priority queue & heaps

data structures
fall 2023

Priority Queue

- queue: FIFO
  - a company call center
  - a new call is added to back of queue
  - calls are taken from the front of the queue
- Sometimes, FIFO is not enough.
  - Air-traffic control center
  - Can’t use FIFO as is, you need to add some factors in:
    - plane’s distance, time spent waiting, amount of remaining fuel
- There are other priorities.

Priority Queue

- Priority queue:
  - Collection of prioritized elements that allows arbitrary element insertion
  - Allows the removal of the element that has first priority
- When an element is added to priority queue, user designated its priority by providing a **key**.
  - The element with minimal key will be the next to be removed.
  - (If 1 is more important than 2)

Priority Queue ADT

- insert(k,v)
  - Creates an entry with key **k** and value **v**
- min()
  - Returns (does not remove) a priority queue entry (k,v) having minimal key.
  - If the queue is empty, returns null.
- removeMin()
  - Removes and *returns* an entry (k,v) having minimal key.
  - Returns null if empty.
- size()
  - Returns the number of entries in the queue
- isEmpty()
  - Returns a boolean indicating whether the queue is empty or not.

Priority Queue

- It is possible for a PQ to have multiple entries with the same key.
  - In that case, *min* and *removeMin* will choose arbitrarily among those.

Method	Return Value	Priority Queue Contents
insert(5,A)		{ (5,A) }
insert(9,C)		{ (5,A), (9,C) }
insert(3,B)		{ (3,B), (5,A), (9,C) }
min( )	(3,B)	{ (3,B), (5,A), (9,C) }
removeMin( )	(3,B)	{ (5,A), (9,C) }
insert(7,D)		{ (5,A), (7,D), (9,C) }
removeMin( )	(5,A)	{ (7,D), (9,C) }
removeMin( )	(7,D)	{ (9,C) }
removeMin( )	(9,C)	{ }
removeMin( )	null	{ }
isEmpty( )	true	{ }

Complexities

- For an **unordered** and **ordered** priority queue:

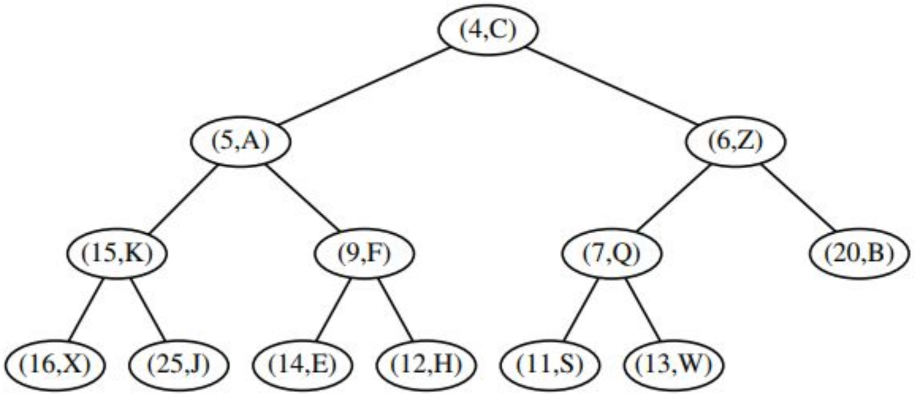
Method	Unsorted List	Sorted List
size	$O(1)$	$O(1)$
isEmpty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min	$O(n)$	$O(1)$
removeMin	$O(n)$	$O(1)$

Heaps

- A way to implement a priority queue.
  - Binary heap
- Allows to perform insertions and removals in logarithmic time
  - An improvement over list-based implementations of priority queue
- It uses a binary tree structure to find a compromise between elements being entirely unsorted and perfectly sorted.
- Has nothing to do with the **memory heap**
  - A dynamically allocated memory that *processes* use to store data like arrays, strings, objects and dynamically allocated data structures.

Heap

- A binary tree **T** stores entries at its positions and satisfies two additional properties:
    - A relational property in terms of the way keys are stored in T
    - A structural property defined in terms of the shape of T.



