

① $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$$\mathbb{R}^3 \rightarrow e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$$

$$\mathbb{R}^2 \rightarrow e'_1 = (1, 0), e'_2 = (0, 1)$$

$$T(e_1) = (1 - 0, 2 \cdot 0) = (1, 0)$$

$$T(e_2) = (0 - 1, 2 \cdot 0) = (-1, 0)$$

$$T(e_3) = (0 - 0, 2 \cdot 1) = (0, 2)$$

$$\therefore [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{\#}$$

② $T: \mathbb{R}^6 \rightarrow \mathbb{R}^4$ defined by $T(a_1, a_2, a_3, a_4, a_5, a_6) = (2a_1 - a_2, a_3 + a_2, 0, 0)$

$$\mathbb{R}^6 \rightarrow e_1 = (1, 0, 0, 0, 0, 0), \dots, e_6 = (0, 0, 0, 0, 0, 1)$$

$$\mathbb{R}^4 \rightarrow e'_1 = (1, 0, 0, 0), e'_2 = (0, 1, 0, 0), e'_3 = (0, 0, 1, 0), e'_4 = (0, 0, 0, 1)$$

$$T(e_1) = (2, 0, 0, 0)$$

$$T(e_2) = (-1, 1, 0, 0)$$

$$T(e_3) = (0, 1, 0, 0)$$

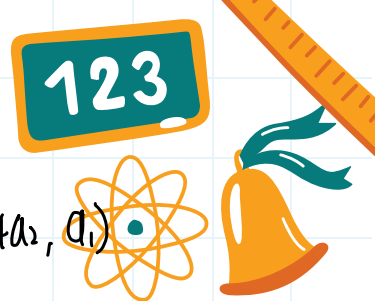
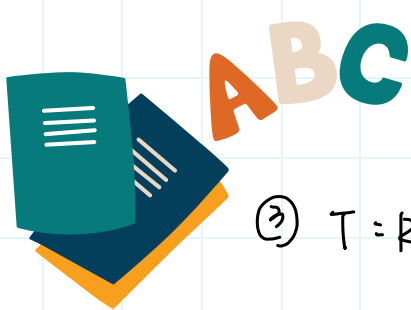
$$T(e_4) = (0, 0, 0, 0)$$

$$T(e_5) = (0, 0, 0, 0)$$

$$T(e_6) = (0, 0, 0, 0)$$

$$\Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\#}$$





③ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

$$\mathbb{R}^2 \rightarrow e_1(1, 0), e_2 = (0, 1)$$

$$\mathbb{R}^3 \rightarrow e'_1(1, 0, 0), e'_2 = (0, 1, 0), e'_3 = (0, 0, 1)$$

$$\begin{aligned} T(e_1) &= (2, 3, 1) \\ T(e_2) &= (-1, 4, 0) \end{aligned} \Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} \#$$

④ $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $T(a_1, \dots, a_n) = (a_n, a_{n-1}, \dots, a_1)$

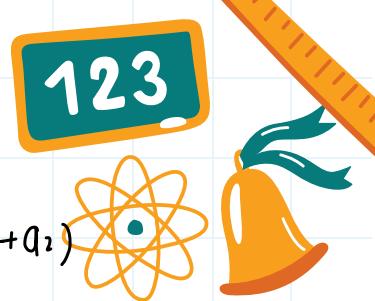
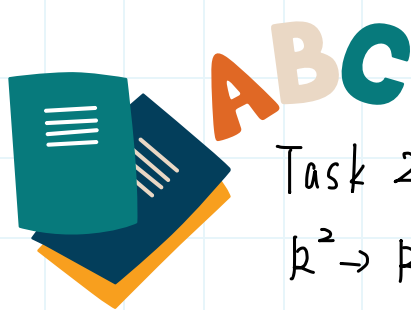
$$T(e_1) = (0, \dots, 0, 1)$$

$$T(e_2) = (0, \dots, 1, 0)$$

\therefore The matrix $[T]_{\beta}^{\gamma} = n \times n$ flip matrix with 1 on the anti-diagonal and 0 elsewhere.

$$\text{like } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \#$$





Task 2

$\mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

Let β be the standard ordered bases for \mathbb{R}^2 and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$
Compute $[T]_{\beta}^{\gamma}$, if $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_{\alpha}^{\gamma}$

$$\beta = \{(1, 0), (0, 1)\} \quad \gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$$

$$T(e_1) = (1, 1, 2)$$

$$T(e_2) = (-1, 0, 1)$$

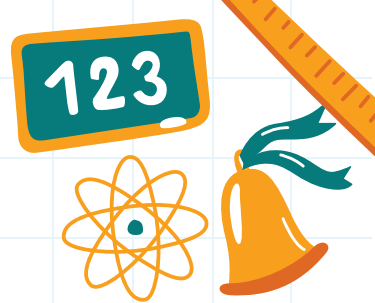
$$c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(2, 2, 3) = (x, y, z)$$

$$\begin{cases} c_1 + 0 + 2c_3 = 1 \\ c_1 + c_2 + 2c_3 = 1 \\ 0 + c_2 + 3c_3 = 2 \end{cases} \quad \begin{matrix} c_2 = 0 \\ c_3 = \frac{2}{3} \\ c_1 = -\frac{1}{3} \end{matrix} \quad [T(e_1)]_{\gamma} = \begin{pmatrix} -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}$$

$$c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(2, 2, 3) = (x, y, z)$$

$$\begin{cases} c_1 + 0 + 2c_3 = -1 \\ c_1 + c_2 + 2c_3 = 0 \\ 0 + c_2 + 3c_3 = 1 \end{cases} \quad \begin{matrix} c_2 = 1 \\ c_3 = 0 \\ c_1 = -1 \end{matrix} \quad [T(e_2)]_{\gamma} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
$$[T]_{\beta}^{\gamma} = \begin{bmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$$





