On the Planar Split Thickness of Graphs

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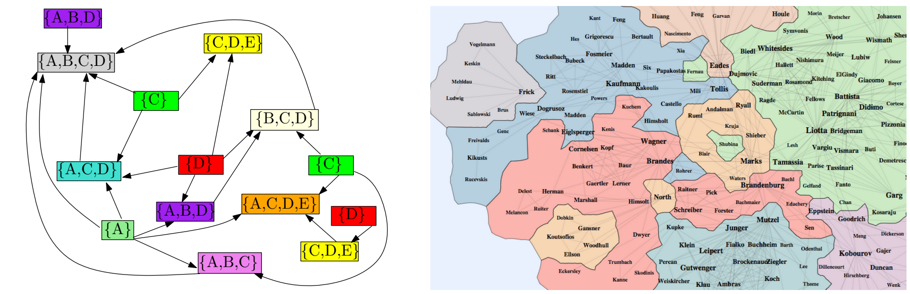
Abstract. Motivated by applications in graph drawing and information visualization, we examine the planar split thickness of a graph, that is, the smallest k such that the graph is k-splittable into a planar graph. A k-split operation substitutes a vertex v by at most k new vertices such that each neighbor of v is connected to at least one of the new vertices.

We first examine the planar split thickness of complete and complete bipartite graphs. We then prove that it is NP-hard to recognize graphs that are 2-splittable into a planar graph, and show that one can approximate the planar split thickness of a graph within a constant factor. If the treewidth is bounded, then we can even verify k-splittablity in linear time, for a constant k.

1 Introduction

Transforming one graph into another by repeatedly applying an operation such as vertex/edge deletion, edge flip or vertex split is a classic problem in graph theory [15]. In this paper, we examine graph transformations under the vertex split operation. Specifically, a *k-split operation* at some vertex v inserts at most k new vertices $v_1, v_2, \ldots, v_k$ in the graph, then, for each neighbor w of v, adds at least one edge (v_i, w) where $i \in [1, k]$, and finally deletes v along with its incident edges.We define a *k-split* of graph G as a graph that is obtained by applying a k-split to each vertex of G at most once. We say that G is *k-splittable* into G^k. If $\mathcal{G}$ is a class of graphs, we say that G is *k-splittable* into a graph of $\mathcal{G}$ (or "*k-splittable* into $\mathcal{G}$") if there is a k-split of G that lies in $\mathcal{G}$. We introduce the $\mathcal{G}$ *split thickness* of a graph G as the minimum integer k such that G is k-splittable into a graph of $\mathcal{G}$.

Graph transformation via vertex splits is important in graph drawing and information visualization. For example, assume that we want to visualize the subset relation among a collection S of n sets. Construct an n-vertex graph G with a vertex for each set and an edge when one set is a subset of another. A planar drawing of this graph gives



(a) (b)

Fig. 1. (a) A 2-split visualization of subset relations among 10 sets. (b) Visualization of a social network. Note the 3 yellow clusters at the lower left of the map.

a nice visualization of the subset relation. Since the graph is not necessarily planar, a natural approach is to split G into a planar graph and then visualize the resulting graph, as illustrated in Figure 1(a). Let's now consider another interesting scenario where we want to visualize a graph G of a social network, see Figure 1(b). First, group the vertices of the graph into clusters by running a clustering algorithm. Now, consider the cluster graph: every cluster is a node and there is an edge between two cluster-nodes if there exists a pair of vertices in the corresponding clusters that are connected by an edge. In general, the cluster graph is non-planar, but we would like to draw the clusters in the plane. Thus, we may need to split a cluster into two or more sub-clusters. The resulting "cluster map" will be confusing if clusters are broken into too many disjoint pieces, which leads to the question of minimizing the planar split thickness.