Relational property

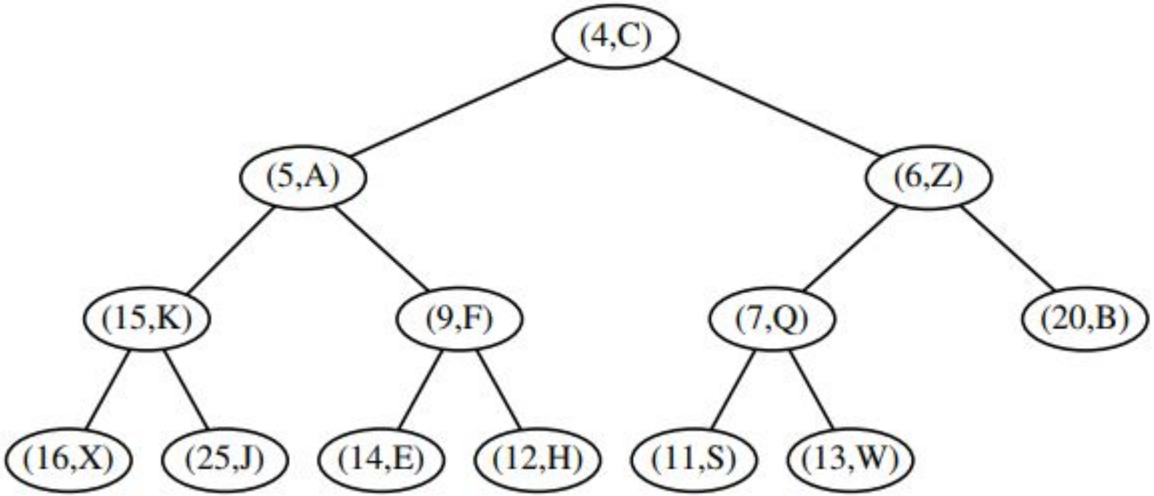
- Heap-order property
  - In a heap T, for every position *p* other than the root, the key stored at *p* is greater than or equal to the key stored at *p*’s parent.
- So, keys encountered on a path from the root to a leaf of T are in **nondecreasing** order.
  - Also, a minimal key is always stored at the root of T.
  - Makes it easy to locate such an entry when *min* or *removeMin* is called.
    - Top of the heap

Structural property

- For the sake of efficiency, we want the heap to have a small height as possible.
- This is enforced by the following requirement:
  - Heap T must be **complete**
- **Complete binary tree property**
  - A heap **T** with height *h* is a **complete** binary tree, if levels 0,1,2,...,h-1 of **T** have the maximal number of nodes possible and the remaining nodes at level *h* reside in the leftmost possible positions at that level.

Completeness

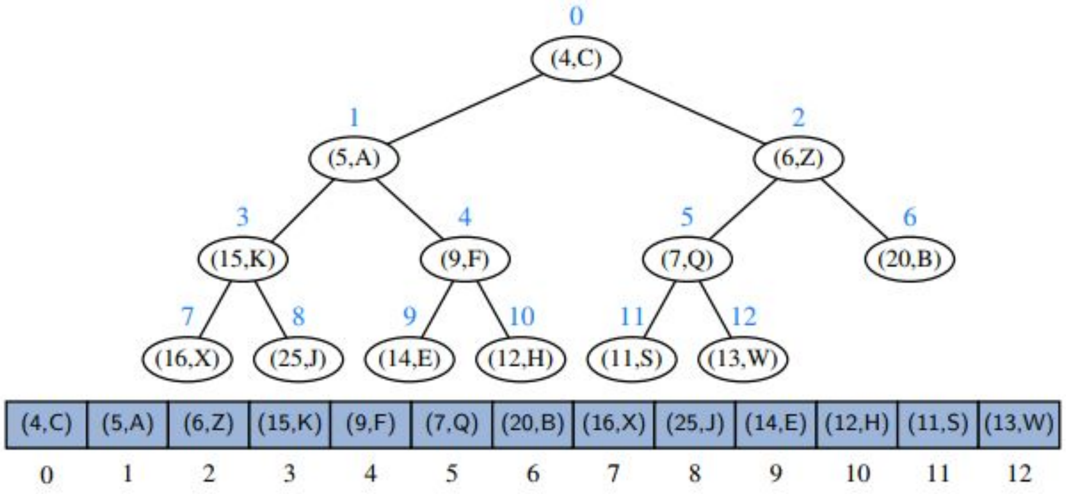
- This below is complete, because levels 0, 1, and 2 are full and six nodes in level 3 are in the **six leftmost possible** positions at that level.



