10. Give an example of a family of intervals $A_n$, n=1,2,..., such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1 \text{ to } \infty}[A_n]$ consists of a single real number. Prove that your example has the stated property.

**Claim:** Let $A_n=[0,1/n]$

$\forall n \in \mathbb{N}[A_{n+1} \subset A_n] \land \cap_{n=1 \text{ to } \infty}[A_n]=\{0\}$

**Proof:** We will separate the claim in two and make two proofs.

**First claim:** $\forall n \in \mathbb{N}[A_{n+1} \subset A_n]$, i.e.,

$\forall n \in \mathbb{N}\ [[0,1/(n+1)] \subset [0,1/n]]$, i.e.

$\forall n \in \mathbb{N} \forall x \in \mathbb{R}[\ (0 \le x \le 1/(n+1) \Rightarrow 0 \le x \le 1/n) \land \neg(0 \le x \le 1/(n+1) \Leftrightarrow 0 \le x \le 1/n)\ ]$

**First proof:** Given and an $n \in \mathbb{N}$ and an $x \in \mathbb{R}$ suppose $0 \le x \le 1/(n+1)$. We know that $1/(n+1) < 1/n$, so $0 \le x \le 1/(n+1) \le 1/n$, hence $0 \le x \le 1/n$. From this we deduce that $A_{n+1} \subseteq A$. Now, to prove the second term, suppose $0 \le x \le 1/n$. If $x=1/(n+0.5)$ then $0 \le x \le 1/n$, but $1/(n+1) < x$, so it is not the case that $0 \le x \le 1/(n+1)$, hence, $A_{n+1}$ and $A_n$ are not equal, so $A_{n+1} \subset A_n$ proving the first claim.

**Second claim:** $\cap_{n=1 \text{ to } \infty}[A_n]=\{0\}$, i.e.,

$\{x|(\forall n)(x \in A_n)\}= \{0\}$, i.e.,

$\forall x \in \mathbb{R}[\forall n \in \mathbb{N}(0 \le x \le 1/n) \Leftrightarrow x=0]$

**Second proof:** Take an arbitrary $x \in \mathbb{R}$.

[⇒] Suppose $\forall n \in \mathbb{N}(0 \le x \le 1/n)$. By way of contradiction suppose $x \neq 0$. By the assumptions, clearly x must be greater than 0. If $x>0$, you can pick an $n \in \mathbb{N}$ so large that makes $1/n<x$, contradicting the assumption, proving the left-to-right part.

[⇐] Suppose $x=0$. Given any $n \in \mathbb{N}$, $1/n$ is always greater than 0, hence, $0 \le x \le 1/n$, proving the right-to-left part.

With both claims proved the original claim gets proved. □