1. Say whether the following is true or false and support your answer by a proof.

$(\exists m\in\mathbb{N})(\exists n\in\mathbb{N})(3m+5n=12)$

Claim: $(\exists m\in\mathbb{N})(\exists n\in\mathbb{N})(3m+5n=12)$ is false

Proof: By cases.

Negating the original claim we get

$(\forall m,n\in\mathbb{N})(3m+5n\neq 12)$

Given $m,n\in\mathbb{N}$

If $m=n=1$ then $3m+5n=8<12$

If $m=2$ and $n=1$ then $3m+5n=11<12$ and

If $m=1$ and $n=2$ then $3m+5n=13>12$.

If $m=n=2$ then $3m+5n=16>12$.

All cases up to here have been not equal than 12, and the last one with $m=n=2$ is greater than 12, so every other combination with $m,n>2$ will be bigger than 12, therefore, for all $m,n\in\mathbb{N}$ $3m+5n\neq 12$, proving the original claim as false. □

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Claim: $\forall a \in \mathbb{Z}[5 | a+(a+1)+(a+2)+(a+3)+(a+4)]$

Proof: Take an arbitrary $a \in \mathbb{Z}$. By algebra we can derive from the expression

$a+(a+1)+(a+2)+(a+3)+(a+4)$

$5a+10$

$5(a+2)$

which is divisible by 5 proving the original claim. □

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number n^2+n+1 is odd.

Claim: $\forall n \in \mathbb{Z}[2 \nmid n^2+n+1]$

Proof: Given an $n \in \mathbb{Z}$, by the Fundamental Theorem of Arithmetic, we know that $2|n$ iff $2|n^2$. We also know that for all $b,c \in \mathbb{Z}$

if $2|b$ and $2|c$ then $2|b+c$, and,

if $2 \nmid b$ and $2 \nmid c$ then $2|b+c$ too.

Set $b=n^2$ and $c=n$. Then, no matter if $2|n$ or $2 \nmid n$, it is always the case that $2|n^2+n$, hence, $2 \nmid n^2+n+1$, proving the original claim. □

4. Prove that every odd natural number is of one of the forms 4n+1 or 4n+3, where n is an integer.

Claim: $\forall m\in\mathbb{N}[2\nmid m \Rightarrow \exists n\in\mathbb{Z}(4n+1=m \vee 4n+3=m)]$

Proof: Given an $m\in\mathbb{N}$ suppose $2\nmid m$, i.e., $\exists p\in\mathbb{Z}[m=2p+1]$.

If $2|p$ then $\exists q\in\mathbb{Z}[p=2q]$, hence, $m=2(2q)+1=4q+1$, hence, $\exists n\in\mathbb{Z}[4n+1=m]$.

If $2\nmid p$ then $\exists q\in\mathbb{Z}[p=2q+1]$, hence, $m=2(2q+1)+1=4q+3$, hence, $\exists n\in\mathbb{Z}[4n+3=m]$.

Therefore, since one of both cases must be true, the original claim is true. □

5. Prove that for any integer n, at least one of the integers n, n+2, n+4 is divisible by 3.

Claim: $\forall n \in \mathbb{Z}[3|n \lor 3|n+2 \lor 3|n+4]$

Proof: Given an $n \in \mathbb{Z}$ we know that it must be true that $3|n$ or $3|n+1$ or $3|n+2$, since one of those 3 consecutive integers must be a multiple of 3. If it is the case that $3|n+1$ then $3|(n+1)+3$, i.e., $3|n+4$. Therefore, the original claim is true. □

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Definitions:

$n \in \mathbb{N}$ is prime iff $\forall d \in \mathbb{N}[1 < d < n \Rightarrow d \nmid n]$

$m,n \in \mathbb{N}$ are twin primes iff m and n are primes and $n=m+2$.

$m,n,o \in \mathbb{N}$ are a prime triple iff m and n are twin primes and n and o are twin primes too.