Binary trees

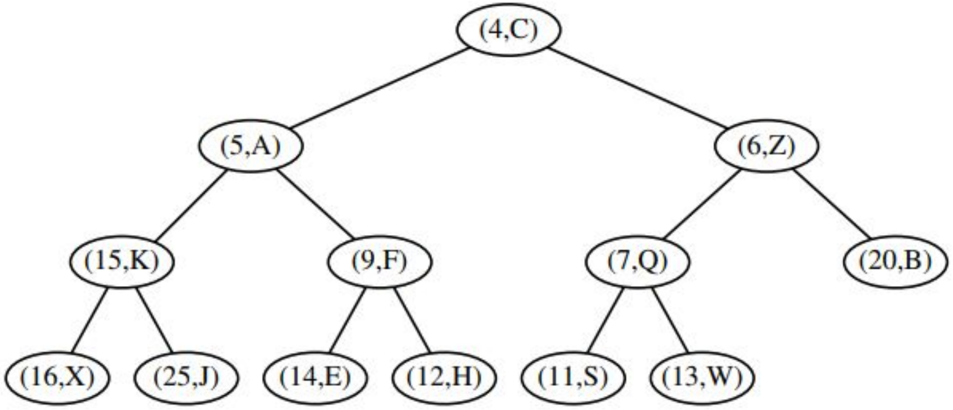
Array-based representation of a complete binary tree

- If *p* is the root, then $f(p)=0$
- If *p* is the left child of pos *q*, then $f(p)=2f(q)+1$
- If *p* is the right child of pos *q*, then $f(p)=2f(q)+2$



min heap & max heap

- The key present at the root node must be less than or equal among the keys present at all of its children.
  - Same property must be recursively true for all sub-trees.
- Inverse is true for **max** heap.



min heap & max heap

- Choosing between these two heaps depend on what you choose as most important and worst
  - For example if you say 0 is the most important (highest priority): use a **min-heap**
  - If you say 0 is the least important, use a **max-heap**.

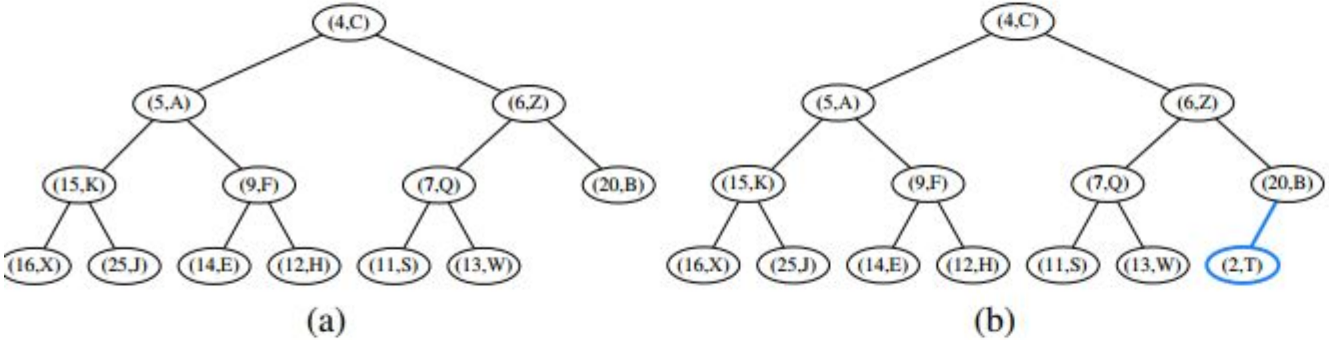
priority queue as a heap

- insert(k,v)
    - We store the pair as an entry at a new node of the tree.
    - We need to maintain the **complete binary tree property**
    - Therefore, new node should be placed at position *p* just beyond the rightmost node at the bottom level of the tree
        - Or as the leftmost position of a new level if the bottom level is already full (or the heap is empty)
    - After adding, the tree may be complete but it is not enough.
        - We also need to satisfy **heap-order property**

up-heap bubbling (bubble-up)

- Occurs when a new element is added or when an element's priority is increased.
- The new or updated element may violate the heap property.
  - It can break the min-heap, max-heap property.
- So, we compare the element to its parent node.
  - If it violates the property, we swap it with the parent.
  - This continues until element is in a position where the heap property is no longer violated.
- This is used when inserting a new element into heap or when an existing element's priority has been increased (for a max heap) or decreased (for a min heap)

