# The Maximum k-Differential Coloring Problem

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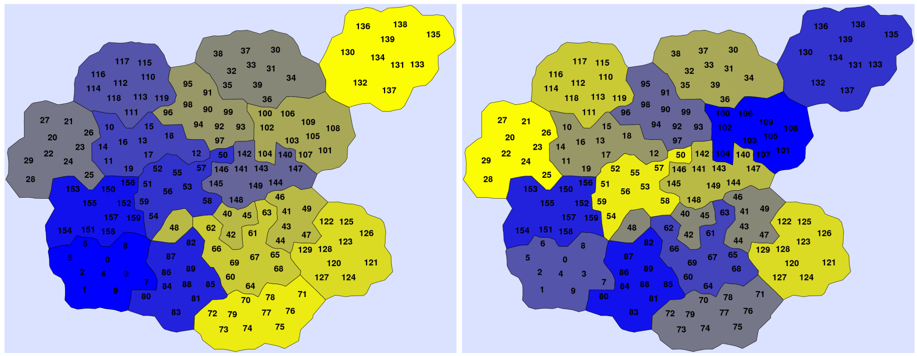
**Abstract.** Given an $n$-vertex graph $G$ and two positive integers $d, k \in \mathbb{N}$, the $(d, kn)$-differential coloring problem asks for a coloring of the vertices of $G$ (if one exists) with distinct numbers from $1$ to $kn$ (treated as *colors*), such that the minimum difference between the two colors of any adjacent vertices is at least $d$. While it was known that the problem of determining whether a general graph is $(2, n)$-differential colorable is NP-complete, our main contribution is a complete characterization of bipartite, planar and outerplanar graphs that admit $(2, n)$-differential colorings. For practical reasons, we consider also color ranges larger than $n$, i.e., $k > 1$. We show that it is NP-complete to determine whether a graph admits a $(3, 2n)$-differential coloring. The same negative result holds for the $(\lfloor 2n/3 \rfloor, 2n)$-differential coloring problem, even in the case where the input graph is planar.

## 1 Introduction

Several methods for visualizing relational datasets use a map metaphor where objects, relations between objects and clusters are represented as cities, roads and countries, respectively. Clusters are usually represented by colored regions, whose boundaries are explicitly defined. The $4$-coloring theorem states that four colors always suffice to color any map such that neighboring countries have distinct colors. However, if not all countries of the map are contiguous and the countries are not colored with unique colors, it would be impossible to distinguish whether two regions with the same color belong to the same country or to different countries. In order to avoid such ambiguity, this necessitates the use of a unique color for each country; see Fig. 1.

However, it is not enough to just assign different colors to each country. Although human perception of color is good and thousands of different colors can be easily distinguished, reading a map can be difficult due to color constancy and color context effects [19]. Dillencourt *et al.* [6] define a good coloring as one in which the colors assigned to the countries are visually distinct while also ensuring that the colors assigned to adjacent countries are as dissimilar as possible. However, not all colors make suitable choices for coloring countries and a "good" color palette is often a gradation of certain map-like colors [3]. In more restricted scenarios, e.g., when a map is printed in gray scale, or when the countries in a given continent must use different shades of a predetermined color, the color space becomes 1-dimensional.

This 1-dimensional fragmented map coloring problem is nicely captured by the *maximum differential coloring problem* [4, 15, 16, 23], which we slightly generalize in



(a) Colored with random assignment of colors (b) Colored with max. differential coloring

Fig. 1. Illustration of a map colored using the same set of colors obtained by the linear interpolation of blue and yellow. There is one country in the middle containing the vertices 40-49 which is fragmented into three small regions.

this paper: Given a map, define the *country graph* $G = (V, E)$ whose vertices represent countries, and two countries are connected by an edge if they share a non-trivial geographic boundary. Given two positive integers $d, k \in \mathbb{N}$, we say that $G$ is $(d, kn)$-differential colorable if and only if there is a coloring of the $n$ vertices of $G$ with distinct numbers from $1$ to $kn$ (treated as *colors*), so that the *minimum color distance* between adjacent vertices of $G$ is at least $d$. The *maximum k-differential coloring* problem asks for the largest value of $d$, called the *k-differential chromatic number* of $G$, so that $G$ is $(d, kn)$-differential colorable. Note that the traditional *maximum differential coloring problem* corresponds to $k = 1$.

A natural reason to study the maximum k-differential coloring problem for $k > 1$ is that using more colors can help produce maps with larger differential chromatic number. Note, for example, that a star graph on $n$ vertices has 1-differential chromatic number (or simply *differential chromatic number*) one, whereas its 2-differential chromatic number is $n + 1$. That is, by doubling the number of colors used, we can improve the quality of the resulting coloring by a factor of $n$. This is our main motivation for studying the maximum k-differential coloring problem for $k > 1$.

**Related Work:** The maximum differential coloring problem is a well-studied problem, which dates back in 1984, when Leung et al. [15] introduced it under the name "separation number" and showed its NP-completeness. It is worth mentioning though that the maximum differential coloring problem is also known as "dual bandwidth" [23] and "anti-bandwidth" [4], since it is the complement of the *bandwidth minimization problem* [17]. Due to the hardness of the problem, heuristics are often used for coloring general graphs, using LP-formulations [8], memetic algorithms [1] and spectral based methods [12]. The differential chromatic number is known only for special graph classes, such as Hamming graphs [7], meshes [20], hypercubes [20, 21], complete binary trees [22], complete $m$-ary trees for odd values of $m$ [4], other special types of trees [22], and complements of interval graphs, threshold graphs and arborescent comparability graphs [14]. Upper bounds on the differential chromatic number are given by Leung et al. [15] for connected graphs and by Miller and Pritikin [16] for bipartite graphs. For a more detailed bibliographic overview refer to [2].

**Our Contribution:** In Section 2, we present preliminary properties and bounds on the $k$-differential chromatic number. One of them guarantees that any graph is $(1, n)$-differential colorable; an arbitrary assignment of distinct colors to the vertices of the input graph guarantees a minimum color distance of one (see Lemma 1). So, the next reasonable question to ask is whether a given graph is $(2, n)$-differential colorable. Unfortunately, this is already an NP-complete problem (for general graphs), since a graph is $(2, n)$-differential colorable if and only if its complement is Hamiltonian [15]. This motivates the study of the $(2, n)$-differential coloring problem for special classes of graphs. In Section 3 we present a complete characterization of bipartite, outer-planar and planar graphs that admit $(2, n)$-differential colorings.

In Section 4, we double the number of available colors. As any graph is $(2, 2n)$-differential colorable (due to Lemma 1; Section 2), we study the $(3, 2n)$-differential coloring problem and we prove that it is NP-complete for general graphs (Theorem 4; Section 4). As a corollary, we show that testing whether a given graph is $(k + 1, kn)$-differential colorable is NP-complete. On the other hand, all planar graphs are $(\lfloor n/3 \rfloor + 1, 2n)$-differential colorable (see Lemma 3; Section 2) and testing whether a given planar graph is $(\lfloor 2n/3 \rfloor, 2n)$-differential colorable is shown to be NP-complete (Theorem 5; Section 4). In Section 5, we provide a simple ILP-formulation for the maximum k-differential coloring problem and experimentally compare the optimal results obtained by the ILP formulation for $k = 1$ and $k = 2$ with GMap, which is a heuristic based on spectral methods developed by Hu et al. [10]. We conclude in Section 6 with open problems and future work.