Related Work. The problem of determining the planar split thickness of a graph G seems to be related to the graph thickness [1], empire-map [12] and k-splitting [15] problem. The *thickness* of a graph G is the minimum integer t such that G admits an edge-partition into t planar subgraphs. One can assume that these planar subgraphs are obtained by applying a t-split operation at each vertex. Hence, thickness is an upper bound on the planar split thickness, e.g., the thickness and thus the planar split thickness of graphs with treewidth ρ and maximum-degree-4 is at most $\lceil \rho/2 \rceil$ [5] and 2 [6], respectively. Analogously, the planar split thickness of a graph is bounded by its *arboricity*, that is, the minimum number of forests into which its edges can be partitioned. We will later show that both parameters also provide an asymptotic lower bound on the planar split thickness.

A k-*pire* map is a k-split planar graph, i.e., each *empire* consists of at most k vertices. In 1890, Heawood [11] proved that every 12 mutually adjacent empires can be drawn as a 2-pire map where each empire is assigned exactly two regions. Later, Ringel and Jackson [19] showed that for every integer $k \geq 2$ a set of $6k$ mutually adjacent empires can be drawn as a k-pire map. This implies an upper bound of $\lceil n/6 \rceil$ on the planar split thickness of a complete graph on n vertices.

A rich body of literature considers the planarization of non-planar graphs via *vertex splits* [7,10,15,16]. Here a *vertex split* is one of our 2-split operations. These results focus on minimizing the *splitting number*, i.e., the total number of vertex splits. Note that upper bounding the splitting number does not necessarily guarantee any good upper bound on the planar split thickness.

Knauer and Ueckerdt [13] studied the *folded covering number* which is equivalent to our problem and stated several results for splitting graphs into star forests, caterpillar forests, or interval graphs, e.g., planar graphs are 4-splittable into a star forest, and planar bipartite graphs as well as outerplanar graphs are 3-splittable into a star forest or a caterpillar forest. It follows from Scheinerman and West [20] that planar graphs are 3-splittable into interval graphs and 4-splittable into a caterpillar forest, while outerplanar graphs are 2-splittable into interval graphs.

Our Contribution. In this paper, we examine the planar split thickness for non-planar graphs. Initially, we focus on splitting the complete and complete bipartite graphs into planar graphs. We then prove that it is NP-hard to recognize graphs that are 2-splittable into a planar graph, while we describe a technique for approximating the planar split thickness within a constant factor. Finally, for bounded treewidth graphs, we present a technique to verify planar k-splittablity in linear time, for any constant k. Because our results are for planar k-splittability, we will drop the word "planar", and use "k-splittable" to mean "planar k-splittable".

2 Planar Split Thickness of K_n and $K_{m,n}$

In this section, we focus on the planar split thickness of K_n and $K_{m,n}$, and on graphs with maximum degree Δ.

2.1 Complete Graphs

Let $f(G)$ be the planar split thickness of the graph G. Recall that Ringel and Jackson [19] showed that $f(K_n) \leq \lceil n/6 \rceil$ for every $n \geq 12$. Since a $(n/6)$-split graph contains at most $n^2/2 - 6$ edges, and the largest complete graph with at most $n^2/2 - 6$ edges is K_n, this bound is tight. Besides, for every $n < 12$, it is straightforward to construct a 2-split graph of K_n by deleting $2(12 - n)$ vertices from the 2-split graph of K_{12}. Hence, we obtain the following theorem.

Theorem 1 (Ringel and Jackson [19]). *If $n \leq 4$, then $f(K_n) = 1$, and if $5 \leq n \leq 12$, then $f(K_n) = 2$. Otherwise, $f(K_n) = \lceil n/6 \rceil$.*

Let K_{12}^2 be any 2-split graph of K_{12}. Then, K_{12}^2 exhibits some useful structure, as stated in the following lemma.