Adding a new entry with key 2 to min heap



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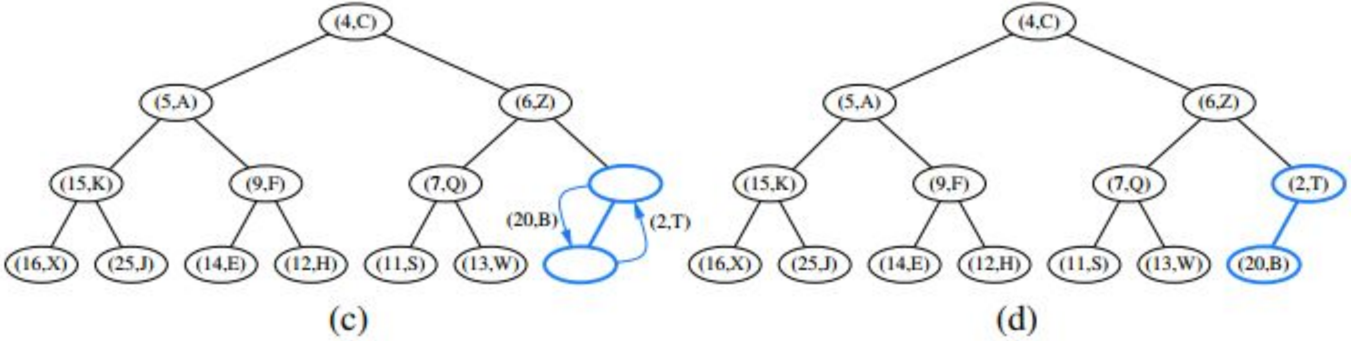

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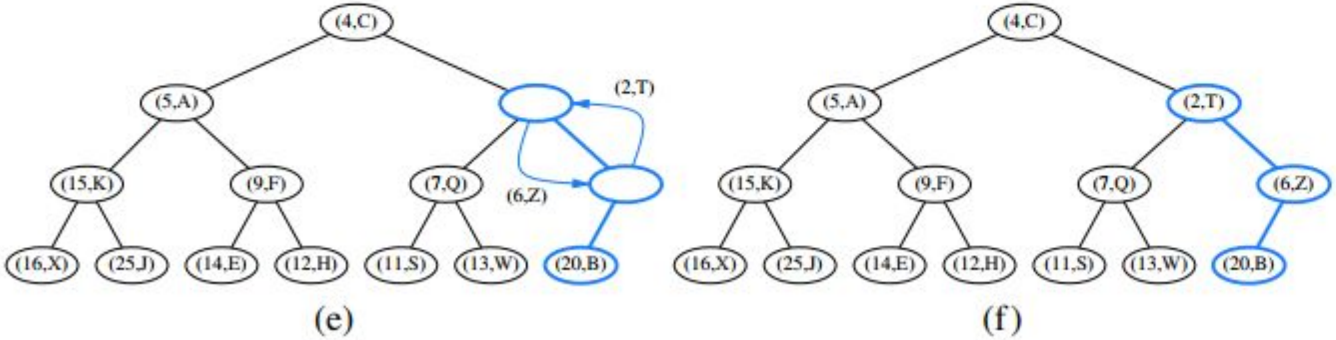