## 2 Preliminaries

The maximum k-differential coloring problem can be easily reduced to the ordinary differential coloring problem as follows: If $G$ is an $n$-vertex graph that is input to the maximum k-differential coloring problem, create a disconnected graph $G'$ that contains all vertices and edges of $G$ plus $(k - 1) \cdot n$ isolated vertices. Clearly, the k-differential chromatic number of $G$ is equal to the 1-differential chromatic number of $G'$. A drawback of this approach, however, is that few results are known for the ordinary differential coloring problem, when the input is a disconnected graph. In the following, we present some immediate upper and lower bounds on the $k$-differential chromatic number for connected graphs.

**Lemma 1.** *The $k$-differential chromatic number of a connected graph is at least $k$.*

*Proof.* Let $G$ be a connected graph on $n$ vertices. It suffices to prove that $G$ is $(k, kn)$-differential colorable. Indeed, an arbitrary assignment of distinct colors from the set $\{k, 2k, \ldots, kn\}$ to the vertices of $G$ guarantees a minimum color distance of $k$. $\square$

**Lemma 2.** *The $k$-differential chromatic number of a connected graph $G = (V, E)$ on $n$ vertices is at most $\lfloor \frac{n}{2} \rfloor + (k-1)n$.*

*Proof.* The proof is a straightforward generalization of the proof of Yixun et al. [23] for the ordinary maximum differential coloring problem. One of the vertices of $G$ has to be assigned with a color in the interval $[\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + (k-1)n]$, as the size of this interval is $(k-1)n + 1$ and there can be only $(k-1)n$ unassigned colors. Since $G$ is connected, that vertex must have at least one neighbor which (regardless of its color) would make the difference along that edge at most $kn - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + (k-1)n$. $\square$

**Lemma 3.** *The $k$-differential chromatic number of a connected $m$-colorable graph $G = (V, E)$ on $n$ vertices is at least $\lfloor \frac{(k-1)n}{m-1} \rfloor + 1$.*

*Proof.* Let $C_i \subseteq V$ be the set of vertices of $G$ with color $i$ and $c_i$ be the number of vertices with color $i$, $i = 1, \ldots, m$. We can show that $G$ is $(\lfloor \frac{(k-1)n}{m-1} \rfloor + 1, kn)$-differential colorable by coloring the vertices of $C_i$ with colors from the following set: $[\,(\sum_{j=1}^{i-1} c_j) + 1 + (i-1)\lfloor \frac{(k-1)n}{m-1} \rfloor,\ (\sum_{j=1}^{i} c_j) + (i-1)\lfloor \frac{(k-1)n}{m-1} \rfloor\,]$ $\square$

## 3 The (2,n)-Differential Coloring Problem

In this section we provide a complete characterization of (i) bipartite graphs, (ii) outerplanar graphs and (iii) planar graphs that admit $(2, n)$-differential coloring. Central to our approach is a result of Leung et al. [15] who showed that a graph $G$ has $(2, n)$-differential coloring if and only if the complement $G^c$ of $G$ is Hamiltonian. Clearly, if the complement of $G$ is disconnected, then $G$ has no $(2, n)$-differential coloring.

In order to simplify our notation scheme, we introduce the notion of *ordered differential coloring* (or simply *ordered coloring*) of a graph, which is defined as follows. Given a graph $G = (V, E)$ and a sequence $S_1 \to S_2 \to \ldots \to S_k$ of $k$ disjoint subsets of $V$, such that $\cup_{i=1}^{k} S_i = V$, an *ordered coloring* of $G$ implied by the sequence $S_1 \to S_2 \to \ldots \to S_k$ is one in which the vertices of $S_i$ are assigned colors from $(\sum_{j=1}^{i-1} |S_j|) + 1$ to $\sum_{j=1}^{i} |S_j|$, $i = 1, 2, \ldots, k$.

**Theorem 1.** *A bipartite graph admits a $(2, n)$-differential coloring if and only if it is not a complete bipartite graph.*

*Proof.* Let $G = (V, E)$ be an $n$-vertex bipartite graph, with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $E \subseteq V_1 \times V_2$. If $G$ is a complete bipartite graph, then its complement is disconnected. Therefore, $G$ does not admit a $(2, n)$-differential coloring. Now, assume that $G$ is not complete bipartite. Then, there exist at least two vertices, say $u \in V_1$ and $v \in V_2$, that are not adjacent, i.e., $(u, v) \notin E$. Consider the ordered coloring of $G$ implied by the sequence $V_1 \setminus \{u\} \to \{u\} \to \{v\} \to V_2 \setminus \{v\}$. As $u$ and $v$ are not adjacent, it follows that the color difference between any two vertices of $G$ is at least two. Hence, $G$ admits a $(2, n)$-differential coloring. $\square$

**Lemma 4.** *An outerplanar graph with $n \geq 6$ vertices, that does not contain $K_{1,n-1}$ as a subgraph, admits a 3-coloring, in which each color set contains at least 2 vertices.*

*Proof.* Let $G = (V, E)$ be an outerplanar graph with $n \geq 6$ vertices, that does not contain $K_{1,n-1}$ as a subgraph. As $G$ is outerplanar, it admits a 3-coloring [18]. Let $C_i \subseteq V$ be the set of vertices of $G$ with color $i$ and $c_i$ be the number of vertices with color $i$, that is $c_i = |C_i|$, for $i = 1, 2, 3$. W.l.o.g. let $c_1 \leq c_2 \leq c_3$. We further assume that each color set contains at least one vertex, that is $c_i \geq 1$, $i = 1, 2, 3$. If there is no set with less than 2 vertices, then the lemma clearly holds. Otherwise we distinguish three cases:

Case 1: $c_1 = c_2 = 1$ and $c_3 \geq 4$. W.l.o.g. assume that $C_1 = \{a\}$ and $C_2 = \{b\}$. As $G$ is outerplanar, vertices $a$ and $b$ can have at most 2 common neighbors. On the other hand, since $G$ has at least 6 vertices, there exists at least one vertex, say $c \in C_3$, which is not a common neighbor of $a$ and $b$. W.l.o.g. assume that $(b, c) \notin E$. Then, vertex $c$ can be colored with color 2. Therefore, we derive a new 3-coloring of $G$ for which we have that $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 3$.

Case 2: $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 3$: W.l.o.g. assume that $C_1 = \{a\}$ and $C_2 = \{b, b'\}$. First, consider the case where there exists at least one vertex, say $c \in C_3$, which is not a neighbor of vertex $a$. In this case, vertex $c$ can be colored with color 1 and a new 3-coloring of $G$ is derived with $c_1 = c_2 = 2$ and $c_3 \geq 3$, as desired. Now consider the more interesting case, where vertex $a$ is a neighbor of all vertices of $C_3$. As $G$ does not contain $K_{1,n-1}$ as a subgraph, either vertex $b$ or vertex $b'$ is not a neighbor of vertex $a$. W.l.o.g. let that vertex be $b$, that is $(a, b) \notin E$. As $G$ is outerplanar, vertices $a$ and $b'$ can have at most 2 common neighbors. Since $G$ has at least 6 vertices and vertex $a$ is a neighbor of all vertices of $C_3$, there exist at least one vertex, say $c \in C_3$, which is not adjacent to vertex $b'$, that is $(b', c) \notin E$. Therefore, we can color vertex $c$ with color 2 and vertex $b$ with color 1 and derive a new 3-coloring of $G$ for which we have that $c_1 = c_2 = 2$ and $c_3 \geq 2$, as desired.