Lemma 1. *Any planar embedding Γ of K_{12}^2 is a triangulation, where each vertex of K_{12} is split exactly once and no two vertices that correspond to the same vertex in K_{12} can appear in the same face.*

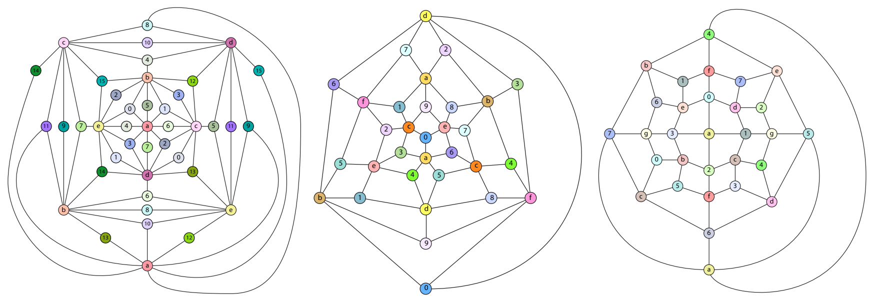


Fig. 2. The 2-split graphs of $K_{5,16}$, $K_{6,10}$ and $K_{7,8}$.

Proof. K_{12} has 66 edges. The 2-split operation produces a graph with at most twice the number of vertices and at least the original number of edges, so any graph K_{12}^2 has 24 vertices and 66 edges, since that is the largest number of edges for a 24-vertex planar graph by Eulers formula. Therefore, if K_{12}^2 is planar, it must be maximal planar, with all faces triangles. If two copies of the same vertex appear on a face, then those copies would not be adjacent and that face could not be a triangle. □

Let H be the graph consisting of 2 copies of K_{12} attached at a common vertex v. Then, H provides an example of a graph that is not 2-splittable even though its edge count does not preclude its possibility of being 2-splittable.

Lemma 2. *The graph H is not 2-splittable.*

Proof. Consider a 2-split graph H' of one copy of K_{12}. By Lemma 2.1, the vertices v_1 and v_2 in H' that correspond to the same vertex in K_{12} cannot appear in the same face. Since v can be split only once, the 2-split graph H'' of the other copy of K_{12} must lie inside some face that is incident to either v_1 or v_2. Without loss of generality, assume that it is incident to some face incident to v_1. Note that both H' and H'' need a copy of v in some face which is not incident to v_1. Since both H' and H'' are triangulations, this would introduce a crossing in any 2-split graph of H. □

2.2 Complete Bipartite Graphs

Hartsfield et al. [10] showed that the splitting number of $K_{m,n}$, where $m, n \geq 2$, is exactly $\lceil (m-2)(n-2)/2 \rceil$. However, their construction does not guarantee tight bounds on the splitting thickness of complete bipartite graphs. For example, if m is an even number, then their construction does not duplicate any vertex of the set A with m vertices, but uses $n + (m/2-1)(n-2)$ vertices to represent the set B of n vertices. Therefore, at least one vertex in the set B is duplicated at least $(n + (m/2-1)(n-2))/n = m/2 - m/n + 2/n \geq 3$ times, for $m \geq 6$ and $n \geq 5$. On the other hand, we show that $K_{m,n}$ is 2-splittable in some of these cases, as stated in the following theorem.

Theorem 2. *The graphs $K_{5,16}$, $K_{6,10}$, and $K_{7,8}$ are 2-splittable, and their 2-split graphs are quadrangulations, which implies that for complete bipartite graphs $K_{m,n}$, where $m = 5, 6, 7$, those are the largest graphs with planar split thickness 2.*

Proof. The sufficiency can be observed from the 2-split construction of $K_{5,16}$, $K_{6,10}$, and $K_{7,8}$, as shown in Figure 2. A planar bipartite graph can have at most $2n - 4$ edges [10]. Since the graphs $K_{5,16}, K_{6,10}$ and $K_{7,8}$ contain exactly $4(m + n) - 4$ edges, their 2-split graphs are quadrangulations, which in turn implies that the result is tight. □

The following theorem gives a necessary condition for a complete bipartite graph to be k-splittable based on the edge count argument.

Theorem 3. *If $d \geq 4k + 4\sqrt{k^2 - 1}$ and $n > \frac{d-\sqrt{d^2-8kd+16}}{2}$, then $K_{n,d-n}$ is not k-splittable.*

Proof. Note that any k-split graph H^k of $K_{n,m}$ must be a planar bipartite graph. Therefore, if p and q are the number of vertices and edges in H^k, respectively, then the inequality $q \leq 2p - 4$ holds.