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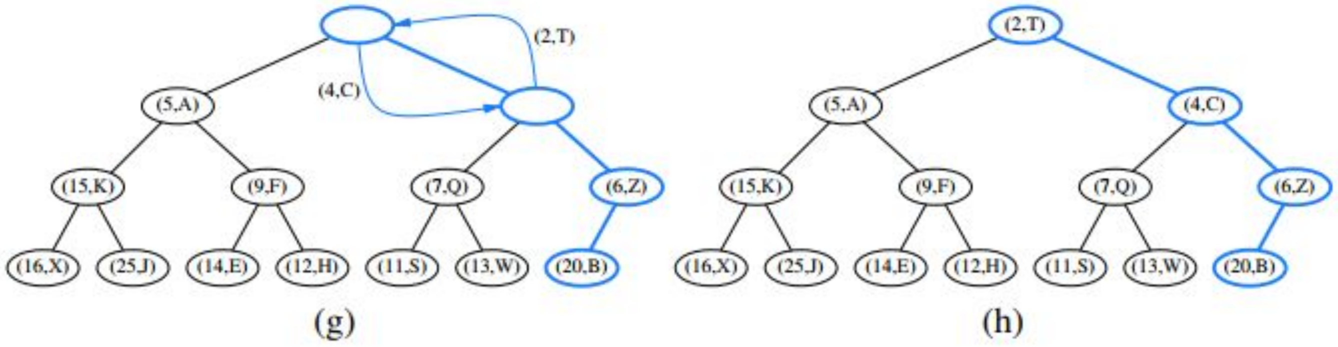
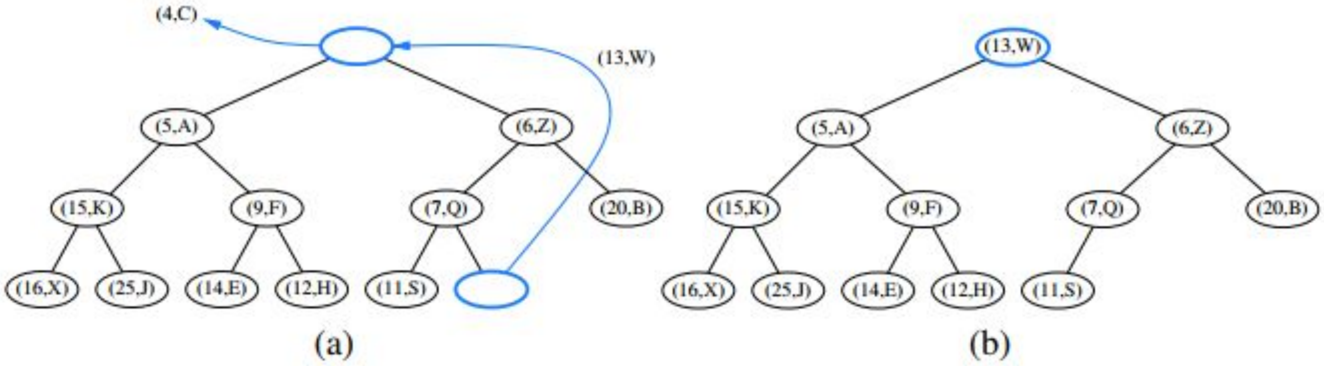


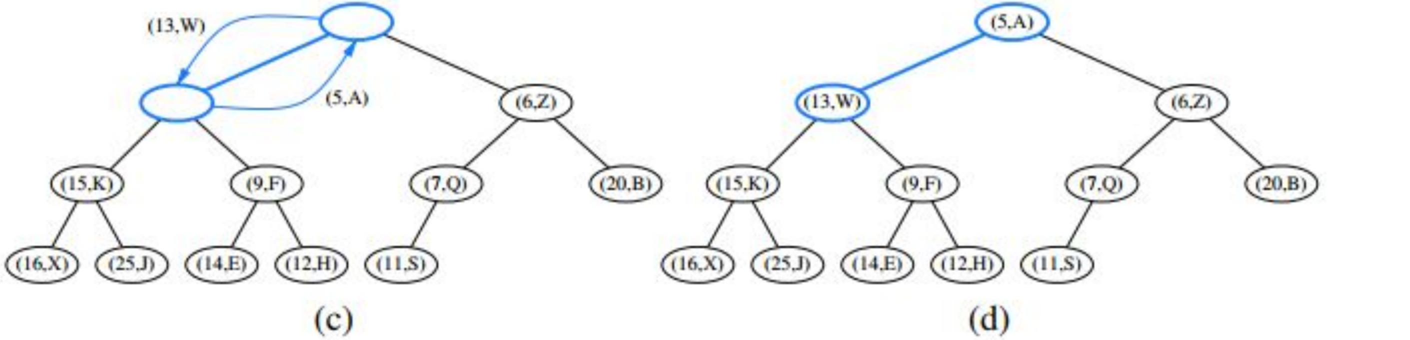
down-heap bubbling

- Used when an element is removed from the heap or an elements priority is decreased.
  - In a min-heap, this typically involves the root element being removed.
  - In max-heap, the largest element.
- After the root is removed, the last element in the heap is moved to the root position.
  - This is compared to its children.
  - If it violates the heap property, it is swapped with one of its children.
  - This is repeated until the heap property is restored.

