit disk representation of G. ■

Corollary 1: It is NP-hard to decide whether a graph has ply number 1.

The following result follows from Theorems 3 and 4.

Corollary 2: Let G be an internally triangulated biconnected planar graph with n vertices. There exists an $O(n \log n)$-time algorithm that decides whether G has a contact representation with unit disks.

III. Drawings with Ply Number Larger Than One

In this section we show that some families of trees admit a drawing with ply number 2. We then use these results to show that general trees have drawings with ply number equal to the height of the tree plus one.

Theorem 5: Every binary tree has ply number 2, which is worst-case optimal.

Proof: It is sufficient to show the statement for complete binary trees. We describe two different constructive methods for creating 2-ply drawings for complete binary trees, which are illustrated in Figures 3(a) and 3(b).

In the first approach we draw the tree in a radial fashion starting with the root. Edges get shorter as go farther from the root, and all edges at the same distance from the root have equal lengths. The initial three edges[2] form $120°$ angles and the same vertex angular resolution is maintained in the layout of the entire tree. At each successive level of the tree, the edges are half the length of the edges in the current level.

[1]The weak dual of a planar graph G is a graph G^* with a vertex for each internal face of G and an edge between two vertices if the corresponding faces share an edge.

[2]The technique is described assuming that the root has three children. In other words, our technique can draw free trees (i.e., acyclic graphs) whose maximum vertex degree is 3.

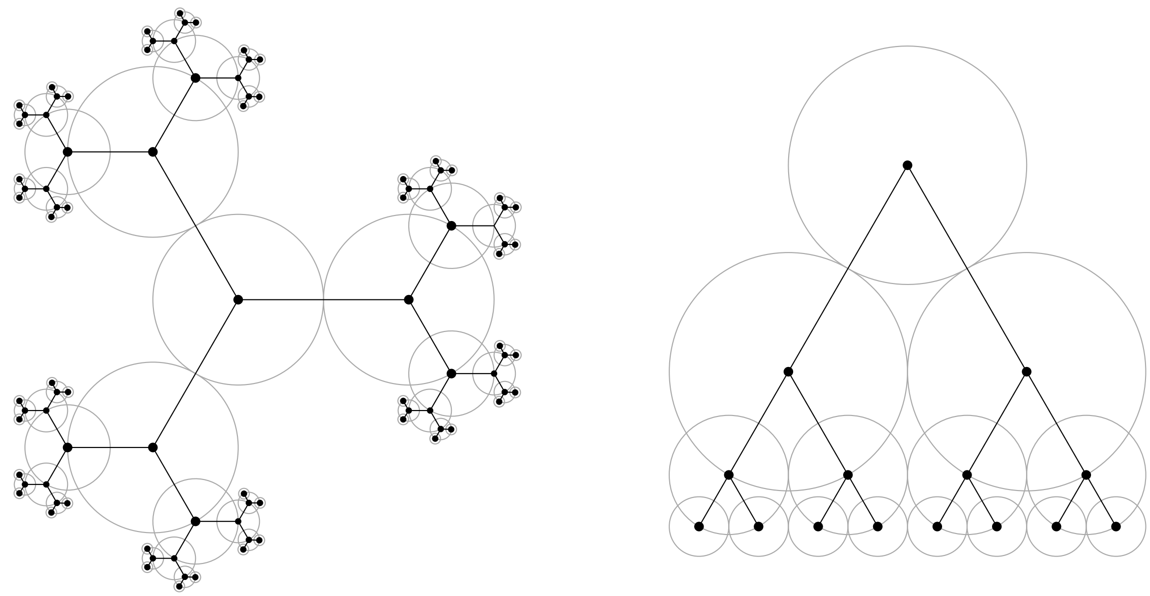


Fig. 3. (a) A drawing of a binary tree with ply number 2. (b) A drawing of a binary tree with ply number 2.

We now show that the resulting drawing has ply number 2. Let w be a vertex of T, let T_w be the subtree rooted at w and let v be the parent of w. We prove that the following property holds for the drawing D_w of T_w.

PC For each vertex u of T_w, each internal point p of C_u is contained in a region R_w as the one shown in Figure 4(a).