Exercise 5 claim: $\forall n \in \mathbb{Z}[3|n \vee 3|n+2 \vee 3|n+4]$

Claim: $\exists! a,b,c \in \mathbb{N}$[a,b,c are a prime triple]

Proof: By contradiction. We know that 3,5,7 is the first prime triple. By way of contradiction, suppose you can pick a,b,c such that they are a prime triple with $3<a<b<c$. Since a,b,c are primes $3 \nmid a,b,c$. We know that $b=a+2$ and $c=a+4$. By Exercise 5 we know that $\forall n \in \mathbb{Z}[3|n \vee 3|n+2 \vee 3|n+4]$. Set $n=a$. So $3|a$ or $3|a+2$ or $3|a+4$, i.e., $3|a$ or $3|b$ or $3|c$, i.e., at least one of a, b or c is not prime, contradicting the assumption. Therefore, the only prime triple is 3,5,7 proving the original claim. □

7. Prove that for any natural number n, $2+2^2+2^3+...+2^n=2^{n+1}-2$

Claim: $\forall n\in\mathbb{N}[2^1+2^2+...+2^n=2^{n+1}-2]$

Proof: By induction.

Base case: We know that $2^1=2^{1+1}-2$, i.e., $2=4-2$, i.e., $2=2$, which is true.

Induction step: Given an $n\in\mathbb{N}$ suppose $2^1+2^2+...+2^n=2^{n+1}-2$. Adding 2^{n+1} to both sides of the equality we get

$2^1+2^2+...+2^n+2^{n+1}=2^{n+1}+2^{n+1}-2$.

The LHS (left hand side) is already in the form of $n+1$. Now, by algebra in the RHS (right hand side):

$2^{n+1}+2^{n+1}-2$

$2(2^{n+1})-2$

$2^{(n+1)+1}-2$

making the RHS in the form of $n+1$. This concludes the induction step and proves the claim. □

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1 \text{ to } \infty}$ tends to limit L as $n \to \infty$, then for any fixed number M>0, the sequence $\{Ma_n\}_{n=1 \text{ to } \infty}$ tends to the limit ML.

Claim: Given $\{a_n\}_{n=1 \text{ to } \infty}$ and $L \in \mathbb{R}$

$(\forall \epsilon>0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n-L|<\epsilon] \Rightarrow (\forall M \in \mathbb{R}>0)(\forall \epsilon>0)(\exists N \in \mathbb{N})(\forall n \geq N)[|Ma_n-ML|<\epsilon]$

Proof: Suppose the antecedent of the claim. Given an $M \in \mathbb{R}>0$, by the assumption we know that we if we take an arbitrary $\epsilon>0$ there will be a $N \in \mathbb{N}$ such that for any given $n \geq N$ it will be the case that $|a_n-L|<\epsilon$. By algebra and since M>0:

$|a_n-L|<\epsilon$

$-\epsilon<a_n-L<\epsilon$

$-M\epsilon<Ma_n-ML<M\epsilon$ (remember M>0)

$-M\epsilon<-\epsilon< Ma_n-ML<\epsilon<M\epsilon$, hence,

$-\epsilon< Ma_n-ML<\epsilon$, thus,

$|Ma_n-ML|<\epsilon$

By the numbers given and chosen, and by the assumptions and reasoning, we can deduce the claim and gets proved. □

9. Given an infinite collection A_n, n=1,2,... of intervals of the real line, their intersection is defined to be

$\cap_{n=1 \text{ to } \infty}[A_n]=\{x|(\forall n)(x\in A_n)\}$

Give an example of a family of intervals A_n, n=1,2,..., such that $A_{n+1}\subset A_n$ for all n and $\cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$. Prove that your example has the stated property.

Claim: Let $A_n=(0,1/n)$

$\forall n\in\mathbb{N}[A_{n+1}\subset A_n] \wedge \cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$

Proof: We will separate the claim in two and make two proofs.

First claim: $\forall n\in\mathbb{N}[A_{n+1}\subset A_n]$, i.e.,

$\forall n\in\mathbb{N}\ [(0,1/(n+1))\subset(0,1/n)]$, i.e.

