Task 1

We need these steps of linear transformation

1. Prove linearity $T(u+v) = T(u) + T(v)$ and $T(cu) = c(Tu)$
2. find $N(T)$, the null space.
3. find $R(T)$, the range.
4. Add $\dim N(T)$ and $\dim(R)$

[1]

$T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

① Linearly $T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1+y_1, x_2+y_2, x_3+y_3)$

$= (x_1+y_1-x_2-y_2, 2(x_3+y_3))$

$T(u)+T(v) = ((x_1-x_2), 2x_3) + ((y_1-y_2), 2y_3) \Rightarrow ((x_1-x_2)+(y_1-y_2), 2x_3+2y_3)$

$\therefore T(u+v) = T(u) + T(v)$

$T(cu) = T(cx_1, cx_2, cx_3) = (cx_1 - cx_2, 2(cx_3))$

$= (c(x_1 - x_2), c(2x_3)) = c((x_1-x_2), 2x_3)$

$\therefore T(cu) = cT(u)$

② $N(T)$

$T(a_1, a_2, a_3) = (0,0) \Rightarrow a_1 - a_2 = 0 \quad a_1 = a_2$

$a_3 = 0$

$\Rightarrow (a, a, 0) \Rightarrow N(T) = \{(a,a,0) : a \in R\} = \mathrm{span}\{(1,1,0)\}$

$\dim = N(T) = 1$ #

③ $R(T)$

$R(T) = \{T(x,y,z) = (x,y,z) \in R^3\}$

$e_1 = (1,0,0) \rightarrow T(e_1) = (1,0)$

$e_2 = (0,1,0) \rightarrow T(e_2) = (-1,0)$

$e_3 = (0,0,1) \rightarrow T(e_3) = (0,2)$

$\because$ $e_2 = -1(e_1) \rightarrow$ It can't provide other information.

We can use $(1,0)$ and $(0,2)$ generate two dim vector.

$\therefore$ $\dim R(T) = 2$

④

$T: V \rightarrow W$

$\dim(V) = \dim N(T) + \dim R(T) = 3$

$\because V = R^3$ $\therefore$ $= \dim 3$ #

⑤

one to one kenerl is $\{0\}$, The kernel is dimesion 1,

so T is not injective,

The range is all of R^2, so T is onto, #

[2] $T: R^6 \to R^4$ defined by $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

①

$T((x_1, \dots, x_6) + (y_1, \dots, y_6)) = T(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5, x_6+y_6)$

$(2(x_1+y_1) - (x_2+y_2), (x_3+y_3) + (x_2, y_2), 0, 0)$

$(2x_1 - x_2, x_2 + x_3, 0, 0) + (2y_1 - y_2, y_2 + y_3, 0, 0)$

$\Rightarrow T(x_1, \dots x_6) + T(y_1 \dots y_6)$

$\therefore T(u+v) = T(u) + T(v)$

$T(c(x_1, \dots x_6)) = T(cx_1, cx_2, cx_3, cx_4, cx_5, cx_6) = (2cx_1 - cx_2, cx_3 + cx_2, 0, 0)$

$= c(2cx_1 - x_2, x_3 + cx_2, 0, 0)$

$\therefore T(cu) = cT(u)$ #

② $N(T)$

$N(T) = \{a_1, \dots, a_6\} \in R^6 : T(a_1, a_2, a_3, a_4, a_5, a_6) = (0, 0, 0, 0)\}$

令 $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0) = (0, 0, 0, 0)$

$2a_1 - a_2 = 0, \quad a_2 = 2a_1$

$a_3 + a_2 = 0 \Rightarrow a_3 + 2a_1 = 0 \Rightarrow a_3 = -2a_1$

$\Rightarrow (a_1, 2a_1, -2a_1, a_4, a_5, a_6)$

$N(T) = \text{span}\{(1, 2, -2, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0),$

$(0, 0, 0, 0, 0, 1)\}$

$\dim(N(T)) = 4$

③ $R(T)$

$R(T) = \{T(a_1, a_2, a_3, a_4, a_5, a_6)\} \in R^4$

$\Rightarrow T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

$T(e_1) = T(1,0,0,0,0,0) = (2,0,0,0)$

$T(e_2) = T(0,1,0,0,0,0) = (-1,1,0,0)$

$T(e_3) = T(0,0,1,0,0,0) = (0,1,0,0)$

$T(e_4) = T(0,0,0,1,0,0) = (0,0,0,0)$

$T(e_5) = T(0,0,0,0,1,0) = (0,0,0,0)$

$T(e_6) = T(0,0,0,0,0,1) = (0,0,0,0)$

$T(e_2) = -\frac{1}{2}(2,0,0,0) + (0,1,0,0) \Rightarrow \text{span}\{T(e_1), T(e_3)\}$