Case 3: $c_1 = 1$, $c_2 \geq 3$ and $c_3 \geq 3$: W.l.o.g. assume that $C_1 = \{a\}$. Then, there exists at least one vertex, say $c \in C_2 \cup C_3$, which is not a neighbor of vertex $a$. In this case, vertex $c$ can be colored with color 1 and a new 3-coloring of $G$ is derived with $c_1 = c_2 = 2$ and $c_3 \geq 3$, as desired.

□

**Lemma 5.** *Let $G = (V, E)$ be an outerplanar graph and let $V'$ and $V''$ be two disjoint subsets of $V$, such that $|V'| \geq 2$ and $|V''| \geq 3$. Then, there exist two vertices $u \in V'$ and $v \in V''$, such that $(u, v) \notin E$.*

*Proof.* The proof follows from the fact that an outerplanar graph is $K_{2,3}$ free [5]. □

**Theorem 2.** *An outerplanar graph with $n \geq 8$ vertices has $(2, n)$-differential coloring if and only if it does not contain $K_{1,n-1}$ as subgraph.*

*Proof.* Let $G = (V, E)$ be an outerplanar graph with $n \geq 8$ vertices. If $G$ contains $K_{1,n-1}$ as subgraph, then the complement $G^c$ of $G$ is disconnected. Therefore, $G$ does not admit a $(2, n)$-differential coloring. Now, assume that $G$ does not contain $K_{1,n-1}$ as subgraph. By Lemma 4, it follows that $G$ admits a 3-coloring, in which each color set contains at least two vertices. Let $C_i \subseteq V$ be the set of vertices with color $i$ and $c_i = |C_i|$, for $i = 1, 2, 3$, such that $2 \leq c_1 \leq c_2 \leq c_3$. We distinguish the following cases:

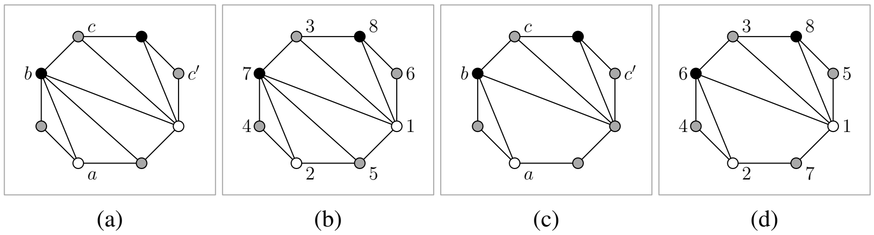



Fig. 2. (a) An outerplanar graph colored with 3 colors, white, black and grey (Case 1 of Thm. 2), and, (b) its $(2, n)$-differential coloring. (c) Another outerplanar graph also colored with 3 colors, white, black and grey (Case 2 of Thm. 2), and, (d) its $(2, n)$-differential coloring.

Case 1: $c_1 = 2$, $c_2 = 2$, $c_3 \geq 4$. Since $|C_1| = 2$ and $|C_3| \geq 4$, by Lemma 5 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly, since $|C_2| = 2$ and $|C_3 \setminus \{c\}| \geq 3$, by Lemma 5 it follows that there exist two vertices $b \in C_2$ and $c' \in C_3$, such that $c \neq c'$ and $(b, c') \notin E$; see Fig. 2a-2b.

Case 2: $c_1 \geq 2$, $c_2 \geq 3$, $c_3 \geq 3$. Since $|C_1| = 2$ and $|C_3| \geq 3$, by Lemma 5 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly, since $|C_2| \geq 3$ and $|C_3 \setminus \{c\}| \geq 2$, by Lemma 5 it follows that there exist two vertices $b \in C_2$ and $c' \in C_3$, such that $c \neq c'$ and $(b, c') \notin E$; see Fig. 2c-2d.

For both cases consider the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{a\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{b\} \to C_2 \setminus \{b\}$. As $(a, c) \notin E$ and $(b, c') \notin E$, it follows that the color difference between any two vertices of $G$ is at least two. Hence, $G$ admits a $(2, n)$-differential coloring. □

The next theorem gives a complete characterization of planar graphs that admit $(2, n)$-differential colorings. Due to space constraints, the detailed proof (which is similar to the one of Theorem 2) is given in Appendix A.

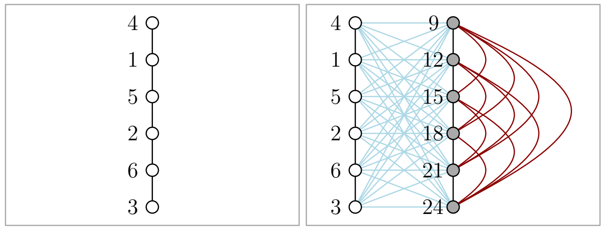
**Theorem 3.** *A planar graph with $n \geq 36$ vertices has a $(2, n)$-differential coloring if and only if it does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$.*

*Sketch of Proof.* It can be shown that a planar graph $G$ with $n \geq 36$ vertices, that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$, admits a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices (Lemma 6 in Appendix A). This together with a property similar to the one presented in Lemma 5 for outerplanar graphs (refer to Lemma 7 in Appendix A) implies that the complement of $G$ is Hamiltonian and hence $G$ has a $(2, n)$-differential coloring. □

## 4 NP-completeness Results

In this section, we prove that the $(3, 2n)$-differential coloring problem is NP-complete. Recall that all graphs are $(2, 2n)$-differential colorable due to Lemma 1.

**Theorem 4.** *Given a graph $G = (V, E)$ on $n$ vertices, it is NP-complete to determine whether $G$ has a $(3, 2n)$-differential coloring.*




(a) Instance $G = (V, E)$ (b) Instance $G' = (V', E')$

Fig. 3. (a) An instance of the $(3, n)$-differential coloring problem for $n = 6$; (b) An instance of the $(3, 2n')$-differential coloring problem constructed based on graph $G$.

*Proof.* The problem is clearly in NP. In order to prove that the problem is NP-hard, we employ a reduction from the $(3, n)$-differential coloring problem, which is known to be NP-complete [15]. More precisely, let $G = (V, E)$ be an instance of the $(3, n)$-differential coloring problem, i.e., graph $G$ is an $n$-vertex graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We will construct a new graph $G'$ with $n' = 2n$ vertices, so that $G'$ is $(3, 2n')$-differential colorable if and only if $G$ is $(3, n)$-differential colorable; see Fig. 3.

Graph $G' = (V', E')$ is constructed by attaching $n$ new vertices to $G$ that form a clique; see the gray colored vertices of Fig. 3b. That is, $V' = V \cup U$, where $U = \{u_1, u_2, \ldots, u_n\}$ and $(u, u') \in E'$ for any pair of vertices $u$ and $u' \in U$. In addition, for each pair of vertices $v \in V$ and $u \in U$ there is an edge connecting them in $G'$, that is $(v, u) \in E'$. In other words, (i) the subgraph, say $G_U$, of $G'$ induced by $U$ is complete and (ii) the bipartite graph, say $G_{U \times V}$, with bipartition $V$ and $U$ is also complete.