The proof is by induction on the height h of T_w. If $h = 1$ then Property **PC** trivially holds. If $h > 1$, then D_w is obtained connecting the roots of two trees T_{w_1} and T_{w_2} of height $h-1$ to a vertex v (see Figure 4(b)). Figure 4(b) shows that Property **PC** holds for the drawing D_w of T_w, while Figure 4(c) shows that the disk C_w intersects only the disk C_v associated with its parent and the disks C_{w_1} and C_{w_2} of its children. Since this is true for every vertex, the ply number of the drawing is 2.

In the second approach we draw the tree in the standard level-by-level fashion, again cutting the edge lengths in half in successive levels. This time we maintain $60°$ angles and basic trigonometry can be used to show that the maximum number of overlapping disks for any point in the plane is 2.

Ply number 2 is worst-case optimal for binary trees. Suppose for the sake of contradiction, that there exists a ply 1 layout for any binary tree. Recall that in such a layout all edges must be the same length. But it is easy to see that in a complete binary tree with edges of uniform length the minimum distance between leaves gets smaller as we add more levels and eventually, the minimum distance between two leaves becomes smaller than the uniform edge length. This necessarily induces overlapping disks, which results in the desired contradiction. ∎

Other families of trees that have ply number 2 are stars and caterpillars as shown in the next two theorems. Recall that a *star* is a tree where all vertices except one are leaves, and a *caterpillar* is a tree such that removing all the leaves we are left with a path, called the *spine* of the caterpillar.

Theorem 6: Every star has ply number 2, which is worst case optimal.

Proof: Let r be the center of the star, and let $v_1, v_2, \ldots, v_k$ be the leaves of the star. Consider any drawing such that the edge (r, v_i) has length $2 \cdot 3^{i-1}$ (see Figure 5(a)). Let p be an internal point of C_{v_i} and let d_p be its distance from r. We have $3^{i-1} < d_p < 3^i$, which implies that, for any v_i and v_j with $i \neq j$, $C_{v_i} \cap C_{v_j} = \emptyset$. On the other hand, C_r has radius 3^{k-1} and therefore $C_r \cap C_{v_i} \neq \emptyset$, for every $i < k$. It follows that the ply number of the drawing is 2.

In order to show that ply number 2 is worst-case optimal, consider a star T with more than six leaves. If T had a drawing with ply number 1, then all the edges should have the same length and the angle between them should be $60°$, but this is impossible because there are more than six leaves. ∎

Theorem 7: Every caterpillar has ply number 2, which is worst case optimal.

Proof: Let T be a caterpillar and let $v_1, v_2, \ldots, v_k$ be the spine of T. Denote by T_i the subtree consisting of v_i and of all the leaves adjacent to v_i ($i = 1, 2, \ldots, k$). Each T_i is a star and can be drawn as described in the proof of Theorem 6. Let m be the maximum number of leaves of all the stars T_i. Combine the drawings of all the stars T_i by placing the roots $v_1, v_2, \ldots, v_k$ (in this order) on a horizontal line so that the distance between v_i and v_{i+1} is $2 \cdot 3^m$. For each v_i ($i = 1, 2, \ldots, k$) the longest incident edges are the edges of the spine. Thus C_{v_i} has radius 3^m and for each leaf v adjacent to v_i, C_v is completely contained in C_{v_i} (the length of the edge (v_i, v) is at most $2 \cdot 3^{m-1}$). This implies that, if v and w are two leaves adjacent to two distinct spine vertices v_i and v_j, then C_v and C_w do not intersect. On the other hand, if v and w are leaves adjacent to the same spine vertex v_i, then C_v and

