9. Given an infinite collection A_n, n=1,2,... of intervals of the real line, their intersection is defined to be

$\cap_{n=1 \text{ to } \infty}[A_n]=\{x|(\forall n)(x\in A_n)\}$

Give an example of a family of intervals A_n, n=1,2,..., such that $A_{n+1}\subset A_n$ for all n and $\cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$. Prove that your example has the stated property.

Claim: Let $A_n=(0,1/n)$

$\forall n\in\mathbb{N}[A_{n+1}\subset A_n] \land \cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$

Proof: We will separate the claim in two and make two proofs.

First claim: $\forall n\in\mathbb{N}[A_{n+1}\subset A_n]$, i.e.,

$\forall n\in\mathbb{N}\ [(0,1/(n+1))\subset(0,1/n)]$, i.e.

$\forall n\in\mathbb{N}\forall x\in\mathbb{R}[\ (0<x<1/(n+1)\Rightarrow 0<x<1/n) \land \neg(0<x<1/(n+1)\Leftrightarrow 0<x<1/n)\ ]$

First proof: Given and an $n\in\mathbb{N}$ and an $x\in\mathbb{R}$ suppose $0<x<1/(n+1)$. We know that $1/(n+1)<1/n$, so $0<x<1/(n+1)<1/n$, hence $0<x<1/n$. From this we deduce that $A_{n+1}\subseteq A$. Now, to prove the second term, suppose $0<x<1/n$. If $x=1/(n+0.5)$ then $0<x<1/n$, but $1/(n+1)<x$, so it is not the case that $0<x<1/(n+1)$, hence, A_{n+1} and A_n are not equal, so $A_{n+1}\subset A_n$ proving the first claim.

Second claim: $\cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$, i.e.,

$\{x|(\forall n)(x\in A_n)\}= \emptyset$, i.e.,

$\neg\exists x\in\mathbb{R}\forall n\in\mathbb{N}[0<x<1/n]$, i.e.,

$\forall x\in\mathbb{R}\exists n\in\mathbb{N}[x\leq 0 \lor 1/n\leq x]$

Second proof: Given an $x\in\mathbb{R}$, if $x\leq 0$ pick any $n\in\mathbb{N}$ and the claim will be true. If $x>0$, pick an $n\in\mathbb{N}$ so large so that $1/n<x$, so, $1/n\leq x$, proving the second claim.

With both claims proved the original claim gets proved. □