6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Definitions:

$n \in \mathbb{N}$ is prime iff $\forall d \in \mathbb{N}[1<d<n \Rightarrow d \nmid n]$

$m,n \in \mathbb{N}$ are twin primes iff m and n are primes and $n=m+2$.

$m,n,o \in \mathbb{N}$ are a prime triple iff m and n are twin primes and n and o are twin primes too.

Exercise 5 claim: $\forall n \in \mathbb{Z}[3|n \vee 3|n+2 \vee 3|n+4]$

Claim: $\exists! a,b,c \in \mathbb{N}$[a,b,c are a prime triple]

Proof: By contradiction. We know that 3,5,7 is the first prime triple. By way of contradiction, suppose you can pick a,b,c such that they are a prime triple with $3<a<b<c$. Since a,b,c are primes $3 \nmid a,b,c$. We know that $b=a+2$ and $c=a+4$. By Exercise 5 we know that $\forall n \in \mathbb{Z}[3|n \vee 3|n+2 \vee 3|n+4]$. Set $n=a$. So $3|a$ or $3|a+2$ or $3|a+4$, i.e., $3|a$ or $3|b$ or $3|c$, i.e., at least one of a, b or c is not prime, contradicting the assumption. Therefore, the only prime triple is 3,5,7 proving the original claim. □