First, suppose that $G$ has a $(3, n)$-differential coloring and let $l : V \to \{1, \ldots, n\}$ be the respective coloring. We compute a coloring $l' : V' \to \{1, \ldots, 4n\}$ of $G'$ as follows: (i) $l'(v) = l(v)$, for all $v \in V' \cap V$ and (ii) $l'(u_i) = n + 3i$, $i = 1, 2, \ldots, n$. Clearly, $l'$ is a $(3, 2n')$-differential coloring of $G'$.

Now, suppose that $G'$ is $(3, 2n')$-differential colorable and let $l' : V' \to \{1, \ldots, 2n'\}$ be the respective coloring (recall that $n' = 2n$). We next show how to compute the $(3, n)$-differential coloring for $G$. W.l.o.g., let $V = \{v_1, \ldots v_n\}$ contain the vertices of $G$, such that $l'(v_1) < \ldots < l'(v_n)$, and $U = \{u_1, \ldots u_n\}$ contains the newly added vertices of $G'$, such that $l'(u_1) < \ldots < l'(u_n)$. Since $G_U$ is complete, it follows that the color difference between any two vertices of $U$ is at least three. Similarly, since $G_{U \times V}$ is complete bipartite, the color difference between any two vertices of $U$ and $V$ is also at least three. We claim that $l'$ can be converted to an equivalent $(3, 2n')$-differential coloring for $G'$, in which all vertices of $V$ are colored with numbers from 1 to $n$, and all vertices of $U$ with numbers from $n + 3$ to $4n$.

Let $U'$ be a maximal set of vertices $\{u_1, \ldots, u_j\} \subseteq U$ so that there is no vertex $v \in V$ with $l'(u_1) < l'(v) < l'(u_j)$. If $U' = U$ and $l'(v) < l'(u_1), \forall v \in V$, then our claim trivially holds. If $U' = U$ and $l'(v) > l'(u_j), \forall v \in V$, then we can safely recolor all the vertices in $V'$ in the reverse order, resulting in a coloring that complies with our claim. Now consider the case where $U' \subsetneq U$. Then, there is a vertex $v_k \in V$ s.t. $l'(v_k) - l'(u_j) >= 3$. Similarly, we define $V' = \{v_k, \ldots, v_l \in V\}$ to be a maximal set of vertices of $V$, so that $l'(v_k) < \ldots < l'(v_l)$ and there is no vertex $u \in U$

with $l'(v_k) < l'(u) < l'(v_l)$. Then, we can safely recolor all vertices of $U' \cup V'$, such that: (i) the relative order of the colors of $U'$ and $V'$ remains unchanged, (ii) the color distance between $v_l$ and $u_1$ is at least three, and (iii) the colors of $U'$ are strictly greater that the ones of $V'$. Note that the color difference between $u_j$ and $u_{j+1}$ and between $v_{k-1}$ and $v_k$ is at least three after recoloring, i.e., $l'(u_{j+1}) - l'(u_j) >= 3$ and $l'(v_k) - l'(v_{k-1}) >= 3$. If we repeat this procedure until $U' = U$, then the resulting coloring complies with our claim. Thus, we obtain a $(3, n)$-differential coloring $l$ for $G$ by assigning $l(v) = l'(v), \forall v \in V$. □

**Corollary 1.** *Given a graph $G = (V, E)$ on $n$ vertices, it is NP-complete to determine whether $G$ has a $(k + 1, kn)$-differential coloring.*

*Sketch of Proof.* Based on an instance $G = (V, E)$ of the $(k+1, n)$-differential coloring problem, which is known to be NP-complete [15], construct a new graph $G' = (V', E')$ with $n' = kn$ vertices, by attaching $n(k - 1)$ new vertices to $G$, as in the proof of Theorem 4. Then, using a similar argument as above, we can show that $G$ has a $(k + 1, n)$-differential coloring if and only if $G'$ has a $(k + 1, kn')$-differential coloring. □

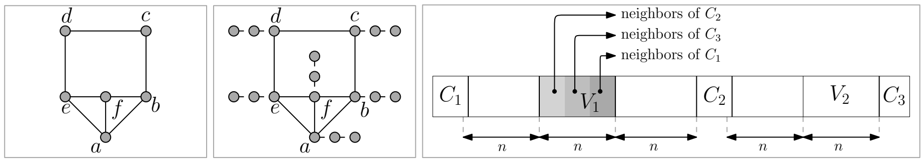
The NP-completeness of 2-differential coloring in Theorem 4 was about general graphs. Next we consider the complexity of the problem for planar graphs. Note that from Lemma 2 and Lemma 3, it follows that the 2-differential chromatic number of a planar graph on $n$-vertices is between $\lfloor \frac{n}{3} \rfloor + 1$ and $\lfloor \frac{3n}{2} \rfloor$ (a planar graph is 4-colorable). The next theorem shows that testing whether a planar graph is $(\lfloor 2n/3 \rfloor, 2n)$-differential colorable is NP-complete. Since this problem can be reduced to the general 2-differential chromatic number problem, it is NP-complete to determine the 2-differential chromatic number even for planar graphs.

**Theorem 5.** *Given an $n$-vertex planar graph $G = (V, E)$, it is NP-complete to determine if $G$ has a $(\lfloor 2n/3 \rfloor, 2n)$-differential coloring.*

*Proof.* The problem is clearly in NP. To prove that the problem is NP-hard, we employ a reduction from the well-known 3-coloring problem, which is NP-complete for planar graphs [11]. Let $G = (V, E)$ be an instance of the 3-coloring problem, i.e., $G$ is an $n$-vertex planar graph. We will construct a new planar graph $G'$ with $n' = 3n$ vertices, so that $G'$ is $(\lfloor 2n'/3 \rfloor, 2n')$-differential colorable if and only if $G$ is 3-colorable.

Graph $G' = (V', E')$ is constructed by attaching a path $v \to v_1 \to v_2$ to each vertex $v \in V$ of $G$; see Fig. 4a-4b. Hence, we can assume that $V' = V \cup V_1 \cup V_2$, where $V$ is the vertex set of $G$, $V_1$ contains the first vertex of each 2-vertex path and $V_2$ the second vertices. Clearly, $G'$ is a planar graph on $n' = 3n$ vertices. Since $G$ is a subgraph of $G'$, $G$ is 3-colorable if $G'$ is 3-colorable. On the other hand, if $G$ is 3-colorable, then $G'$ is also 3-colorable: for each vertex $v \in V$, simply color its neighbors $v_1$ and $v_2$ with two distinct colors different from the color of $v$. Next, we show that $G'$ is 3-colorable if and only if $G'$ has a $(\lfloor 2n'/3 \rfloor, 2n')$-differential coloring.

First assume that $G'$ has a $(\lfloor 2n'/3 \rfloor, 2n')$-differential coloring and let $l : V' \to \{1, \ldots, 2n'\}$ be the respective coloring. Let $u \in V'$ be a vertex of $G'$. We assign a color $c(u)$ to $u$ as follows: $c(u) = i$, if $2(i - 1)n + 1 \leq l(u) \leq 2in$, $i = 1, 2, 3$. Since $l$ is a $(\lfloor 2n'/3 \rfloor, 2n')$-differential coloring, no two vertices with the same color are adjacent. Hence, coloring $c$ is a 3-coloring for $G'$.