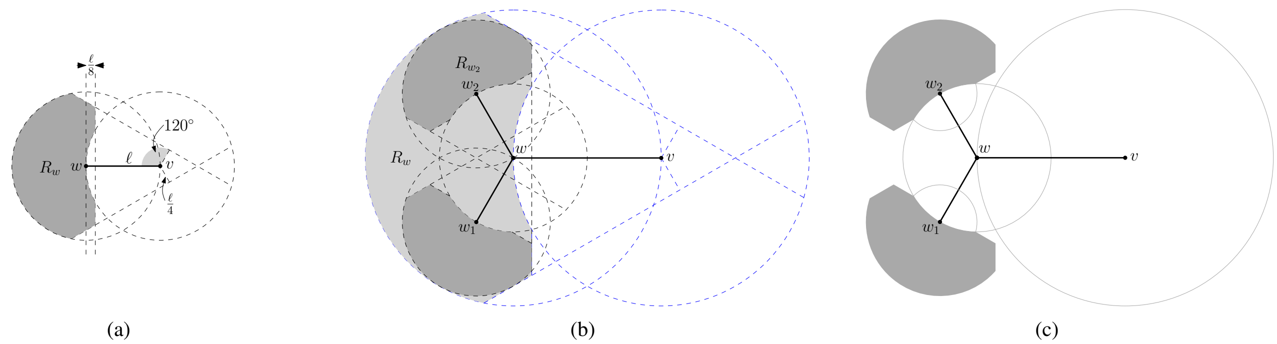


Fig. 4. Illustration for the proof of Theorem 5. (a) Illustration of property **PC**. (b) Property **PC** is preserved. (c) The drawing has ply number 2.

C_w do not intersect as shown in the proof of Theorem 6. It follows that the drawing has ply number 2. This is worst-case optimal, since caterpillars include stars. ∎

The result about stars can also be used to prove the following result about general trees.

Theorem 8: Every rooted tree T with height h has ply number at most $h + 1$.

Proof: The proof is by induction on the height h. If $h = 1$, then T is a star and it admits a drawing with ply number 2 by Theorem 6. If $h > 1$, let r be the root of T, let $v_1, v_2, \dots, v_k$ be the children of r and let T_i be the subtree of T rooted at v_i ($i = 1, 2, \dots, k$). By induction, each T_i has a drawing whose ply number is at most h. Let r_i be the radius of the smallest disk centered at v_i that contains the whole drawing of T_i and let l_i be the maximum length of an edge in the drawing of T_i. Let $l = \max_i\{r_i + l_i\}$. Construct a drawing of T such that the edge (r, v_i) has length $2 \cdot l \cdot 3^{i-1}$. The edge (r, v_i) is the longest edge incident to v_i (any other edge has length at most l_i) and therefore the radius of C_{v_i} is $l \cdot 3^{i-1}$. This implies that if v is a vertex of the subtree T_i, then C_v is completely contained in C_{v_i}. Then v is contained in a disk of radius r_i centered at v_i and the radius of C_v is at most $l_i/2$. Hence, every internal point of C_v is at distance at most $r_i + l_i/2 < l$ from v_i. Furthermore, if p is an internal point of C_{v_i} and d_p is its distance from r, we have $l \cdot 3^{i-1} < d_p < l \cdot 3^i$. This implies that for any v_i and v_j with $i \neq j$, $C_{v_i} \cap C_{v_j} = \emptyset$. Therefore if v and w are two vertices of two distinct subtrees T_i and T_j, then C_v and C_w are disjoint. In summary, the drawings of all subtrees T_i (for $i = 1, 2, \dots, k$) are disjoint and so are the disks centered at their vertices. The longest edge incident to r is the edge (r, v_k), whose length is $2 \cdot l \cdot 3^{k-1}$. Thus the disk C_r has radius $l \cdot 3^{k-1}$ and intersects all disks C_{v_i} for $i < k$, and therefore all drawings of T_i for $i < k$. Since each of these drawings has ply number at most h the drawing of T has ply number at most $h + 1$. ∎

IV. Discussion and Future Work

We studied graphs with low ply number. We proved that the class of graphs with ply number 1 coincide with class of graphs that have unit disk contact representations, and some example classes graphs that have ply number 1. Then, we showed some classes of trees that have ply number 2, and proved that, for general trees, the ply number is bounded by the height. Our initial study of this problem opens up several interesting research questions.

First note that the constructive techniques in the proof of Theorem 5 produce drawings with polynomial area for balanced binary trees, but exponential area is required if the tree is not balanced. Analogously, the drawing technique for stars and caterpillars produces drawings with exponential area. This naturally motivates the following question:

Problem 1: Is it possible to draw a binary tree, a star, or a caterpillar in polynomial area with ply number 2?

Another natural problem about trees is the following:

Problem 2: Is it possible to draw ternary trees with ply number 2?

An interesting observation about low-ply drawings is that non-planar drawings may have significantly smaller ply number than planar ones. In Figure 5(b) and 5(c) two drawings of the same graph are shown. The first one is planar, but its ply number is $\Omega(n)$; the second one is non-planar, but its ply number is 2. This motivates the study of the following research question.