Consider a complete bipartite graph $K_{n,d-n}$ that is k-splittable. The number of edges in this graph is $n \times (d - n)$. Since any k-split graph of $K_{n,d-n}$ can have at most kd vertices, we have

$$n(d - n) \leq 2kd - 4 \Leftrightarrow n^2 - nd + 2kd - 4 \geq 0 \tag{1}$$

The factorization of the previous polynomial (1) gives

$$n^2 - nd + 2kd - 4 = \left(n - \frac{d - \sqrt{d^2 - 8kd + 16}}{2}\right)\left(n - \frac{d + \sqrt{d^2 - 8kd + 16}}{2}\right),$$

when $d \geq 4k + 4\sqrt{k^2 - 1}$. Therefore, Equation (1) holds if $n \leq \frac{d-\sqrt{d^2-8kd+16}}{2}$ or $n \geq \frac{d+\sqrt{d^2-8kd+16}}{2}$. □

2.3 Graphs with Maximum Degree Δ

Recall that the planar split thickness of a graph is bounded by its arboricity. By definition, any maximum-degree-Δ graph has degeneracy at most Δ and, thus, arboricity at most Δ. Hence, the planar split thickness of a maximum-degree-Δ graph is bounded by Δ.

Moreover, since every 2-regular graph is planar, the planar split thickness of any graph with maximum degree Δ is bounded by $\lceil \Delta/2 \rceil$. Therefore, the planar split thickness of a maximum-degree-5 graph is at most 3. The following theorem states that this bound is tight.

Theorem 4. *For any nontrivial minor-closed property P, there exists a graph G of maximum degree five whose P split thickness is at least 3.*

Proof. This follows from a combination of the following observations:

1. There exist arbitrarily large 5-regular graphs with girth $\Omega(\log n)$ [17].

[9] A graph G is k-degenerate if every subgraph of G contains a vertex of degree at most k.

2. Splitting a graph cannot decrease its girth.
3. For every h, the K_h-minor-free n-vertex graphs all have at most $O(nh\sqrt{\log h})$ edges [21].
4. Every graph with n vertices, m edges, and girth g has a minor with $O(n/g)$ vertices and $m - n + O(n/g)$ edges [2].

Thus, let h be large enough that K_h does not have property P. If G is a sufficiently large n-vertex 5-regular graph with logarithmic girth (Observation 1), then any 2-split of G will have $2n$ vertices and $5n/2$ edges. By Observation 4, this 2-split will have a minor whose number of edges is larger by a logarithmic factor than its number of vertices, and for n sufficiently large this factor will be large enough to ensure that a K_h minor exists within the 2-split of G (by Observation 3). Thus, G cannot be 2-split into a graph with property P. □

3 NP-hardness and Approximation

Faria et al. [7] showed that determining the splitting number of a graph is NP-hard, even when the input is restricted to cubic graphs. Since cubic graphs are 2-splittable, their hardness proof does not readily imply the hardness of 2-splittable graph recognition. In this section, we show that it is NP-hard to recognize graphs that are 2-splittable into a planar graph. We then show that the arboricity of k-splittable graphs is bounded by $3k + 1$ and that testing k-splittability is fixed-parameter tractable in the treewidth of the given graph.

3.1 NP-hardness of 2-Splittability

The reduction is from planar 3-SAT with a cycle through the clause vertices [14]. Specifically the input is an instance of 3-SAT with variables X and clauses C such that the following graph is planar: the vertex set is $X \cup C$; we add edge (x, c) if variable x appears in clause c; and we add a cycle through all the clause vertices. Kratochvíl et al. [14] showed that this version of 3-SAT remains NP-complete.

For our construction, we will need to restrict the splitting options for some vertices. For a vertex v, *attaching* K_{12} *to* v means inserting a new copy of K_{12} into the graph and identifying v with a vertex of this K_{12}. A vertex that has a K_{12} attached will be called a “K-vertex”.