$\therefore \dim(R(T)) = 2$

④ $\dim(R^6) = 6$, $\dim N(T) = 4$, $\dim R(T) = 2$

$\therefore 4 + 2 = 6$ #

⑤ one to one kernel is $\{0\}$, but this $\dim N(T) = 4$

not one to one #

but $\dim R(T) = 2 \rightarrow$ not to R^4

$\therefore$ not onto,

[3]

① $T: M_{1\times 2}(R) \to M_{1\times 3}(R)$ defined by $T([a_1, a_2]) = T([a_1+a_2, 0, 2a_1-a_2])$

$\Rightarrow M_{1\times 2} \Rightarrow R^2,\ M_{1\times 3} \Rightarrow R^3$

$T(x_1+y_1,\ x_2+y_2) = (x_1+y_1+x_2+y_2,\ 0,\ 2(x_1+y_1)-(x_2+y_2))$

$\to T(u+v)$

$\Rightarrow ((x_1+x_2)+(y_1+y_2),\ 0,\ (2x_1-x_2)+(2y_1-y_2))$

$T(x_1, x_2) = (x_1+x_2,\ 0,\ 2x_1-x_2)$

$T(y_1, y_2) = (y_1+y_2,\ 0,\ 2y_1-y_2)$ $+$ $\Rightarrow$ $(x_1+x_2)+(y_1+y_2),\ 0,\ (2x_1-x_2)+(2y_1-y_2)$

$\Rightarrow$ 相同

$c(x_1, x_2) = (cx_1, cx_2)$

$T(cx_1, cx_2) = (cx_1+cx_2,\ 0\ \ 2(cx_1)-cx_2)$

$\therefore T(cx_1, cx_2) = cT(x_1, x_2)$

② $N(T)$

$(a_1+a_2,\ 0,\ 2a_1-a_2) = (0,0,0) \Rightarrow$ $a_1+a_2=0$, $2a_1-a_2=0$ $\therefore a_1=a_2=0$

$N(T) = \{0\}$, $\dim N(T) = 0$ #

③ $(a_1+a_2,\ 0,\ 2a_1-a_2)$

R^3 構成一個 2維子空間 $\dim R(T)=2$

④ 定義 dim = 2, $0+2=2 \to$ 符合

⑤ one to one knerl 為 0, so match one to one.

Task 2

① $T(a_1,a_2)=(1,a_2) \Rightarrow$ have to mapping $(0,0)$

$T(0,0)=(1,0)\neq(0,0)$, so not linear.

② $T(a_1,a_2)=(a_1,a_1^2)$

$T(ca_1,ca_2)=cT(a_1,a_2)$

$T(ca_1,ca_2)=T(ca_1,(ca_1)^2)=T(ca_1,c^2a_1^2)$

$cT(a_1,a_2)=c(a_1,(a_1)^2)=ca_1,ca_1^2$

$\Rightarrow$ only $c=a_1$, so not linear.

③ $T(a_1,a_2)=(|a_1|,a_2)$

$T(-1,0)=(1,0)$ but $-T(1,0)=(-1,0)$

$T(-1,0)\neq -T(1,0)$ not linear.

Task 3

$T(1,0) = (1,4)\ ,\ T(1,1) = (2,5)$

$T(0,1) = (1,1) \Rightarrow T(x,y) = xT(1,0) + yT(0,1)$

$$T(2,3) = 2T(1,0) + 3T(0,1) = 2(1,4) + 3(1,1)$$
$$= (5,11)$$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x+y\ ,\ 4x+y)$ and $T(x,y) = (0,0)$

$\begin{cases} x+y = 0 \\ 4x+y = 0 \end{cases}$ $\quad x = -y$ $\quad x = 0,\ y = 0$ $\quad \Rightarrow N(T) = \{(0,0)\}$

T is one to one, #

Task 4

$T(1,2,1) = (1,1)$

$T(3,6,3) = (2,1) \Rightarrow 3T(1,2,1)$

$3(1,1) = (3,3) \neq (2,1)$

So, it's not linear transformation.

Task 5.

$N(T) = \{(x,y) \in R^2 : T(x,y) = (0,0)\}$

To any (x,y), $T(x,y)$ outcome have to $(x,0)$

$\Rightarrow X = 0$, y no limit

$\Rightarrow N(T) = \{(0,y) : y \in R\}$

$T(x,y) = (0,0) \Rightarrow (x,0) = (0,0)$

$x = 0$

#