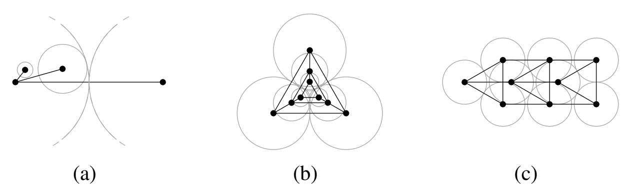
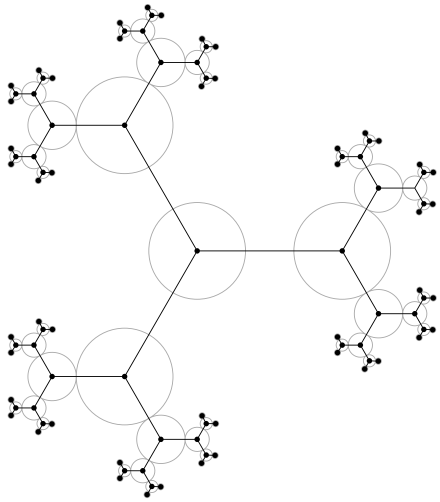


Fig. 5. (a) Drawing with ply number 2 of a star. (b) A planar drawing of a graph G with ply number $O(n)$. (c) A non-planar drawing of G with ply number 2.

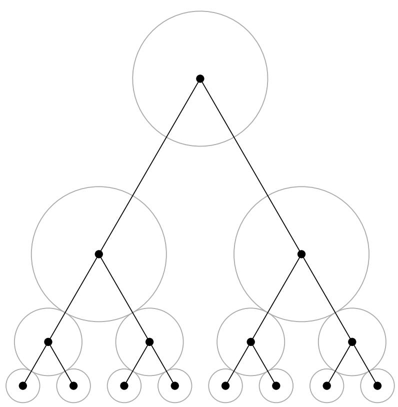
Problem 3: Is there a relationship between the number of edge crossings and the ply number? What planar graphs can be drawn in a non-planar fashion with low ply?

We finally suggest the study of some generalizations of notion of ply and ply number. An *empty-ply drawing* is a drawing of a graph such that the disk C_v of each vertex v contains no other vertices. In other words, in an empty-ply drawing, intersections of disks are accepted if no vertex is inside the disk of another vertex. Our intuition is that empty-ply drawings have small ply number. The following problems can be investigated.

Problem 4: Is the ply number of empty-ply drawings constant?



(a)



(b)

Fig. 6. (b) A drawing of a binary tree with α-ply number 1, for $\alpha = \frac{1}{3}$. (c) A drawing of a binary tree with α-ply number 1, for $\alpha = \frac{1}{3}$.

Problem 5: Characterize those graphs that admit an empty-ply drawing.

Consider another generalization of the notion of ply. For each vertex $v \in D$, let C_v^α be the open disk centered at v and whose radius r_v^α is α times the length of the longest edge incident to v. Let $S_q^\alpha = \{C_v^\alpha \mid \|v - q\| < r_v^\alpha\}$. The *$\alpha$-ply number* of D, denoted by $\alpha\text{-}\mathsf{pn}(D)$, is defined as

$$\alpha\text{-}\mathsf{pn}(D) = \max_{q \in \mathbb{R}^2} |S_q^\alpha|.$$

With this definition we consider smaller or larger disks around each vertex depending on the value of α (the "standard" definition of ply number is obtained for $\alpha = 0.5$), thus reducing or increasing the ply number of a drawing. For example, the drawings produced by the techniques behind Theorem 5 have α-ply number 1 if we choose $\alpha \leq \frac{1}{3}$ (see Figures 6(a) and 6(b) for an illustration).

Theorem 9: Every binary tree has a drawing D such that $\alpha\text{-}\mathsf{pn}(D) = 1$, for $\alpha \leq \frac{1}{3}$.

The definition of α-ply number opens up many research directions. Drawings with different values of the α-ply number for different values of α can be studied. A specific problem suggested by Theorem 9 is the following.

Problem 6: For what value of α does every d-ary tree have an $\alpha\text{-}\mathsf{pn}(D) = 1$ drawing?

Acknowledgments

Work on this problem began at the NII Shonan Meeting *Big Graph Drawing: Metrics and Methods*, Jan. 12-15, 2015. Research supported in part by the MIUR project AMANDA: Algorithmics for MAssive and Networked DAta, prot. 2012C4E3KT_001. We thank the anonymous referees for their suggestions that helped us to improve this paper.

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