Lemma 3. *If C is a cycle of K-vertices then in any planar 2-split, the cycle C appears intact, i.e. for each edge of C there is a copy of the edge in the 2-split such that the copies are joined in a cycle.*

Proof. Let v be a vertex of cycle C. We will argue that the two edges incident to v in C are incident to the same copy of v in the planar 2-split. This implies that the cycle appears intact in the planar 2-split.

Suppose the vertices of C are $v = c_0, c_1, c_2, \ldots, c_t$ in that order, with an edge (v, c_t). As noted earlier in the paper, a planar 2-split of K_{12} must split all vertices, and no two copies of a vertex share a face in the planar 2-split. Furthermore, any planar 2-split of K_{12} is connected.

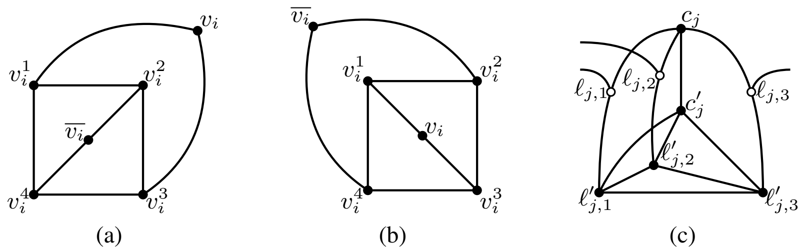


Fig. 3. (a) A variable gadget shown in the planar configuration corresponding to $v_i =$ *true* and (b) in the planar configuration corresponding to $v_i =$ *false*. (c) A clause gadget—a K_5 with added subdivision vertices $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ corresponding to the literals in the clause. The half-edges join the corresponding variable vertices.

Let H_i be the induced planar 2-split of the K_{12} incident to c_i. Let v^1 and v^2 be the two copies of v in H_0. Suppose that the copy of edge (v, c_1) in the planar 2-split is incident to v^1. Our goal is to show that the copy of edge (v, c_t) in the planar 2-split is also incident to v^1. H_1 must lie in a face F of H_0 that is incident to v^1. Since there is an edge (c_1, c_2), H_2 must also lie in face F of H_0. Continuing in this way, we find that H_t must also lie in the face F. Therefore, the copy of the edge (c_t, v) must be incident to v^1 in the planar 2-split. □

Note that the Lemma extends to any 2-connected subgraph of K-vertices.

Given an instance of planar 3-SAT with a cycle through the clause vertices, we construct a graph as follows. We will make a K-vertex c_j for each clause c_j, and join them in a cycle as given in the input instance. By the Lemma above, this "clause" cycle will appear intact in any planar 2-split of the graph.

Let T be any other cycle of K-vertices, disjoint from the clause cycle. T will also appear intact in any planar 2-split, so we can identify the "outside" of the cycle T as the side that contains the clause cycle. The other side is the "inside".

For each variable v_i, we create a vertex gadget as shown in Figures 3(a)–(b) with six K-vertices: two special vertices v_i and $\bar{v}_i$ and four other vertices forming a "variable cycle" $v_i^1, v_i^2, v_i^3, v_i^4$ together with two paths v_i^1, v_i, v_i^3 and $v_i^2, \bar{v}_i, v_i^4$. Observe that, in an embedding of any planar 2-split, the vertex gadget will appear intact, and exactly one of v_i and $\bar{v}_i$ must lie inside the variable cycle and exactly one must lie outside the variable cycle. Our intended correspondence is that the one that lies outside is the one that is set to *true*.

For each clause c_j with literals $\ell_{j,k}$, $k = 1, 2, 3$, we create a K_5 clause gadget, as shown in Figure 3(c), with five K-vertices: two vertices c_j, c'_j and three vertices $\ell'_{j,k}$. Furthermore, we subdivide each edge $(c_j, \ell'_{j,k})$ by a vertex $\ell_{j,k}$ that is *not* a K-vertex. If literal $\ell_{j,k}$ is v_i, then we add an edge $(v_i, \ell_{j,k})$ and if literal $\ell_{j,k}$ is $\bar{v}_i$, then we add an edge $(\bar{v}_i, \ell_{j,k})$. Figure 4 shows an example of the construction.

