Task 1

① $T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$R^3 \to e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$R^2 \to e_1' = (1,0)$, $e_2' = (0,1)$

$T(e_1) = (1-0, 2\cdot 0) = (1,0)$

$T(e_2) = (0-1, 2\cdot 0) = (-1,0)$

$T(e_3) = (0-0, 2\cdot 1) = (0,2)$

$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ #

② $T: R^6 \to R^4$ defined by $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

$R^6 \to e_1 = (1,0,0,0,0,0)$,,,,, , $e_6 = (0,0,0,0,0,1)$

$R^4 \to e_1' = (1,0,0,0)$, $e_2' = (0,1,0,0)$, $e_3' = (0,0,1,0)$, $e_4' = (0,0,0,1)$

$T(e_1) = (2,0,0,0)$

$T(e_2) = (-1,1,0,0)$

$T(e_3) = (0,1,0,0)$ $\Rightarrow$ $[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ #

$T(e_4) = (0,0,0,0)$

$T(e_5) = (0,0,0,0)$

$T(e_6) = (0,0,0,0)$

③ $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

$R^2 \to e_1(1,0)$, $e_2 = (0,1)$

$R^3 \to e'_1(1,0,0)$, $e'_2 = (0,1,0)$, $e'_3 = (0,0,1)$

$T(e_1) = (2,3,1)$

$T(e_2) = (-1,4,0)$

$$\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$ #

④ $T: R^n \to R^n$ defined by $T(a_1, \dots, a_n) = (a_n, a_{n-1}, \dots, a_1)$

$T(e_1) = (0, \dots, 0, 1)$

$T(e_2) = (0, \dots, 1, 0)$

$\therefore$ The matrix $[T]_\beta^\gamma$ = $n \times n$ flip matrix with 1 on the anti-diagonal and 0 elsewhere.

like $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ #

Task 2

$R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

Let β be the standard ordered bases for R^2 and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$,

Compute $[T]_\beta^\gamma$, if $\alpha = \{(1,2),(2,3)\}$, compute $[T]_\alpha^\gamma$

$\beta = \{(1,0),(0,1)\}$ $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$

$T(e_1) = (1,1,2)$

$T(e_2) = (-1,0,1)$

$C_1(1,1,0) + C_2(0,1,1) + C_3(2,2,3) = (x,y,z)$

$$\begin{cases} C_1 + 0 + 2C_3 = 1 \\ C_1 + C_2 + 2C_3 = 1 \\ 0 + C_2 + 3C_3 = 2 \end{cases} \qquad \begin{aligned} C_2 &= 0 \\ C_3 &= \frac{2}{3} \\ C_1 &= -\frac{1}{3} \end{aligned} \qquad [T(e_1)]_\gamma = \begin{pmatrix} -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}$$

$C_1(1,1,0) + C_2(0,1,1) + C_3(2,2,3) = (x,y,z)$

$$\begin{cases} C_1 + 0 + 2C_3 = -1 \\ C_1 + C_2 + 2C_3 = 0 \\ 0 + C_2 + 3C_3 = 1 \end{cases} \qquad \begin{cases} C_2 = 1 \\ C_3 = 0 \\ C_1 = -1 \end{cases} \qquad [T(e_2)]_\gamma = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[T]_\beta^\gamma = \begin{bmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$$

$\alpha = \{(1,2),(2,3)\}$ 求 $[T]^{\gamma}_{\alpha}$

$e_1=(1,2)$, $e_2=(2,3)$

$T(e_1) = (-1,1,4)$

$T(e_2) = (-1,2,7)$

$c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3) = (-1,1,4)$

$$\begin{cases} c_1 + 0 + 2c_3 = -1 \\ c_1 + c_2 + 2c_3 = 1 \\ 0 + c_2 + 3c_3 = 4 \end{cases} \qquad c_2 = 2,\quad c_3 = \frac{2}{3},\quad c_1 = -\frac{7}{3}$$

$$[T(1,2)]_{\gamma} = \begin{pmatrix} -\frac{7}{3} \\ 2 \\ \frac{2}{3} \end{pmatrix}$$

$$\begin{cases} c_1 + 0 + 2c_3 = -1 \\ c_1 + c_2 + 2c_3 = 2 \\ 0 + c_2 + 3c_3 = 7 \end{cases} \qquad c_2 = 3,\quad c_3 = \frac{4}{3},\quad c_1 = -\frac{11}{3}$$

$$[T(2,3)]_{\gamma} = \begin{pmatrix} -\frac{11}{3} \\ 3 \\ \frac{4}{3} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

#

Task 3

$$tr(AB) = \sum_{i}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji}$$

$$\sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ij} = \sum_{j=1}^{n} (BA)_{jj} = tr(BA) \quad \#$$

example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} = 50 + 19 = 69$$

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \qquad 23 + 46 = 69$$

$$\therefore tr(AB) = tr(BA) \quad \#$$

Task 4

①

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$$E_{11} : T(E_{11}) = E_{11}^t = E_{11}$$

$$E_{12} : T(E_{12}) = E_{12}^t = E_{21} \quad \Rightarrow \quad [T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{21} : T(E_{21}) = E_{21}^t = E_{12}$$

$$E_{22} : T(E_{22}) = E_{22}^t = E_{22}$$

$$[T(A)]_\alpha, \quad A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \quad \rightarrow \quad T(A) = A^t = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow 1 \cdot E_{11} + (-1) E_{12} + 4 \cdot E_{21} + 2 E_{22}$$

$$[TA]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix} \#$$

(2)

Define $T: P_2(R) \to R$ by $T(f(x)) = f(2)$

Compute $[T]_\beta^\gamma$ and $[T(f(x))]_\beta^\gamma$, where $f(x) = 4x^2 - 2x + 1$

$\beta = \{1, x, x^2\}$ $\quad \gamma = \{1\}$

$\dim P_2(R) = 3 \quad \dim(R) = 1$

$T(1) = 1$

$T(x) = 2$ $\quad [T]_\beta^\gamma = (1, 2, 4)$ #

$T(x^2) = 4$

$[T(f(x))]_\beta^\gamma$ for $f(x) = 4x^2 - 2x + 1$

$T(f(x)) = f(2)$

$4 \cdot 2^2 - 2 \cdot 2 + 1 = 13$ #