Removing the entry with smallest key



Removing the entry with smallest key



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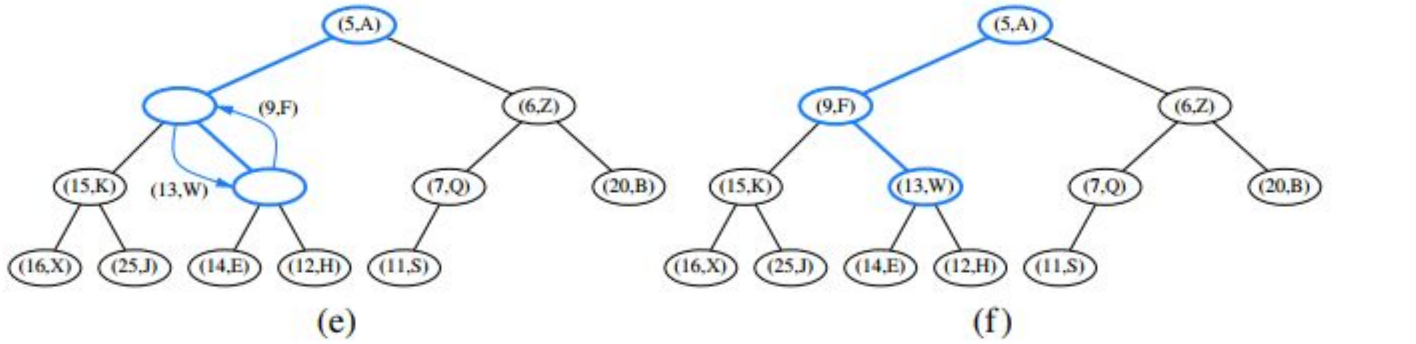
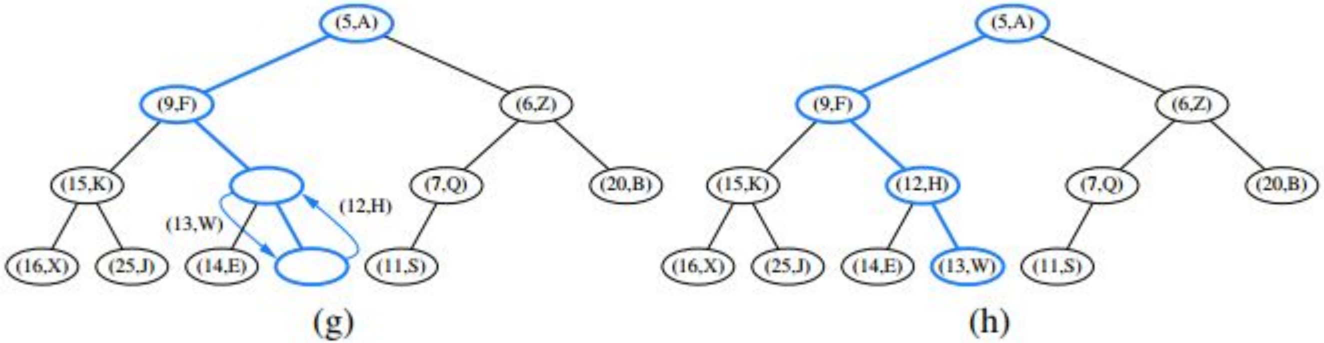
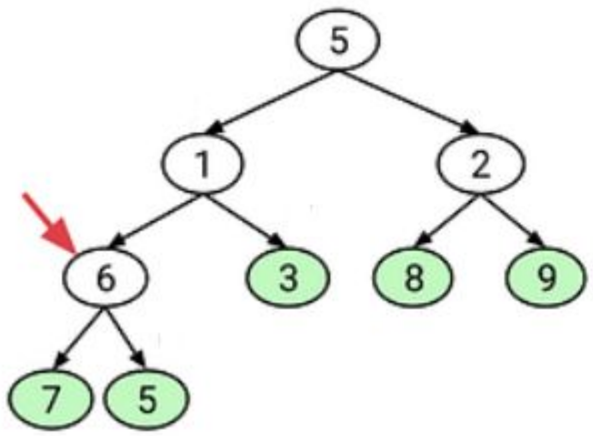