Note that the only non-K-vertices are the $\ell_{j,k}$'s, which have degree 3 and can be split in one of three ways as shown in Figures 5(a)–(c). In each possibility, one edge

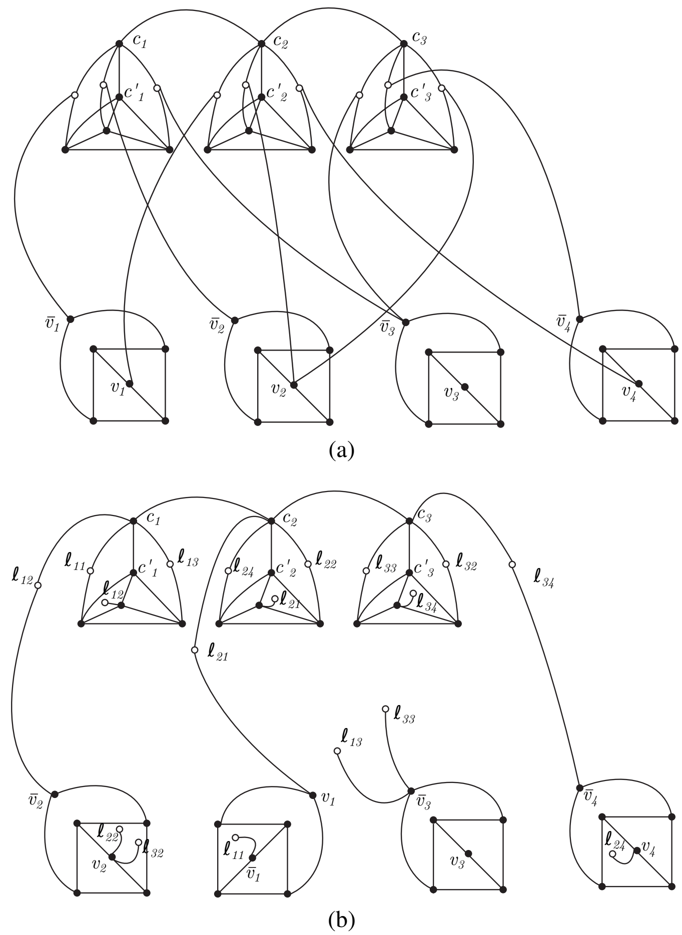


Fig. 4. (a) A graph that corresponds to the 3-SAT instance $\phi = (\bar{v_1} \vee \bar{v_2} \vee \bar{v_3}) \wedge (v_1 \vee v_2 \vee v_4) \wedge (v_2 \vee \bar{v_3} \vee \bar{v_4})$. (b) A planarization of the graph in (a) that satisfies ϕ: $v_1 =$ true, $v_2 = v_3 = v_4 =$ false

incident to $\ell_{j,k}$ is "split off" from the other two. If the edge to the variable gadget is split off from the other two, we call this the *F-split*.

Observe that if, in the clause gadget for c_j, all three of $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ use the F-split (or no split), then we effectively have edges from c_j to each of $\ell'_{j,1}, \ell'_{j,2}, \ell'_{j,3}$, so the clause gadget is a K_5 which must remain intact after the 2-split and is not planar. This means that in any planar 2-split of the clause gadget, at least one of $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ must be split with a non-F-split.

