Task 1

determine whether T is invertible

① $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

If T is invertible must satisfy injective and surjective.

But the dim is two, can't surjective to R^3

$\therefore$ T is not invertible.

② $T: R^3 \to R^3$

$T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$

$$\begin{cases} 3a_1 - 2a_3 = 0 \\ a_2 = 0 \\ 3a_1 + 4a_2 = 0 \end{cases} \qquad \begin{matrix} a_2 = 0 \\ a_1 = 0 \\ a_3 = 0 \end{matrix}$$

$\because$ $\dim(V) = \dim(W) = 3$

injective $\to$ surjective

$\therefore$ T is invertible.

③ $T: M_{2\times2}(R) \to P_2(R)$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + 2bx + (c+d)x^2$

$\because$ $M_{2\times2}(R) \to \dim$ is 4, $P_2(R)$ dim is 3

$\therefore$ T is not invertible.

Task 2

① R^3 and $P_3(R)$

$R^3 \rightarrow$ dim is 3,

$P_3(R) \rightarrow$ dim is 4 $\ni \{1, x, x^2, x^3\}$

$\because$ the dimensions are different, so they can't isomorphic,

② R^4 and $P_3(R)$

$R^4 \rightarrow$ dim is 4,

$P_3(R) \rightarrow$ dim is 4,

In finite-dimensional vector spaces, if two spaces have the same dimension, the isomorphism exits. #

③ $V = \{A \in M_{2\times2}(R) : tr(A) = 0\}$ vs R^3

$M_{2\times2}(R) \ni$ dim is 4,

$\because$ $tr(A) = 0 \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{matrix} a+d=0 \\ d=-a \end{matrix}$, so we can know the dim is 3,

$R^3 \rightarrow$ dim 3

The two space have the same dimension

$\therefore$ They are isomorphic #

Task 3

Let A be an $n\times n$ matrix and 0 means zero matrix.

① Suppose that $A^2=0$. Prove that A is not invertible.

Suppose A were invertible, From $A^2=0$

$A^{-1}\cdot A^2 = 0\cdot A^{-1} = A = 0$

$\rightarrow$ A is zero matrix

$\therefore$ A can't be invertible.

②

Suppose A is invertible, From $AB=0$

$AB=0 \Rightarrow B = A^{-1}(AB) = A^{-1}\cdot 0 = 0$

$\therefore B=0$

so A can't be invertible.

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Task 4

Let V and W be n-dimensional vector spaces, and let $T: V \to W$ be a linear transformation. Suppose that β is basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

When $V = R^2$, $W = R^2$, In V, $\beta = \{(1,0)\ (0,1)\}$

In linear transformation: $T: R^2 \to R^2$

$T(1,0) = (2,1)$, $T(0,1) = (1,2)$

$\Rightarrow$ T is invertible $\Rightarrow \{T(1,0), T(0,1)\}$ is R^2 basis.

$T(\beta) = \{(2,1), (1,2)\}$. For it to be a basis of R^2, it must be both linearly independent and span R^2.

$$\begin{cases} 2\alpha + \beta = 0 \\ 1\alpha + 2\beta = 0 \end{cases} \quad \alpha = 0,\ \beta = 0 \quad \Rightarrow \text{linearly independent.}$$

$\therefore \{(2,1), (1,2)\}$ is basis of R^2.

Injective: $T(1,0)$ and $T(0,1)$ can't combine to zero except with zero cofficients, thus $\ker(T) = \{0\}$

Surjective: Because $T(\beta)$ spans R^2 for any $y \in R^2$ and $x \in R^2$

so $T(x) = y$

An injective and surjective map from R^2 to R^2 bijection, thus invertible.

T is invertible $\Longleftrightarrow$ $T(\beta)$ is a basis for W.