(a) Instance $G$ (b) Graph $G'$ (c) Differential coloring

Fig. 4. (a) An instance of the 3-coloring problem; (b) An instance of the $(\lfloor 2n'/3 \rfloor, 2n')$-differential coloring problem constructed based on graph $G$; (c) The $(\lfloor 2n'/3 \rfloor, 2n')$-differential coloring of $G'$, in the case where $G$ is 3-colorable.

Now, consider the case where $G'$ is 3-colorable. Let $C_i \subseteq V$ be the set of vertices of the input graph $G$ with color $i$, $i = 1, 2, 3$. Clearly, $C_1 \cup C_2 \cup C_3 = V$. We compute a coloring $l$ of the vertices of graph $G'$ as follows (see Fig. 4c):

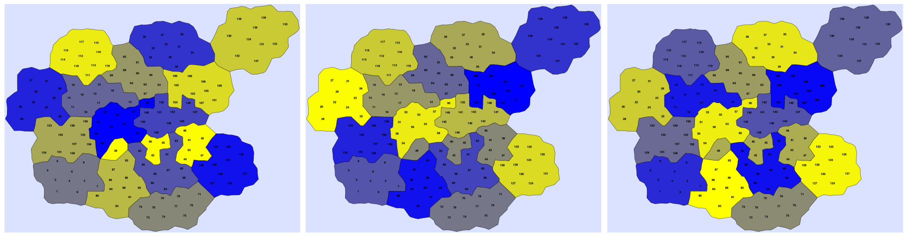
- Vertices in $C_1$ are assigned colors from 1 to $|C_1|$.
- Vertices in $C_2$ are assigned colors from $3n + |C_1| + 1$ to $3n + |C_1| + |C_2|$.
- Vertices in $C_3$ are assigned colors from $5n+|C_1|+|C_2|+1$ to $5n+|C_1|+|C_2|+|C_3|$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_1$, $l(v_1) = l(v) + 2n$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_2$, $l(v_1) = l(v) - 2n$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_3$, $l(v_1) = l(v) - 4n$.
- For a vertex $v_2 \in V_2$ that is a neighbor of a vertexx $v_1 \in V_1$, $l(v_2) = l(v_1)+3n+|C_2|$.

From the above, it follows that the color difference between (i) any two vertices in $G$, (ii) a vertex $v_1 \in V_1$ and its neighbor $v \in V$, and (iii) a vertex $v_1 \in V_1$ and its neighbor $v_2 \in V_2$, is at least $2n = \lfloor \frac{2n'}{3} \rfloor$. Thus, $G'$ is $(\lfloor 2n'/3 \rfloor, 2n')$-differential colorable. □

## 5 An ILP for the Maximum k-Differential Coloring Problem

In this section, we describe an integer linear program (ILP) formulation for the maximum k-differential coloring problem. Recall that an input graph $G$ to the maximum k-differential coloring problem can be easily converted to an input to the maximum 1-differential coloring by creating a disconnected graph $G'$ that contains all vertices and edges of $G$ plus $(k-1) \cdot n$ isolated vertices. In order to formulate the maximum 1-differential coloring problem as an integer linear program, we introduce for every vertex $v_i \in V$ of the input graph $G$ a variable $x_i$, which represents the color assigned to vertex $v_i$. The 1-differential chromatic number of $G$ is represented by a variable $OPT$, which is maximized in the objective function. The exact formulation is given below. The first two constraints ensure that all vertices are assigned colors from 1 to $n$. The third constraint guarantees that no two vertices are assigned the same color, and the forth constraint maximizes the 1-differential chromatic number of the graph. The first three constraints also guarantee that the variables are assigned integer values.

$$
\begin{array}{lll}
\textbf{maximize} & OPT & \\
\textbf{subject to} & x_i \quad\quad\;\; \leq n & \forall v_i \in V \\
& x_i \quad\quad\;\; \geq 1 & \forall v_i \in V \\
& |x_i - x_j| \geq 1 & \forall (v_i, v_j) \in V^2 \\
& |x_i - x_j| \geq OPT & \forall (v_i, v_j) \in E
\end{array}
$$



(a) (b) (c)

Fig. 5. A map with 16 countries colored by: (a) GMap [10], (b) ILP-n, (c) ILP-2n.

Note that a constraint that uses the absolute value is of the form $|X| \geq Z$ and therefore can be replaced by two new constraints: (i) $X + M \cdot b \geq Z$ and (ii) $-X + M \cdot (1-b) \geq Z$, where $b$ is a binary variable and $M$ is the maximum value that can be assigned to the sum of the variables, $Z + X$. If $b$ is equal to zero, then the two constraints are $X \geq Z$ and $-X + M \geq Z$, with the second constraint always true. On the other hand, if $b$ is equal to one, then the two constraints are $X + M \geq Z$ and $-X \geq Z$, with the first constraint always true.

Next we study two variants of the ILP formulation described above: ILP-n and ILP-2n, which correspond to $k = 1$ and $k = 2$, and compare them with GMap, which is a heuristic based on spectral methods developed by Hu et al. [10].

**Experiment's Setup:** We generate a collection of $1,200$ synthetic maps and analyze the performance of ILP-n and ILP-2n, on an Intel Core i5 1.7GHz processor with 8GB RAM, using the CPLEX solver [13]. For each map a country graph $G_c = (V_c, E_c)$ with $n$ countries is generated using the following procedure. (1) We generate $10n$ vertices and place an edge between pairs of vertices (i,j) such that $\lfloor \frac{i}{10} \rfloor = \lfloor \frac{j}{10} \rfloor$, with probability 0.5, thus resulting in a graph $G$ with approximately $n$ clusters. (2) More edges are added between all pairs of vertices with probability $p$, where $p$ takes the values $1/2, 1/4 \ldots 2^{-10}$. (3) Ten random graphs are generated for different values of $p$. (4) Graph $G$ is used as an input to a map generating algorithm (available as the `Graphviz` [9] function `gvmap`), to obtain a map $M$ with country graph $G_c$. A sample map generated by the aforementioned procedure is shown in Fig. 5.

Note that the value of $p$ determines the "fragmentation" of the map $M$, i.e., the number of regions in each country, and hence, also affects the number of edges in the country graph. When $p$ is equal to $1/2$, the country graph is a nearly complete graph, whereas for $p$ equal to $2^{-10}$, the country graph is nearly a tree. To determine a suitable range for the number of vertices in the country graph, we evaluated real world datasets, such as those available at gmap.cs.arizona.edu. Even for large graphs with over $1,000$ vertices, the country graphs tend to be small, with less than $16$ countries.