Lemma 4. *If the formula is satisfiable, then the graph has a planar 2-split.*

Proof. For every literal $\ell_{j,k}$ that is set to *false*, we do an F-split on the vertex $\ell_{j,k}$. For every literal $\ell_{j,k}$ that is set to *true*, we split off the edge to $\ell'_{j,k}$; see Figure 5(b). For any K-vertex v incident to edges E_v outside its K_{12}, we split all vertices of the K_{12} as

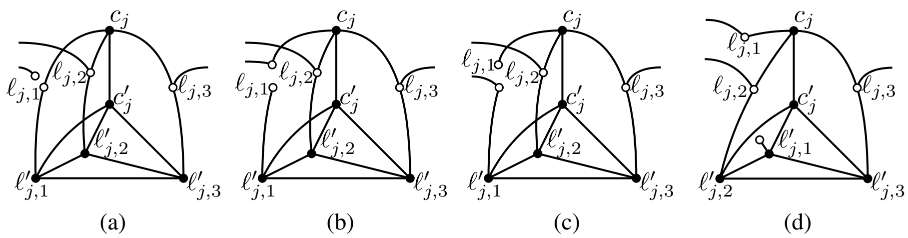


Fig. 5. (a)–(c) The three ways of splitting $\ell_{j,1}$; (a) is the F-split. (d) A planar drawing of the clause gadget when literal $\ell_{j,1}$ is set to *true* and the split of vertex $\ell_{j,1}$ results in a dangling edge to $\ell'_{j,1}$.

required for a planar 2-split of K_{12} but we keep the edges of E_v incident to the same copy of v, which we identify as the "real" v.

If variable v_i is set to *true*, we place (real) vertex v_i outside the variable cycle and we place vertex $\bar{v}_i$ and its dangling edges inside the variable cycle. If variable v_i is set to *false*, we place vertex $\bar{v}_i$ outside the variable cycle and we place vertex v_i and its dangling edges inside the variable cycle.

Consider a clause c_j. It has a true literal, say $\ell_{j,1}$. We have split off the edge from $\ell_{j,1}$ to $\ell'_{j,1}$ which cuts one edge of the K_5 and permits a planar drawing of the clause gadget as shown in Figure 5(d), with $\ell'_{j,1}$ and its dangling edge inside the cycle $c', \ell'_{j,2}, \ell'_{j,3}$.

Because we started with an instance of planar 3-SAT with a cycle through the clause vertices, we know that the graph of clauses versus variables plus the clause cycle is planar. We make a planar embedding of the split graph based on this, embedding the variable and clause gadgets as described above. The resulting embedding is planar. □

Lemma 5. *If the graph has a planar 2-split, then the formula is satisfiable.*

Proof. Consider a planar embedding of a 2-split of the graph. As noted above, in each clause gadget, say c_j, at least one of the vertices $\ell_{j,k}$, $k = 1, 2, 3$, must be split with a non-F-split. Suppose that vertex $\ell_{j,k}$ is split with a non-F-split. If literal $\ell_{j,k}$ is v_i then we will set variable v_i to *true*; and if literal $\ell_{j,k}$ is $\bar{v}_i$ then we will set variable v_i to *false*. We must show that this is a valid truth-value setting. Suppose not. Then, for some i, vertex v_i is joined to vertex $\ell_{j,k}$ that is split with a non-F-split, and vertex $\bar{v}_i$ is joined to vertex $\ell_{r,s}$ that is split with a non-F-split. But then we essentially have an edge from v_i to a vertex of the c_j clause gadget and an edge from $\bar{v}_i$ to a vertex of the c_r clause gadget. Because each clause gadget is a connected graph of K-vertices, and the clause gadgets are joined by the clause cycle, this gives a path of K-vertices from v_i to $\bar{v}_i$. Then the 6 vertices of the variable gadget for v_i form a subdivided $K_{3,3}$ of K-vertices. This must remain intact under 2-splits and is non-planar. Contradiction to having a planar 2-split of the graph. □

Theorem 5. *It is NP-hard to decide whether a graph has planar split thickness 2.*

3.2 Approximating Split Thickness

In this section, we need the concept of arboricity. The *arboricity* $a(G)$ of a graph G is the minimum integer such that G admits a decomposition into $a(G)$ forests. By definition, the planar split thickness of a graph is bounded by its arboricity. We now show that the arboricity of a k-splittable graph approximates its planar split thickness within a constant factor.