$$\alpha = \{(1, 2), (2, 3)\} \text{ 求 } [T]_{\alpha}^r$$

$$e_1 = (1, 2), e_2 = (2, 3)$$

$$T(e_1) = (-1, 1, 4)$$

$$T(e_2) = (-1, 2, 7)$$

$$C_1(1, 1, 0) + C_2(0, 1, 1) + C_3(2, 2, 3) = (-1, 1, 4)$$

$$\begin{cases} C_1 + 0 + 2C_3 = -1 \\ C_1 + C_2 + 2C_3 = 1 \\ 0 + C_2 + 3C_3 = 4 \end{cases}$$

$$C_2 = 2$$

$$C_3 = \frac{2}{3}$$

$$C_1 = -\frac{7}{3}$$

$$[T(1, 2)]_r = \begin{pmatrix} -\frac{7}{3} \\ 2 \\ \frac{2}{3} \end{pmatrix}$$

$$\begin{cases} C_1 + 0 + 2C_3 = -1 \\ C_1 + C_2 + 2C_3 = 2 \\ 0 + C_2 + 3C_3 = 7 \end{cases}$$

$$C_2 = 3$$

$$C_3 = \frac{4}{3}$$

$$C_1 = -\frac{11}{3}$$

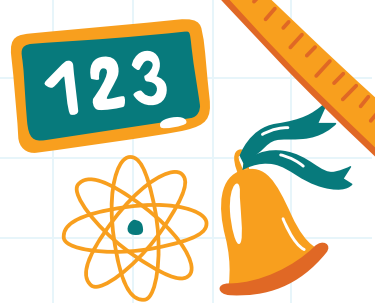
$$[T(2, 3)]_r = \begin{pmatrix} -\frac{11}{3} \\ 3 \\ \frac{4}{3} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \#$$





Task 3



$$\text{tr}(AB) = \sum_i^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ji}$$

$$\sum_{i=1}^n \sum_{j=1}^n B_{ji} A_{ij} = \sum_{j=1}^n (BA)_{jj} = \text{tr}(BA) \quad \#$$

example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

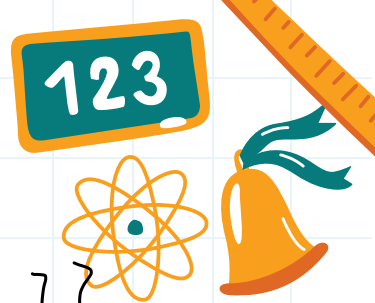
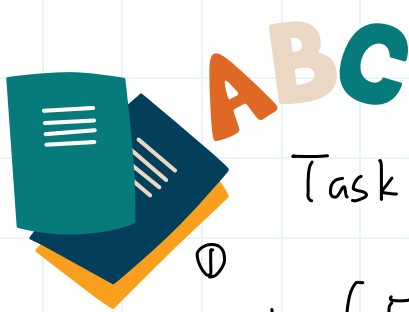
$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \Rightarrow 50 + 19 = 69$$

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$23 + 46 = 69$$

$$\therefore \text{tr}(AB) = \text{tr}(BA) \quad \#$$





Task 4

①

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$$E_{11} = T(E_{11}) = E_{11}^t = E_{11}$$

$$E_{12} = T(E_{12}) = E_{12}^t = E_{21} \Rightarrow [T]_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{21} = T(E_{21}) = E_{21}^t = E_{12}$$

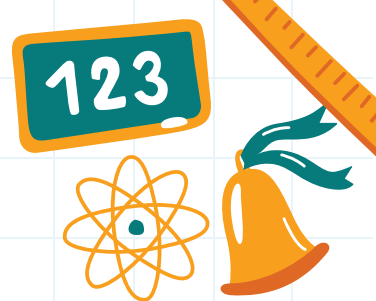
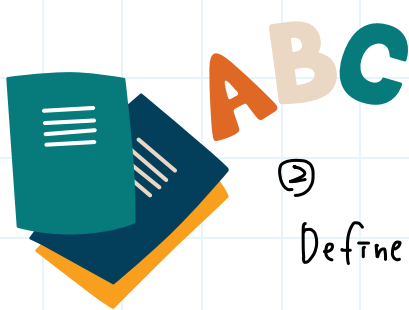
$$E_{22} = T(E_{22}) = E_{22}^t = E_{22}$$

$$[T(A)]_{\alpha}, \quad A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow T(A) = A^t = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow 1 \cdot E_{11} + (-1) E_{12} + 4 \cdot E_{21} + 2 E_{22}$$

$$[TA]_{\alpha} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}_{\#}$$





②

Define $T: P_2(\mathbb{R}) \rightarrow \mathbb{R}$ by $T(f(x)) = f(2)$

Compute $[T]_{\beta}^{\gamma}$ and $[T(f(x))]_{\beta}^{\gamma}$, where $f(x) = 4x^2 - 2x + 1$

$$\beta = \{1, x, x^2\} \quad \gamma = \{1\}$$

$$\dim P_2(\mathbb{R}) = 3 \quad \dim(\mathbb{R}) = 1$$

$$T(1) = 1$$

$$T(x) = 2$$

$$T(x^2) = 4$$

$$[T]_{\beta}^{\gamma} = (1, 2, 4) \quad \#$$

$$[T(f(x))]_{\beta}^{\gamma} \text{ for } f(x) = 4x^2 - 2x + 1$$

$$T(f(x)) = f(2)$$

$$4 \cdot 2^2 - 2 \cdot 2 + 1 = 13 \quad \#$$