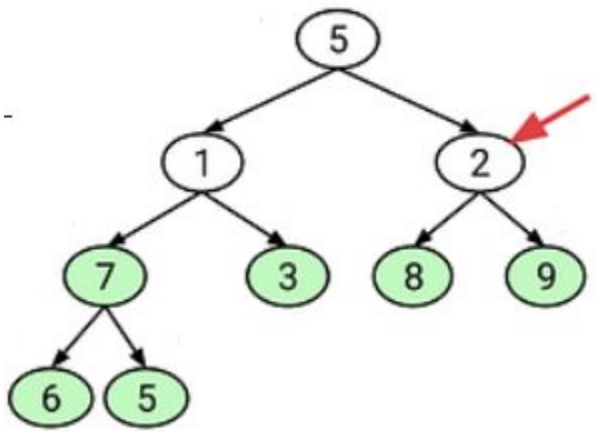
Removing the entry with smallest key



Creating a heap with bottom up method

- Create a *max heap* with bottom up method:
- {5,1,2,6,3,8,9,7,5}
- Lets go step by step





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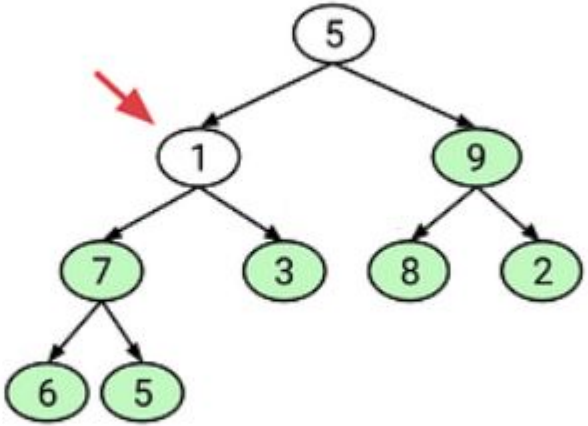
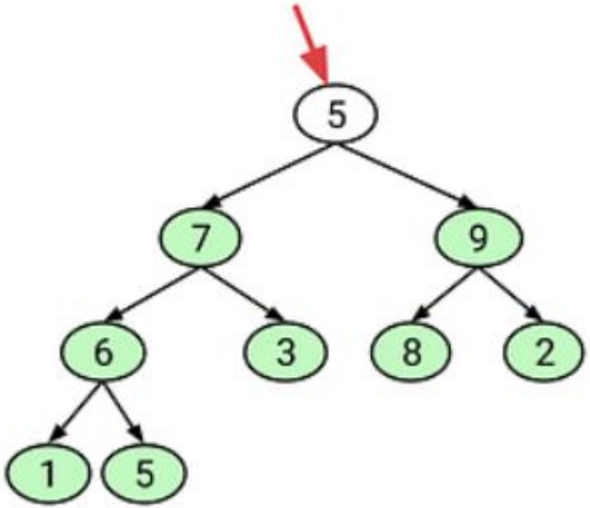
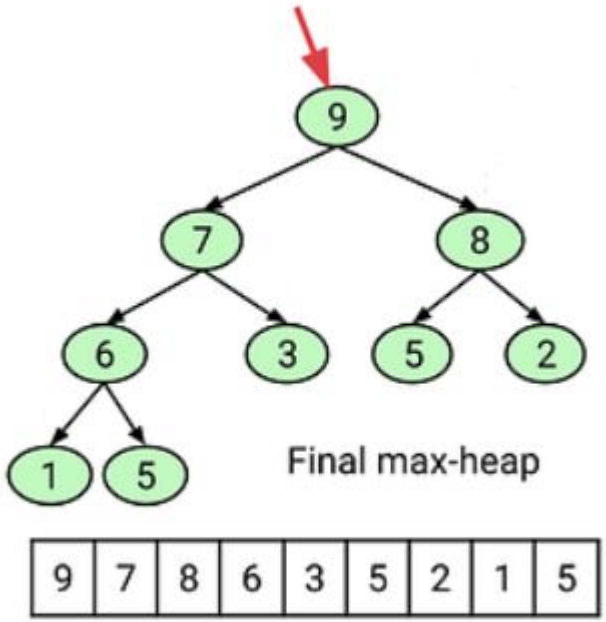




Final



I expect you to know...

- Which bubbling is used when an element's priority is decreased?
- Which bubbling is used when an element is inserted to a priority queue?
- Given a tree, can you say whether it is min-heap or max-heap?
- Given a tree, write it in array form.
- Removing a tree from a heap tree
- Adding an element to a heap tree
- Create a min-heap or max-heap from a given unsorted array