**Evaluation Results:** Fig. 6 summarizes the experimental results. Since $n$ is ranging from $5$ to $16$, the running times of both ILP-n and ILP-2n are reasonable, although still much higher than GMap. The color assignments produced by ILP-n and GMap are

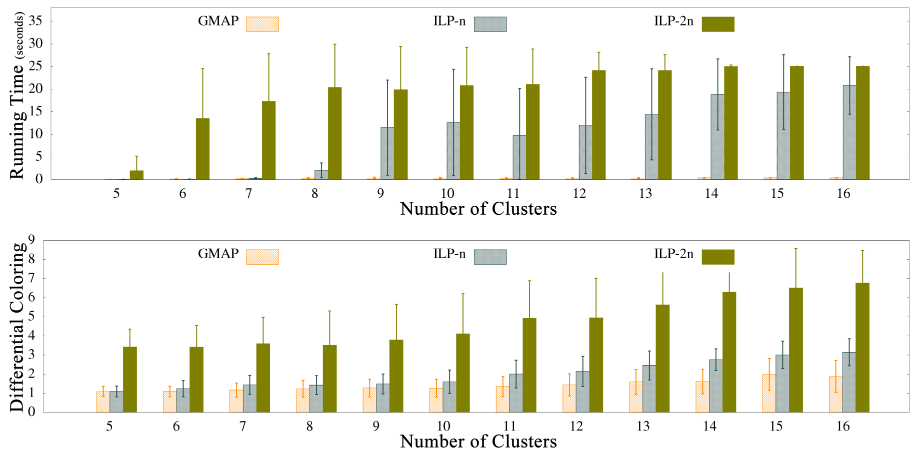



Fig. 6. Illustration of: (a) running time results for all algorithms of our experiment and (b) differential coloring performance of algorithms GMap, ILP-n and ILP-2n.

comparable, while the color assignment of ILP-2n results in the best minimum color distance. It is worth mentioning, though, that in the presence of twice as many colors as the graph's vertices, it is easier to obtain higher color difference between adjacent vertices. However, this high difference comes at the cost of assigning pairs of colors that are more similar to each other for non-adjacent vertices, as it is also the case in our motivating example from the Introduction where $G$ is a star.

## 6 Conclusion and Future Work

Even though the $(2, n)$-differential coloring is NP-complete for general graphs, in this paper, we gave a complete characterization of bipartite, outerplanar and planar graphs that admit $(2, n)$-differential colorings. Note that these characterizations directly lead to polynomial-time recognition algorithms. We also generalized the differential coloring problem for more colors than the number of vertices in the graph and showed that it is NP-complete to determine whether a general graph admits a $(3, 2n)$-differential coloring. Even for planar graphs, the problem of determining whether a graph is $(\lfloor 2n/3 \rfloor, 2n)$-differential colorable remains NP-hard.

Several related problems are still open: (i) Is it possible to characterize which bipartite, outerplanar or planar graphs are $(3, n)$-differential colorable? (ii) Extend the characterizations for those planar graphs that admit $(2, n)$-differential colorings to 1-planar graphs. (iii) Extend the results above to $(d, kn)$-differential coloring problems with larger $k > 2$. (iv) As all planar graphs are $(\lfloor \frac{n}{3} \rfloor + 1, 2n)$-differential colorable, is it possible to characterize which planar graphs are $(\lfloor \frac{n}{3} \rfloor + 2, 2n)$-differential colorable? (v) Since it is NP-complete to determine the 1-differential chromatic number of a planar graph [2], a natural question to ask is whether it is possible to compute in polynomial time the corresponding chromatic number of an outerplanar graph.

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# Appendix

## A The $(2, n)$-Differential Coloring Problem for Planar Graphs

**Lemma 6.** *A planar graph with $n \geq 36$ vertices, that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$, admits a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices.*

*Proof.* Let $G = (V, E)$ be a planar graph with $n \geq 36$ vertices, that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$. As $G$ is planar, it admits a 4-coloring [5]. Let $C_i \subseteq V$ be the set of vertices of $G$ with color $i$ and $c_i$ be the number of vertices with color $i$, that is $c_i = |C_i|$, for $i = 1, 2, 3, 4$. W.l.o.g. let $c_1 \leq c_2 \leq c_3 \leq c_4$. We further assume that each color set contains at least one vertex, that is $c_i \geq 1$, $i = 1, 2, 3, 4$. We distinguish the following cases:

Case 1: $c_1 = 1, c_2 = 1, c_3 \leq 7$. Since $G$ has at least 36 vertices, it follows that $c_4 \geq 27$. Observe that a vertex of $C_4$ cannot be assigned a color other than 4 if and only if it is adjacent to at least one vertex in $C_1$, one vertex in $C_2$ and one vertex in $C_3$. Let $C_4^* \subseteq C_4$ be the set of such vertices and $c_4^* = |C_4^*|$. We claim that $c_4^* \leq 14$. To prove the claim, we construct an auxiliary bipartite graph $G_{aux} = (V_{aux}, E_{aux})$ where $V_{aux} = C_1 \cup C_2 \cup C_3 \cup C_4^*$ and $E_{aux} = E \cap (C_4^* \times (C_1 \cup C_2 \cup C_3))$. Clearly, $G_{aux}$ is planar, since it is subgraph of $G$. Since all vertices of $C_4^*$ have degree at least 3, it holds that $3c_4^* \leq |E_{aux}|$. On the other hand, it is known that a planar bipartite graph on $n$ vertices cannot have more that $2n-4$ edges. Therefore, $|E_{aux}| \leq 2(c_1+c_2+c_3+c_4^*)-4$, which implies that $3c_4^* \leq 2(c_1+c_2+c_3+c_4^*)-4$. As $c_1 + c_2 + c_3 \leq 9$, it follows that our claim indeed holds. A a consequence, we can change the color of several vertices being currently in $C_4$ and obtain a new 4-coloring of $G$ in which $C_4$ has exactly 14 vertices. This implies that the number of vertices in $C_1$, $C_2$ and $C_3$ is at least 22 or equivalently that one out of $C_1$, $C_2$ and $C_3$ must contain strictly more that 7 vertices, say $C_3$. So, in the new coloring it holds that $c_1 \geq 1$, $c_2 \geq 1$ and $c_3 \geq 8$; a case that is covered in the following.

Case 2: $c_1 = 1$, $c_2 = 1$, $c_3 \geq 8$. Assume w.l.o.g. that $C_1 = \{a\}$ and $C_2 = \{b\}$. Since $G$ does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex, say $c \in C_3 \cup C_4$, which is not neighboring either $a$ or $b$, say $b$: $(b, c) \notin E$. Then, vertex $c$ can be safely colored with color 2. In the new coloring, it holds that $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 7$; refer to Case 5.

Case 3: $c_1 = 1$, $c_2 = 2$, $c_3 \leq 6$. Since $G$ has at least 36 vertices, it follows that $c_4 \geq 27$. Now, observe that $c_1 + c_2 + c_3 \leq 9$. Hence, following similar arguments as in Case 2, we can prove that there is a 4-coloring of $G$ in which $C_4$ has exactly 14 vertices. So, again the number of vertices in $C_1$, $C_2$ and $C_3$ is at least 22 and consequently at least one out of $C_1$, $C_2$ and $C_3$ must contain strictly more that 7 vertices, say $C_3$. In the new coloring, it holds that $c_1 \geq 1$, $c_2 \geq 2$ and $c_3 \geq 8$; refer to Case 6.

Case 4: $c_1 = 1, c_2 = 2, c_3 \geq 7$. Assume w.l.o.g. that $C_1 = \{a\}$ and $C_2 = \{b, b'\}$. First, consider the case where there exists at least one vertex, say $c \in C_3 \cup C_4$, which is not neighboring with vertex $a$. In this particular case, vertex $c$ can be colored with color 1 and a new 3-coloring of $G$ is derived for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 6$, as desired. Now consider the more interesting case, where vertex $a$ is neighboring with all vertices of $C_3 \cup C_4$. As $G$ does not contain $K_{1,n-1}$ as subgraph, vertex $a$ is not a neighbor of vertex $b$ or vertex $b'$ or both. In the latter case, vertex $a$ can be colored with color 2. Therefore, if we select two arbitrary vertices of $C_3$ and color them with color 1, we derive a new 3-coloring of $G$ for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 5$, as desired. On the other hand, if vertex $a$ is neighboring exactly one out of $b$ and $b'$, say w.l.o.g. $b$, that is $(a, b) \in E$ and $(a, b') \notin E$, then there exists a vertex $c \in C_3 \cup C_4$ such that $(b, c) \notin E$; as otherwise $K_{a,b,C_3 \cup C_4}$ forms a $K_{1,1,n-3}$. So, if we color vertex $b'$ with color 1 and vertex $c$ with color 2, we obtain a new coloring of $G$ for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 6$, as desired.