$\forall n\in\mathbb{N}\forall x\in\mathbb{R}[\ (0<x<1/(n+1)\Rightarrow 0<x<1/n) \wedge \neg(0<x<1/(n+1)\Leftrightarrow 0<x<1/n)\ ]$

First proof: Given and an $n\in\mathbb{N}$ and an $x\in\mathbb{R}$ suppose $0<x<1/(n+1)$. We know that $1/(n+1)<1/n$, so $0<x<1/(n+1)<1/n$, hence $0<x<1/n$. From this we deduce that $A_{n+1}\subseteq A$. Now, to prove the second term, suppose $0<x<1/n$. If $x=1/(n+0.5)$ then $0<x<1/n$, but $1/(n+1)<x$, so it is not the case that $0<x<1/(n+1)$, hence, A_{n+1} and A_n are not equal, so $A_{n+1}\subset A_n$ proving the first claim.

Second claim: $\cap_{n=1 \text{ to } \infty}[A_n]=\emptyset$, i.e.,

$\{x|(\forall n)(x\in A_n)\}= \emptyset$, i.e.,

$\neg\exists x\in\mathbb{R}\forall n\in\mathbb{N}[0<x<1/n]$, i.e.,

$\forall x\in\mathbb{R}\exists n\in\mathbb{N}[x\leq 0 \vee 1/n\leq x]$

Second proof: Given an $x\in\mathbb{R}$, if $x\leq 0$ pick any $n\in\mathbb{N}$ and the claim will be true. If $x>0$, pick an $n\in\mathbb{N}$ so large so that $1/n<x$, so, $1/n\leq x$, proving the second claim.

With both claims proved the original claim gets proved. □

10. Give an example of a family of intervals A_n, n=1,2,..., such that $A_{n+1}\subset A_n$ for all n and $\cap_{n=1 \text{ to } \infty}[A_n]$ consists of a single real number. Prove that your example has the stated property.

Claim: Let $A_n=[0,1/n]$

$\forall n\in\mathbb{N}[A_{n+1}\subset A_n] \wedge \cap_{n=1 \text{ to } \infty}[A_n]=\{0\}$

Proof: We will separate the claim in two and make two proofs.

First claim: $\forall n\in\mathbb{N}[A_{n+1}\subset A_n]$, i.e.,

$\forall n\in\mathbb{N}\ [[0,1/(n+1)]\subset[0,1/n]]$, i.e.

$\forall n\in\mathbb{N}\forall x\in\mathbb{R}[\ (0\leq x\leq 1/(n+1)\Rightarrow 0\leq x\leq 1/n) \wedge \neg(0\leq x\leq 1/(n+1)\Leftrightarrow 0\leq x\leq 1/n)\ ]$

First proof: Given and an $n\in\mathbb{N}$ and an $x\in\mathbb{R}$ suppose $0\leq x\leq 1/(n+1)$. We know that $1/(n+1) <1/n$, so $0\leq x\leq 1/(n+1)\leq 1/n$, hence $0\leq x\leq 1/n$. From this we deduce that $A_{n+1}\subseteq A$. Now, to prove the second term, suppose $0\leq x\leq 1/n$. If $x=1/(n+0.5)$ then $0\leq x\leq 1/n$, but $1/(n+1)<x$, so it is not the case that $0\leq x\leq 1/(n+1)$, hence, A_{n+1} and A_n are not equal, so $A_{n+1}\subset A_n$ proving the first claim.

Second claim: $\cap_{n=1 \text{ to } \infty}[A_n]=\{0\}$, i.e.,

$\{x|(\forall n)(x\in A_n)\}= \{0\}$, i.e.,

$\forall x\in\mathbb{R}[\forall n\in\mathbb{N}(0\leq x\leq 1/n)\Leftrightarrow x=0]$

Second proof: Take an arbitrary $x\in\mathbb{R}$.

[$\Rightarrow$] Suppose $\forall n\in\mathbb{N}(0\leq x\leq 1/n)$. By way of contradiction suppose $x\neq 0$. By the assumptions, clearly x must be greater than 0. If $x>0$, you can pick an $n\in\mathbb{N}$ so large that makes $1/n<x$, contradicting the assumption, proving the left-to-right part.

[$\Leftarrow$] Suppose $x=0$. Given any $n\in\mathbb{N}$, $1/n$ is always greater than 0, hence, $0\leq x\leq 1/n$, proving the right-to-left part.

With both claims proved the original claim gets proved. □