Let G be a k-splittable graph with n vertices and let G^k be a k-split graph of G. Since G^k is planar, it has at most $3kn - 6$ edges. Therefore, the number of edges in an n-vertex graph is also at most $(3k+1)(n-1)$: for n at most $6k$, this follows simply from the fact that any n-vertex graph can have at most $n(n-1)/2$ edges, and for larger n this modified expression is bigger than $3kn - 6$. But Nash-Williams [18] showed that the arboricity of a graph is at most $a(G)$ if and only if every n-vertex subgraph has at most $a(G)(n-1)$ edges. Using this characterization and the bound on the number of edges, the arboricity is at most $3k + 1$.

Theorem 6. *The arboricity of a k-splittable graph is bounded by $3k + 1$, and therefore approximates its planar split thickness within factor $3 + 1/k$.*

Note that the thickness of a graph is bounded by its arboricity, and thus also approximates the planar split thickness within factor $3 + 1/k$.

3.3 Fixed-Parameter Tractability

Although k-splittability is NP-complete, we show in this section that it is solvable in polynomial time for graphs of bounded treewidth. The result applies not only to planarity, but to many other graph properties.

Theorem 7. *Let P be a graph property, such as planarity, that can be tested in monadic second-order graph logic, and let k and w be fixed constants. Then it is possible to test in linear time whether a graph of treewidth at most w is k-splittable into P in linear time.*

Proof. We use Courcelle's theorem [3], according to which any monadic second-order property can be tested for bounded-treewidth graphs in linear time. We modify the formula for P into a formula for the graphs k-splittable into P.

To do so, we need to be able to distinguish the two endpoints of each edge of our given graph G, within the modified formula. Thus, we wrap the formula in existential quantifiers for an edge set T and a vertex r, and we form the conjunction of the formula with the conditions that every partition of the vertices into two subsets is crossed by an edge, that every nonempty vertex subset includes at least one vertex with at most one neighbor in the subset, and that, for every edge e that is not part of T, there is a path in T starting from r whose vertices include the endpoints of e. These conditions ensure that T is a depth-first search tree of the given graph, in which the two endpoints of each edge of the graph are related to each other as ancestor and descendant; we can orient each edge from its ancestor to its descendant [4].

With this orientation in hand, we wrap the formula in another set of existential quantifiers, asking for k^2 edge sets, and we add conditions to the formula ensuring that these sets form a partition of the edges of the given graph. If we number the split copies of each vertex in a k-splitting of the given graph from 1 to k, then these k^2 edge sets determine, for each input edge, which copy of its ancestral endpoint and which copy of its descendant endpoint are connected in the graph resulting from the splitting.

Given these preliminary modifications, it is straightforward but tedious to modify the formula for P itself so that it applies to the graph whose splitting is described by the above variables rather than to the input graph. To do so, we need only replace every vertex set variable by k such variables (one for each copy of each vertex), expand the formula into a disjunction or conjunction of k copies of the formula for each individual vertex variable that it contains, and modify the predicates for vertex-edge incidence within the formula to take account of these multiple copies. $\square$

4 Conclusion

In this paper, we have explored the split thickness of graphs while transforming them to planar graphs. We have proved some tight bounds on the planar split thickness of complete and complete bipartite graphs. In general, we have proved that recognizing 2-splittable graphs is NP-hard, but it is possible to approximate the planar split thickness of a graph within a constant factor. Furthermore, if the treewidth of the input graph is bounded, then for any fixed k, one can decide k-splittability into planar graphs in linear time.

Splitting number has been examined also on the projective plane [9] and torus [8]. Hence, it is natural to study split thickness on different surfaces. We observed that any graph that can be embedded on the torus or projective plane is 2-splittable. For the projective plane, use the hemisphere model of the projective plane, in which points on the equator of the sphere are identified with the opposite point on the equator; then expand the hemisphere to a sphere with two copies of each point, and choose arbitrarily which of the two copies to use for each edge. For the torus, draw the torus as a square with periodic boundary conditions, make two copies of the square, and when an edge crosses the square boundary connect it around between the two squares.

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