Case 5: $c_1 = 1, c_2 \geq 3, c_3 \geq 3$. Assume w.l.o.g. that $C_1 = \{a\}$. As $G$ does not contain $K_{1,n-1}$ as subgraph, there is a vertex $a' \in C_2 \cup C_3 \cup C_4$ which is not neighboring with $a$. Hence, $a'$ can be colored with color 1. In the new coloring, it holds that $c_1 \geq 2$, $c_2 \geq 2$, $c_3 \geq 2$; refer to Case 6.

Case 6: $c_1 \geq 2$, $c_2 \geq 2$, $c_3 \leq 4$. Since $G$ has at least 36 vertices, it follows that $c_4 \geq 24$. Since $c_1 + c_2 + c_3 \leq 12$, similarly to Case 2 we can prove that there is a 4-coloring of $G$ in which $C_4$ has exactly 20 vertices. So, the number of vertices in $C_1$, $C_2$ and $C_3$ is at least 16 and consequently at least one out of $C_1$, $C_2$ and $C_3$ must contain strictly more that 5 vertices, say $C_3$. In the new coloring, it holds that $c_1 \geq 2$, $c_2 \geq 2$ and $c_3 \geq 6$, as desired.

From the above case analysis, it follows that $G$ has a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices. □

**Lemma 7.** *Let $G = (V, E)$ be a planar graph and let $V'$ and $V''$ be two disjoint subsets of $V$, such that $|V'| \geq 3$ and $|V''| \geq 3$. Then, there exists two vertices $u \in V'$ and $v \in V''$, such that $(u, v) \notin E$.*

*Proof.* The proof follows directly from the fact that a planar graph does not contain $K_{3,3}$ as subgraph. □

**Theorem 3.** *A planar graph with $n \geq 36$ vertices has a $(2, n)$-differential coloring if and only if it does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$.*

*Proof.* Let $G = (V, E)$ be an $n$-vertex planar graph, with $n \geq 36$. If $G$ contains $K_{l,n-l}$ as subgraph, then the complement $G^c$ of $G$ is disconnected, $l = 1, 2 \ldots, \lceil n/2 \rceil$. Therefore, $G$ has no $(2, n)$-differential coloring. Note, however, that since $G$ is planar, it cannot contain $K_{l,n-l}$, $l = 3, 4 \ldots, \lceil n/2 \rceil$, as subgraph. On the other hand, if $G$ contains $K_{1,1,n-3}$ as subgraph, then $G^c$ has two vertices of degree one with a common neighbor. Hence, $G^c$ is not Hamiltonian and as a consequence $G$ has no $(2, n)$-differential coloring, as well.

Now, assume that $G$ does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$. By Lemma 6, it follows that $G$ admits a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices. Let $C_i \subseteq V$ be the set of vertices with color $i$ and $c_i = |C_i|$, for $i = 1, 2, 3, 4$, such that $c_1 \leq c_2 \leq c_3 \leq c_4$, $c_1, c_2 \geq 2$ and $c_3, c_4 \geq 5$. We distinguish the following cases:

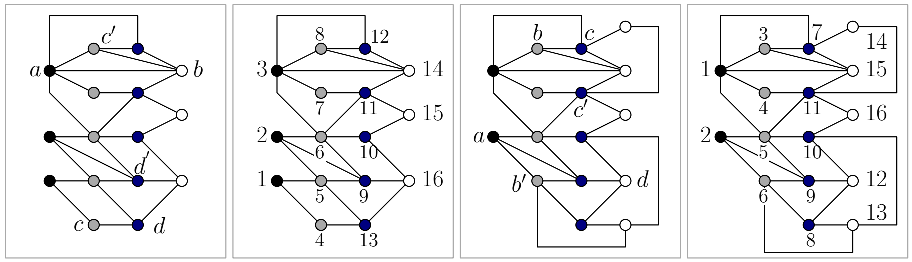


Fig. 7. (a) A planar graph colored with 4 colors, black ($C_1$), white ($C_2$), grey ($C_3$) and blue ($C_4$); Case 1 of Thm. 3, and, (b) its $(2, n)$-differential coloring. (c) Another planar graph also colored with 4 colors, black, grey, blue and white; Case 2 of Thm. 3, and, (d) its $(2, n)$-differential coloring.

Case 1: $c_1 \geq 3$, $c_2 \geq 3$, $c_3 \geq 5$, $c_4 \geq 5$. Since $|C_1| \geq 3$ and $|C_3| \geq 5$, by Lemma 7 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly, since $|C_2| \geq 3$ and $|C_4| \geq 5$, by Lemma 7 it follows that there exist two vertices $b \in C_2$ and $d \in C_4$, such that $(b, d) \notin E$. Finally, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 7 it follows that there exist two vertices $c' \in C_3$ and $d' \in C_4$, such that $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. Now consider the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{a\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d'\} \to C_4 \setminus \{d, d'\} \to \{d\} \to \{b\} \to C_2 \setminus \{b\}$. As $(a, c) \notin E$, $(b, d) \notin E$ and $(c', d') \notin E$, it follows that the color difference between any two vertices of G is at least two. Hence, $G$ has a $(2, n)$-differential coloring.

Case 2: $c_1 = 2$, $c_2 \geq 4$, $c_3 \geq 5$, $c_4 \geq 5$. Since $G$ does not contain $K_{2,n-2}$ as subgraph, there exists two vertices, say $a \in C_1$ and $b \in C_2 \cup C_3 \cup C_4$ such that $(a, b) \notin E$. Assume w.l.o.g. that $b \in C_2$. Since $|C_2 \setminus \{b\}| \geq 3$ and $|C_3| \geq 5$, by Lemma 7 it follows that there exist two vertices $b' \in C_2$ and $c \in C_3$, such that $b' \neq b$ and $(b', c) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4| \geq 5$, by Lemma 7 it follows that there exist two vertices $c' \in C_3$ and $d \in C_4$, such that $c' \neq c$ and $(c', d) \notin E$. Similarly, to the previous case the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{b\} \to C_2 \setminus \{b\} \to \{b'\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d\} \to C_4 \setminus \{d\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

Case 3: $c_1 = 2$, $c_2 = 3$, $c_3 \geq 5$, $c_4 \geq 5$. Assume w.l.o.g. that $C_1 = \{a, a'\}$ and $C_2 = \{b, b', b''\}$. We distinguish two sub cases:

- The subgraph of $G$ induced by $C_1 \cup C_2$ is $K_{2,3}$, that is $C_1 \times C_2 \subseteq E$. Since $G$ does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex of $C_1$, say vertex $a$, that is not neighboring with a vertex of $C_3 \cup C_4$, say vertex $c \in C_3$, that is $(a, c) \notin E$. Since $|C_2| = 3$ and $|C_4| \geq 5$, by Lemma 7 it follows that there exist a vertex of $C_2$, say vertex $b$, and a vertex of $C_4$, say vertex $d$, that are not adjacent, that is $(b, d) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 7 it follows that there exist two vertices $c' \in C_3$ and $d' \in C_4$, such that $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. So, in the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{a\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d'\} \to$

$C_4 \setminus \{d, d'\} \rightarrow \{d\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$, it holds that the color difference between any two vertices of $G$ is greater or equal to two. Hence, $G$ has a $(2, n)$-differential coloring.

- The subgraph of $G$ induced by $C_1 \cup C_2$ is not $K_{2,3}$. Assume w.l.o.g. that $(a, b) \notin E$. Since $|C_3| \geq 5$ and $|C_4| \geq 5$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $c$, and a vertex of $C_4$, say vertex $d$, that are not adjacent, that is $(c, d) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|\{a', b', b''\}| = 3$, by Lemma 7 it follows that there exist a vertex of $C_3 \setminus \{c\}$, say vertex $c'$, and a vertex of $\{a', b', b''\}$, say vertex $x$, that are not adjacent, that is $(x, c') \notin E$. First consider the case where $x = a'$. In this case, the ordered coloring implied by the sequence $\{b'\} \rightarrow \{b''\} \rightarrow \{b\} \rightarrow \{a\} \rightarrow \{a'\} \rightarrow \{c'\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d\}$ guarantees that $G$ has a $(2, n)$-differential coloring. Now consider the case, where $x \in \{b', b'\}$. As both cases are symmetric, we assume w.l.o.g. that $x = b'$. In this case, the ordered coloring implied by the sequence $\{a'\} \rightarrow \{a\} \rightarrow \{b\} \rightarrow \{b''\} \rightarrow \{b'\} \rightarrow \{c'\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

Case 4: $c_1 = 2$, $c_2 = 2$, $c_3 \geq 5$, $c_4 \geq 5$. Assume w.l.o.g. that $C_1 = \{a, a'\}$ and $C_2 = \{b, b'\}$. Again, we distinguish two sub cases:

- The subgraph of $G$ induced by $C_1 \cup C_2$ is $K_{2,2}$, that is $C_1 \times C_2 \subseteq E$. Since $G$ does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex of $C_1$, say vertex $a$, that is not neighboring with a vertex of $C_3 \cup C_4$, say vertex $c \in C_3$, that is $(a, c) \notin E$. Similarly, there exists a vertex of $C_2$, say vertex $b$, that is not neighboring with a vertex of $C_3 \cup C_4$, say vertex $w$, that is $(b, w) \notin E$.

  First, consider the case where the subgraph of $G$ induced by $C_2 \cup C_4$ is not $K_{2,c_4}$ that is $w \in C_4$ . Since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{w\}| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $c'$, and a vertex of $C_4$, say vertex $d$, that are not adjacent, that is $c \neq c'$, $w \neq d$ and $(c', d) \notin E$. In this case, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d, w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

  Now consider the more interesting case where the subgraph of $G$ induced by $C_2 \cup C_4$ is $K_{2,c_4}$, that is $w \in C_3$. We distinguish two sub cases:

  a. $w \neq c$. Since $|C_3 \setminus \{c, w\}| \geq 3$ and $|C_4| \geq 5$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $p$, and a vertex of $C_4$, say vertex $q$, that are not adjacent, that is $c \neq w \neq p$, and $(p, q) \notin E$. Similarly, since $|C_3 \setminus \{p, w\}| \geq 3$ and $|C_4 \setminus \{q\}| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $p'$, and a vertex of $C_4$, say vertex $q'$, that are not adjacent, that is $p \neq w \neq p'$, $q \neq q'$ and $(p', q') \notin E$. If $p' \neq c$, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow \{p'\} \rightarrow \{q'\} \rightarrow C_4 \setminus \{q, q'\} \rightarrow \{q\} \rightarrow \{p\} \rightarrow C_3 \setminus \{p, p', c, w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that $G$ has a $(2, n)$-differential coloring. If $p' = c$, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{p'\} \rightarrow \{q'\} \rightarrow C_4 \setminus \{q, q'\} \rightarrow \{q\} \rightarrow \{p\} \rightarrow C_3 \setminus \{p, p', w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

  b. $w = c$. Since the subgraph of $G$ induced by $C_2 \cup C_4$ is $K_{2,c_4}$ and $|C_2 \cup \{a\}| = 3$ and $|C_4| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_4$, say vertex $q$, not adjacent to vertex $a$, that is $(a, q) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{q\}| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_4$, say vertex $q'$, and a vertex of $C_3$, say vertex $p'$, that are not adjacent, that is $q' \neq q$, $p' \neq c$ and $(p', q') \notin E$. Then the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{a\} \to \{q\} \to C_4 \setminus \{q, q'\} \to \{q'\} \to \{p'\} \to C_3 \setminus \{p', c\} \to \{c\} \to \{b\} \to C_2 \setminus \{b\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

– The subgraph of $G$ induced by $C_1 \cup C_2$ is not $K_{2,2}$. Assume w.l.o.g. that $(a, b) \notin E$.
  First, consider the case where the subgraph of $G$ induced by $\{a', b'\} \cup C_3 \cup C_4$ is $K_{2,c_3+c_4}$ that is $\{a', b'\} \times C_3 \cup C_4 \subseteq E$. Since $|\{a, a', b'\}| = 3$ and $|C_3| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $c$, not adjacent to vertex a, that is $(a, c) \notin E$. Similarly since, $|\{b, a', b'\}| = 3$ and $|C_4| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_4$, say vertex $d$, not adjacent to vertex b, that is $(b, d) \notin E$. Since $|C_3 \setminus \{c\}| \geq 3$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $c'$, and a vertex of $C_4$, say vertex $d'$, that are not adjacent, that is $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. Then the ordered coloring implied by the sequence $C_1 \setminus \{a\} \to \{a\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d'\} \to C_4 \setminus \{d, d'\} \to \{d\} \to \{b\} \to C_2 \setminus \{b\}$ guarantees that $G$ has a $(2, n)$-differential coloring.
  Now we consider the case where the subgraph of $G$ induced by $\{a', b'\} \cup C_3 \cup C_4$ is not $K_{2,c_3+c_4}$. There exist a vertex of $C_3 \cup C_4$, say vertex $c$, and a vertex of $\{a', b'\}$, say vertex $p$, that are not adjacent, that is $(p, c) \notin E$. Assume w.l.o.g $c \in C_3$. Since $|C_3 \setminus \{c\}| \geq 3$ and $|C_4| \geq 4$, by Lemma 7 it follows that there exist a vertex of $C_3$, say vertex $c'$, and a vertex of $C_4$, say vertex $d$, that are not adjacent, that is $c \neq c'$ and $(c', d') \notin E$. If $p = b'$ the ordered coloring implied by the sequence $\{a'\} \to \{a\} \to \{b\} \to \{b'\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d'\} \to C_4 \setminus \{d'\}$ guarantees that $G$ has a $(2, n)$-differential coloring. If $p = a'$ the ordered coloring implied by the sequence $\{b'\} \to \{b\} \to \{a\} \to \{a'\} \to \{c\} \to C_3 \setminus \{c, c'\} \to \{c'\} \to \{d'\} \to C_4 \setminus \{d'\}$ guarantees that $G$ has a $(2, n)$-differential coloring.

From the above case analysis, it follows that $G$ is $(2, n)$-differential colorable, as desired. □